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Dynamic Properties of the Nash Equilibrium

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DYNAMIC PROPERTIES OF THE NASH EQUILIBRIUM

ABSTRACT

In this paper the authors examine the games with well-defined reaction functions. The focus is on the stability property of the Nash equilibria, i.e. the convergency in the strategy profile space to a Nash equilibrium when, beginning with some initial strategy choices in a neighborhood, players take turn to make improvements. Some interesting propositions on the dynamic properties have been established, which offer a kind of explanation as to why in general the outcomes of games and the economic dynamic process can be rather diversified.

Keywords: Nash equilibrium, asymptotic stability, strategic market games

JEL Classification: C72, D43

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DYNAMIC PROPERTIES OF THE NASH EQUILIBRIUM

1. Introduction

The concept of the Nash equilibrium (NE) is a static concept in the sense that at the same instant of time every player of the game has to choose his strategy as a best response to the choices of the others'. Thus one of the weakness of this concept is, the criterion for the "best response" depends on the "conjecture" of what the other players are playing. As a result, in games with multiple Nash equilibria, usually it is very difficult to offer any acceptable justification as to which particular NE should be chosen as a "solution" of the game. Therefore it is of great interests to make clear whether or not it is possible to justify an NE solution by examining some dynamic process in some neighborhood of the NE. To some extent this is similar to what has been done for the justification of a cooperative bargaining solution by examining the convergency of some sequential bargaining process.

In this paper we examine the above mentioned dynamic property of the Nash equilibrium of a class of games, namely, games with real-valued reaction functions. Though most of the examples we discuss here are games with pure strategies to be able being described by real numbers. The reader can observe from the example of a sell-all market game that the stability criterions developed in this paper can be also applied to games with more complicated strategies. We want to emphasize here, while our basic results in this paper can be applied to all games with well-defined reaction functions, they are particularly useful for those games with incomplete information, where a player knows only the reaction function of himself. Traditionally it is assumed that in a game with incomplete information, a player can assign some probabilities for his opponents being of some possible types (or having some possible payoffs), and then play some Bayesian equilibrium strategy. But in the reality, equilibrium is more likely being achieved through a dynamic process, in which players make improvements step by step according to their reaction functions.

While we examine the dynamic process, we only consider the situations where only one player is allowed to make an improvement each time. As pointed out by Professor Shapley when he examines potential games [refer to Shapley (1994)], in the case with simultaneous improvements, even the Cournot equilibrium with one homogeneous product with three or more firms can be not stable. Actually, with one-by-one alternate improvements, the convergency of the dynamic process may still depend on the order to make improvement. The reader can observe this interesting phenominon from Example 2 in the next section.

Section 2 is mainly devoted to the basic definitions and the fundamental result for games with linear reaction functions. Some simple examples can be found there for the explanation of the basic concepts. In Section 3 we discuss the stability of the Bertrand equilibrium with n firms and with differentiate products. We have proved that with constant marginal costs and linear market demands, the Bertrand equilibrium is globally asymptotically stable. In Section 4, we give a nonstable example with linear reaction functions. In Section 5 we generalize our results in Section 2 for games with non-linear reaction functions. An example of a strategic market game is discussed in Section 6. One can see there, even with linear separable utility functions, the NEs of strategic market games may or may not be asymptotically stable. Finally we conclude the paper in Section 7.

We point out here, while the nonstability property of a market equilibrium may not be welcome by the economists who always want to choose an equilibrium as a "solution", it really reflects the complicated economic phenomina of the real world, where people can observe more of cyclings, chaos than of equilibria. Therefore it is not necessary for us always to choose an equilibrium as a solution. Sometimes, a disequilibrium path may be even more welcome by every party being involved in the game.

2. Nash Equilibrium with Linear Reaction Functions

Consider an n-person game G in which the strategy of Player i can be expressed by a real number $x_i \in \mathbf{R}$. Assume that for $x_{-i} = (x_1, ..., x_{i-1}, , x_{i+1}, ..., x_n)$ chosen by all other players, the unique best response of Player i is determined by

$$x_i = f_i(x_{-i}) = C_i + \sum_{j\neq i} a_{ij}x_j ; \quad (i = 1, ..., n) \tag{1}$$

Then we say that G is a game with linear reaction functions.

Assume that the linear system of equations of (1) has a unique solution $x^* = (x_1^*, ..., x_n^*)$. Thus x^* is the unique Nash equilibrium of G. Now we want to examine the following dynamic property of x^* -- Imagine that at the beginning the players choose an arbitrary strategy profile $x_0 = (x_{10}, ..., x_{n0})$. Then Player 1 computes his best response x_{11} to $(, x_{20}, ..., x_{n0})$ according to (1), and then Player 2 computes his best response x_{21} to $(x_{11}, , x_{30} ..., x_{n0})$, ... Continue the above computation to the infinite horizon, for every i, we then have a sequence

$$x_{i0}, x_{i1}, ..., x_{ik}, ..., ... \tag{2}$$

We want to examine that, under what conditions, we have for every i,

$$\lim_{k\to\infty} x_{ik} = x_i^* \tag{3}$$

Here we give

Definition 1. The Nash equilibrium x* of G is said to be globally asymtotically stable under one-by-one optimal improvement in the natural order, if beginning with any initial strategy profile x_0, (3) holds for every i. If the stability property is not affectted by the ordering of the players, (i.e. the order of making improvements), then x* is said to be globally asymptotically stable under one-by-one optimal improvement. If (3) holds for all initial strategy profile x contained in some neighborhood of x*, then we say that x* is locally asymptotically stable under one-by-one improvement (in the natural order).

To explain the above definition, we examine the following

Example 1. In the Bertrand competition of two firms, assume that Firm i has a zero fixed cost, and a constant marginal cost c_i. Assume that the market demands are given by $q_i = Q_i - p_i + a_{ij}p_j$, (when the right-hand-side is negative, we agree that actually q_i = 0), where p_i is the price charged by Firm i, p_j is the price charged by the rival, q_i is Firm i's quantity demanded; and Q_i and a_{ij} are constants with $Q_i > 0$ and $0 \leq a_{ij} < 1$. (i, j = 1, 2; i ≠ j).

It is easy to verify that the reaction function for Firm i is

$$p_i = 0.5(Q_i + c_i + a_{ij}p_j) \quad (4)$$

and the Bertrand equilibrium is given by

$$p_i{}^* = (4-a_{ij}a_{ji})[2(Q_i+c_i) + a_{ij}(Q_j+c_j)] \quad (5)$$

According to the description of the construction of the sequence $\{x_{ik}\}$ in (2), we have

$$p_{1k+1} = 0.5(Q_1 + c_1 + a_{12}p_{2k}) \; ; \quad p_{2k+1} = 0.5(Q_2 + c_2 + a_{21}p_{1k+1}) \qquad (6)$$

From (4) and (6) one derives

$$p_{1k+1} - p_1{}^* = 0.5a_{12}(p_{2k} - p_1{}^*) \; ; \quad p_{2k+1} - p_2{}^* = 0.5a_{21}(p_{1k+1} - p_1{}^*) \qquad (7)$$

Thus

$$p_{1k+1} - p_1{}^* = 0.25a_{12}a_{21}(p_{1k} - p_1{}^*) \; ; \quad p_{2k+1} - p_2{}^* = 0.25a_{12}a_{21}(p_{2k} - p_2{}^*) \qquad (8)$$

From (8) it is obvious that $\lim_k (p_{ik} - p_i{}^*) = 0$, or equivalently $\lim_k p_{ik} = p_i{}^*$ for i = 1, 2. It is also easy to verify the above results do not depend on the ordering of the two players. Thus the Bertrand equilibrium is globally asymptotically stable under one-by-one optimal improvement.

We point out here that for n > 2, the argument for the stability of a Nash equilibrium is not so straight forward as that in the above example. Here we consider the general situation. In the construction of the sequence in (2) we have

$$x_{ik+1} = C_i + a_{i1}x_{1,k+1} + \ldots + a_{ii-1}x_{i-1,k+1} + a_{ii+1}x_{i+1,k} + \ldots + a_{in}x_{n,k} \; ; \quad (i = 1, \ldots, n) \qquad (9)$$

Combine (9) with (1), we have

$$x_{ik+1}-x_i{}^* = a_{i1}(x_{1,k+1}-x_1{}^*)+\ldots+a_{ii-1}(x_{i-1,k+1}-x_{i-1}{}^*)+a_{ii+1}(x_{i+1,k}-x_{i+1}{}^*)+\ldots+a_{in}(x_{nk}-x_n{}^*) \; ; \quad (i = 1, \ldots, n) \qquad (10)$$

Let

$y_{ik} = x_{ik} - x_i^*$, $(i = 1, ..., n; k = 1, 2, ...)$; $y_k = (y_{1k}, ..., y_{nk})^t$

and

$$A = \begin{pmatrix} 1 & 0 & \dots\dots & 0 \\ -a_{21} & 1 & 0 \dots & 0 \\ \dots & \dots & \dots & \\ -a_{n1} & \dots -a_{nn-1} & & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & a_{12} & \dots\dots & a_{1n} \\ 0 & 0 & a_{23} \dots & a_{2n} \\ \dots & \dots & \dots & \\ 0 & \dots & \dots\dots & 0 \end{pmatrix}$$

We then have a system of linear difference equations

$$Ay_{k+1} = By_k \tag{11}$$

Obviously $\det A = 1$, and A is invertible. Let $M = A^{-1}B$. Then (11) is equivalent to

$$y_{k+1} = My_k \tag{12}$$

Note that the Nash equilibrium corresponds to the zero solution of (12). Thus we have the following

Theorem 1. The Nash equilibrium of G is globally asymptotically stable under one-by-one optimal improvement in the natural order if and only if every eigenvalue of $M = A^{-1}B$ has an absolute value less than 1.

Proof. The above conclusions directly follow from the structure of the fundamental solution matrix of (12). [refer to, for example, Miller (1968)] For the "if" part, just observe that every solution (every real solution in particular) y_k of (12) can be expressd in the form PY^ky_0, where P is a nonsingular constant matrix with complex entries and $\|P\| = 1$, y_0 is the constant column n-vector equal to x_0-x^*, and Y^k is a block matrix $[Y^k_1 \oplus \ldots \oplus Y^k_m]$ with Y^k_v being defined by

$$Y^k_v = \begin{pmatrix} \lambda_v^k & C_k^1\lambda_v^{k-1} & \ldots\ldots\ldots & C_k^{r-1}\lambda_v^{k-r+1} \\ 0 & \lambda_v^k & C_k^1\lambda_v^{k-1}\ldots & C_k^{r-2}\lambda_v^{k-r+2} \\ \ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots \\ 0 & \ldots\ldots\ldots & \ldots\ldots\ldots & \lambda_v^k \end{pmatrix}$$

Here we assume that in the Jordan normal form $P^{-1}MP$ of M, the eigenvalue λ_v corresponds to an r×r Jordan block

$$\begin{pmatrix} \lambda_v & 1 & & & \\ & \lambda_v & 1 & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & \lambda_v & 1 \\ & & & & \lambda_v \end{pmatrix}$$

Obviously, when all the eigenvalues are with absolute value less than 1, for any solution y_k of (12), we have

$$\|y_k\| = \|(PY^k)y_0\| \leq \|P\|\cdot\|Y^k\|\cdot\|y_0\| = \|Y^k\|\cdot\|y_0\|$$

Since $\lim_k \|Y^k\| = 0$, we thus have $\lim_k \|y_k\| = 0$.

In case with some $|\lambda| \geq 1$, we can find a (complex) solution z_k of (12) with $\lim \|z_k\| \neq 0$. Let $z_k = y_k + iw_k$ with y_k being the real part, and w_k the imaginary part of z_k. Then y_k and w_k are all <u>real</u> solution of (12). Moreover, at least one of $\lim \|y_k\| = 0$ and $\lim \|w_k\| = 0$ is violated. Thus the zero solution of (12) is not asymptotically stable.

Theorem 1 is thus proved. o

Let us now examine an example with different stability properties in different order of making improvements.

Example 2. Consider a three person game with reaction functions given by

$$x_1 = 0.01x_2 + 0.1x_3\,; \quad x_2 = x_1 + 0.01x_3\,; \quad x_3 = x_1 + 20x_2$$

We have

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -20 & 1 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 21 & 20 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0.01 & 0.1 \\ 0 & 0 & 0.01 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & 0.01 & 0.1 \\ 0 & 0.01 & 0.11 \\ 0 & 0.21 & 2.3 \end{pmatrix}$$

It is easy to check that M has an eigenvalue greater than 1. Thus the NE <0,0,0> is not stable.

Now we interchange the order of Player 2 and Player 3. We than have the reaction functions

$$x'_1 = 0.01x'_2 + 0.1x'_3\,; \quad x'_2 = x'_1 + 20x'_3\,; \quad x'_3 = x'_1 + 0.01x'_2$$

Thus

$$A' = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -0.01 & 1 \end{pmatrix} \qquad A'^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1.01 & 0.01 & 1 \end{pmatrix}$$

$$B' = \begin{pmatrix} 0 & 0.01 & 0.1 \\ 0 & 0 & 20 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M' = \begin{pmatrix} 0 & 0.01 & 0.1 \\ 0 & 0.01 & 20.1 \\ 0 & 0.0101 & 0.301 \end{pmatrix}$$

This time all the eigenvalue of M' are with absolute value less than 1. Thus the NE <0,0,0> is asymptotically stable.

We have compute the data for the first several rounds for the above two examples, from which one can observe the trend of convergence and that of divergence.

$x_{10} = 1$	$x'_{10} = 1$
$x_{20} = 1$	$x'_{20} = 1$
$x_{30} = 1$	$x'_{30} = 1$
$x_{11} = 0.11$	$x'_{11} = 0.11$
$x_{21} = 0.12$	$x'_{21} = 20.11$
$x_{31} = 2.51$	$x'_{31} = 0.3111$
$x_{12} = 0.2642$	$x'_{12} = 0.23221$
$x_{22} = 0.5403$	$x'_{22} = 6.45421$
$x_{32} = 11.0702$	$x'_{32} = 0.2967521$
..., ...	..., ...

3. Bertrand Equilibrium with n Firms

As an application of Theorem 1, let us examine the Bertrand competition with n firms. Assume that Firm i produces product i with a zero fixed cost and a constant marginal cost c_i. Assume that the market demands are given by

$$q_i = \max\ \{0, Q_i - p_i + \textstyle\sum_{j\neq i} a_{ij} p_j\ \} \tag{13}$$

where p_l is the price charged by Firm l, q_l is Firm l's quantity demanded, and $Q_l > c_l$, and the a_{ij} are nonnegative constants satisfying

$$\textstyle\sum_{j\neq i} a_{ij} < 1\ , \quad i = 1, ..., n \tag{14}$$

It is not difficult to prove that there exists a unique Bertrand equilibrium $p^* = (p_1^*, ..., p_n^*)$ under the price competition. (Refer to Yao [2]) The reaction function for Firm i is given by

$$p_i = 0.5(Q_i + c_i + \textstyle\sum_{j\neq i} a_{ij} p_j) \tag{15}$$

Construct the sequences $\{p_{ik}\}$. If we write $y_{ik} = p_{ik} - p_i^*$, and $y_k = (y_{1k}, ..., y_{nk})^t$, we have

$$A y_{k+1} = B y_k \tag{16}$$

Here

$$A = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -0.5a_{21} & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ -0.5a_{n1} & \dots & -0.5a_{nn-1} & & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0.5a_{12} & \dots & \dots & 0.5a_{1n} \\ 0 & 0 & 0.5a_{23} & \dots & 0.5a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

To show that p^* is globally asymptotically stable under one-by-one optimal improvement, it suffices to show that $A^{-1}B$ has no eigenvalue with absolute value greater than 1.

Let λ be an eigenvalue of $A^{-1}B$, and let v be the associate eigenvector. We then have $A^{-1}B\text{v} = \lambda\text{v}$, or, equivalently, $(\lambda A\text{-}B)\text{v} = \mathbf{0}$. Because v is not a zero vector, we must have $\det(\lambda A\text{-}B) = 0$, which is equivalent to $\det(A\text{-}\lambda^{-1}B) = 0$ when $\lambda \neq 0$. Thus to obtain the required conclusion, we need only show that for any λ with $|\lambda| \geq 1$,

$$\det(A\text{-}\lambda^{-1}B) \neq 0 \qquad (17)$$

It is easy to verify that (14) guarantees $(A\text{-}\lambda^{-1}B)$ being a dominant diagonal matrix for any λ with $|\lambda| \geq 1$, and thus being nonsingular. The required conclusion thus follows.

Thus we have shown

Proposition 1. The Bertrand equilibrium with linear market demands and constant marginal costs is globally asymptotically stable if all the inequalities in (14) hold.

4. A Nonstable Example

In the section we examine an example with nonstable Nash equilibriums.

Example 3. Two people, 1 and 2, simultaneously choose a real number. Let r be chosen by 1, and s be chosen by 2. Then the payoffs are given by

$$\pi_1 = -(r - s)^2 ; \quad \pi_2 = 1 - (0.5s - r + 0.25)^2$$

Note that the reaction functions are given by

$$r = s ; \quad s = 2r - 0.5$$

The unique Nash equilibrium is $\langle r^*,s^* \rangle$ $= \langle 0.5,0.5 \rangle$. The dynamic system is given by

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} r_{k+1} \\ s_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r_k \\ s_k \end{pmatrix}$$

or, equivalently

$$\begin{pmatrix} r_{k+1}-0.5 \\ s_{k+1}-0.5 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} r_k-0.5 \\ s_k-0.5 \end{pmatrix}$$

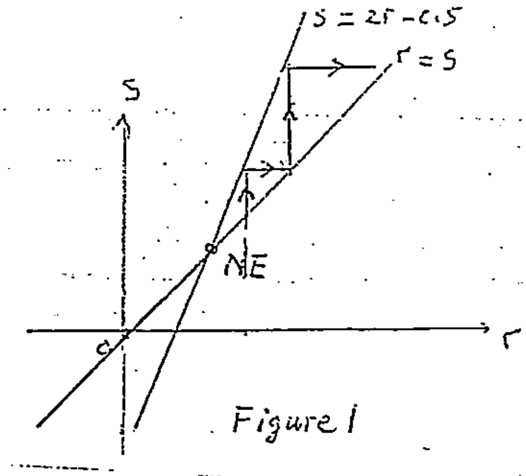

Figure 1

Since $\lambda = 2$ is an eigenvalue of the coefficient matrix in (17), the Nash equilibrium is not stable. In fact, from Figure 1 one can see that along any trajectory starting with $s_0 > 0.5$, we always have $\lim_k r_k = \lim_k s_k = \infty$.

Remark. If we introduce the constraints $0 \le r, s \le 1$ in the above example, then it is easy to show that, in addition to the Nash equilibrium <0.5,0.5>, we have other two Nash equilibria <0,0> and <1,1>. Moreover, in a small neighbourhood of (1,1), the reaction functions are given by r = s, s = 1, the dynamic system in this neighbourhood is given by

$$\begin{pmatrix} r_{k+1}-1 \\ s_{k+1}-1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r_k-1 \\ s_k-1 \end{pmatrix}$$

of which the coefficient matrix has eigenvalues all equal to 0. Thus <1,1> is (locally) asymptoically stable. In fact for any trajectory starting with $s_0 > 0.5$, we always have $\lim_k r_k = \lim_k s_k = 1$. Similarly we can argue <0,0> is also locally asymptotically stable with all trajectories starting with a $s_0 < 0.5$ tending to (0,0). (Figure 2) Now if this were a game of incomplete information with each player only knows the payoff of himself, generically any dynamic playing process with one-by-one optimal improvement must be with probability 1 finally reach to a "solution" either of <0,0> or of <1,1>. This support our arguments in Section 1.

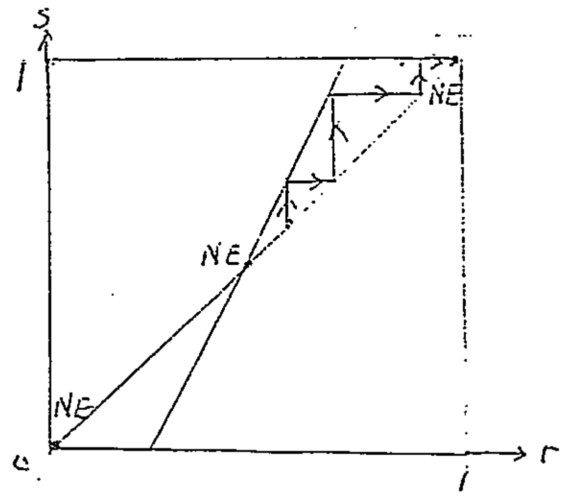

Figure 2

5. Games with General Reaction Functions

We now consider the n-person games with each player having **R** as his strategy set, having a well-defined reaction function which may not be linear. We still write the reaction functions as

$$x_i = f_i(x_{-i}) \; ; \quad (i = 1, ..., n) \qquad (18)$$

Assume that x^* is a Nash equilibrium. Therefore

$$x_i^* = f_i(x_{-i}^*) \qquad (19)$$

For any given $x = (x_1, ..., x_n)$, we define

$$x_{ik+1} = f_i(x_{1k+1}, ..., x_{i-1k+1}, , x_{i+1k}, ... x_{nk}) \; ; \quad (i = 1, ..., n) \qquad (20)$$

We now want to examine the dynamic system (20). Assume that all the f_i are C^1. Subtracting (19) from (20) one can derive

$$x_{ik+1} - x_i^* = \Sigma_{j<i} \partial_j f_i(z_j)(x_{jk+1} - x_j^*) + \Sigma_{j>i} \partial_j f_i(z_j)(x_{jk} - x_j^*) \qquad (21)$$

where

$$\partial_j f_i = \partial f_i / \partial x_j$$

$$z_j = (x_1^* , ..., x_{j-1}^*, x_j^* + \theta_j(x_{jk+1} - x_j^*), x_{j+1k+1}, ..., x_{i-1k+1}, , x_{i+1k}, ..., x_{nk}) , \quad (j < i) \; ;$$

$$z_j = (x_1^*, ..., x_{i-1}^*, , x_{i+1}^*, ... x_{j-1}^*, x_j^* + \theta_j(x_{jk} - x_k^*), x_{j+1k}, ..., x_{nk}) , \quad (j > i) \; ;$$

$$0 < \theta_j = \theta_i(x_k) < 1 , \quad (j \neq i)$$

We introduce the following notations

$A(x_k) = (a_{ij}(x_k))_{n\times n}$; $a_{ij}(x_k) = -\partial_j f_i(z_j)$ if $i > j$; $a_{ij}(x_k) = 1$ if $i = j$; and $a_{ij}(x_k) = 0$ if $i < j$

$B(x_k) = (b_{ij}(x_k))_{n\times n}$; $b_{ij}(x_k) = 0$ if $i \geq j$; $b_{ij}(x_k) = \partial_j f_i(z_j)$ if $i < j$

$y_{ik} = x_{ik} - x_i^*$, $y_k = (y_{1k}, \dots, y_{in})^t$

Then (21) is equilivalent to

$$A(x_k)y_{k+1} = B(x_k)y_k$$

Since $A(x_k)$ is nonsingular, we obtain

$$y_{k+1} = M(x^*+y_k)y_k \qquad (22)$$

where $M(x) = [A(x)]^{-1}B(x)$

Definition 2. Given the f_i as in (18), we define a function $f = f(f_1, \dots, f_n) : \mathbf{R}^n \to \mathbf{R}^n$ by

$$f(x_1, \dots, x_n) = (z_1, \dots, z_n)$$

where

$$z_1 = f_1(x_2, \dots, x_n),\ z_2 = f_2(z_1, x_3, \dots, x_n),\ \dots,\ z_n = f(z_1, \dots, z_{n-1})$$

Theorem 2. Assume that the f_i are all C^1. Then (i). if $M(x^*)$ has all eigenvalues with absolute value less than 1, then the zero solution of (22) and hence the Nash equilibrium x^* of (18) is locally asymptotically stable under one-by-one improvements in the natural order;

(ii). if there exist a natural number m, some natural number $i \leq n$, a real number $\delta > 0$, such that $(u_1, ..., u_n)^t \equiv [M(x^*+y_m)...M(x^*+y_1)]v$ has the property that $|u_i| \geq |v_i|$ for any y_1 with $\|y_1\| < \delta$ and any column n-vector $v = (v_1, ..., v_n)^t$, and if the function $f(f_1, ..., f_n)$ defined in definition 2 has the propertiy that the set $S = \{y \in \mathbf{R}^n : f^k(y)$ has nonzero ith component for every $k \in \mathbf{N}\}$ is dense in $\mathbf{R}^n$, then the zero solution of (22) and hence the Nash equilibrium x^* of (18) is not assymptotically stable.

The proof of Theorem is a bit lengthy, we leave it in the Appendix.

6. Strategic Market Games

Here we use Theorem 2 to examine a simple example of a two trader sell-all market game. [For the general descriptions of strategic market games the reader can refer to Dubey and Shubik (1978), Sahi and Yao (1989), Amir, Sahi, Shubik and Yao (1990)]. Assume Trader i receives at the beginning an initial endowment

$$a_i = (a_{i1}, ...a_{im}; b_i) \tag{23}$$

where a_{ij} is the amount of commodity j, j = 1, ..., m; and b_i is the amount of commodity money which will be used for the payments. For simplicity we assume in this example that all the a_{ij} and b_j are positive, and we also assume that each trader has a linear separable utility function $u_i : \mathbf{R}^{m+1} \dashrightarrow \mathbf{R}_+$ defined by

$$u_i(x_{i1}, \ldots, x_{im}; y_i) = c_{i1}x_{i1} + \ldots + c_{im}x_{im} + d_i y_i \, ; \quad i = 1, 2 \tag{24}$$

where x_{ij} is i's final holding of good j after the trade (and hence is the amount he consumes), and y_i is his final holding of the commodity money; and all the c_{ij} and d_i are positive constants.

We now describe the trading mechanism. After they receive the endowments, each trader must send all of them except the money good to the market, one commodity to one trading post. Then each trader announces an amount of money r_{ij} for the bidding of commodity j. The price of commodity j is computed by

$$p_j = (r_{1j} + r_{2j})/(a_{1j} + a_{2j}) \tag{25}$$

The final holdings are given by

$$x_{ij} = \begin{cases} r_{ij}/p_j \, , & \text{if } p_j > 0 \\ a_{ij} \, , & \text{if } p_j = 0 \end{cases} \tag{26}$$

and

$$y_i = b_i - \Sigma_j r_{ij} + \Sigma_j a_{ij} p_j \tag{27}$$

Thus we have a strategic market game G. The two traders are the players. The strategy set for Player i is

$$S_i = \{r_i = (r_{i1}, \ldots, r_{im}) : r_{ij} \geq 0, \text{ and } \Sigma_j r_{ij} \leq b_i\} \tag{28}$$

The payoffs are given by

$$\pi_i(r_1, r_2) = u_i(x_{i1}, ..., x_{im}; y_i)\ , \quad i = 1, 2 \tag{29}$$

A Nash equilibrium of G is said to be a strategic sell-all market equilibrium.

For simplicity we want that the b_i are sufficiently large so that we always have an interior equilibrium. i.e. an equilibrium $<r_1^*,r_2^*>$ with $\sum_j r_{ij}^* < b$ for every i. For this purpose we assume that for every Trader i:

$$b_i > \max_j \{mc_{ij}(a_{1j}+a_{2j})^2 a_{kj}^{-1} d_i^{-1}\} \tag{30}$$

here k stands for the oponent of i. To see that (30) guarantees any equilibrium to be interior, it suffices to show that at any equilibrium $r_{ij}^* \leq b_i/m$ for any (i,j). If not, say, $r_{11}^* > b_1/m \geq c_{11}(a_{11}+a_{12})^2 a_{21}^{-1} d_1^{-1}$. Imagine Trader 1 reduces his bid r_{11}^* by an amount $\Delta < r_{11}^*$. By calculation the reduction of x_{11} is <u>less than</u> $(a_{11}+a_{21})\Delta(r_{11}^*+r_{21}^*)^{-1} < a_{21}d_1\Delta [c_{11}(a_{11}+a_{21})]^{-1}$. But at the same time the increment of his y_1 is precisely equal to $a_{21}\Delta (a_{11}+a_{21})^{-1}$. Hence the increment in his utility is <u>greater than</u> $d_1 a_{21}\Delta(a_{11}+a_{21})^{-1} - c_{11}a_{21}d_1\Delta [c_{11}(a_{11}+a_{21})]^{-1} = 0$. This contradicts the fact that $<r_1^*,r_2^*>$ is a Nash equilibrium.

Please note that (30), though not a necessary condition, makes it easy to argue that at any equilibrium we must have all the $r_{ij}^* > 0$. In fact we cannot have $r_{1j}^* = r_{2j}^* = 0$ for any j. Otherwise any trader can make an improvement by bidding with any small amount of money for good j and receives all good j. We can neither have, say, $r_{1j}^* = 0$ but $r_{2j}^* > 0$. Otherwise Trader 2 can reduce r_{2j}^* a little bit to make an improvement.

We will see that with (30) different trading posts can be regarded as separate markets, since in a small neighbourhood of a equilibrium any trader's decision at any trading post is only affected by his opponent's decision at the same trading post. To compute the reaction functions, we differentiate (29) with respect to r_{ij}. The first order condition is

$$c_{ij}(a_{1j}+a_{2j})r_{hj}(r_{1j}+r_{2j})^{-2} - d_i a_{2j}(a_{1j}+a_{2j})^{-1} = 0\ ;\quad i, h = 1, 2, i \neq h$$

If we write $\alpha_{ij} = c_{ij}d_i^{-1}$, $A_j = a_{1j}+a_{2j}$, from the above equation we can solve

$$r_{ij} = (A_j\alpha_{ij}^{1/2}a_{hj}^{-1/2})r_{hj}^{1/2} - r_{hj}\ ;\quad i, h = 1, 2\ ; i \neq h$$

Thus the reaction functions at the jth trading post are given by

$$r_{ij} = f_{ij}(r_{hj}) = \max\ \{0, (A_j\alpha_{ij}^{1/2}a_{hj}^{-1/2})r_{hj}^{1/2} - r_{hj}\ \}\ ;\quad i, h = 1, 2, i \neq h \tag{31}$$

The unique Nash equilibrium is given by

$$r_{ij}^* = \alpha_{ij}^2\alpha_{hj}a_{ij}A_j^2(\alpha_{1j}a_{1j}+\alpha_{2j}a_{2j})^{-2}\ ;\quad j = 1, ..., m,\ i, h = 1, 2, i \neq h \tag{32}$$

Let us now examine the dynamic property of the equilibrium. At the NE r*, in the jth trading post, we have

$$A(r_j^*) = \begin{pmatrix} 1 & 0 \\ -0.5(A_j\alpha_{2j}^{1/2}a_{1j}^{-1/2})r_{1j}^{*-1/2}+1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (2\alpha_{1j}a_{1j})^{-1}(\alpha_{1j}a_{1j}-\alpha_{2j}a_{2j}) & 1 \end{pmatrix}$$

$$B(r_j^*) = \begin{pmatrix} 0 & 0.5(A_j\alpha_{ij}^{1/2}a_{kj}^{-1/2})r_2^{*\,1/2}-1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (2\alpha_{2j}a_{2j})^{-1}(\alpha_{1j}a_{1j}-\alpha_{2j}a_{2j}) \\ 0 & 0 \end{pmatrix}$$

And

$$M(r_j^*) = \begin{pmatrix} 0 & (2\alpha_{2j}a_{2j})^{-1}(\alpha_{1j}a_{1j}-\alpha_{2j}a_{2j}) \\ 0 & -(4\alpha_{1j}\alpha_{2j}a_{1j}a_{2j})^{-1}(\alpha_{1j}a_{1j}-\alpha_{2j}a_{2j})^2 \end{pmatrix}$$

Now it is easy to see that the two eigenvalues of $M(r_j^*)$ are

$$\lambda_1 = 0, \quad \lambda_2 = -(4\alpha_{1j}\alpha_{2j}a_{1j}a_{2j})^{-1}(\alpha_{1j}a_{1j}-\alpha_{2j}a_{2j})^2$$

Therefore we have

Proposition 2. In the above sell-all market game G, the Nash equilibrium is asymptotically stable under one-by-one improvements if $D_j = (4\alpha_{1j}\alpha_{2j}a_{1j}a_{2j})^{-1}(\alpha_{1j}a_{1j}-\alpha_{2j}a_{2j})^2 < 1$ holds for every j. The Nash equilibrium is not asymptotically stable if $D_j = (4\alpha_{1j}\alpha_{2j}a_{1j}a_{2j})^{-1}(\alpha_{1j}a_{1j}-\alpha_{2j}a_{2j})^2 > 1$ holds for at least one j.

Proof. The proof of the first assertion directly follows from Theorem 2 (i). To verify the second assertion, assume that $D_j > 1$ for some j. Let $r_{1j} = (r_{11j}, r_{12j})$ be the initial bids in the jth trading post, where r_{1ij} is the bid by trader i. let $\{ r_1, r_2, \ldots, r_k, \ldots\}$ be the trajectory starting from r_1. From (31), in a small neighborhood of r*, we have

$$A(r_{1j}) = \begin{pmatrix} 1 & 0 \\ \eta(r_{1j}) & 1 \end{pmatrix}, \qquad B(r_{1j}) = \begin{pmatrix} 0 & \kappa(r_{1j}) \\ 0 & 0 \end{pmatrix}$$

where $\eta(r_{1j}) = -0.5(A_j\alpha_{2j}^{1/2}a_{1j}^{-1/2})r^*_{1j}{}^{-1/2}+1+o(\|r_1-r^*\|)$, and $\kappa(r_{1j}) =$ $0.5(A_j\alpha_{ij}^{1/2}a_{kj}^{-1/2})r_2^{*\,1/2}-1+o(\|r_1-r^*\|)$. Hence we have

$$M(r_{1j}) = \begin{pmatrix} 0 & \kappa(r_{1j}) \\ 0 & -\eta(r_{1j})\kappa(r_{1j}) \end{pmatrix}$$

Now we see that for any 2-column vector $v = (x,y)^t$ and for any r_1 sufficiently close to r^*, the absolute value of the second component of $M(r_{1j})v$, i.e. the absolute value of $-\eta(r_{1j})\kappa(r_{1j})y$, is sufficiently close to $y(4\alpha_{1j}\alpha_{2j}a_{1j}a_{2j})^{-1}(\alpha_{1j}a_{1j}-\alpha_{2j}a_{2j})^2$, which is greater than or equal to $|y|$. On the other hand, for any r_{21} which is transcendental with respect to all those r^*_{i1}, A_j, a_{ij}, and α_{ij}, (note that the set of all 2-vectors with these r_{21} is dense in $\mathbf{R}^2$), we have r_{2k} remaining to be transcendental with respect to them and $r_{2k} - r^*_{21}$ never equals to zero for any k. Thus the NE r^* is not assymptotically stable according to Theorem 2 (ii). ®

We now examine two numerical examples.

Example 4. $m = 1$, $a_1 = a_2 = 1$, $A = 2$, $\alpha_1 = 4, \alpha_2 = 1$, $b_1 = b_2 = 3$; $D = 9/16 < 1$. (We omit the subscript $j = 1$.)

It is not difficult to determine the strategic market equilibrium with $<r_1^*, r_2^*> =$ $<2.56, 0.64>$. According to Proposition 2, this NE must be asymptotically stable. The reaction functions in some neighborhood of this NE look like

$r_1 = 4r_2^{1/2} - r_2 ; \quad r_2 = 2r_1^{1/2} - r_1$

Let us choose $r_{10} = 2$, and $r_{20} = 0.828427225$ (the best response to r_{10}). Then we have

$r_{11} = 2.81229176$
$r_{21} = 0.541686033$
$r_{12} = 2.402286884$
$r_{22} = 0.697577619$
$r_{13} = 2.643262081$
$r_{23} = 0.608360333$
$r_{14} = 2.511537955$
$r_{24} = 0.658028548$
$r_{15} = 2.586729805$
$r_{25} = 0.629932941$
$r_{16} = 2.544799655$
$r_{26} = 0.645685985$
$r_{17} = 2.568497545$
$r_{27} = 0.636809021$
... ...

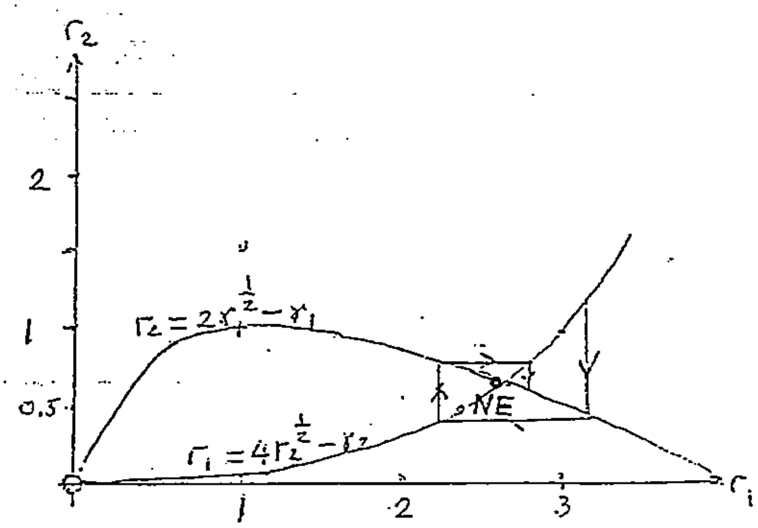

Figure 3

It is not difficult to observe the trend of convergence. (Figure 3)

Example 5. $m = 1$, $a_1 = a_2 = 1$, $A = 2$, $\alpha_1 = 9, \alpha_2 = 1$, $b_1 = b_2 = 10$; $D = 16/9 > 1$.

It is not difficult to determine the strategic market equilibrium with $<r_1^*, r_2^*> = <3.24, 0.36>$. According to Proposition 2, this Ne must be nonstable. The reaction functions in some neighborhood of this NE look like

$r_1 = 6r_2^{1/2} - r_2 ; \quad r_2 = 2r_1^{1/2} - r_1$

Let us choose $r_{10} = 1$, and $r_{20} = 0.35$. Then we have

$r_{11} = 3.1996...$
$r_{21} = 0.3779...$
$r_{12} = 3.3104...$
$r_{22} = 0.3286...$
$r_{13} = 3.1103...$
$r_{23} = 0.4169...$
$r_{14} = 3.4572$
$r_{24} = 0.2615...$
$..., ..., ...$

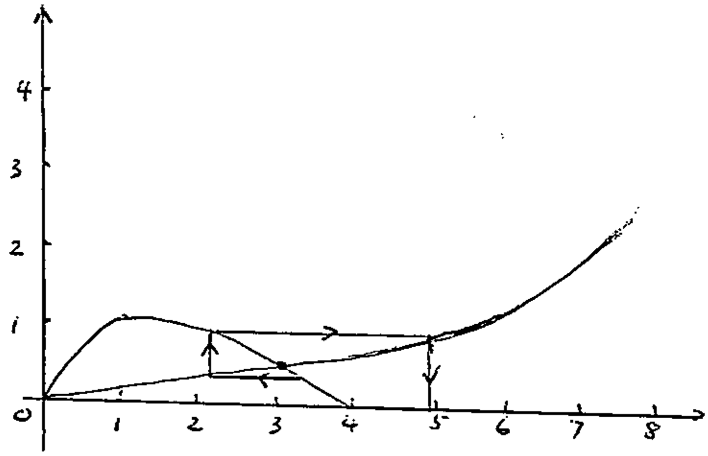

Figure 4

We thus can observe the trend of divergence. (Figure 4)

7. Concluding Remarks

Through the above discussions one can see that, although the Nash equilibrium concept has been chosen as one of the very important solution concepts in economic games, Nash equilibria obtained in different setings may have rather different dynamic properties. Under the situation that deviations being inevitable and firms or players can adjust their strategies simultaneously from time to time, many Nash eqilibrium could be not stable, and, as a result, what people can observe is not the eqilibrium but some cyclical path or even the chao phenominon.

Appendix

Proof of Theorem 2. Given $\varepsilon > 0$, since $M(x^*)$ has its eigenvalues all less than 1, there must exist a sufficiently large K, such that $\|[M(x^*)]^K\| < \varepsilon$. In particular we can determine such a K for $\varepsilon = 1/4$.

On the other hand, by

$$y_2 = M(x^*+y_1)y_1, \ldots, y_K = M(x^*+y_{K-1})y_{K-1}$$

It is easy to see that the product matrix

$$M(x^*+y_{K-1}) \ldots M(x^*+y_1)$$

is continuously dependent on y_1. When $\|y_1\|$ tends to the zero vector, this product matrix tends to $[M(x^*)]^K$. Consequently there exist some $\delta > 0$, such that

$$\|M(x^*+y_{K-1}) \ldots M(x^*+y_1) - [M(x^*)]^K\| < 1/4 \,, \quad \forall\ y_1 \text{ with } \|y_1\| < \delta$$

Hence we have

$$\|M(x^*+y_{K-1}) \ldots M(x^*+y_1)\| < 1/2 \,, \forall\ y_1 \text{ with } \|y_1\| < \delta$$

Let

$$\Delta = \max \{\|M(x^*+y_{k-1}) \dots M(x^*+y_1)\| : k = 2, \dots, K \text{ and } \|y_1\| < \delta\}$$

For any natural number n we can express n as $n = mK+r$, where m is some natural number, and r is some nonnegative integer less than K. Now for $\forall\ y_1$ with $\|y_1\| < \delta$, and $y_n = M(x^*+y_{n-1}) \dots M(x^*+y_1)$, it is easy to deduce that

$$\|y_n\| \leq \| M(x^*+y_{n-1}) \dots M(x^*+y_{n-r+2})\|\| M(x^*+y_{n-r+1}) \dots M(x^*+y_1)y_1\|$$

$$\leq (1/2)^m \Delta \|y_1\|$$

For $n \to \infty$, we also have $m \to \infty$. Thus we obtain

$$\lim_{n\to\infty} \|y_n\| = 0\ ,\ \forall\ y_1 \text{ with } \|y_1\| < \delta$$

The first statement (i) in Theorem 2 has been proved.

Now consider the case in (ii). Choose some y_1 in the set S, then the trajectory $\{y_1, y_2, \dots, y_k, \dots\}$ can never tend to 0. In fact, if this trajectory tended to 0, then there should exist some sufficiently large K, such that $\|y_k\| < \delta$ for all $k \geq K$. But then by our assumptions, the ith component of y_K is not zero and the absolute values of the ith components of y_K, y_{K+m}, y_{K+2m}, ...form an increasing sequence, and thus contradicts with $\{y_1, y_2, \dots, y_k, \dots\}$ tending to 0.

Theorem 2 is thus proved. o

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