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1. Introduction

While the properties of constant elasticity of substitution technologies have been studied extensively (see McFadden (1963), Hanoch (1978) and the references cited therein) the same attention does not seem to have been accorded constant own-price elasticity demand functions. These demand functions have been widely used in empirical work, and in the study of the welfare implications of price stabilisation (see Turnovsky (1976), and Newbery and Stiglitz (1981), for example). Economic literature has considered the implications of systems of constant own-price demand (and income elasticities)[2], but the restrictions imposed by the assumption that a subset of demand functions of a system has constant own-price elasticities have not been presented. The importance of such restrictions lies in the question: do such systems allow relaxation of the restrictive nature of full constant elasticity systems? This paper points out that if a system of firm or consumer demand functions has one demand function with a constant own-price elasticity then, in order to be integrable, significant restrictions must be placed upon the system and that elasticity. Here the term integrable is used to denote the case where unconditional, and conditional firm (consumer) demand functions imply and are implied by profit (indirect utility) functions, without significant restrictions on the nature and domains of these functions.

Constant own-price elasticities are often used because of their apparent simplicity. They greatly simplify expressions which determine the outcome of price stabilisation, for example. Newbery and Stiglitz (1981, pp. 26-27) assume that the demand function facing agricultural producers has a constant elasticity and based on this they show that whether variation in farm income increases or decreases when fluctuating prices are stabilised depends upon whether or not the absolute elasticity is greater or less than 1/2. This turns out to be an uninteresting question if the demand function is that of a firm[3], because the constant own price elasticity must be elastic. In another example Turnovsky (1976, p. 142) argues that the desirability of stabilisation for producers depends upon whether or not a constant elasticity consumer demand function is elastic. This paper demonstrates that if this elasticity is to be derived from an ordinary demand function in which consumer's surplus yields exact measures of welfare change then such an

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[2] For the case of consumer demand functions see Byron (1970) and the ensuing papers - Basmann et al (1973), and Byron (1973).

[3] Newbery and Stiglitz (1981) make extensive use of constant elasticity demand functions when exploring price stabilisation, and buffer stock schemes. Generally they presume that the agricultural commodities are sold directly to final consumers. A wide class of these commodities are sold to manufacturing firms who process the agricultural output before selling it to consumers. The findings of this paper with respect to firm demand functions are germane to commodities which are purchased by firms.

elasticity must be elastic; under these conditions ruling out the comparison of Turnovsky *op. cit.*[4]

Constant own-price elasticity demand functions are widely used in computable general equilibrium models. These demand functions are very restrictive, although they are consistent with optimising behaviour where they are derived explicitly from particular - usually Cobb-Douglas and CES (cf Mansur and Whalley (1984, pp. 88-90)) - production and utility functions. As is well known, where they have been specified on an *ad hoc* basis these demand functions may not be in accord with optimising choices by economic agents. Galper, Lucke, and Toder (1988, pp. 91-108) employ a general equilibrium model to assess budget and welfare effects of the US Tax Reform Act 1986. The model presumes that the demand elasticity for capital is unitary with respect to its own price (user cost). It is shown below that this particular capital demand function - and any constant, elastic cost-minimising demand function - is not integrable, and thus not consistent with the optimising decisions of firms. While Whalley (1984, p. 1033) identifies difficulties with elasticities in computable general equilibrium models he does not mention the issue analysed in this paper.

It is the purpose of this paper to point out that restricting one own-price elasticity to be constant imposes significant constraints on the forms of the demand functions and on the magnitudes of the elasticities. The restrictions on the elasticities are implied by the requirement of integrability of the demand functions. The next section considers the demand functions of firms, and the third section explores consumer demand functions. These hitherto unappreciated findings have implications for the use of constant elasticity functions to place structure on theoretical investigations, and for empirical work, and they are discussed in the fourth and final section.

2. Firm Demand Functions

The firm's profit maximising demand function for input k is described thus

$$x_k(\mathbf{p},\mathbf{w}) = f^k(\mathbf{p},\mathbf{w}^{-k})w_k^{\alpha_k} \tag{1}$$

where x_k denotes input k,

$f^k(\mathbf{p},\mathbf{w}^{-k}) > 0$,

p is the vector of strictly positive output prices,

w is the vector of strictly positve input prices,

[4] Turnovsky (1976) also uses constant elasticity demand functions to study consumer welfare changes resulting from stabilisation.

$\mathbf{w}^{-k}$ is the vector of all input prices but the kth,

and α_k is a constant.

The profit-maximising input demand functions, namely (1), are the focus of this section of the paper. The establishment of the properties of these functions draws upon

Proposition 1: *The kth input demand function will have a constant own-price elasticity if and only if it has the form (1).*

This proposition is demonstrated in Appendix A, and it is reported at least as early as Wold and Jureen (1953, 105), and it will apply to consumer demand functions as well as those of firms. It implies that input k has a constant own-price elasticity of demand if and only if its demand function has the form (1). In order to ascertain restrictions on the magnitude of the constant own-price elasticity it is useful to consider the integrability of the demand function, and this is done by drawing on the fact that the demand function's existence is implied by, and implies, the existence of a profit function. This relationship between functional forms of the demand function, (1), and the profit function is described in

Proposition 2: *Given that the profit function is twice continuously differentiable, any profit-maximising input demand function - say, the kth - will have an own-price constant elasticity if and only if the concomitant profit function has the form*

$$\pi(p,w) = \pi_1(p,w^{-k})w_k^{\alpha_k^*} + \pi_2(p,w^{-k}) \tag{2}$$

where $\pi_2(p,w^{-k}) \equiv \pi(p,w^{-k},w_k \rightarrow \infty)$.

Proof: Differentiability of the profit function means that Hotelling's lemma provides the identity

$$x_k(\mathbf{p},\mathbf{w}) \equiv -\,\partial\pi(\mathbf{p},\mathbf{w}^{-k},w_k)/\partial w_k.$$

and hence that the profit function of this proposition implies that the demand function is

$$\begin{aligned} x_k(\mathbf{p},\mathbf{w}^{-k},w_k) &= -\,\alpha_k^*\pi_1(\mathbf{p},\mathbf{w}^{-k})w_k^{\alpha_k^*-1} \\ &\equiv f^k(\mathbf{p},\mathbf{w}^{-k})w_k^{\alpha_k} \end{aligned} \tag{3}$$

where $\alpha_k = \alpha_k^*-1$, and $f^k(\mathbf{p},\mathbf{w}^{-k}) > 0$. To establish that (3) implies the profit function of this proposition notice that, given that $\mathbf{p}$, and $\mathbf{w}^{-k}$ are held constant, integration yields

$$\begin{aligned} \pi(\mathbf{p},\mathbf{w}) &= \pi(\mathbf{p},\mathbf{w}^{-k},w_k) \\ &= \int_a^{w_k} [\partial\pi(\mathbf{p},\mathbf{w}^{-k},s)/\partial w_k]ds + h^k(\mathbf{p},\mathbf{w}^{-k}) \\ &= \int_a^{w_k} -\,f^k(\mathbf{p},\mathbf{w}^{-k})s^{\alpha_k}\,ds + h^k(\mathbf{p},\mathbf{w}^{-k}) \\ &= f^k(\mathbf{p},\mathbf{w}^{-k})\int_{w_k}^{a} s^{\alpha_k}\,ds + h^k(\mathbf{p},\mathbf{w}^{-k}) \end{aligned}$$

$$= f^k(\mathbf{p},\mathbf{w}^{-k})\Big[\, 1/(\alpha_k + 1)s^{\alpha_k+1} \Big]_{w_k}^{a} + h^k(\mathbf{p},\mathbf{w}^{-k})$$

$$= -1/(\alpha_k + 1)f^k(\mathbf{p},\mathbf{w}^{-k})w_k^{\alpha_k + 1} + h^k(\mathbf{p},\mathbf{w}^{-k})$$
$$\equiv \pi_1(\mathbf{p},\mathbf{w}^{-k})w_k^{\alpha_k^*} + \pi_2(\mathbf{p},\mathbf{w}^{-k})$$

letting $a \to \infty$, assuming $\alpha_k < -1$, and where $\alpha_k^* \equiv \alpha_k + 1$, and $h^k(\mathbf{p},\mathbf{w}^{-k}) = \pi(\mathbf{p},\mathbf{w}^{-k},w_k \to \infty)$.

The profit function, when $w_k \to \infty$, leads naturally to the following definition.

<u>Definition:</u> $x_k(\mathbf{p},\mathbf{w})$ is termed an essential input if and only if $\pi(\mathbf{p},\mathbf{w}^{-k},w_k \to \infty) = 0$.

If x_k is essential then the profit maximising level of the input does not fall quickly enough as $w_k \to \infty$ for positive profits to be produced.

In order to elaborate the requirement that $\alpha_k < -1$ consider the profit function for a given a. It is

$$\pi(\mathbf{p},\mathbf{w},a) = \begin{cases} f^k(\mathbf{p},\mathbf{w}^{-k})/(\alpha_k + 1)[a^{\alpha_k+1} - w_k^{\alpha_k+1}] + h^k(\mathbf{p},\mathbf{w}^{-k}) & \text{for } \alpha_k + 1 \neq 0 \quad (4.1) \\ f^k(\mathbf{p},\mathbf{w}^{-k})[\ln a - \ln w_k] + g^k(\mathbf{p},\mathbf{w}^{-k}) & \text{for } \alpha_k + 1 = 0 \quad (4.2) \end{cases}$$

Considering only (4.1), the profit function property of homogeneity of degree 1 in prices (cf McFadden (1978)) means that

$$\begin{aligned} \pi(\lambda\mathbf{p},\lambda\mathbf{w},a) &= f^k(\lambda\mathbf{p},\lambda\mathbf{w}^{-k})1/(\alpha_k + 1)[a^{\alpha_k+1} - \lambda w_k^{\alpha_k+1}] + h^k(\lambda\mathbf{p},\lambda\mathbf{w}^{-k}) \\ &= \lambda(f^k(\mathbf{p},\mathbf{w}^{-k})1/(\alpha_k + 1)[a^{\alpha_k+1} - w_k^{\alpha_k+1}] + h^k(\mathbf{p},\mathbf{w}^{-k})) \\ &= \lambda\pi(\mathbf{p},\mathbf{w},a). \end{aligned}$$

for $\lambda > 0$. The second equality requires setting a^{α_k+1} to zero. The requirement $\alpha_k+1 < 0$ and $a \to \infty$ sets a^{α_k+1} to zero, and this is the procedure of Proposition 2. The only other possibility is that $\alpha_k+1 > 0$ and $a \to 0$. Thus, the profit function would then be

$$\pi(\mathbf{p},\mathbf{w}) = -1/(\alpha_k + 1)f^k(\mathbf{p},\mathbf{w}^{-k})w_k^{\alpha_k + 1} + h^k(\mathbf{p},\mathbf{w}^{-k}) \quad \text{for } \alpha_k + 1 > 0$$

where $h^k(\mathbf{p},\mathbf{w}^{-k}) \equiv \pi(\mathbf{p},\mathbf{w}^{-k},w_k \to 0)$. The profit function must be convex, and non-negative, and for the latter it is required that

$$1/(\alpha_k + 1)f^k(\mathbf{p},\mathbf{w}^{-k})w_k^{\alpha_k + 1} \leq h^k(\mathbf{p},\mathbf{w}^{-k}) \qquad (5)$$

for all $\mathbf{p},\mathbf{w}^{-k}$, and, in particular, for $w_k > 0$, which cannot be the case if $f^k(\mathbf{p},\mathbf{w}^{-k})$, and $h^k(\mathbf{p},\mathbf{w}^{-k})$ are finite. If either of these functions is not finite then changes in w_k (and x_k) will have no effect on profits and hence study of the behaviour of x_k will not be interesting. When $\alpha_k+1 > 0$ the demand function may be locally integrable over the set {$\mathbf{p}$, $\mathbf{w}$ such that (5) is satisfied} although it is not generally integrable. From this point the term "integrable" is used to mean globally integrable. It is noteworthy from (5) that for $\alpha_k + 1 > 0$ it must be that

$$\pi(\mathbf{p},\mathbf{w}^{-k},w_k \to 0) \equiv \pi^*_2(\mathbf{p},\mathbf{w}^{-k}) > 0.$$

Turning now to (4.2) it is apparent that this function is not homogeneous of degree 1 in all prices for any $a > 0$. Thus a profit function does not exist if a demand function has a constant elasticity equal to 1.

This discussion of the rationale for Proposition 2 establishes that the profit function is not well behaved - and hence that the kth demand function is not generally integrable - if $\alpha_k \geq -1$. This conclusion yields the central point of this section, and it is summarised in

__Proposition 3:__ If the profit function is twice continuously differentiable, a demand function which has a constant own-price elasticity must be own-price elastic.

A parallel finding for output conditioned, cost-minimising input demand functions is given in:

__Proposition 4:__ If the cost function is twice continuously differentiable and a conditional input demand function $x_i = x_i(\mathbf{q},\mathbf{w})$ - where $\mathbf{q}$ is a vector of outputs - has a constant own-price elasticity then

i) the cost function is of the form

$$C(\mathbf{q},\mathbf{w}) = c_1(\mathbf{q},\mathbf{w}^{-k})w_k^{\delta_k^*} + c_2(\mathbf{q},\mathbf{w}^{-k}), \text{ and}$$

ii) the demand function must be inelastic.

the establishment of which parallels the approach for Proposition 3, and is set out in Appendix A. This proposition suggests

__Definition:__ $x_k(\mathbf{q},\mathbf{w})$ is termed potentially dominant if and only if $C(\mathbf{q},\mathbf{w}^{-k},w_k \to 0) = 0$.

which is used subsequently. All inputs in a Cobb-Douglas production function are potentially dominant in that, as the price of any one of them approaches zero, this input can be substituted for all others to the extent that production can be maintained at virtually zero cost.

The constant own-price elasticity has implications for the form and nature of cross price elasticities. These are indicated in [5]

Proposition 5: *(i) If the kth input is essential and its profit maximising demand function is given by (1) then* [6]

$$\eta^j{}_k(p,w) \equiv \frac{\partial x_j(p,w)}{\partial w_k}\frac{w_k}{x_j} = \alpha_k^* \quad j \neq k, j = 1, \ldots, n \qquad \textit{inputs}$$

and

$$\eta^{qj}{}_k(p,w) \equiv \frac{\partial q_j(p,w)}{\partial w_k}\frac{w_k}{q_j} = \alpha_k^* \quad j = 1, \ldots, m \qquad \textit{outputs}$$

(ii) If the kth input is potentially dominant and its cost minimising demand function has a constant own-price elasticity then

$$\eta^j{}_k(q,w) \equiv \frac{\partial x_j(q,w)}{\partial w_k}\frac{w_k}{x_j} = \delta_k^* \quad j \neq k, j = 1, \ldots, n \qquad \textit{inputs}$$

where δ_k^* *is that of the cost function in Proposition 4.*[7]

Thus all cross elasticities with respect to the input price w_k are constant, equal and negative (positive) if the kth input is essential (potentially dominant) and has a profit maximising (cost minimising) demand function with a constant own-price elasticity.

There are other restrictions implied by the assumption of a constant elasticity of demand. Consider the way the effect of an input price change is readily divided into separate substitution and output effects. Using the ith demand function $x_i(\mathbf{p},\mathbf{w}) \equiv x_i(\mathbf{q}(\mathbf{p},\mathbf{w}),\mathbf{w})$

$$\frac{\partial x_i(\mathbf{p},\mathbf{w})}{\partial w_i} = \frac{\partial x_i(\mathbf{q},\mathbf{w})}{\partial w_i} + \sum_{j=1}^{m} \frac{\partial x_i(\mathbf{q},\mathbf{w})}{\partial q_j}\frac{\partial q_j}{\partial w_i}$$

which means that, where there are m outputs, the profit maximising own-price elasticity is

$$\eta^i{}_i(\mathbf{p},\mathbf{w}) = \eta^i{}_i(\mathbf{q},\mathbf{w}) + \sum_{j=1}^{m} \eta^i{}_{qj}(\mathbf{q},\mathbf{w})\eta^{qj}{}_i(\mathbf{p},\mathbf{w}). \qquad (5)$$

which is a sort of Slutsky equation for the firm. If the kth input has a constant own-price profit-maximising demand function is essential then, using Proposition 5 part (i),

$$\alpha_k = \eta^k{}_k(\mathbf{q},\mathbf{w}) + \sum_{j=1}^{m} \eta^k{}_{qj}(\mathbf{q},\mathbf{w})\alpha_k^*$$

[5] From the profit function (2) - where because the kth input is essential - $\pi_2(\mathbf{p},\mathbf{w}^{-k}) = 0$ - $\partial x_j(\mathbf{p},\mathbf{w})/\partial w_k = -\partial^2\pi_1(\mathbf{p},\mathbf{w}^{-k})w_k^{\alpha_k^*}/\partial w_j\partial w_k = -\alpha_k^*[\partial\pi_1(\mathbf{p},\mathbf{w}^{-k})w_k^{\alpha_k^*}/\partial w_j]/w_k = \alpha_k^* x_j/w_k$ (by Hotelling's lemma) which establishes $\eta^j{}_k(\mathbf{p},\mathbf{w}) = \alpha_k^*$. The proof of $\eta^{qj}{}_k(\mathbf{p},\mathbf{w}) = \alpha_k^*$ follows analogous steps.

[6] $\eta^i{}_j$ denotes the elasticity of i with respect to input price j.

[7] The proof of part (ii) of Proposition 5 is analogous to that of part (i) with the exception that it uses the cost function rather than the profit function.

which entails

$$\sum_{j=1}^{m} \eta^k_{qj}(q,w) = \kappa$$

a constant, when $\eta^k_k(q,w)$ is constant. In particular, for the single output case this implies $\eta^k_q(q,w) = \kappa$ and hence that in this case the cost function must be of the form $q^{1/\mu}c(w)$ where $\kappa = 1/\mu$. Because the cost function is of this form if, and only if, the production function is homogeneous of degree μ (see Diewert (1982)), this sketches the argument for

Proposition 6: *Suppose input k has a constant own-price profit maximising elasticity of demand, and is essential and that there is a single output, then constancy of the own-price cost minimising input demand implies that the production function must be homogeneous.*

The generality of Propositions 3 and 4 is surprising. The severe restrictions of these propositions are required in order that a constant own-price elastic demand function be part of an integrable system, and they are present for very general forms of the profit function. The Cobb-Douglas profit function is self-dual, in that it implies and is implied by a Cobb-Douglas production function. For this functional form these restrictions of the propositions are readily shown algebraically (see Appendix B). However, the restrictions are applicable to any constant elasticity demand function, even those which admit multiple outputs.

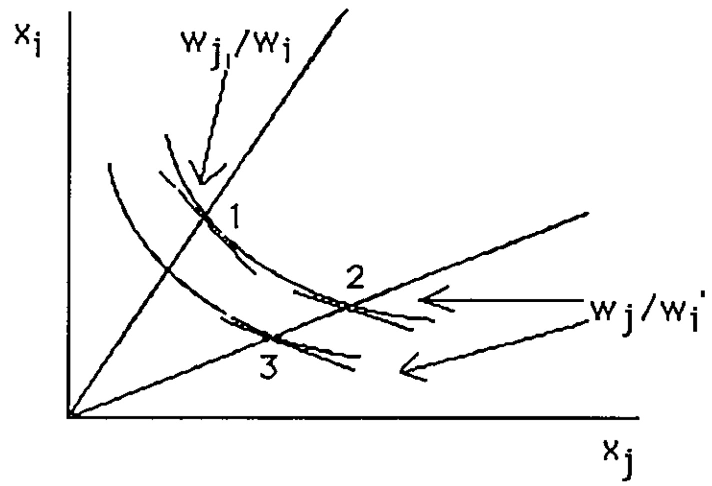
In the following diagram is depicted the case where the profit maximising and cost minimising demand functions for a single output technology each have constant own-price elasticities of demand. From Proposition 6 the production function will be homogeneous, and (5) takes the form

$$\eta^i_i(p,w) = \eta^i_i(q,w) + 1/\mu\eta^q_i(p,w) \qquad (6)$$

where the degree of homogeneity is $\mu < 1$.

In the diagram $w_i' > w_i$, and the consequent profit-maximising change from 1 to 3 is depicted for the homogeneous production function. When the ith price is raised the price of the ith input changes relative to that of input j leading to a movement along the isoquant from 1 to 2. The ith inputs own-price (point) elasticity for this movement is $\eta^i_i(q,w)$, and if it is constant then it must be inelastic by Proposition 4. The ith price also changes relative to the output price yielding the movement from 2 to 3 in the diagram. The (point) measure of this output effect on the profit-maximisation elasticity, $\eta^i_i(p,w)$, in (6) being $1/\mu\eta^q_i(p,w)$. From Proposition 3, if the ith profit-maximising demand function has a constant own-price elasticity then these substitution and output effects combine to ensure that the profit-maximising demand function for that input is own-price elastic. Thus the change in x_i in the move from 1 to 3 is elastic despite the fact that the movement from 1 to 2 will have an own-price elasticity which is inelastic. These characteristics of (6) for constant profit-maximising and cost-minimising input demand functions are readily demonstrated algebraically for the Cobb-Douglas production function using the elasticities of Appendix B.

Output and Input Responses to an Increase In the Price of the ith Input



While the technical reasons for the consequences of constant own-price elasticities are clear the economic rationale is not obvious. Some indication of this can be obtained by consideration of the Morishima elasticity of substitution for the single-output technology between inputs. Blackorby and Russell (1989) develop a cogent argument for this elasticity being the appropriate measure of ease of substitution between inputs when the number of these inputs is greater than 2. The Morishima elasticity for a change in the ith price is defined by Blackorby and Russell (1981) to be

$$M_{ij}(q,w) = \eta_{ji}(q,w) - \eta_{ii}(q,w).$$

Applying the general approach which established Proposition 5 to a constant own-price cost minimising input demand and its concomitant cost function yields[8]

Proposition 7: *If the kth cost minimising demand function has a constant own price elasticity of demand and is potentially dominant then*

$$M_{kj}(q,w) = 1 \qquad \text{for} \quad j \neq k, j = 1, \ldots, n.$$

[8] Using Proposition 5 part (ii) $M_{kj}(q,w) = \eta_{jk}(q,w) - \eta_{kk}(q,w) = \delta_k^* - (\delta_k^* - 1) = 1$.

The restriction to a constant own-price elasticity has the effect of restricting the elasticity of substitution between that input and others to a constant elasticity equal to one. This is the magnitude of the Cobb-Douglas elasticity of substitution, and for this production function all own-price elasticities are constant. However, Proposition 7 does not suggest a Cobb-Douglas functional form for the underlying production function. The Morishima elasticity is not symmetric, and if k is the only input with a constant own-price elasticity $M_{jk}(q,w) \neq 1$ for $j \neq k$, $j = 1, \ldots, n$ which violates the requirements for a Cobb-Douglas production function.[9] Nevertheless, the constant elasticity restriction generates Cobb-Douglas like substitution elasticities with respect to the particular input on which the restriction is imposed.

3. Consumer Demand Functions

The Hicksian demand functions in which real income is held constant are considered first. For these demand functions to be integrable it must be that they imply, and are implied by, the existence of an expenditure function. Drawing the obvious parallel between cost and expenditure functions Proposition 4 can be invoked to immediately obtain

Proposition 8: *If the consumer expenditure function is twice continuously differentiable and a conditional commodity demand function $x_i = x_i(u,\mathbf{p})$ - where u denotes utility level and* ***p*** *a vector of consumption goods prices - has a constant own-price elasticity then*

i) the expenditure function is of the form

$$E(u,\mathbf{p}) = E_1(u,\mathbf{p}^{-i})p_i^{\delta_i} + E_2(u,\mathbf{p}^{-i}), \text{ and}$$

ii) the demand function must be inelastic.

Furthermore, the one-to-one relationship between the expenditure function and the indirect utility function leads naturally to

Proposition 9: *If the consumer expenditure function is twice continuously differentiable and a conditional input demand function $x_i = x_i(u,\mathbf{p})$ has a constant own-price elasticity and is potentially dominant then*

i) the indirect utility function is of the form $V(I,\mathbf{p}) = V(Ip_i^{-\delta_i}, \mathbf{p}^{-i})$

ii) the Marshallian demand function $x_i = x_i(I,\mathbf{p}) = (\delta_i I)/p_i$

where I is consumer income.

Proof: From Proposition 8 and potential dominance $E(u,\mathbf{p}) = E_1(u,\mathbf{p}^{-i})p_i^{\delta_i}$. The requirement that expenditure equal income, and hence

$$E_1(u,\mathbf{p}^{-i}) = Ip_i^{-\delta_i}.$$

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[9] See the discussion of Blackorby and Russell (1989, p.885)

Inversion of the expenditure function with respect to its u argument (see Barten and Bohm (1982, p. 413)) yields

$$V(I,\mathbf{p}) = u = E_I^{-1}(Ip_i^{-\delta_i},\mathbf{p}^{-i}) = V(Ip_i^{-\delta_i},\mathbf{p}^{-i})$$

which establishes part (i). Application of Roy's lemma establishes part (ii).

Thus the assumptions that the Hicksian demand is constant own-price elastic and potentially dominant places most severe restrictions on the form of the Marshallian demand function. In particular it implies that the Marshallian demand has unitary elasticity with respect to income and its own price. These characterise the demand functions of Cobb-Douglas utility functions. Another feature of the form of the indirect utility function in i) means that $\partial V/\partial I$ will be a function of p_i and hence consumer surplus calculated on the basis of the demand curve in ii) will not provide an exact measure of welfare change. Incidentally, it is noteworthy that the corollary of part (i) also holds. That is, if $V(I,\mathbf{p}) = V(Ip_i^{-\delta_i},\mathbf{p}^{-i})$ then the ith good is potentially dominant and has an inelastic constant own-price Hicksian elasticity of demand.

Finally let us consider the constant own-price elasticity Marshallian demand function. Here use is made of the fact that integrability means that existence of the the demand function is necessary and sufficient for the existence of an indirect utility function. Application of the reasoning leading to Proposition 1 yields the conclusion that the ordinary demand function will be constant own-price elastic if and only if it has the general form

$$x_k(I,\mathbf{p}) = g^k(I,\mathbf{p}^{-k})p_k^{\alpha_k}$$

which, when conjoined with Roy's lemma (1943) results in

$$x_k(I,\mathbf{p}) = \frac{-\partial V(I,\mathbf{p}^{-k},p_k)/\partial p_k}{\partial V(I,\mathbf{p}^{-k},p_k)/\partial I} = g^k(I,\mathbf{p}^{-k})p_k^{\alpha_k}.$$

from which it is straight forward to infer [10] that

$$\partial V(I,\mathbf{p}^{-k},p_k)/\partial p_k = h^k(I,\mathbf{p})z^{p_k}(I,\mathbf{p}^{-k})p_k^{\nu_k}.$$

and $$\partial V(I,\mathbf{p}^{-k},p_k)/\partial I = h^k(I,\mathbf{p})z^I(I,\mathbf{p}^{-k})p_k^{\varpi_k}.$$

where $\alpha_k = \nu_k - \varpi_k$, and $h^k()$, $z^{p_k}()$, and $z^I()$ may be different functions.

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[10] The inference is conducted by taking logarithms of the left- and right-hand sides of the second equality and differentiating with with respect to the logarithm of p_k. Integrating back the resulting expression, $\partial \ln[-\partial V(I,\mathbf{p}^{-k},p_k)/\partial p_k]/\ln p_k - \partial \ln[\partial V(I,\mathbf{p}^{-k},p_k)/\partial I]/\ln p_k = \alpha_k$ with respect to $\ln p_k$ yields the functional form in the text.

The particular case where the demand function is employed to indicate welfare change which result from price changes is of particular interest because of the wide use of a constant own-price elasticity in this assessment. If the demand function is such that it provides an exact measure of consumer surplus then the following proposition is applicable.

Proposition 10: *(i) If the indirect utility function is twice continuously differentiable and has the property that consumer surplus calculated for changes in the price of the kth good is an accurate measure of welfare change then the kth demand function will have an own-price constant elasticity*

(i) if and only if the concomitant indirect utility function has the form

$$V(I,p) = v_1(p^{-k})p_k^{\alpha_k^*} + v_2(I,p^{-k})$$

where $v_2(I,p^{-k}) \equiv V(I,p^{-k},p_k \to \infty)$, *and*

(ii) the good is own-price elastic.

Proof: Application of Roy's lemma to the indirect utility function of (i) will obviously yield a constant elasticity of demand of the form (7).

The assumption that for changes in the kth price the consumer surplus is an accurate measure of welfare change requires that the indirect utility function has the form $\partial^2 V(I,\mathbf{p})/\partial I \partial p_k = 0$ (see for example Turnovsky, Shalit, and Schmitz (1980, p.140)). Starting with the constant elasticity demand function, application of the argument leading to Proposition 1 together with differentiability of the indirect utility function - which ensures that Roy's (1943) lemma is operational - provide

$$x_k(I,\mathbf{p}) \equiv \frac{-\partial V(I,\mathbf{p}^{-k},p_k)/\partial p_k}{\partial V(I,\mathbf{p}^{-k},p_k)/\partial I} = g^k(I,\mathbf{p}^{-k})p_k^{\alpha_k}$$

Now $\partial^2 V(I,\mathbf{p})/\partial I \partial p_k = 0$, implies

$$\partial V(I,\mathbf{p}^{-k},p_k)/\partial I = \partial V(I,\mathbf{p}^{-k})/\partial I$$

and $\quad \partial V(I,\mathbf{p}^{-k},p_k)/\partial p_k = \partial V(\mathbf{p}^{-k},p_k)/\partial p_k \equiv h^k(\mathbf{p}^{-k})p_k^{\alpha_k}$ [11]

which means that

$$x_k(I,\mathbf{p}) \equiv \frac{-\partial V(\mathbf{p}^{-k},p_k)/\partial p_k}{\partial V(I,\mathbf{p}^{-k})/\partial I} = g^k(I,\mathbf{p}^{-k})p_k^{\alpha_k}$$

Hence the indirect utility function of this proposition can be obtained, recognising that, I, and $\mathbf{p}^{-k}$ are held constant, as

$$\begin{aligned} V(I,\mathbf{p}) &= V(I,\mathbf{p}^{-k},p_k) \\ &= \int_a^{p_k} [\partial V(I,\mathbf{p}^{-k},s)/\partial p_k]ds + z^k(I,\mathbf{p}^{-k}) \\ &= \int_a^{p_k} \partial V(\mathbf{p}^{-k},s)/\partial p_k]ds + z^k(I,\mathbf{p}^{-k}) \\ &= \int_a^{p_k} h^k(\mathbf{p}^{-k})s^{\alpha_k}ds + z^k(I,\mathbf{p}^{-k}) \end{aligned}$$

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[11] That is, the derivative of the indirect utility function with respect to p_k must be mutiplicative in $p_k^{\alpha_k}$ (because of the form of the demand function, and the fact that $\partial V(I,\mathbf{p}^{-k},p_k)/\partial I$ is not a function of p_k), and not a function of I (because $\partial^2 V(I,\mathbf{p})/\partial I \partial p_k = 0$).

$$= - h^k(p^{-k})\Big[1/(\alpha_k + 1)s^{\alpha_k+1} \Big]_{p_k}^{a} + z^k(I,p^{-k})$$
$$= - 1/(\alpha_k + 1)h^k(p^{-k})p_k^{\alpha_k + 1} + z^k(I,p^{-k})$$
$$\equiv v_1(p^{-k})p_k^{\alpha_k^*} + v_2(I,p^{-k})$$

letting $a \to \infty$, and assuming $\alpha_k^* \equiv \alpha_k + 1 < 0$. This establishes part i), and part ii) is established from the inequality $\alpha_k + 1 < 0$.

Again the restriction of a constant elasticity has significant implications for the characteristics of demand functions and their concomitant primal and dual functions. Propositions 9 and 10 together indicate that very different restrictions are implied by the assumptions of Hicksian and Marshallian constant own-price demand functions, when the Marshallian function is such that consumer's surplus is an accurate measure of welfare change. The assumption on the Hicksian demand function entails that the own-price elasticity being inelastic, but on the Marshallian demand function it implies that the demand function must be elastic. In the presence of potential dominance, the Hicksian version implies that the Marshallian demand function has unitary elasticity, and that the ordinary, or Marshallian, demand function will not yield consumer's surpluses which exactly measure welfare change.[12] If it is desired to measure changes in welfare precisely, and to confine attention to constant own-price elasticities, then either the expenditure function can be used to compute compensating or equivalent variations, or the indirect utility function (which yields a constant elasticity Marshallian demand function which provides exact measures of consumer's surplus) can be used to calculate consumer's surplus: but the variations and consumer's surplus approaches will not be consistent. This is not unexpected because the particular expenditure and indirect utility functions are not self-dual.

4. Comment

The assumption of a constant own-price elasticity on just one demand function in a demand system places stiff restrictions on the nature of the demand functions and on the tastes or technology they represent. The general implication of this assumption for firms is that profit-maximising (cost-minimising) demand functions must be elastic (inelastic). For consumers it is that ordinary (compensated) demand functions must be elastic (inelastic), if the demand function is to represent consumer's surplus as a precise measure of welfare. Various other restrictions flow from the constant elasticity

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[12] Turnovsky (1976, 142-144) uses constant elasticity demand functions to study consumer welfare changes resulting from stabilisation, although it does not recognise that for consumer's surplus to be measured accurately the ordinary demand function's constant elasticity must be elastic, nor that this elasticity will not be 1 in an integrable system. Also, it's specification of a Cobb-Douglas utility function (p. 137) is not in accord with measuring consumer's surplus accurately under a demand curve.

assumption. It is noteworthy that different functional forms of the dual functions - that is, indirect utility, profit, cost, and expenditure functions - are implied by elasticities which differ according to whether they denote elastic or inelastic demand functions.

If global integrability is not required, then restricting the nature of profit or utility functions and their domain can allow the implied restrictions on the magnitudes of the elasticities to be relaxed.

The restrictions implied by the constant elasticity assumption are properly part of the integrability conditions of the associated demand system. Hence, the specification of a constant own-price demand function in empirical work should include the implications of this assumption as maintained integrability restrictions or it may be used as a specification test. The restrictions have significant implications for empirical work. The example of a translog profit function serves to describe the empirical issues raised by the constant elasticity assumption for the firm. This function and its concomitant share equations are:

$$\ln\pi = \alpha_0 + \Sigma_i\, \alpha_i \ln q_i \;+\; \Sigma_i\, \Sigma_j\, \alpha_{ij} \ln q_i \ln q_j \;+\; \varepsilon_0$$

$$\frac{\partial \ln\pi}{\partial \ln q_f} = \alpha_f + \Sigma_j\, \alpha_{fj} \ln q_j \;+\; \varepsilon_f \qquad f = 1, .., m+n$$

where $\Sigma_i\, \alpha_i = 1$, $\alpha_{fj} = \alpha_{jf}$, $\Sigma_j \alpha_{fj} = 0$ $f = 1, .., m+n$, and the elements of $\mathbf{q} = (\mathbf{p}', \mathbf{w}')'$, that is, q_f, are the input and output prices. This function may be convex over a subset of the domain of prices. Let $q_k = w_k$ then hypothesis that the kth demand function has a constant elasticity and is essential is

$$\alpha_k < -1, \text{ and } \alpha_{fk} = 0 \quad f = 1, .., m+n.$$

Recall from section 2 that the profit function is

$$\pi(\mathbf{p},\mathbf{w}) = \begin{cases} \pi_1(\mathbf{p},\mathbf{w}^{-k}) w_k^{\alpha_k^*} + \pi_2(\mathbf{p},\mathbf{w}^{-k}) & \text{for} \quad \alpha_k^* = \alpha_k + 1 < 0 \\ \pi_1(\mathbf{p},\mathbf{w}^{-k}) w_k^{\alpha_k^*} + \pi^*{}_2(\mathbf{p},\mathbf{w}^{-k}) & \text{for} \quad \alpha_k^* = \alpha_k + 1 > 0 \end{cases}$$

and that $\pi_2(\mathbf{p},\mathbf{w}^{-k}) = 0$ if the k^{th} input is essential, but $\pi^*{}_2(\mathbf{p},\mathbf{w}^{-k}) > 0$. It is immediately apparent that the translog cannot incorporate the case where input k has an own price elasticity which is constant and which is an inessential input to production: $\alpha_{fjk} = 0$ $f = 1, .., m+n$ is required to generate a constant elasticity, and yet this implies that input k is essential. If input k is essential then by the discussion of

Proposition 2 it cannot be -1, and it must be elastic; thus yielding the hypothesis cited above.

While their use should be avoided, constant own-price elasticity demand functions are convenient to use. They may be used to approximate more general demand functions in economic theory. However, their implications for the remaining demand functions and for the nature of tastes or technology should be recognised if the generality of the theory is to be properly appreciated and if inconsistencies are to be avoided.

Appendix A:

Proposition 1: It is clear from the multiplicative functional form that the demand function for input k has a constant elasticity. In establishing the reverse implication for the consider the constant elasticity for the kth input

$$\frac{\partial \ln x_k}{\partial \ln w_k} = \alpha_k$$

Because $d\ln x_k = \frac{\partial \ln x_k}{\partial \ln w_k} d\ln w_k$ integrating with respect to w_k yields

$$\int d\ln x_k = \alpha_k \int d\ln w_k \quad + \quad \ln f^k(\mathbf{p},\mathbf{w}^{-k})$$

where $\mathbf{p}$ and $\mathbf{w}^{-k}$ are held constant. Consequently,

$$\ln x_k = \alpha_k \ln w_k \quad + \quad \ln f^k(\mathbf{p},\mathbf{w}^{-k})$$

which establishes that an input demand function has constant own-price elasticities if and only if it has the form (1). Notice that $f^k(\mathbf{p},\mathbf{w}^{-k}) > 0$.

Proposition 4: In establishing Proposition 4, differentiability of the cost function means that Shephard's lemma provides the identity

$$x_k(\mathbf{q},\mathbf{w}) \equiv - \partial C(\mathbf{q},\mathbf{w}^{-k},w_k)/\partial w_k.$$

and hence that the cost function of this proposition implies that the demand function is

$$x_k(\mathbf{q},\mathbf{w}^{-k},w_k) = \delta_k^* c_1(\mathbf{q},\mathbf{w}^{-k}) w_k^{\delta_k^*-1}$$

$$\equiv f^k(\mathbf{q},\mathbf{w}^{-k}) w_k^{\delta_k} \qquad \text{(A3)}$$

where $\delta_k = \delta_k^*-1$. To establish that (A3) implies the cost function of this proposition notice that, given that q, and w^{-k} are held constant, integration yields

$$C(\mathbf{q},\mathbf{w}) = C(\mathbf{q},\mathbf{w}^{-k},w_k)$$

$$= \int_a^{w_k} [\partial C(\mathbf{q},\mathbf{w}^{-k},s)/\partial w_k] ds \; + \; z^k(\mathbf{q},\mathbf{w}^{-k})$$

$$= \int_a^{w_k} f^k(\mathbf{q},\mathbf{w}^{-k}) s^{\delta_k} \, ds \; + \; z^k(\mathbf{q},\mathbf{w}^{-k})$$

$$= f^k(\mathbf{q},\mathbf{w}^{-k}) \int_a^{w_k} s^{\delta_k} \, ds \; + \; z^k(\mathbf{q},\mathbf{w}^{-k})$$

$$= f^k(\mathbf{q},\mathbf{w}^{-k}) \Big[1/(\delta_k + 1) s^{\alpha_k+1} \Big]_a^{w_k} \; + \; z^k(\mathbf{q},\mathbf{w}^{-k})$$

$$= 1/(\delta_k + 1) f^k(\mathbf{q},\mathbf{w}^{-k}) w_k^{\delta_k + 1} \; + \; z^k(\mathbf{q},\mathbf{w}^{-k})$$

$$\equiv c_1(\mathbf{q},\mathbf{w}^{-k}) w_k^{\delta_k^*} \; + \; c_2(\mathbf{q},\mathbf{w}^{-k})$$

letting $a \rightarrow 0$, assuming $\delta_k > -1$, and where $\delta_k^* \equiv \delta_k + 1$, and $z^k(\mathbf{q},\mathbf{w}^{-k}) = c(\mathbf{q},\mathbf{w}^{-k},w_k \rightarrow 0)$. The restriction $\delta_k > -1$ is required for integrability and it implies that the cost-minimising demand function will be inelastic if it is own-price constant.

From analogous reasoning to that for Proposition 1, the kth conditional demand function will have an own-price constant elasticity if, and only if, it has the general form $x_k(q,w^{-k},w_k) = f^k(q,w^{-k})w_k^{\delta_k}$.

Appendix B: The Cobb-Douglas Case

It is well known that, because the Cobb-Douglas production function is self dual the profit function is Cobb-Douglas if and only if the production function is Cobb-Douglas. Consider the Cobb-Douglas production function

$$q = \prod_{i=1}^{n} x_i^{\alpha_i}.$$

It has the concomitant cost and profit functions (cf. Lau (1978))

$$c(q,\mathbf{w}) \equiv q^{1/\mu}c(\mathbf{w}) \equiv q^{1/\mu}\prod_{i=1}^{n}(w_i/\alpha_i)^{\alpha_i/\mu}$$

and

$$\pi(p,\mathbf{w}) \equiv (1-\mu)p^{1/(1-\mu)}\prod_{i=1}^{n}(w_i/\alpha_i)^{-\alpha_i/(1-\mu)}$$

respectively, where $c(\mathbf{w})$ is the unit cost function

and $\mu \equiv \sum_{i=1}^{n} \alpha_i < 1.$

The α_i must be positive, and their sum satisfy this restriction if the profit function is to exist. Consequently, the own-price elasticities

$$\eta^k{}_k(p,\mathbf{w}) = (\mu - \alpha_k - 1)/(1-\mu), \text{ and}$$

$$\eta^k{}_k(q,\mathbf{w}) = (\alpha_k - \mu)/\mu$$

respectively, are in accord with Propositions (3) and (4).

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