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Harrod's Growth Model:

An illumination using the multiplier-accelerator model

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ABSTRACT

It is shown that Multiplier-Accelerator models under plausible parameter assumptions manifest the two Harrod Propositions; a Unique Warranted Rate of growth and a Knife Edge.

This paper shows how Multiplier-Accelerator models of the type developed by Hicks (1950) and Sameulson (1939) can be used to demonstrate the central claims of Harrod's theory of the dynamics of national income.

These claims are two.

First, there is only one rate of growth of national income which is sustainable (or "warranted"), and this rate of growth equals s/c : the propensity to save divided by the capital/output ratio. This first claim might be called the Unique Warranted Rate thesis.

Second, the actual rate of growth will not converge on the sustainable ("warranted") growth rate, s/c. If the actual growth does not equal s/c, it will move further and further away from s/c over time. This second claim is the "Knife Edge"thesis.

This two claims are entirely clear. Unfortunately, the arguments which Harrod (1939) offered in favour of them are not; they are loose, "incompletely worked out" (Solow 1988, 307) and "seemed to depend on vague generalisations" (Solow 1988, 307). Consequently, "there are a host of possible formalisations which claim to capture the spirit and essence of Harrod's contribution" (Jones 1975, 43), which are "sometimes conflicting" (Sen 1970, 10). Some of the better known formalisations include those by Alexander (1950), Baumol (1951), Hahn and Matthews (1964), Jorgenson (1960), and Rose (1959).

It is not the purpose of this note to suggest that this literature is wrong or misguided. But it is our purpose to suggest it is superfluous. Our grounds for this assertion will lie in this note's demonstration that Multiplier-Accelerator (henceforth MA) models manifest the two Harrod propositions. Consequently, the Unique Warranted Rate and Knife Edge propositions do not need special demonstrations, special models, or its very own interpretative literature. The familiar MA model is an entirely adequate vehicle for the exposition of Harrod's theory.

Despite the celebrity of MA models, we are aware of only one paper (Alexander 1949) which has anticipated our identification of the Warranted Rate and Knife Edge within this class of model. This unawareness of the Harrodian properties of MA models is, perhaps, a result of a tendency to place MA models and the Harrod model under different subject headings. Although Harrod describes his theory as a "marriage of the "acceleration principle" and the "multiplier" theory" (1939, 14), MA models are generally seen as part of the study of short run fluctuations, while the Harrod literature is seen as part of the study of long run trends. Hicks provides one exception to this separation of Harrod's model and the MA models. In *The Trade Cycle* (1950) Hicks proposes an interpretation of Harrod's theory in terms of an MA model. However, we will show the "warranted rate" of this model has no relation to the warranted rate of Harrod.

The first section of the paper shows how a typical MA model manifests the two Harrod propositions. The second section shows how other models of the MA class also exhibit these propositions. The third section discusses earlier assessments of the relationship between the Harrod model and MA models.

I A SIMPLE MULTIPLIER-ACCELERATOR MODEL

Consider the following model of national income of the Multiplier-Accelerator class;

$$Y_t = C_t + I_t \tag{1}$$

$$C_t = (1-s)\ Y_t \tag{2}$$

$$I_t = c\ (Y_{t-1} - Y_{t-2}) \tag{3}$$

This model has the characteristic equation,

$$R^2 - R\ c/s + c/s = 0 \tag{4}$$

The two roots satisfy,

$$R_{1,2} = \frac{c/s \pm \sqrt{ (c/s)^2 - 4(c/s) }}{2} \tag{5}$$

The roots are real if and only if;

$$s/c < 1/4 \tag{6}$$

The stylised facts for major western economies suggests that s is no larger than 1/4 and that c is unlikely to be less than 2 for annual data. So it would seem likely (6) is easily satisfied. [1]

Assuming (6) is satisfied, and both roots are real, the complete solution is;

$$Y(t) = A_1 R_1^t + A_2 R_2^t \qquad (7)$$

(7) shows that, special circumstances aside, Y will *not* grow at a steady rate. However, there are some special circumstances in which it will.

Consider the definitised solution for Y ,

$$Y(t) = \frac{(R_2 Y(0) - Y(1))R_1^t + (Y(1) - R_1 Y(0))R_2^t}{R_2 - R_1} \qquad (8)$$

Inspection of (8) reveals, there are two initial income paths which will lead to steady rates of growth.

1. *If* $Y(1) = R_1 Y(0)$ *then* Y will grow at a rate of $R_1 - 1$ in all periods thereafter, ie if the initial rate of growth is $R_1 - 1$, then the rate of growth will be $R_1 - 1$ in all periods thereafter. We will denote $R_1 - 1$ as g_1.

2. *If* $Y(1) = R_2 Y(0)$ *then* Y will grow at a rate of $R_2 - 1$ in all periods thereafter. ie if the initial rate of growth is $R_2 - 1$ then the rate of growth will be $R_2 - 1$ in all periods thereafter. Denote $R_2 - 1$ as g_2.

So it would seem there are *two* sustainable rates of growth. What can we say about these rates of growth ? We can prove that if both roots are real, then both roots are greater than 1. [2] This means the two sustainable rates of growth are positive; ie there is no *negative* warranted rate of growth.

We can say something more definite about the smaller of these two warranted growth rates. *The smaller growth rate is approximately s/c,* ie the smaller growth rate roughly conforms to the formula of the warranted growth rate provided by Harrod. Table 1 shows the value of the smaller growth rate implied by nine plausible combinations of s and c. For the sake of comparison, the value of s/c is supplied in brackets.

Table 1: The Value of The Smaller Warranted Growth Rate

per cent per period
(s/c in brackets)

propensity to save	capital/output ratio 3	4	5
0.1	3.57 (3.33)	2.63 (2.5)	2.08 (2.0)
0.2	7.74 (6.66)	5.57 (5.0)	4.35 (4.0)
0.3	12.7 (10)	8.89 (7.5)	6.85 (6.0)

The correspondence between the model's smaller growth rate and Harrod's warranted growth rate is very close.

The discrepancy which does exist is easily understood.

It reflects the difference between investment expressed as a fraction of $Y_{t-1} - Y_{t-2}$, and investment expressed as a fraction of $Y_{t+1} - Y_t$. Investment in period t is by assumption equal to c of $Y_{t-1} - Y_{t-2}$. But investment as a fraction of $Y_{t+1} - Y_t$ will be lower than c since $Y_{t+1} - Y_t$ is larger than $Y_{t-1} - Y_{t-2}$ by a factor of $(1+g_1)^2$. Consequently, the effective or "ex post" investment coefficient is smaller by $(1+g_1)^2$, and therefore the growth rate is larger than that predicted by the s/c formula by a factor of $(1+g_1)^2$.

More formally, we can write, merely as a matter of identities,

$$g_1 = \frac{I/Y}{I/\Delta Y}$$

But (2) allows us to rewrite this as,

$$g_1 = \frac{s}{I/\Delta Y}$$

But, owing to the lag in the investment function,

$$I = c\ \Delta Y\ /(1+g_1^{2})$$

Therefore,

$$g_1 = (1+g_1)^2\ s/c \qquad (9)$$

(Inspection of Table 1 will confirm the accuracy of this conclusion).

(9) implies that for growth rates of less than 5 per cent per period, the proportionate discrepancy between the smaller warranted rate and the s/c formula will be less than about 1/10.

The value of the larger of the two warranted rates is very different from s/c; it is approximately (1 -2s/c)/(s/c); ie not very different from the *reciprocal* of s/c. Proof: Let R_1 denote the smaller growth rate. ie $R_1 \cong 1$. But $R_1 + R_2 = c/s$. Therefore $R_2 \cong c/s - 1$. Therefore, $R_2 - 1 \cong c/s - 2 = (1 - 2s/c)/(s/c) \cong 1/(s/c)$. [3]

We may infer from this that the larger of the two warranted rates will, for plausible values of s and c, be extremely large. For example, if c = 4 and s = 0.1, the larger rate will be 3797% per period. Thus it seems reasonable to say that, once we have added a full employment constraint, the larger warranted rate is no longer feasible. The model is left with only one warranted rate of growth; the minor rate, which approximately equals s/c.

What we have established so far is that our simple Multiplier-Accelerator model exhibits the Unique Warranted Rate property. We will now show it also exhibits the Knife Edge property.

Define an initial rate of growth g by,

$$1 + g = Y(1)/Y(0) \tag{10}$$

Then (8) can be rewritten as;

$$Y(t) = \frac{Y(0)}{R_2 - R_1} \{ (g_2 - g)R_1^t + (g - g_1)R_2^t \} \tag{11}$$

Taking R_2 to be the dominant root we can safely assume the coefficient on R_1^t is positive. (Proof: If R_2 is the dominant root then $R_2 - R_1$ must be positive. Consequently the sign of the coefficient is the same as the sign of $g_2 - g$. But $g_2 = R_2 - 1$, which, as we have shown, will be enormous for plausible parameter values. Therefore $g_2 - g$ is positive).

The coefficient on the R_2^t term depends on the value of $g - g_1$. We can distinguish three cases;

1. If $g - g_1$ is zero then the coefficient on R_2^t is zero, and the economy will grow at g_1 in all periods thereafter. We have already noticed this result; it is the existence of the smaller warranted rate.

2. If $g > g_1$ then the coefficient on R_2 is positive. Consequently, the economy's growth rate will tend towards the growth rate g_2. This is the Knife Edge; an upward disturbance to the growth rate will be amplified over time instead of diminished.

3. If $g < g_1$ then the coefficient on R_2 is negative. Consequently, national income will grow more slowly, and then fall, and by larger and larger amounts. This is the "other side" of the Knife Edge; a downward disturbance to the growth rate will be amplified over time instead of diminished.

Thus we have proved that the simple Multiplier-Accelerator model exhibits the Knife Edge property, as well as the Unique Warranted Rate property.

II OTHER MULTIPLIER-ACCELERATOR MODELS

In this section we show that variations of the MA model also manifest the Harrod Propositions.

Consider the model which Hicks developed in *The Trade Cycle* (1950). It is identical to the model of section II except that consumption depends on income lagged one period, rather than current income.

$$Y_t = C_t + I_t \tag{12}$$

$$C_t = (1-s)\ Y_{t-1} \tag{13}$$

$$I_t = c\ (Y_{t-1} - Y_{t-2}) \tag{14}$$

The roots of its characteristic equation are real if and only if

either $\sqrt{c} < 1 - \sqrt{s}$ (15)

or $\sqrt{c} > 1 + \sqrt{s}$ (16)

It seems unlikely that (15) will be satisfied. This would require $c < 1$, and that implies the expense of capital equipment is less than the income it produces in a single period. So we will concentrate on the second condition, (16).

It can be shown that if (16) is satisfied then both roots are greater than 1 . Further, it can be shown, by the same method as was used in Section I, that each of these roots sustain a warranted rate of growth equal to the value of the root, less 1. Thus, as in the previous model, there are two sustainable positive growth rates (assuming (16) is satisfied). As before, the value of the smaller of the sustainable growth rates is in the region of s/c. Table 2 shows the value of the smaller growth rate corresponding to nine plausible combinations of s and c. The value of s/c is supplied in brackets.

Table 2: The Value of The Smaller Warranted Growth Rate When There is a One Period Lag Between Consumption and Income

per cent per period
(s/c in brackets)

	capital/output ratio		
propensity to save	3	4	5
0.1	5.4 (3.33)	3.49 (2.5)	2.58 (2.0)
0.2	11.89 (6.66)	7.33 (5.0)	5.33 (4.0)
0.3	20.0 (10.0)	11.61 (7.5)	8.29 (6.0)

The correspondence is rougher than before. This is explained by the lag in consumption. This lag implies the ratio of consumption to current income is smaller than (1-s) by a factor of $1 + g_1$. This means that saving as a fraction of current income is increased, and so the rate of growth is increased relative to s/c. It is easy to check that g_1 satisfies

$$g_1 = (1+g_1)s/c + (1+g_1)g_1/c \quad > \quad s/c \qquad (17)$$

The second warranted rate, g_2, will be so large that a full employment constraint will make it unfeasible. Consequently, there is only one warranted rate. And, as in the model of Section I, this rate is unstable.

The original Samuelson specification of the Multiplier - Accelerator (1939) also manifests in certain situations the two Harrod properties of a Unique Warranted Rate and a Knife Edge.

In the Samuelson specification investment depends on the lagged change in consumption.

$$Y_t = C_t + I_t \tag{18}$$

$$C_t = (1-s)\ Y_{t-1} \tag{19}$$

$$I_t = c\ (C_t - C_{t-1}) \tag{20}$$

The roots of this system are real if and only if

$$(1-s) > 4c/(1+c)^2 \tag{21}$$

Under the fairly plausible assumption that $c > 1$ this can be equivalently expressed as

$$c > \frac{1 + \sqrt{s}}{1 - \sqrt{s}} \tag{22}$$

For any propensity less than 0.25 this will be satisfied for any c greater than 3. Thus, if the decision period is 1 year, it would seem likely that this condition is satisfied. The literature on the Samuelson model has concentrated on the possibility of complex roots. Here we will restrict ourselves to the real roots case.

Assuming c > 1 then both roots, if real, exceed 1. [5] As before there is a major and a minor root. A relationship between the minor root and s/c still exists, but the discrepancies can be substantial in proportional terms.

Table 3: The Value of The Smaller Warranted Growth Rate When There is a One Period Lag Between Consumption And Income And Investment Depends on the Change in Consumption

per cent per period
(s/c in brackets)

marginal propensity to save	capital/output ratio 3	4	5
0.1	6.5 (3.33)	4.07 (2.5)	2.97 (2.0)
0.2	20.0 (6.66)	10.55 (5.0)	7.33 (4.0)
0.3	(complex roots)	23.76 (7.5)	14.61 (6.0)

The much rougher correspondence is explained by the fact that investment responds to the lagged change in consumption not the current change in income. This reduces the *effective* ratio of investment to income change, making actual growth still larger than s/c. It can be checked from the Table that g_1 satisfies

$$g_1 = (1+g_1)\ s/c(1-s) \ + \ g_1(1+g_1)/c(1-s) \quad > \quad s/c \qquad (23)$$

III EARLIER LITERATURE

We have shown how MA models under plausible assumptions possess the distinguishing characteristics of Harrod's model. There has been some earlier discussion of the connection between the two types of model. Hicks (1949) suggested a variant of an MA model would capture the Harrod propositions. This model was identical to equations (12) to (14) of Section II , except that it included an element of autonomous investment. Crucially, the autonomous component of investment was assumed to grow at some exogenous rate, g_a. Consequently, unlike the MA models dealt with in this paper, the particular integral of national income is not zero; it is positive and it grows at a rate g_a. Thus the existence and growth of autonomous investment implies a "moving equilibrium" for national income. If income in the initial two periods equals the moving equilibrium value provided by the particular integral, income will remain equal to its particular integral value, growing at g_a. However, if in the second period income exceeds its moving equilibrium value, its value will move further and further apart from the moving equilibrium.

This set up seems to be like Harrod's, and it is not surprising Hicks thought he had "captured" Harrod's model by it. However, there is a vital difference between the two models. Hicks' moving equilibrium grows at a rate of g_a. But g_a is the rate of growth in autonomous investment; it does not bear any relationship with s/c. Consequently, Hicks moving equilibrium need not conform even roughly to Harrod's formula for the warranted rate.

To make the contrast another way; the source of steady growth in Hicks and Harrod are entirely different. Hicks' moving equilibrium is driven by autonomous investment, not by the accelerator. But Harrod's warranted growth rate is driven by the accelerator. To Hicks the accelerator's only function is to provide the Knife Edge; in Harrod it also provides the Unique Warranted Rate.

Alexander (1949) pointed out how an MA model could produce two steady growth rates, one of which was small, and one very large.

He also pointed out that the small growth rate was unstable. However, he did not note the relationship of the smaller growth rate and s/c. Indeed, he asserted that MA models differed in "essential respects" from Harrod's model. The position we have pressed in the paper is exactly the reverse; in our view MA models and Harrod's model are in *essential respects exactly the same*.

FOOTNOTES

1.

In recent years the average propensity to save has been below 0.25 in all major western economies. In 1987 the ratio of net saving to national income was 0.03 in the United States, 0.21 in Japan, 0.13 in Germany and 0.06 in the United Kingdom. (Source: *National Accounts, 1975-1987. Volume II*, OECD, Paris 1989). Estimates of the capital/output ratio for broad sectors of the economy are scarce, but an estimate pertaining to the UK exists. It is estimated that in 1987 the ratio of capital to value added of "Non-Financial Corporate and Quasi-Corporate Enterprises" was 2.44. (Source: Table 14, *National Accounts, 1975-1987. Volume II*, OECD, Paris 1989).

2.

Proof: The characteristic equation is $R^2 - R\,c/s + c/s = 0$. Therefore, $R_1R_2 = c/s$ *and* $R_1 + R_2 = c/s$. But the satisfaction of (6) requires $c/s > 4$. Therefore, if (6) is satisfied, $R_1R_2 > 4$, and at least one root must be greater than 1. Let R_2 be the larger root, and therefore necessarily greater than 1. Let $R_1 = 1 + \varepsilon$. Consequently, $(1+\varepsilon)R_2 = c/s$ *and* $1 + \varepsilon + R_2 = c/s$. Consequently, $\varepsilon R_2 = 1 + \varepsilon$; ie $\varepsilon = -1/(1-R_2)$. But $R_2 > 1$. Therefore $\varepsilon > 0$. Therefore, $R_1 > 1$.

3.

A rationale for this second growth is as follows. As a matter of identities it is true that the rate of growth equals the I/Y divided by I/ΔY, where I is actual investment. I/Y is parametrical; it equals s. Consequently, the lower I/ΔY, the higher the growth rate. Further, in this model a higher the growth rate lowers I/ΔY, since I equals c of ΔY lagged one period, which is smaller than ΔY. This means there is some very high rate of growth which makes I/ΔY so low that the high rate of growth is sustained.

4.

The characteristic equation is $R^2 - (1 - s + c)R + c = 0$. Therefore, $R_1R_2 = c$ *and* $R_1 + R_2 = 1 - s + c$. But satisfaction of (16) requires $c > 1$. Thus if (16) is satisfied then $R_1R_2 > 1$, and at least one root must be greater than 1. Let R_2 be the greater root. Let $R_1 = 1 + \varepsilon$. Therefore, $(1+\varepsilon)R_2 = c$ *and* $1 + \varepsilon + R_2 = 1 - s + c$. Consequently, $\varepsilon(1-R_2) = -s$. But $R_2 > 1$, therefore $\varepsilon > 0$. Therefore $R_1 > 1$.

5.

The characteristic equation is $R^2 - (1-s)(1+c)R + (1-s)c = 0$. Therefore, $R_1R_2 = (1-s)c$ *and* $R_1 + R_2 = (1-s)(1 + c)$. But satisfaction of (22) requires $c > (1 + \sqrt{s})/(1-\sqrt{s})$; the satisfaction of which implies $(1-s)c > 1$. Consequently, if (22) is satisfied then $R_1R_2 > 1$, and at least one root must be greater than 1. Let R_2 be the greater root. Let $R_1 = 1 + \varepsilon$. Therefore, $(1+\varepsilon)R_2 = (1-s)c$ *and* $1 + \varepsilon + R_2 = 1 - s + (1-s)c$. Consequently, $\varepsilon(1-R_2) = -s$. But $R_2 > 1$. Therefore $\varepsilon > 0$. Therefore $R_1 > 1$.

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