Partial valuations hide the contaminating value

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Abstract

Partial truth assignments give rise to Boolean-valued semantics for both paracomplete and paraconsistent weak Kleene logic. To accommodate partiality, the semantic consequence relation of classical propositional logic is adjusted in two natural ways, linked by a duality principle.

Keywords. Propositional logic · partial valuation · weak Kleene logic

1 Introduction

Situations involving vagueness, ambiguity, incomplete information or ongoing computation motivate the consideration of partial truth assignments, a theme that has been explored from a variety of perspectives in philosophical logic, mathematics, proof theory and theoretical computer science [9, 10, 12, 13, 15, 16, 18, 21].[1]

This note revisits two particularly direct ways to adjust the satisfaction relation of classical propositional logic in order to accommodate partiality. While originally conceived with an eye towards Kleene's *strong* tables [18], it appears that their relevance for propositional logics based on Kleene's *weak* tables has gone largely unnoticed, at least explicitly so: partial truth assignments give rise, in a natural and dual manner, to Boolean-valued semantics for both *paracomplete weak Kleene logic* (B_3) and *paraconsistent weak Kleene logic* (PWK), as well as for mixed forms.[2]

[1] This list is by no means meant to be exhaustive.

[2] For a comprehensive account of the logics in question I refer the reader to [5], which is likely to become the standard reference. B_3 and PWK have in the last decade garnered a considerable amount of attention [1, 2, 4, 5, 7, 8, 19, 20, 22].—Again, this list of references is by no means meant exhaustive.

Nonetheless, partial assignments can be treated quite independently of this connection, and I will begin with such a treatment that does not require any additional truth value except for the classical ones. The link to Kleene's weak tables will be made explicit later.

2 Semantics of partial valuations

Let Form denote the set of formulas in the language of classical propositional logic, with a countable set Var of propositional variables $p, q, \ldots$, and the propositional connectives of conjunction $\wedge$, disjunction $\vee$ and negation $\neg$. Given $\varphi \in \mathrm{Form}$, we write $\mathrm{Var}\,\varphi$ for the set of propositional variables occurring in φ, while for $\Delta \subseteq \mathrm{Form}$ we put $\mathrm{Var}\,\Delta = \bigcup_{\varphi\in\Delta} \mathrm{Var}\,\varphi$. Throughout, let $\vDash$ denote classical semantical consequence. Classical metalogic will be freely employed.

Let X and Y be sets. A *partial function* $f : X \rightharpoonup Y$ is an assignment of exactly one element $f(x) \in Y$ to each x in some subset $\mathrm{dom}\, f \subseteq X$, the *domain* of f. In other words, a partial function is given by a (total) mapping $f : \mathrm{dom}\, f \to Y$. Equivalently, a partial function amounts to a functional binary relation $f \subseteq X \times Y$. For instance, the empty set is a partial function $X \rightharpoonup Y$. It is common to use the harpoon arrow to display a partial function.

Henceforth, by a *partial truth assignment*

$$v : \mathrm{Var} \rightharpoonup \mathbf{2}$$

we understand a partial function on the set of propositional variables, taking values in the Boolean algebra $\mathbf{2} = \{\,0, 1\,\}$. A partial truth assignment on variables naturally extends to a *partial valuation* on compound formulas. In fact, maintaining the classical interpretation of the connectives, any $v : \mathrm{Var} \rightharpoonup \mathbf{2}$ extends uniquely to

$$v^* : \mathrm{Form} \rightharpoonup \mathbf{2}, \quad \text{with } \mathrm{dom}\, v^* = \{\, \varphi \in \mathrm{Form} \mid \mathrm{Var}\,\varphi \subseteq \mathrm{dom}\, v \,\}.$$

This extension is defined by recursion, exactly as in the conventional "total" case (cf., e.g., [24, Theorem 1.2.2]), except for the restriction to subsets of Form, as follows:

(i) if $p \in \mathrm{dom}\, v$, then $p \in \mathrm{dom}\, v^*$ and $v^*(p) = v(p)$;

(ii) if $\varphi, \psi \in \mathrm{dom}\, v^*$, then $\varphi \wedge \psi \in \mathrm{dom}\, v^*$ and $v^*(\varphi \wedge \psi) = \min(v^*(\varphi), v^*(\psi))$;

(iii) if $\varphi, \psi \in \mathrm{dom}\, v^*$, then $\varphi \vee \psi \in \mathrm{dom}\, v^*$ and $v^*(\varphi \vee \psi) = \max(v^*(\varphi), v^*(\psi))$;

(iv) if $\varphi \in \mathrm{dom}\, v^*$, then $\neg\varphi \in \mathrm{dom}\, v^*$ and $v^*(\neg\varphi) = 1 - v^*(\varphi)$.

Thus, v^* behaves just like an ordinary valuation, but is only defined for formulas whose variables receive truth values by v. However, depending on the satisfaction relation to be considered below, the set of formulas that are modelled by a given partial truth assignment v can indeed be larger than the domain of v^*. For instance, even if $p \notin \mathrm{Var}\, v$, the assignment v may still be considered to satisfy p on the "tolerant" interpretation of Section 2.2 below.

Note that every partial valution can be extended to a total one. This can be done arbitrarily, unless the assignment v was total to begin with. For what follows, it will suffice to know that some such total extension exists, by fiat.

We consider relations $\succ$ between partial assignments v and formulas φ, and read $v \succ \varphi$ as "v models/satisfies φ". There are various ways to define such a relation. Any given $\succ$ extends to a relation between assignments and *sets* Γ of formulas by stipulating, in the usual way, that $v \succ \Gamma$ iff $v \succ \psi$ for every $\psi \in \Gamma$. Moreover, let

$$\Gamma \succ \varphi \quad \text{iff} \quad (\forall v : \mathrm{Var} \rightharpoonup \mathbf{2})(\, v \succ \Gamma \Rightarrow v \succ \varphi\,). \tag{1}$$

We will write $\Gamma \not\succ \varphi$ for the classical negation of (1), which is to say that there exists a partial assignment $v : \mathrm{Var} \rightharpoonup \mathbf{2}$ such that $v \succ \Gamma$ yet $v \not\succ \varphi$, i.e., *not* $v \succ \varphi$.

For instance, according to van Fraassen's principle of *supervaluation* [10], we may stipulate that $v \succ \varphi$ iff $\alpha \vDash \varphi$ *for every extension* $\alpha \supseteq v$. Instead, we are going to invoke variable inclusion requirements on the domains of partial assignments. The approach in the present paper can thus be traced back at least to Nait Abdallah's treatment [18], as well as to earlier independent work of Langholm [16].

2.1 Strict interpretation

To accommodate partiality, we adjust the semantic consequence relation of classical logic in two different ways. A first and presumably natural approach is to require that all variables of a given formula receive truth values to begin with. Accordingly, we define

$$v \blacktriangleright \varphi \quad \text{iff} \quad \mathrm{Var}\,\varphi \subseteq \mathrm{dom}\, v \ \underline{\text{and}}\ v^*(\varphi) = 1. \tag{2}$$

Nait Abdallah has put this idea under scrutiny already, yet with an eye towards Kleene's *strong* tables [18, Chapter 2]. We deviate from [18, Section 2.1.2.4] in that we require a modest language only, track the domains of partial assignments explicitly, and work towards a systematization of *weak* Kleene logics.

Through (1) for the case of (2) we obtain a consequence relation $\blacktriangleright$. Note that classical semantical consequence $\vDash$ corresponds to the case in which we quantify over all and only the *total* assignments v.

It is instructive to consider cases in which a partial assignment v fails to satisfy a formula φ, which happens already if v stays quiet about at least one of the variables of φ, i.e., if $\mathrm{Var}\,\varphi \not\subseteq \mathrm{dom}\,v$. In particular, the logic at hand does not have any tautologies $\emptyset \blacktriangleright \varphi$, for if $p \in \mathrm{Var}\,\varphi$, then any partial assignment that avoids p will fail to model φ. In particular, excluded middle $\emptyset \blacktriangleright \varphi \vee \neg\varphi$ is not valid, which renders the logic at hand *paracomplete*.

It turns out that $\blacktriangleright$ amounts to nothing but paracomplete weak Kleene logic B_3 (cf. the discussion below). To this end, we pass through Urquhart's characterisation [23, Theorem 4] of B_3, which is the second item in the following proposition.

Proposition 1. *For any* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Form}$*, the following are equivalent.*

1. $\Gamma \blacktriangleright \varphi$.
2. $\Gamma \vDash \varphi$*, and if* Γ *is classically consistent, then* $\mathrm{Var}\,\varphi \subseteq \mathrm{Var}\,\Gamma$.

Proof. We concentrate on the contrapositives.

(1) $\Rightarrow$ (2) Suppose that either (a) $\Gamma \nvDash \varphi$ or (b) that Γ is consistent and $\mathrm{Var}\,\varphi \not\subseteq \mathrm{Var}\,\Gamma$. As regards case (a), if $\Gamma \nvDash \varphi$, then there is a total valuation $\alpha : \mathrm{Form} \to \mathbf{2}$ such that $\alpha(\psi) = 1$ for every $\psi \in \Gamma$, yet $\alpha(\varphi) = 0$, and this α witnesses $\Gamma \not\blacktriangleright \varphi$ right away. Thus we move on to case (b). Accordingly, since Γ is consistent, there is a total valuation $\alpha : \mathrm{Form} \to \mathbf{2}$ such that $\alpha(\psi) = 1$ for every $\psi \in \Gamma$. Now consider the restriction $v = \alpha\restriction_{\mathrm{Var}\,\Gamma}$. We claim that this v witnesses $\Gamma \not\blacktriangleright \varphi$. In fact, on the one hand, if $\psi \in \Gamma$, then $\mathrm{Var}\,\psi \subseteq \mathrm{Var}\,\Gamma = \mathrm{dom}\,v$ and $v^*(\psi) = \alpha(\psi) = 1$, and so $v \blacktriangleright \Gamma$. On the other hand, since $\mathrm{Var}\,\varphi \not\subseteq \mathrm{dom}\,v = \mathrm{Var}\,\Gamma$, this already suffices to see that $v \not\blacktriangleright \varphi$.

(2) $\Rightarrow$ (1) Suppose that there is a partial assignment $v : \mathrm{Var} \rightharpoonup \mathbf{2}$ such that $v \blacktriangleright \Gamma$ yet $v \not\blacktriangleright \varphi$. This is to say that (a) $\mathrm{Var}\,\Gamma \subseteq \mathrm{dom}\,v$ and $v^*(\psi) = 1$ for every $\psi \in \Gamma$, as well as that (b) if $\mathrm{Var}\,\varphi \subseteq \mathrm{dom}\,v$ then $v^*(\varphi) = 0$. We need to show that either (i) $\Gamma \nvDash \varphi$, or that (ii) Γ is consistent as well as that $\mathrm{Var}\,\varphi \not\subseteq \mathrm{Var}\,\Gamma$. To do so, we distinguish two cases. First, if $\mathrm{Var}\,\varphi \subseteq \mathrm{dom}\,v$, then $v^*(\varphi) = 0$ according to (b). Now any extension of v^* to a total valuation witnesses $\Gamma \nvDash \varphi$, whence the conclusion (i). Next we consider the case in which $\mathrm{Var}\,\varphi \not\subseteq \mathrm{dom}\,v$. It follows that $\mathrm{Var}\,\varphi \not\subseteq \mathrm{Var}\,\Gamma$ by (a). Now it remains to check that Γ is consistent, but this follows from the assumption that $v \blacktriangleright \Gamma$, passing again to some extension of v to a total valuation. □

2.2 Tolerant interpretation

A second way to account for partiality is to relax the requirement that all variables of a formula be assigned a truth value. On this interpretation, a formula will be satisfied unless explicitly falsified. Thus, we stipulate

$$v \rhd \varphi \quad \text{iff} \quad \operatorname{Var}\varphi \subseteq \operatorname{dom} v \ \underline{\text{implies}}\ v^*(\varphi) = 1. \tag{3}$$

Nait Abdallah has coined his form of (3) "potential truth" [18, Section 2.1.2.5]. Again, we deviate from [18] in that we track the domains of partial assignments explicitly, and follow down a rather different route anyway.

Through the corresponding case of (1) for (3), we again obtain a consequence relation $\rhd$. As above, the case in which we quantifiy over all and only the total valuations gives back classical semantical consequence.

Dually to the relation $\blacktriangleright$ of Section 2.1, a partial assignment v now validates a formula φ already if $\operatorname{Var}\varphi \nsubseteq \operatorname{dom} v$, since in this case $v \rhd \varphi$ is vacuously true. However, $\rhd$ fails *ex falso.* In fact, let $p \in \operatorname{Var}$, stipulate $v(p) = 0$, and leave v undefined elsewhere. Then, for every propositional variable $q \neq p$, this v provides a counterexample to $q \wedge \neg q \rhd p$, which renders the logic at hand *paraconsistent.*

It turns out that $\rhd$ amounts to nothing but the paraconsistent weak Kleene logic PWK (cf. the discussion below) on account of Ciuni and Carrara's characterisation [7, Theorem 3.8] (in the streamlined form of [5, Theorem 1.3.2]), which is the second item in the following proposition.

Proposition 2. *For any* $\Gamma \cup \{ \varphi \} \subseteq \operatorname{Form}$*, the following are equivalent.*

1. $\Gamma \rhd \varphi$.

2. *There is* $\Delta \subseteq \Gamma$ *such that* $\operatorname{Var}\Delta \subseteq \operatorname{Var}\varphi$ *and* $\Delta \vDash \varphi$.

Proof. (1) $\Rightarrow$ (2) We concentrate on the contrapositive. Accordingly, suppose that for every $\Delta \subseteq \Gamma$, if $\operatorname{Var}\Delta \subseteq \operatorname{Var}\varphi$, then $\Delta \nvDash \varphi$. In particular, this is the case for $\Delta_0 := \{ \psi \in \Gamma \mid \operatorname{Var}\psi \subseteq \operatorname{Var}\varphi \}$, whence we obtain a total valuation α such that $\alpha(\psi) = 1$ for every $\psi \in \Delta_0$, yet $\alpha(\varphi) = 0$. The restriction $v := \alpha\restriction_{\operatorname{Var}\varphi}$ is the sought-after counterexample that witnesses $\Gamma \not\rhd \varphi$. Indeed, on the one hand we have that $v \rhd \Gamma$, for if $\psi \in \Gamma$ is such that $\operatorname{Var}\psi \subseteq \operatorname{dom} v = \operatorname{Var}\varphi$, then $\psi \in \Delta_0$, and therefore $v^*(\psi) = \alpha(\psi) = 1$; on the other hand, of course $\operatorname{Var}\varphi \subseteq \operatorname{dom} v$, yet $v^*(\varphi) = \alpha(\varphi) = 0$, which is to say that $v \not\rhd \varphi$.

(2) $\Rightarrow$ (1) Suppose that there is $\Delta \subseteq \Gamma$ such that (a) $\operatorname{Var}\Delta \subseteq \operatorname{Var}\varphi$ and (b) $\Delta \vDash \varphi$. To see that $\Gamma \rhd \varphi$, consider a partial assignment v, and assume that (c) $v \rhd \Gamma$. We need to check $v \rhd \varphi$, whence we suppose that $\operatorname{Var}\varphi \subseteq \operatorname{dom} v$. Now (a) yields $\operatorname{Var}\Delta \subseteq \operatorname{dom} v$, which according to (c) implies $v^*(\psi) = 1$ for every $\psi \in \Delta$. Extend v to a total valuation $\alpha : \operatorname{Var} \to \mathbf{2}$. Since $\alpha(\psi) = v^*(\psi) = 1$ for every $\psi \in \Delta$, in view of (b) it follows that $v^*(\varphi) = \alpha(\varphi) = 1$, as required. □

3 Duality

Unfolding the definitions, it is readily seen that $\rhd$ can be expressed in terms of $\blacktriangleright$, and vice versa (cf. [18, Section 2.1.2.3]), viz.

$$v \rhd \varphi \text{ iff } v \not\blacktriangleright \neg\varphi \quad \text{and} \quad v \blacktriangleright \varphi \text{ iff } v \not\rhd \neg\varphi.$$

A slightly more fine-grained approach explains the duality of certain failures of classical principles that set B_3 and PWK apart both from classical propositional logic, as well as from each other. To this end, let us extend (1) to allow for multiple conclusions, disjunctively understood, by setting

$$\Gamma \succ \Delta \quad \text{iff} \quad (\forall v : \mathrm{Var} \rightharpoonup \mathbf{2})(\, v \succ \Gamma \Rightarrow (\exists \varphi \in \Delta)\, v \succ \varphi \,).$$

Standard shorthand will be used, so that $\varphi \rhd \Delta$ stands for $\{\, \varphi \,\} \rhd \Delta$, etc.

Furthermore, we employ the well-known *duality mapping* $^d : \mathrm{Form} \to \mathrm{Form}$, recursively defined by

$$\begin{aligned} p^d &= p \text{ for } p \in \mathrm{Var}, \\ (\varphi \wedge \psi)^d &= \varphi^d \vee \psi^d, \\ (\varphi \vee \psi)^d &= \varphi^d \wedge \psi^d, \\ (\neg\varphi)^d &= \neg\varphi^d, \end{aligned}$$

for which see, e.g., [24]. A simple inductive argument shows that $(\varphi^d)^d = \varphi$ for every formula φ. In the following, let $\Gamma^d = \{\, \psi^d \mid \psi \in \Gamma \,\}$.

Let v be a partial truth assignment. Define $\neg v : \mathrm{Var} \rightharpoonup \mathbf{2}$ pointwise through Boolean complementation, i.e., such that $(\neg v)(p) = 1 - v(p)$, where of course $\mathrm{dom}\, \neg v = \mathrm{dom}\, v$. Induction yields

$$(\neg v)^*(\varphi) = 1 - v^*(\varphi^d), \tag{4}$$

for every formula φ with $\mathrm{Var}\, \varphi \subseteq \mathrm{dom}\, v$.

Proposition 3. *For any* $\Gamma, \Delta \subseteq \mathrm{Form}$*, the following are equivalent.*

1. $\Gamma \rhd \Delta$.
2. $\Delta^d \blacktriangleright \Gamma^d$.

Proof. First observe that $v \not\rhd \varphi$ iff $\mathrm{Var}\, \varphi \subseteq \mathrm{dom}\, v$ and $v^*(\varphi) = 0$, which, by (4) and since $\varphi^{dd} = \varphi$, is the case precisely when $\mathrm{Var}\, \varphi \subseteq \mathrm{dom}\, \neg v$ and $(\neg v)^*(\varphi^d) = 1$. Therefore $v \not\rhd \varphi$ iff $\neg v \blacktriangleright \varphi^d$. Equivalently: $v \rhd \varphi$ iff $\neg v \not\blacktriangleright \varphi^d$. It follows that

$$\begin{aligned} \Gamma \rhd \Delta \quad &\text{iff} \quad (\forall v : \mathrm{Var} \rightharpoonup \mathbf{2})(\, (\forall \psi \in \Gamma)\, \neg v \not\blacktriangleright \psi^d \;\Rightarrow\; (\exists \varphi \in \Delta)\, \neg v \not\blacktriangleright \varphi^d \,) \\ &\text{iff} \quad (\forall v : \mathrm{Var} \rightharpoonup \mathbf{2})(\, (\forall \varphi \in \Delta)\, \neg v \blacktriangleright \varphi^d \;\Rightarrow\; (\exists \psi \in \Gamma)\, \neg v \blacktriangleright \psi^d \,) \\ &\text{iff} \quad (\forall v : \mathrm{Var} \rightharpoonup \mathbf{2})(\, (\forall \delta \in \Delta^d)\, \neg v \blacktriangleright \delta \;\Rightarrow\; (\exists \gamma \in \Gamma^d)\, \neg v \blacktriangleright \gamma \,) \\ &\text{iff} \quad (\forall v : \mathrm{Var} \rightharpoonup \mathbf{2})(\, (\forall \delta \in \Delta^d)\, v \blacktriangleright \delta \;\Rightarrow\; (\exists \gamma \in \Gamma^d)\, v \blacktriangleright \gamma \,) \end{aligned}$$

since every partial truth assignment can be written in the form $v = \neg w$, where $w := \neg v$. □

Proposition 4. *For any* $\Gamma, \Delta \subseteq \mathrm{Form}$*, the following are equivalent.*

1. $\Gamma \blacktriangleright \Delta$.

2. $\Delta^d \triangleright \Gamma^d$.

Proof. Similar to the proof of Proposition 3. □

Here is an example. The logic of $\blacktriangleright$ does not validate disjunctive addition, which is the principle $\varphi \blacktriangleright \varphi \vee \psi$. For instance, if p and q are distinct propositional variables, then the assignment v_p which is undefined except for $v(p) = 1$ shows that $p \not\blacktriangleright p \vee q$. Proposition 4 yields that $(p \vee q)^d \not\triangleright p^d$, which is to say that $p \wedge q \not\triangleright p$. Therefore, $\triangleright$ does not validate conjunctive simplification, which is the principle $\varphi \wedge \psi \triangleright \varphi$. See, e.g., [5, Lemma 1.2.3-4] for the corresponding statements for B_3 and PWK, to which we turn our attention briefly in Section 5.

4 Mixed consequence

Let us now return to the single-conclusion setting. Analogous to the approach of *mixed consequence logics* [3,6], we break the symmetry as to how in (1) "truth" under partial assignments is supposed to pass from assumptions to the conclusion.[3] In the present setting, we may do so as follows. Let $\succ_1$ and $\succ_2$ be relations between partial assignments and formulas. We stipulate

$$\Gamma \succ_{i|j} \varphi \quad \text{iff} \quad (\forall v : \mathrm{Var} \rightharpoonup \mathbf{2})(\, v \succ_i \Gamma \Rightarrow v \succ_j \varphi \,), \tag{5}$$

where $i, j \in \{\, 1, 2 \,\}$.

The cases of interest to us are those in which $\succ_1$ and $\succ_2$ are again given by $\blacktriangleright$ and $\triangleright$ of Sections 2.1 and 2.2, respectively. Apparently $\succ_{1|1}$ amounts to $\blacktriangleright$, while $\succ_{2|2}$ is nothing but $\triangleright$. It will next be seen that the mixed case $\succ_{1|2}$ recovers classical semantic consequence $\vDash$, while the dual mixed form $\succ_{2|1}$ calls for an additional constant to be possible at all.

Rather than $\succ_{i|j}$, let us instead write

$$\begin{aligned} \Gamma \mathrel{\blacktriangleright\!\!\triangleright} \varphi \quad &\text{iff} \quad (\forall v : \mathrm{Var} \rightharpoonup \mathbf{2})(\, v \blacktriangleright \Gamma \Rightarrow v \triangleright \varphi \,), \\ \Gamma \mathrel{\triangleright\!\!\blacktriangleright} \varphi \quad &\text{iff} \quad (\forall v : \mathrm{Var} \rightharpoonup \mathbf{2})(\, v \triangleright \Gamma \Rightarrow v \blacktriangleright \varphi \,). \end{aligned}$$

Proposition 5. *For any* $\Gamma \cup \{\, \varphi \,\} \subseteq \mathrm{Form}$*, the following are equivalent.*

[3] I am grateful to Allard Tamminga for bringing this to my attention.

1. $\Gamma \blacktriangleright\!\!\triangleright \varphi$.

2. $\Gamma \vDash \varphi$.

Proof. (1) $\Rightarrow$ (2) Suppose that $\Gamma \blacktriangleright\!\!\triangleright \varphi$. Consider a valuation $\alpha : \mathrm{Form} \to \mathbf{2}$ such that $\alpha(\psi) = 1$ for every $\psi \in \Gamma$. The restriction $v := \alpha{\restriction}_{\mathrm{Var}}$ is a (total) assignment such that $v \blacktriangleright \psi$ for every $\psi \in \Gamma$. It follows that $v \triangleright \varphi$. Since $\mathrm{Var}\,\varphi \subseteq \mathrm{dom}\,v$, we obtain $\alpha(v) = v^*(\varphi) = 1$, as required.

(2) $\Rightarrow$ (1) Suppose that $\Gamma \vDash \varphi$. Consider an assignment $v : \mathrm{Var} \rightharpoonup \mathbf{2}$ for which $v \blacktriangleright \Gamma$. To see that $v \triangleright \varphi$, suppose that $\mathrm{Var}\,\varphi \subseteq \mathrm{dom}\,v$. Consider some extension of v to a total valuation α. For every $\psi \in \Gamma$, since $v \blacktriangleright \psi$, we know that $\mathrm{Var}\,\psi \subseteq \mathrm{dom}\,v$ and thus $\alpha(v) = v^*(\psi) = 1$. Since $\Gamma \vDash \varphi$, this implies that $v^*(\varphi) = \alpha(\varphi) = 1$. □

Let us next consider the dual mixed form. As it stands, $\Gamma \triangleright\!\!\blacktriangleright \varphi$ can never be the case. To see this, consider the empty assignment $v_0 : \mathrm{Var} \rightharpoonup \mathbf{2}$, i.e., the assignment with $\mathrm{dom}\,v_0 = \emptyset$. Clearly $v_0 \triangleright \Gamma$, since $\mathrm{Var}\,\psi \neq \emptyset$ for every $\psi \in \Gamma$. Now, if it were the case that $\Gamma \triangleright\!\!\blacktriangleright \varphi$, then we would be able to infer that $v_0 \blacktriangleright \varphi$. In particular $\mathrm{Var}\,\varphi = \emptyset$, which the present setting does not provide for.

Thus, to make this work, and for the remainder of this section only, we introduce an additional nullary connective $\bot$ (*falsum*), of course with the convention that $v^*(\bot) = 0$ for every assignment v. This yields a supply of formulas without propositional variables, namely $\mathrm{Form}_0 = \{\, \varphi \in \mathrm{Form} \mid \mathrm{Var}\,\varphi = \emptyset \,\}$. Let

$$T = \{\, \varphi \in \mathrm{Form}_0 \mid \emptyset \vDash \varphi \,\} \quad \text{and} \quad F = \{\, \varphi \in \mathrm{Form}_0 \mid \emptyset \vDash \neg\varphi \,\}.$$

Proposition 6. *For any* $\Gamma \cup \{\, \varphi \,\} \subseteq \mathrm{Form}$*, the following are equivalent.*

1. $\Gamma \triangleright\!\!\blacktriangleright \varphi$.

2. Either $\varphi \in T$ *or there is* $\psi \in \Gamma \cap F$.

Proof. (1) $\Rightarrow$ (2) Suppose that $\Gamma \triangleright\!\!\blacktriangleright \varphi$. We assume $\Gamma \cap F = \emptyset$ and show that $\varphi \in T$. To this end, we consider the empty assignment $v_0 : \emptyset \to \mathbf{2}$. We claim that $v_0 \triangleright \Gamma$. In fact, for every $\psi \in \Gamma$, either $\mathrm{Var}\,\psi = \emptyset$, in which case $\psi \in T$ and thus $v_0^*(\psi) = 1$, or else $\mathrm{Var}\,\psi \neq \emptyset = \mathrm{dom}\,v_0$, in which case again $v_0 \triangleright \varphi$. It follows that $v_0 \blacktriangleright \varphi$, and this is to say that $\mathrm{Var}\,\varphi = \emptyset$ as well as $v_0^*(\varphi) = 1$. Now, since $\varphi \in \mathrm{Form}_0$, it is clear that $\alpha(\varphi) = v_0^*(\varphi) = 1$ for every total valuation α, i.e., $\emptyset \vDash \varphi$.

(2) $\Rightarrow$ (1) Suppose that either (a) $\varphi \in T$ or that (b) there is $\psi \in \Gamma \cap F$. To see that this implies $\Gamma \triangleright\!\!\blacktriangleright \varphi$, consider a partial assignment v for which $v \triangleright \Gamma$. We need to check that $v \blacktriangleright \varphi$. Now, if indeed $\varphi \in T$, then $v \blacktriangleright \varphi$ is trivial. But under the present assumptions, this is the only

case that may occur, for if (b) there were indeed some $\psi \in \Gamma \cap F$, then, since $v \rhd \psi$, we had $v^*(\psi) = 1 = \alpha(\psi)$ for any valuation α, contrary to the assumption that $\psi \in F$. □

5 Discussion

Two ways to deal with partial truth assignments have been discussed: a "tolerant" approach, in which formulas are satisfied unless refuted, and a "strict" one, in which formulas are satisfied only when put into consideration properly. Let us now make the connection with weak Kleene logic explicit.

Kleene supplied a third value u to the classical two-valued tables in order to explain a "weak sense" in which the propositional connectives apply to partial recursive predicates [14, p. 334]. From an algebraic point of view, this means to adjoin an absorbing element to the Boolean algebra **2**, viz.

$\vee$	0	u	1
0	0	u	1
u	u	u	u
1	1	u	1

$\wedge$	0	u	1
0	0	u	0
u	u	u	u
1	0	u	1

$\neg$	
0	1
u	u
1	0

Both B_3 and PWK are matrix logics based on this algebra **WK**, but they differ in their respective choice of designated values, as follows. Every (total) assignment $w : \mathrm{Var} \to \mathbf{WK}$ extends uniquely to a valuation $w^* : \mathrm{Form} \to \mathbf{WK}$, and we stipulate, with regard to B_3,

$$w \vDash_{B_3} \varphi \quad \text{iff} \quad w^*(\varphi) = 1$$

as well as, for PWK,

$$w \vDash_{\mathrm{PWK}} \varphi \quad \text{iff} \quad w^*(\varphi) \in \{ u, 1 \} .$$

Now $\vDash_{B_3}$ and $\vDash_{\mathrm{PWK}}$ denote the consequence relations that correspond to the respective cases of (1), but where we now quantify over all the total assignments $w : \mathrm{Var} \to \mathbf{WK}$ rather than the partial Boolean ones.[4] Passing again through Urquhart's and Ciuni and Carrara's results cited above, Propositions 1 and 2 now offer simple two-valued semantics for both B_3 and PWK, if at the price of hiding the undefined value behind

[4]There is a one-to-one correspondence between partial assignments $\mathrm{Var} \rightharpoonup \mathbf{2}$ on the one hand and total assignments $\mathrm{Var} \to \mathbf{WK}$ on the other, which in fact is an isomorphism of *involutive bisemilattices* (for the latter see [4]) [17]. This observation prompted the present note.

chunks of undefined variables.[5] The philosophical issue about the proper interpretation of u thus merely shifts, though I do not intend to put this at stake here. Incidentally, however, Proposition 1 appears to be quite in line with Beall's off-topic interpretation, according to which B_3 "as a logic concerns not simply truth-preservation but *truth-and-topic preservation*" [1, p. 140]. It might not be a stretch to think of partial assignments as mapping out certain topics of discourse, so that we read

$v \blacktriangleright \varphi$ as "φ is on-topic and true",

which explains, for instance, the failure of disjunctive addition discussed above. In comparison, PWK handles topicality in a different manner: writing the conditional of (3) as a disjunction, we read, once again under Beall's interpretation,

$v \vartriangleright \varphi$ as "φ is off-topic or (on-topic and) true",

which of course corresponds to the choice of designated values. By contraposition, PWK can also be understood through *on-topic backward falsity preservation* [22]. Duality provides yet another reading of PWK, namely as a logic of *backward dual truth-and-topic splitting*: if the dual of the conclusion of a PWK-consequence is on-topic and true, then so is the dual of at least one of the premises.

Partial valuations thus readily account for topicality, but remain one-dimensional in that they do not divide propositions that are off-topic further into *true and off-topic* and *false and off-topic*. Such a two-dimensional refinement has recently been developed by Song *et al.* [22], who propose a Herzberger-style semantics [11] that accommodates B_3, PWK and classical logic within a single framework, and thereby generalize Beall's interpretation.[6] Exploring how partial valuations can be employed within such multi-dimensional frameworks could be an interesting venue for further research.

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[5]Incidentally, the *ad hoc* terminology of this paper now aligns with the conventional one, in that "strictness" corresponds with preservation of the value 1, while the "tolerant" approach requires the values $u, 1$ to be preserved under logical consequence.

[6]I am grateful to the anonymous referee for bringing this to my attention.

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