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On generalising Engel's Law: Commodity expenditure shares in hierarchic demand systems

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Abstract

This paper establishes some simple new properties of hierarchic demand systems as defined in Jackson[1984]. In these systems the number of commodities purchased increases with income. Within hierarchic demand systems with piecewise linear engel curves it is shown that every commodity has the property that the commodity budget share is a decreasing proportion of (i) the expenditure on that commodity and all commodities which enter the budget after that commodity and (ii) total expenditure after the commodity enters the budget. These general propositions give predictions about commodity budget shares in general and as a special case within commodity groups. It is shown that the famous empirical regularity known as Engel's law can be regarded as a special case of these propositions by applying them to the first commodity.

KEYWORDS: DEMAND ANALYSIS; BUDGET SHARES; NON-NEGATIVE DEMANDS; ENGEL'S LAW.

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On Generalising Engel's Law:

Commodity Expenditure Shares in Hierarchic Demand Systems

Engel's law that the proportion of expenditure devoted to food falls as total expenditure increases, has been the most famous empirical generalization in demand theory, but it has remained that and never had a clear theoretical basis. Houthakker [1961] commented that it was such an important generalisation that attention should be paid to deriving theoretical structures which would generate this property. While fitted demand systems have had sufficient parameter flexibility to include this property no general simple theoretical structure has been proposed.

The failure to develop such a theoretical structure may be due to the neglect of what is surely a central problem in demand theory, for at low incomes the range of commodities which are purchased is very limited. It is shown below that taking this aspect of demand into account provides a very simple model which leads to Engel's Law.

It was observed in Jackson [1989] that the first commodity to enter the budget in the hierarchic linear expenditure system always showed Engel's Law-like behaviour and that if food was the first commodity to enter the budget it would exhibit Engel's law. This paper shows that Engel's Law is a property of all members of the class of additive preference functions with linear engel curves under two conditions. First, the preference function generates changes in the set of commodities purchased as income increases and second, food is recognised as the first commodity to enter the budget set.

The hierarchic linear expenditure system structure used in Jackson[1984] and [1989] has been estimated for two small systems. Wales and Woodland[1983] estimated a three commodity model of the demand for meat products, and Lee and Pitt[1986] estimated the demand for food commodity groups using Indonesian data. In both of these models the structure ensures that the first commodity to enter the budget must have a declining expenditure share. The Lee and Pitt model is particularly instructive because the first commodity to enter the budgets of Indonesian households, cereal foods, has a declining budget share as total food expenditure increases, thus illustrating Engel's Law-like behaviour in an intuitively appealing overall food demand structure. This example suggests a general proposition, that within a commodity group, the first commodity to enter the budget will have a declining budget share as expenditure on the group increases.

Since the grouping of commodities is arbitrary, if the grouping chosen is that of all commodities in the sequence of commodities entering the household budget beginning with the commodity of interest, a new general proposition would follow. The expenditure on any commodity would be a declining share of the expenditure on that commodity and all succeeding commodities in the household's purchasing sequence. In this paper we prove the two basic propositions listed above for the class of preferences with piecewise linear engel curves.

The class of additive utility functions with linear engel curves was characterised by Pollak[1971]. He did not discuss the imposition of the non-negativity restrictions other than to point out that they lead to kinks in the engel curves. Jackson[1984] derived closed form expressions for the engel curves in the HLES case.

ENGEL CURVES WITH QUASI-HOMOTHETIC PREFERENCES.

For each of the additive preference functions which can generate engel curves locally linear in income the Kuhn-Tucker conditions generate the sequence in which the commodities enter the purchased set. For any given income level the conditions define a set J of purchased commodities and are given for each of the possible cases defined by Pollak[1971] in the Appendix.

Using translation of the origin of the preference map into the strictly negative quadrant as in Jackson[1984], the solution functions for the linear segments given by Pollak[1971] are modified so that they apply over an income range $y_J<y<y_{J+1}$, where y_J is the income at which the J'th commodity enters. Houthakker[1960] called these the critical incomes. Note that $y_1 = 0$. Within each segment summation over commodities applies only to commodities in the set of commodities up to J and the marginal budget shares are normalised to sum to one. We refer to these as hierarchic preference systems.

With the changes above, all hierarchic additive preference functions with linear engel curves have demand functions written as

$$(1)\qquad \begin{aligned} p_i q_i &= a_{K,i} + b_{K,i}\, y && \text{for } y_K \le y \le y_{K+1} \text{ and all K.} \\ & && i \le K \\ &= 0 && \text{otherwise} \end{aligned}$$

Further

$$\begin{aligned} a_{K,i} &= 0 && K < i \\ b_{K,i} &= 0 && K < i \\ b_{K,i} &\ge b_{K+1,i} && \text{for all K and equality only occurring if commodities K and K+1 enter at the same income.} \end{aligned}$$

The marginal budget shares $b_{K,i}$ are of the form

$$b_{K,i} = \beta_i \Big/ \sum_{j=1}^{K} \beta_j \qquad (2)$$

with each β_i a function of the price of the commodity and the parameters involving commodity i in the canonical form of the preference function in the sense of Pollak[1971].

It was shown in Jackson[1989] that the linear structure (1) permits a wide variety of engel curve shapes, including forms where the expenditure share on the commodity is not monotonic.

Consider the expenditure $e_{K,n}$ which is the expenditure on good K, at income y_{n+1}. Denote

$$sb(k,n) = \sum_{j=k}^{n} \beta_j$$

and the length of the income interval over which exactly n commodities are purchased by

$$\Delta y_n = y_{n+1} - y_n$$

Then

$$e_{K,n} = \sum_{j=K}^{n} \frac{sb(K,K)}{sb(1,j)} \Delta y_j$$

The expenditure for the group g consisting of commodity K and those which enter later, is of the form

$$e_{g,n} = \sum_{j=K}^{n} \frac{sb(K,j)}{sb(1,j)} \Delta y_j$$

Hence

$$e_{K,n+1} = e_{K,n} + \frac{sb(K,K)}{sb(1,n+1)} \Delta y_{n+1}$$

$$e_{g,n+1} = e_{g,n} + \frac{sb(K,n+1)}{sb(1,n+1)} \Delta y_{n+1} \qquad (3)$$

Define $s_{K,g,n} = e_{K,n} / e_{g,n}$

as the share of commodity K in total expenditure on group g at income level y_{n+1}.

Proposition 1. In a hierarchic additive utility model with piecewise linear engel curves, the share of commodity K in total expenditure on commodity K and commodities entering later than K is a monotonically declining function.

First consider the values of $s_{K,g,n}$ at the income levels at which commodities enter the purchased set.

Then it is simple to show that

$$s_{K,g,n+1} < s_{K,g,n}$$

iff

$$\frac{sb(K,K)}{sb(K,n+1)} \frac{1}{s_{K,g,n}} < 1 \qquad (4)$$

But

$$s_{K,g,n} = \frac{\sum_{j=K}^{n} \frac{sb(K,K)}{sb(1,j)} \Delta y_j}{\sum_{j=K}^{n} \frac{sb(K,j)}{sb(1,j)} \Delta y_j} \tag{5}$$

$$= \frac{sb(K,K)}{\overline{sb}(K,n)}$$

where the weighted mean of the terms $sb(K,j)$ is represented by

$$\overline{sb}(K,n)$$

Since the weighted mean must lie between the extreme values, the inequality (4) must be true.

Then note that the proposition does not depend on the value of Δy_{n+1} and therefore holds for an arbitrary sequence of critical incomes y_n.

Between the critical incomes, the sign of the first derivative of the expenditure share depends on the inequality (4), hence the proposition is true for all y.

Proposition 2. In an hierarchic additive utility model with piecewise linear engel curves the share of commodity K in total expenditure less the expenditure at which commodity K entered the budget is a monotonically declining function of total expenditure.

Define

$$s_{K,a,n} = \frac{e_{K,n}}{y_{n+1} - y_K}$$

where the index a refers to all commodities as the reference group.

Then

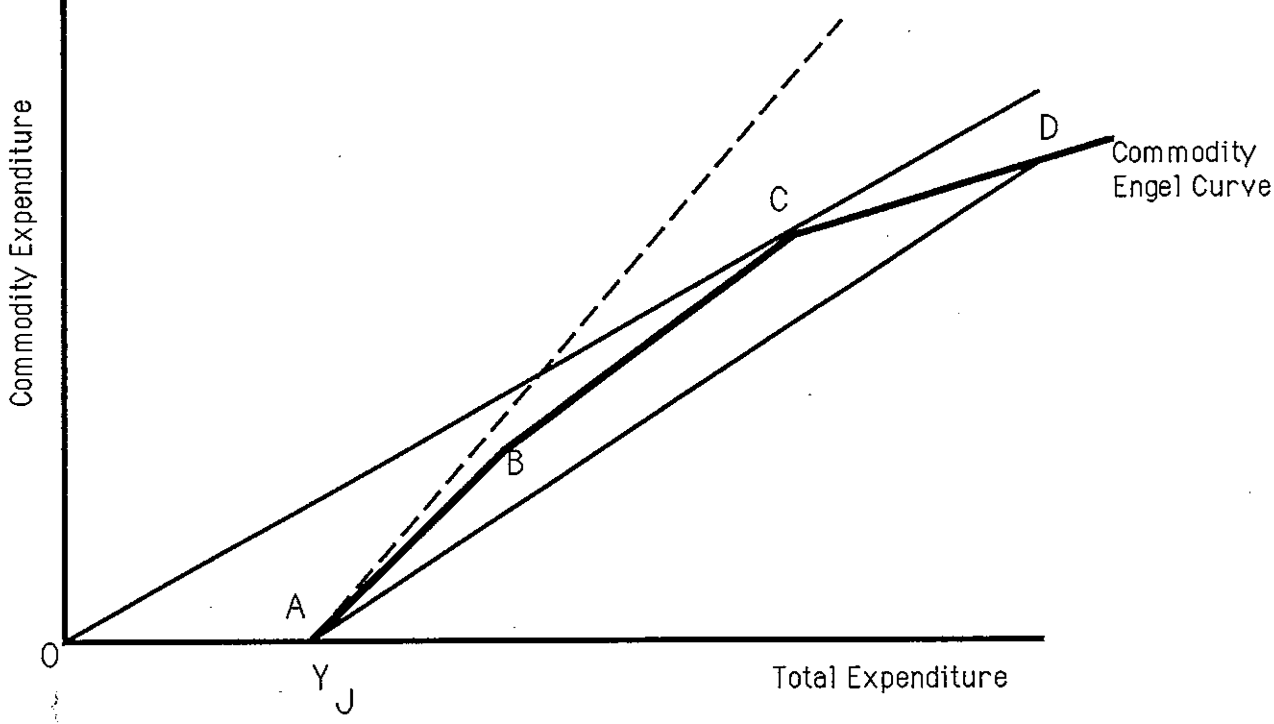
$$s_{K,a,n+1} = \sum_{j=K}^{n+1} \frac{sb(K,K)}{sb(1,j)} \frac{\Delta y_j}{y_{n+2} - y_K}$$

$$= \frac{y_{n+1} - y_K}{y_{n+2} - y_K} s_{K,a,n} + \frac{\beta_K}{sb(1,n+1)} \frac{y_{n+2} - y_{n+1}}{y_{n+2} - y_K}$$

Using an argument similar to before, $s_{K,a,n}$ is a weighted mean, and therefore is larger than its smallest term. But $\beta_K/sb(1,n+1)$ is smaller than its smallest term, so $s_{K,a,n+1}$ must be smaller than $s_{K,a,n}$ and the proposition is proved.

Proposition 2 is illustrated in Figure 1. The engel curve for commodity J is given by ABCD. It lies below AM, the line which would assign all marginal expenditure above the expenditure level Y_J to commodity J. The additivity and linearity imply that at each critical expenditure level the slope of ABCD decreases. The concavity of the engel curve means that the gradient of the line AD, which is just the share of commodity J in total expenditure expenditure less the expenditure at which J enters the budget,
must be continuously decreasing. The gradient of OC gives the maximal budget share for the commodity expressed as a share of total expenditure.

FIGURE 1



Much analysis is done for a group of commodities. A simple specialisation of Proposition 1 leads to an application within groups.

Proposition 3. For an arbitrary group of commodities, of which the first to enter the budget is commodity K, the share of commodity K in expenditure on commodities in the group is a monotonically declining function of income.

The proof of Proposition 1 above also suffices for this case when the sum sb(K,j) in the denominator of (5) is replaced by the sum of the β's for the commodities in the group which are in the household budget at each income level, and K is the first commodity in the group to enter.

ENGEL'S LAW.

By taking the first commodity as the commodity of interest in Proposition 1, we have

Proposition 4. Given the conditions of Proposition 1, the first commodity has a monotonically declining budget share.

This follows immediately from Proposition 1, with K=1.

It is interesting to note that this result also follows from the application of Proposition 2 with K=1. Thus for the first commodity in the budget both Propositions apply.

It is obvious that the first commodity to enter the budget initially has an expenditure share of 1. As the k'th commodity enters the budget, the marginal budget share of the first commodity must fall by the factor sb(1,k-1)/sb(1,k). Hence as the second and later commodities enter the budget the marginal share decreases and the average share must also be decreasing.

Identifying food as the first commodity to enter the budget (in common with a long tradition of recognising a hierarchy of needs, dating at least as early as Aristotle) we have

Proposition 5. (A restricted version of Engel's Law) If a food commodity is the first commodity to enter the budget, it will have a continuously declining budget share.

The result will be unchanged if several food commodities are distinguished but they all enter the budget prior to other commodities entering the budget.

There are two obvious limitations to this result. While applied work initially used additive models it has increasingly moved away from them because of the need for curvature in the expenditure shares. Wales and Woodland[1983] and Lee and Pitt[1986] maintained additivity but obtained curvature in the engel curves by imposing non-negativity restrictions. In spite of the perceived limitations of additive models imposing the non-negativity restrictions can give shares which are non-linear in income and a number of other appealing properties. Such non-linearity is therefore not compelling evidence against the use of additive preference functions with locally linear engel curves.

The other limitation arises because in practise there are many food commodities. The repeated confirmation of Engel's law suggests that none of them have a sufficiently large marginal budget share at entry to overcome the dominant effect of the food commodities entering at low incomes.

This analysis suggests a reason for the somewhat puzzling phenomena that there has been no similar general and reproducible property for other commodities than food. Engel's Law in this model depends on the property that food is the first commodity entering the budget. For later commodities both the order in the hierarchy and the marginal budget shares mean their behaviour is more complex and likely to vary between countries or consumer groups. Being first clearly generates distinctive behaviour.

References

Houthakker, H.S. The capacity method of quadratic programming. *Econometrica*, 28, 1960, 62-87.

Houthakker, H.S. The Present State of Consumption Theory, *Econometrica*, 29, 1961, 704-740.

Jackson, L.F. Hierarchic Demand and the Engel Curve for Variety, *The Review of Economics and Statistics*, LXVI, 1984, 8-15.

Jackson, L.F. Zero Demands, Commodity Choice and Hierarchic Purchasing Models. *University of Sydney, Econometrics Discussion Paper*, 89-04. 1989.

Lee, Lung-Fei and Pitt, Mark M. Microeconometric Demand systems with binding non-negativity constraints: the dual approach. *Econometrica*, 54, 1986, 1237-1242.

Pollak, R.A. Additive Utility Functions and Linear Engel Curves. *Review of Economic Studies*, 38, 1971, 401-414.

Wales, T.J. and Woodland, A.D. Estimation of Consumer Demand Systems with Binding Non-Negativity Constraints. *Journal of Econometrics*, 21, 1983, 263-285.

APPENDIX

Pollak[1971] showed that there were only five general classes of additive preference functions which generate locally linear engel curves. This appendix specifies the conditions which generate the order in which commodities enter the budget for each of those cases. In the following we assume that the preference map is translated into the strictly negative quadrant. The changes for other cases are straightforward.

The general problem is to maximize

$$u(q) = \Sigma\, u^k(q_k + \gamma_k)$$

subject to

$$\Sigma\, p_k q_k \leq y \qquad \text{and} \quad q_k \geq 0 \quad \text{for all } k$$

The Kuhn-Tucker conditions are $q_k \left(\frac{\partial u^k}{\partial q_k} - \lambda p_k \right) = 0$ for all k.

The five cases are:

A.1 $u^k(q_k + \gamma_k) = b_k \log (q_k + \gamma_k)$

$\Rightarrow \dfrac{b_k}{p_k c_k} = \lambda$ at entry, and values of the left had side determine the order in which entry occurs.

A.2 $u^k(q_k + \gamma_k) = - b_k (q_k + \gamma_k)^c \qquad r<0,\ b_k>0$

$\Rightarrow \quad (-c\ b_k\ (\gamma_k)^{c-1})/p_k$ defines the entry order.

A.3 $u^k(q_k + \gamma_k) = b_k (q_k + \gamma_k)^r \qquad 0<r<1,\ b_k>0$

$\Rightarrow \quad (c\ b_k\ (\gamma_k)^{c-1})/p_k$ defines the entry order.

A.4 $u^k(q_k + \gamma_k) = - b_k (\gamma_k - q_k)^c \qquad r>1,\ b_k>0$

$\Rightarrow \quad (c\ b_k\ (\gamma_k)^{c-1})/p_k$ defines the entry order.

A.5 $u^k(q_k + \gamma_k) = - b_k \exp ((\gamma_k - q_k)/b_k) \qquad b_k>0$

$\Rightarrow \quad - \log p_k + b_k/\gamma_k$ defines the entry order.

The engel curves are given by Pollak[1971] (1.10b), (1.13), (1.13), (1.13), (1.18) respectively. Summation over k=1,n must be replaced by summation over k=1,J and the corresponding curve applies only to the interval $y_J \leq y \leq y_{J+1}$. The income level y_J can be found by constructing the set of engel curves over the interval and calculating the value y_J at which $q_J = 0$.