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# A test for fractionally integrated time series

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# A TEST FOR FRACTIONALLY INTEGRATED TIME SERIES*

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ABSTRACT

Recent work by Sowell (1990) and Diebold and Rudebusch (1991a) show that Dickey-Fuller unit root tests can have low power under fractionally integrated alternatives. This paper proposes a locally best test designed particularly to detect such alternatives. The test is based on the asymptotic likelihood function in frequency domain. A Monte Carlo experiment shows that the test is quite powerful, even in small samples.

KEY WORDS: ARFIMA process, Unit root, Locally best test, Periodogram.

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## 1. Introduction

In recent years there have be an increase in interest in applying strongly dependent process models to economic time series. One such model that is often used is the autoregressive fractionally integrated moving average (ARFIMA(p,d,q)) process, which can be expressed in general as

$$\Phi(L)(1-L)^d(X_t - \mu - \gamma t) = \Theta(L)\varepsilon_t \tag{1}$$

where d can takes on integer or non-integer values, and

$$\Phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$$

and

$$\Theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \quad ,$$

have roots that lie outside of the unit circle, $L$ is the lag operator such that $LX_t = X_{t-1}$, $\mu$ is an unknown constant, $\gamma$ is the unknown coefficient of time trend t and $\varepsilon_t \sim (0, \sigma^2)$ is white noise. Properties of ARFIMA(p,d,q) processes have been studied, for example, by Granger and Joyeux (1980) and Hosking (1981). Among others, Geweke and Porter-Hudak (1983) suggest a method of estimation of d which is asymptotically robust to the specifications of the polynomials $\Phi(z)$ and $\Theta(z)$. Sowell (1992a) provides procedures of maximum likelihood estimation of d, $\phi = (\phi_1, \cdots, \phi_p)'$, $\theta = (\theta_1, \cdots, \theta_q)'$ and $\sigma^2$. Recent applications of the ARFIMA(p,d,q) model includes Diebold and Rudebush (1989, 1991b) and Sowell (1992b). For further details, see Brockwell and Davis (1987), Diebold and Nerlove (1990).

The most important feature of the ARFIMA(p,d,q) model is that, by allowing d to take non-integer values, a wide range of long-run dependent time series can be modelled. The value of d captures quantitative extent of the persistence of a shock to the time series in that estimates (and therefore confidence intervals of estimates and forecasts) of parameters involved in (1) converge at a rate which is dependent on the value of d. Many studies, for example those mentioned above, indicate that the range $0.5 < d < 1.5$ is possible and of interest for many economic time series. The unit root hypothesis $d = 1$ is a special case in the ARFIMA(p,d,q) framework.

If an ARMA or ARIMA model is fitted to an ARFIMA process, the estimates of parameters of the model are usually inconsistent. Consider, for example, fitting an ARIMA(1,1,0) model to an ARFIMA(1,1.2,0) series. In practice, we usually start with taking first differences of the data, so effectively we are fitting an ARMA(1,0,0) model to an ARFIMA(1,0.2,0) series. The differenced series may be written as $y_t = \tau + \phi_1 y_{t-1} + u_t$, $u_t = (1-L)^{-0.2}\varepsilon_t$. Obviously $u_t$'s are serially correlated, so least square estimates of $\tau$ and $\phi_1$, ignoring this fact, are inconsistent. Thus needs arise as to testing the value of d. Sowell (1990) indicates that the Dickey-Fuller unit root tests under ARFIMA alternatives can have low power. This is supported by Diebold and Rudebusch (1991a) in their Monte Carlo simulation. This paper constructs a test which is particularly designed to detect ARFIMA alternatives. This is a locally best test based on the frequency domain asymptotic likelihood function of the periodogram.[1] Hypothesis testing based on this kind of likelihood function is discussed, for example, in Harvey (1989).

In the next section we introduce the test. In the construction of the test, some properties of ARFIMA processes will be used. Section 3 reports the results of a Monte Carlo study on the power of the test. Concluding discussion is found in section 4.

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[1]Therefore values of p and q are assumed to be known. For testing values of d when p and q are unknown, see Wu (1992).

## 2. The Test

Let $Y_t = (1-L)X_t$ and $\tilde{d} = 1-d$. Then (1) can be written as

$$\Phi(L)(1-L)^{\tilde{d}}(Y_t - \gamma) = \Theta(L)\varepsilon_t \quad , -0.5 < \tilde{d} < 0.5. \tag{2}$$

The series $\{Y_t\}$ is stationary. Its moving average representation takes the form

$$Y_t = \gamma + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} \quad , \tag{3}$$

where $\sum_{i=0}^{\infty}|\psi_i| < \infty$ if $-0.5 < \tilde{d} \le 0$, and $\sum_{i=0}^{\infty}|\psi_i| = \infty$ if $0 < \tilde{d} < 0.5$ which indicates the long memory in $\{Y_t\}$. Typically for an ARFIMA(0,$\tilde{d}$,0) series, the covariances can be expressed as

$$\gamma(k) = \frac{\Gamma(\tilde{d}+k)\ \Gamma(1-2\tilde{d})}{\Gamma(1-\tilde{d})\ \Gamma(k-\tilde{d}+1)\ \Gamma(\tilde{d})}\ \sigma^2 \quad . \tag{4}$$

This expression will be used in the next section to generate the ARFIMA(0,$\tilde{d}$,0) series.

In the frequency domain, $\{Y_t\}$ has the spectral density

$$f(\omega) = \frac{\sigma^2}{2\pi}\ |1-e^{-i\omega}|^{-2\tilde{d}}\ \frac{|\Theta(e^{-i\omega})|^2}{|\Phi(e^{-i\omega})|^2} \tag{5}$$

where $\omega$ is the frequency. It is assumed that $|\Theta(e^{i\omega})|^2/|\Phi(e^{i\omega})|^2$ is bounded from above and away from zero. Note that (5) becomes the familiar spectral density of an ARMA(p,q) process when $\tilde{d} = 0$. Also $\lim_{\omega\to 0} f(\omega) = \infty$ when $0 < \tilde{d} < 0.5$ and $\lim_{\omega\to 0} f(\omega) = 0$ when $-0.5 < \tilde{d} < 0$.

The hypothesis of interest is that of

$$H_0:\quad \tilde{d} = 0 \qquad \text{against} \qquad H_a:\quad \tilde{d} < 0, \tag{6}$$

or

$$H_0:\quad \tilde{d} = 0 \qquad \text{against} \qquad H_a^-:\quad \tilde{d} > 0. \tag{7}$$

Consider hypothesis (6) first. Let the periodogram be

$$I(\omega_j) = \frac{1}{n}\left|\sum_{t=1}^{n}(X_t - \bar{X})e^{-i\omega_j t}\right|^2 \tag{8}$$

where

$$\omega_j = 2\pi j/n, \qquad j = 0,\ 1,\ \cdots,\ [n/2] \quad .$$

are Fourier frequencies and [x] is the integer part of x. When $-0.5 < \tilde{d} \le 0$ the coefficients of the moving average representation (3) satisfy $\sum_{i=0}^{\infty}|\psi_i| < \infty$. Thus for $j = 0, 1, \cdots, m$ where $m < [n/2]$, $I(\omega_j)$ converge in distribution to random variables $D_j$, which are independent and exponentially distributed (see Theorem 10.3.2 of Brockwell and Davis, 1987, p.337). Specifically, $D_j$ has the density

$$g(D_j) = \frac{1}{2\pi f(\omega_j)} \exp\left(- \frac{1}{2\pi f(\omega_j)} D_j\right) \tag{9}$$

where $f(\omega_j)$ is the spectral density as in (5). Obviously when $j = 0$ (9) is not well defined as $\lim_{\omega\to 0} f(\omega) = 0$. So we exclude $j = 0$ from our observations of $I(\omega_j)$. From (9) the asymptotic log-likelihood function of $\tilde{d}$ is

$$\log L = - [n/2] \log 2\pi - \sum_{j=1}^{m} \log f(\omega_j) - \sum_{j=1}^{m} \frac{I(\omega_j)}{2\pi f(\omega_j)} \quad . \tag{10}$$

Although we are considering the range of $-0.5 < \tilde{d} \le 0$ , as we will see in the end of this section, the above likelihood function is also valid for $0 < \tilde{d} < 0.5$.

Given this likelihood function, one-sided LM tests can be constructed. One option is the Kuhn-Tucker test. However, this test involves inequality restricted maximum likelihood estimation of $\tilde{d}$ which can be computationally very demanding. Instead, we construct an asymptotic locally best (ALB) test. In general, for testing $H_0$: $\beta = 0$ against $H_a$: $\beta < 0$ (or $\beta > 0$), the one-sided locally best critical region takes the form

$$\left.\frac{\partial \log L}{\partial \beta}\right|_{\beta=0} < c \quad (\text{or} > c'),$$

where c (or c′)is a constant such that the test has a prescribed size (see Ferguson, (1967, p.235)). Applying this to (10), the ALB critical region is given by

$$S_m = \left.\frac{\partial \log L}{\partial \tilde{d}}\right|_{\tilde{d}=0}$$

$$= \sum_{j=1}^{m} \frac{f'(\omega_j)}{2\pi f(\omega_j)^2} \left( I(\omega_j) - 2\pi f(\omega_j) \right)\Big|_{\tilde{d}=0}$$

$$= -2 \sum_{j=1}^{m} \log\left(2\sin(\omega_j/2)\right)\left(\sigma^{-2} I(\omega_j)\frac{|\Phi(e^{-i\omega_j})|^2}{|\theta(e^{-i\omega_j})|^2} - 1\right) \tag{11}$$

$$< c \quad .$$

Note that we can also write

$$\frac{\partial \log L}{\partial \tilde{d}} = \sum_{j=1}^{m} \frac{f'(\omega_j)}{f(\omega_j)} z_j$$

$$= \sum_{j=1}^{m} \left( -2\log(2\sin\omega_j/2) \right) z_j$$

where

$$z_j = \frac{1}{2\pi f(\omega_j)} \left( I(\omega_j) - 2\pi f(\omega_j) \right)$$

with $E(z_j) = 0$ and $V(z_j) = 1$. Let

$$s_m^2 = \sum_{j=1}^{m} \left( -2\log(2\sin\omega_j/2) \right)^2 \quad . \tag{12}$$

Since $s_m^2$ is a constant for fixed m, the ALB test can be equivalently expressed as rejecting $H_0$ for small values of

$$w = \frac{S_m}{s_m} \quad . \tag{13}$$

To establish the asymptotic distribution of w, let

$$Z_j = \frac{1}{2\pi f(\omega_j)} \left( D_j - 2\pi f(\omega_j) \right) \quad .$$

Since $I(\omega_j) \Rightarrow D_j$ (we use $\Rightarrow$ to denote convergence in distribution), we have $z_j \Rightarrow Z_j$. Then using the Cramer-Wold device, $w \Rightarrow W$ with

$$W = \left( \sum_{j=1}^{m} \frac{f'(\omega_j)}{f(\omega_j)} Z_j \Big|_{\tilde{d}=0} \right) / \ s_m$$

$$= \sum_{j=1}^{m} \left( -2\log(2\sin\omega_j/2) \right) Z_j \Big|_{\tilde{d}=0} / \ s_m \quad .$$

This argument holds for any fixed m and $n \to \infty$. Thus it is valid for any arbitrary large m as long as $n/m \to \infty$ as $n \to \infty$. Therefore we only need to establish the asymptotic distribution of W as $m \to \infty$ with $m/n \to 0$. Noticing that $|\Theta(e^{i\omega_j})|^2/|\Phi(e^{i\omega_j})|^2$ is bounded away from zero, it can be shown that the Lyapounov's condition (see Billingsley, (1985, p.371), and the appendix)

$$\lim_{m\to\infty} \sum_{j=1}^{m} \frac{1}{s_m^{2+\delta}} E\left( \left| \left( -2\log(2\sin\omega_j/2) \right) Z_j \Big|_{\tilde{d}=0} \right|^{2+\delta} \right) = 0 \tag{14}$$

holds for $\delta = 2$. Thus from the central limit theorem (Billingsley, *op cit*) W, and therefore w, has asymptotically the standard normal distribution.

In practice, $\phi$, $\theta$ and $\sigma^2$ in w of (13) may be replaced by their consistent estimates under $H_0$, using the standard procedure of fitting an ARMA(p,q) model. The asymptotic normality of w still remains. However as can be seen from (11), the ALB test basically involves comparing the estimated spectral density $I(\omega_j)$ with the theoretical density of an ARMA(p,q) process under the null hypothesis. But fitting an ARMA(p,q) model with data being generated from an ARFIMA(p,$\tilde{d}$,q) process will give biased estimates of $\phi$ and $\theta$ such that the fitted model is forced to behave like an ARFIMA one. This can damage the power of the test. So it is preferred that $\phi$, $\theta$ and $\sigma^2$ in the test statistic (13) be replaced by their consistent estimates from fitting an unrestricted ARFIMA(p,$\tilde{d}$,q) model. Such estimates are available, see Sowell (1992a) and references therein.

Consider now the hypothesis (7). As shown in Yajima (1989), the result of Theorem 10.3.2 of Brockwell and Davis (1987, p.337) still holds for $0 < \tilde{d} < 0.5$,[2] despite $\sum_{i=0}^{\infty}|\psi_i|$ not being bounded. Thus, all the arguments for the case $-0.5 < \tilde{d} \le 0$ are still valid for $0 < \tilde{d} < 0.5$. Eventually, the ALB test for (7) is to reject $H_0$ for large values of w in (13).

## 3. Empirical Power Evidence

In this section, we report an empirical power result of the ALB test from a Monte Carlo experiment. The test is applied to the ARFIMA(0,$\tilde{d}$,0) series, which corresponds to differenced data in applied work. To generate such series, the Cholesky decomposition of the covariance matrix based on (4) is found by the IMSL subroutine LFTDS. Then N independent Gaussian ARFIMA(0,$\tilde{d}$,0) series, each with length n, are generated by the subroutine RNMVN. To mimic the situations found in practice, each series has 2 added. Thus each series has mean 2 and covariances given in (4) with $\sigma^2 = 1$. The above ARFIMA(0,$\tilde{d}$,0) series were generated in N = 5000 replications for each of the sample sizes n = 50, 100, 200, 400. For each sample size, three values of m were chosen with $m = n^{\alpha}$, $\alpha$ = 0.4, 0.5, 0.6. This is an attempt to check changes in power of the test when the rate of increase of m relative to n changes. The power was

[2] Note that the periodogram at $\omega = 0$ is excluded from the likelihood function.

evaluated at the points $\tilde{d}$ = -0.4, -0.2, -0.05, 0.0, 0.05, 0.2, 0.4. In calculating the test statistic, the only nuisance parameter $\sigma^2$ involved is simply replaced by sample variance $\Sigma(Y_t-\bar{Y})^2/n$. This obviously is a valid estimate of $\gamma(0)$ in (4) under $H_0$, but a biased estimate of $\sigma^2$ when $\tilde{d} \neq 0$. As discussed before, this can distort the power of the ALB test. But since the bias is bounded compared to the magnitude of $I(\omega_j)$ which has no upper limit for $\tilde{d} > 0$ and can be arbitrarily close to zero for $\tilde{d} < 0$, we may expect that the extent of any distortion is limited.

In comparison to the ALB test, power of the augmented Dickey-Fuller (ADF) test is also simulated. The ADF test is the t-test ($\hat{\tau}_\tau$) of $\rho = 1$ in the AR(g) regression

$$X_t = \gamma + \upsilon t + \rho X_{t-1} + \sum_{i=1}^{g-1} \beta_i \Delta X_{t-i} + v_t \quad .$$

The ARFIMA(0,d,0) series $X_t$ with $d = 1 + \tilde{d}$ are generated by $X_t = X_{t-1} + 2 + Y_t$, $X_0 = 2$. The orders of g are chosen as $g = [4(n/100)^{1/4}]$, [x] being the integer part of x. The critical values for n = 50, 100 are from Fuller (1976, p.372). Critical values for n = 200, 400 are obtained using interpolation.

The results of the ALB test are summarized in Table 1 on the next page. Consider first the case of testing against $\tilde{d} > 0$ alternatives. The ALB test has good sizes and remarkably good power. The sizes are generally only slightly larger than the nominal size of 0.05. In terms of the high power of the test, this difference between nominal and real size is negligible. The sizes do not seem stable over the range of sample sizes n = 50 to n = 200, but they have tendency to approach the nominal size. A referable feature of the result is that the ALB test is very powerful even in small samples. For example when n = 50, the test rejects the null hypothesis ($\tilde{d} = 0$) 10% of the time at $\tilde{d} = 0.05$, a point which is very close to the null hypothesis. At $\tilde{d}$ = 0.2, the power is above 0.4. As sample sizes increase, power also increases.

Table 1. Power of the ALB tests against the ARFIMA(1, $\tilde{d}$, 0) process*
(one-sided tests against $\tilde{d} > 0$ and $\tilde{d} < 0$)

| n | m | $\tilde{d}$ = .0 | .05 | .2 | .4 | $\tilde{d}$ = .0 | -.05 | -.2 | -.4 |
|---|---|---|---|---|---|---|---|---|---|
| 50 | 5 | .058 | .118 | .430 | .820 | .003 | .005 | .020 | .141 |
| | 7 | .054 | .117 | .434 | .824 | .002 | .003 | .029 | .201 |
| | 10 | .056 | .118 | .431 | .826 | .003 | .004 | .035 | .197 |
| 100 | 6 | .061 | .149 | .657 | .968 | .008 | .018 | .162 | .766 |
| | 10 | .056 | .149 | .686 | .980 | .009 | .026 | .246 | .899 |
| | 16 | .052 | .146 | .693 | .983 | .008 | .025 | .272 | .925 |
| 200 | 8 | .065 | .210 | .848 | .999 | .015 | .045 | .542 | .998 |
| | 14 | .060 | .214 | .894 | 1.000 | .016 | .066 | .710 | 1.000 |
| | 24 | .055 | .209 | .908 | 1.000 | .014 | .070 | .782 | 1.000 |
| 400 | 11 | .063 | .293 | .966 | 1.000 | .022 | .102 | .920 | 1.000 |
| | 20 | .058 | .309 | .987 | 1.000 | .024 | .143 | .979 | 1.000 |
| | 36 | .053 | .319 | .994 | 1.000 | .023 | .164 | .994 | 1.000 |

* Critical values are 1.645 against $\tilde{d} > 0$, −1.645 against $\tilde{d} < 0$ alternatives, respectively. Maximum width of 95% confidence interval of estimates of power is smaller than ∓0.014.

For example when sample size increased from n = 50 to n = 100, power improved from about 0.43 to 0.66 at $\tilde{d} = 0.2$. Another feature of the test is that the power does not differ much as $\alpha$ (or m, with $m = n^{\alpha}$) varies from 0.4 to 0.6. In the Monte Monte Carlo study, other values such as $\alpha = 0.1$, 0.9 have also been attempted. But these choices of $\alpha$ (therefore m) produced very bad sizes, so the result is not reported here for the conciseness of the paper. Finally, from Table 2 on the next page, we notice that the ADF test has low power against fractional integrated alternatives, which echoes the finding of other authors mentioned in section 1. The power advantage of the ALB test over the ADF test in this case is obvious and remarkable.

Next we examine the power of the ALB test against $\tilde{d} < 0$ alternatives. The

Table 2. Power of the ADF tests against the ARFIMA(0, $\tilde{d}$, 0) process*
(one-sided tests against $\tilde{d} > 0$ and $\tilde{d} < 0$)

| n | $\tilde{d}$ = .0 | .05 | .2 | .4 | $\tilde{d}$ = .0 | -.05 | -.2 | -.4 |
|---|---|---|---|---|---|---|---|---|
| 50 | .064 | .087 | .117 | .165 | .046 | .052 | .083 | .162 |
| 100 | .058 | .066 | .121 | .175 | .048 | .052 | .100 | .279 |
| 200 | .049 | .084 | .158 | .233 | .051 | .062 | .186 | .613 |
| 400 | .050 | .082 | .175 | .260 | .045 | .068 | .272 | .848 |

* Critical values against $\tilde{d} > 0$ alternatives are -0.87, -0.90, -0.9133, -0.926 for n = 50, 100, 200, 400, respectively. For against $\tilde{d} < 0$ alternatives, they are -3.50, -3.45, -3.4376, -3.424, respectively. Maximum width of 95% confidence interval of estimates of power is smaller than ∓0.014.

sizes of the test are well below the nominal size. However, as is expected from the theory, real sizes approach the nominal size as sample size becomes larger. Due to the lower real size, the test is not as powerful as against $\tilde{d} > 0$ alternatives. This is particularly true uniformly over different sample sizes at $\tilde{d} = -0.05$, which is a very close alternative to the null hypothesis. For large samples and large absolute values of $\tilde{d}$, the powers in the two cases are compatible. Comparing the power of the ALB test to that of the ADF test, the ALB test dominates for n = 200, 400. For n = 100 where the real sizes of the ALB test are below 0.01, the test is still more powerful for $\tilde{d} \le -0.2$. As to the dependence of power on the choice of m, the situation is largely the same as in the case of against $\tilde{d} > 0$ alternatives. Although it seems from Table 1 that larger values of $\alpha$, like $\alpha = 0.6$, may give slightly better power than the other two values of $\alpha$, our Monte Carlo study shows (not reported here) that the choice of $\alpha$ values either close to zero or close to one produces severe distortion of the size of the test.

## 4. Further Discussion

The evidence of low power of unit root tests against fractionally integrated time series suggests the need for tests designed particularly for these sorts of alternatives. This paper proposes one such test for ARFIMA(p,d,q) processes. The limited Monte Carlo experiment for the case of ARFIMA(0,d,0) indicates that the test can be very powerful, even in small samples. Of course in actual applied situations, general ARFIMA(p,d,q) processes are more likely. In these cases, from its construction, the test is expected to continue to have its strong power as long as the nuisance parameters are estimated under the unrestricted model. One may start with identifying the polynomial orders p and q, following, for example, the procedure of Sowell (1992b). Then the ALB test can be applied to determine whether the unit root hypothesis is adequate to capture the long-run dependency of the time series.

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Appendix

To verify (14), we note that $2\pi f(\omega_j) = \sigma^2|\Theta(e^{i\omega_j})|^2/|\Phi(e^{i\omega_j})|^2$ under the null hypothesis and therefore is bounded. Since $D_j$ follows the exponential distribution with mean $2\pi f(\omega_j)$, $E(Z_j^4|\tilde{d}=0)$ is also bounded. Thus we only need to establish

$$\lim_{m\to\infty} \sum_{j=1}^{m} \frac{1}{s_m^4} \left( -2\log(2\sin\omega_j/2) \right)^4 = 0 \quad .$$

Let u = m/n. Then $u \to 0$ as $n \to \infty$. Noting that $\omega_j = 2\pi j/n$, and $\lim_{m\to\infty}$ is the same as $\lim_{n\to\infty}$, for the numerator,

$$\lim_{n\to\infty} \sum_{j=1}^{m} \frac{1}{n} \left( -2\log(2\sin\pi j/n) \right)^4 = \lim_{u\to 0} \int_0^u \left(-2\log(2\sin\pi x)\right)^4 dx. \qquad \text{(A1)}$$

and for the denominator,

$$\lim_{n\to\infty} \frac{1}{n} s_m^4 = \lim_{n\to\infty} n \left( \frac{1}{n} \sum_{j=1}^{m} \left( -2\log(2\sin\pi j/n) \right)^2 \right)^2$$

$$= \lim_{u\to 0} \frac{m}{u} \left( \int_0^u \left(-2\log(2\sin\pi x)\right)^2 dx \right)^2 \qquad \text{(A2)}$$

Using the Lhopital's rule repeatedly, it is easy to verify that the right hand side of (A1) divided by the right hand side of (A2) is of order O(1/m).

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