

Model-Theoretic Inferentialism and Paraconsistency in Multiple-valued Structures

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Abstract

Carnap [5] observed in 1943 that there are non-normal valuations which do not follow the usual truth conditions of classical connectives, yet preserve all classical inferences. This finding reveals an asymmetry between the syntax and semantics of classical logic and might be counted as a challenge to the program of inferentialism. While several solutions have been proposed for classical logic, the problem persists for some well-known paraconsistent logics and none of the existing solutions work properly. To address this issue, we propose a method to *read off* [model-theoretic] semantics from the deductive behavior of these logics in a *uniform* way. We examine this method for three paraconsistent logics chosen from different traditions: **LP**, **J₃** and **mbC**. We aim to show that the syntax-first approach of inferentialism to logic can still be maintained for these paraconsistent logics, without losing any semantics.

Keywords— Inferentialism, Model-Theoretic Inferentialism, Paraconsistent Logics, Categoricity, Carnap’s problem.

1 Introduction

There have been two main approaches to the semantics of logical connectives: proof-theoretic semantics and model-theoretic semantics. According to the model-theoretic tradition, semantics is a recursive definition of truth on a model, and the focus is on concepts such as truth, reference, and validity. In contrast, in the proof-theoretic

tradition, such concepts are avoided. Instead, proof-theoretic semantics proposes to define meaning through syntactic concepts such as rules of inference and proofs.

On the other hand, we have the *Inferentialists*, who claim that the meaning of expressions in a language is determined by their inferential role (see Brandom [15] and Peregrin [12] for more information about inferentialism). They usually argue that concepts such as “interpretation,” “reference,” and “truth” (which are typically used in model-theoretic semantics) do not play a constitutive role in determining meaning—or if they do, their role is not fundamental. Proof-theoretic semantics appears to be a natural choice for inferentialists who wish to define meaning in terms of inferential patterns.

However, there is also a moderate approach taken by Garson [9], known as *Model-Theoretic Inferentialism*, according to which we do not need to avoid the notion of a model or insist on defining meaning *entirely* in proof-theoretic terms. Similar distinctions between *strict (or strong) inferentialism* and *moderate inferentialism* have also been discussed by Tennant [21] and Speitel [22]. In model-theoretic inferentialism, our candidate for semantics is still a model-theoretic construction, but it is the role of inferential rules to determine its exact structure. Roughly speaking, if U is a set of model-theoretic constructions that contain the semantics for a particular logic (and possibly include meaningless model-theoretic structures), and R is a set of inference rules, we want to see whether R can somehow pick out the intended semantics from U , or whether there is a way to associate what R selects with the intended semantics. This is our main perspective on the meaning of connectives in what follows.

However, it is not necessary for inferentialists to adopt this particular perspective, which we call Model-Theoretic Inferentialism. The following technical results might be interesting for all inferentialists because they support the claim that “... however you define a logical operator, there should be some kind of inferential way to it,” as Peregrin [12, p.186] puts it.¹

Historically, Carnap’s findings in 1943 about non-normal valuations in propositional classical logic can be regarded as the first challenge to model-theoretic inferentialism. He observed that there exist valuations that do not respect the usual truth conditions of classical connectives, yet satisfy every inference $\Gamma \vdash A$ that is derivable in classical logic. For instance, consider the *trivial* valuation v that assigns 1 to every sentence of the language. It is clear that v satisfies every valid inference of classical logic, but it obviously violates the standard truth condition for negation, since $v(A) = v(\neg A) = 1$.

In what follows, we ask whether model-theoretic inferentialists can explain the semantics of deviant logics, particularly paraconsistent logics. Is there an inferentialist way to derive the semantics of certain paraconsistent logics from their deductive behavior? To answer this question, we consider three famous paraconsistent logics: Asenjo-Priest’s Logic of Paradox **LP** [13], [23]; D’Ottaviano and da Costa’s logic **J₃** [11]; and the logic **mbC** (also known as **B**) [20], which is a base logic for a large family of paraconsistent logics known as Logics of Formal Inconsistency (LFIs).

The common solutions for classical logic do not apply uniformly here. For example, the solution proposed by Bonnay and Westerståhl [4], based on accepting the principles of *compositionality* and *non-triviality*, does not work, simply because the

¹Although Garson (2013) frequently employs the term “Model-Theoretic Inferentialism,” it is important to note that our approach diverges from his. For example, we are not going to develop Garson’s idea of *Natural Semantics*, and unlike him, we adopt Multiple-Conclusion logics.

logic the semantics of **mbC** is not compositional (see [6, p.43]). Likewise, accepting *non-triviality* as a semantic constraint is not a good solution, because the trivial model that assigns the third value (the *glutty* value) to all formulas is a normal model in **LP** (see Definition 2.8 below). Therefore, non-triviality and non-compositionality cannot be accepted here. Accepting either would essentially prevent us from considering the intended semantics of these logics (either **LP** or **mbC**) as a legitimate semantics at all. However, there are semantic constraints specific to **LP** that can eliminate non-normal models. For instance, Tabakci [18] proposes the constraint of closure on the third value (this constraint is called *i to i* in his paper) and demonstrates that this constraint can effectively eliminate non-normal models (at least in multiple-conclusion systems). Tabakci [18] has also proposed other solutions, such as fixing the interpretation of negation. However, accepting these semantic constraints is not entirely in line with the spirit of the inferentialist program, since it was supposed that all semantic properties would be derived from the syntax and inferential behavior of the logics, rather than being accepted in advance as prior principles.

Adopting multiple-conclusion proof systems only works partially. We will show that it works with respect to the bi-valuational² semantics of **mbC**, but the multiple-conclusion sequent systems of **J₃** and **LP** still admit non-normal models. The existence of non-normal models for **LP** has already been shown by Tabakci [18]. In fact, it will be shown that there is nothing special about paraconsistency or about a specific presentation of a logic in sequent style. Multiple-conclusion systems, in principle, cannot provide a categorical³ characterization of multi-valued matrices with more than two values (see Proposition 3.3).

Since a multiple-conclusion system is not sufficient for a categorical axiomatization, going beyond this machinery to find one would be too artificial, complicated, or semantically loaded. However, some attempts in this direction have been made. For example, Trafford [19] proposes a three-sided sequent calculus for **LP** that determines its intended semantics. Rumfitt [26] addresses the problem for logics with gappy values (but not particularly **LP**, which has a glutty value). A more general solution for many-valued logics has been proposed by Hjortland [10], which provides a categorical axiomatization of these logics by means of an *n*-sided sequent calculus. Hjortland defends his approach from an inferentialist position called *multilateralism*, according to which we have more than one type of assertion and denial in the language. However, multilateralism is not a stance that every inferentialist would be willing to adopt. Aside from this, in our view, his method has other issues as well. The *n*-sided sequents seem very complex and artificial, as if they have no connection to the inferential behavior of language users. Another issue is that the definition of validity in any *n*-valued matrix will depend on that matrix. In fact, we will not have a single, unified definition of validity. All of these factors make this approach somewhat unsatisfactory from an inferentialist perspective. Our claim is not that the above methods cannot solve the categoricity problem for these paraconsistent logics. All of these solutions technically eliminate non-normal models for some of these logics. We aim to present a method that is more compatible with inferentialist standards. For example, it should not depend on accepting semantic constraints, nor should it require the adoption of more complex proof systems.

The proposed solution in this paper is still based on multiple-conclusion systems.

²We use “bi-valuation” and “2-valued valuation” interchangeably.

³The term ‘categorical’ has been used with different meanings in the literature. To distinguish between these, see the discussion after Proposition 3.3.

However, as noted before, this is not sufficient. We propose to change the *type of model*. Instead of 3-valued valuations, we return to bi-valuations. This change would be more neutral and natural for inferentialists: an inferentialist does not know *prima facie* too much about the intended semantics. He just assumes that some sentences are true and some are false. Only after that does he find out, through the deductive behavior of his system, that there are true sentences along with their negations, or equivalently, that there are inconsistent sentences and therefore possibly inconsistent states or values.

The next step is to make sense of these models. Let V be the set of 2-valued models of a paraconsistent logic $\mathbf{L} \in \{\mathbf{LP}, \mathbf{J}_3, \mathbf{mbC}\}$. We propose a straightforward construction of 3-valued models $V/3\top$ over V and show that our constructed $V/3\top$ induces the intended semantics of our logic $\mathbf{L}$ in each case. Interestingly, in the case of the logic $\mathbf{mbC}$, our construction induces an alternative semantics (a totally different creature): a 3-valued non-deterministic matrix (or Nmatrix), which is sound and complete for it. We use the term “multiple-valued structures” to refer to matrices, regardless of whether they are deterministic or non-deterministic. Our main claim is that the semantics of some paraconsistent logics characterized by these structures can be recovered directly from their deductive behavior.

The multiple-conclusion proof systems, when combined with the method we introduce, in a sense allow one to read semantics off the syntax. In response to the criticisms directed at multiple-conclusion systems (e.g., [27]), this very success can serve as a defense of why multiple conclusions provide a better tool for representing inferences. The multiple-conclusion framework is a better tool because it allows for the reconstruction (or recovery) of semantics.

2 Preliminaries

We identify a propositional language with a set of connectives of fixed arity. Let $\mathcal{L}^c$ be the common classical propositional language, i.e. $\mathcal{L}^c = \{\wedge, \vee, \supset, \neg\}$. There are two other connectives: the falsum constant $\mathbf{F}$, which should be treated as a 0-place operator, and the consistency operator $\circ$, which should be treated as a one-place operator. The classical languages extended by these connectives are denoted by $\mathcal{L}_F^c$ and $\mathcal{L}_\circ^c$, respectively. The formulas for each language are generated by the usual formation rules from a (countably infinite) set of primitives $\{p_n : n \in \mathbb{N}\}$, called the propositional variables. The set of all formulas of a given language $\mathcal{L}$ is denoted by $\mathcal{W}(\mathcal{L})$. Finite subsets of $\mathcal{W}(\mathcal{L})$ are denoted by $\Gamma, \Delta, \Gamma', \Delta', \dots$, and their members by $\alpha, \beta, \gamma, \dots$, etc.

In a given language $\mathcal{L}$, we refer to a structure of the form $\Gamma \vdash \alpha$ as an *inference* and $\Gamma \Rightarrow \Delta$ as a *sequent*. A single-conclusion consequence $\vdash$ relation is a set of inferences, i.e., $\vdash \subseteq \mathcal{P}(\mathcal{W}(\mathcal{L})) \times \mathcal{W}(\mathcal{L})$. A multiple-conclusion consequence relation $\Rightarrow$ is a set of sequents, i.e., $\Rightarrow \subseteq \mathcal{P}(\mathcal{W}(\mathcal{L})) \times \mathcal{P}(\mathcal{W}(\mathcal{L}))$. Additional conditions are usually added to treat a relation as a consequence relation, such as being Tarskian (monotonic, reflexive, and transitive). However, for the purpose of this paper, these properties are not relevant, and the minimal definition is sufficient. Nevertheless, every logic mentioned in this paper is Tarskian (or Scottian in the multiple-conclusion cases).

The consequence relations introduced in this paper are defined either proof-theoretically by Gentzen-type systems or model-theoretically, either by (N)matrices or valuations semantics. In any case, whether we define a consequence relation model-theoretically or proof-theoretically, when we want to say that a certain sequent $\Gamma \Rightarrow \Delta$ (or inference

$\Gamma \vdash \alpha$) holds in a certain consequence relation $\Rightarrow$ (or $\vdash$), we write: $\Gamma \Rightarrow \Delta \in \Rightarrow$ (or $\Gamma \vdash \alpha \in \vdash$).

Note that we use the consequence relation symbol in two ways: once to represent an individual sequent or inference, regardless of whether it holds or not, and once to represent a specific set of inferences or sequents—that is, the set of inferences or sequents that hold. The distinction between the two is clear from the context and should not cause confusion.

2.1 Semantics

Definition 2.1 (matrix). A *matrix* (or simply matrix) for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

1. $\mathcal{V}$ is a non-empty set of truth values (domain);
2. $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$ (set of designated values);
3. $\mathcal{O}$ is a set of functions containing exactly an n -ary function $\bar{\diamond} : \mathcal{V}^n \rightarrow \mathcal{V}$ for every n -ary connective $\diamond$ of $\mathcal{L}$.

We write $\mathcal{V}_{\mathcal{M}}$ to emphasize that $\mathcal{V}$ belongs to the matrix $\mathcal{M}$. When $|\mathcal{V}| = n$, we say that $\mathcal{M}$ is an n -valued Nmatrix. Several matrices will be introduced later. However, we consider a special 2-valued matrix for each possible propositional language $\mathcal{L}$ and denote it by $\mathcal{M}_2^{\mathcal{L}}$:

$$\mathcal{M}_2^{\mathcal{L}} = \langle \{0, 1\}, \{1\}, \bar{\mathcal{L}} \rangle$$

where $\bar{\mathcal{L}} = \{\bar{\diamond} : \diamond \in \mathcal{L}\}$ and $\bar{\diamond}$ is a constant function associated with the n -ary connective $\diamond$ such that $\bar{\diamond}(a_1, \dots, a_n) = 1$. At first glance, this definition may seem a bit confusing. However, it will later help us uniformly define a particular class of models of a consequence (see Definition 2.11 below). Specifically, this notation allows us to discuss the two-valued models of any logic in this paper, even if that logic does not have a two-valued matrix-based semantics. For this purpose, the behaviour of the operators is irrelevant; therefore, instead of a constant function that assigns 1 to every input of connectives, we could consider any other arbitrary function. The only relevant thing is that 1 is the designated value.

In the next section, we introduce these non-deterministic matrices, which are important semantical tools for **mbC** and other LFIs.

Definition 2.2 (Nmatrix). A *non-deterministic matrix*, or *Nmatrix*, for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ that satisfies items (1) and (2) of Definition 2.1 and the following condition:

1. $\mathcal{O}$ is a set of functions containing exactly an n -ary function $\bar{\diamond} : \mathcal{V}^n \rightarrow (\mathcal{P}(\mathcal{V}) - \emptyset)$ for every n -ary connective $\diamond$ of $\mathcal{L}$ (every operation on elements of $\mathcal{V}$ returns a non-empty subset of $\mathcal{V}$).

By definition of Nmatrices, matrices are a special case of Nmatrices. In fact, matrices are those Nmatrices whose operators return only a singleton of the domain. In other words, if each operator $\bar{\diamond}$ of the Nmatrix satisfies $|\bar{\diamond}(a_1, \dots, a_n)| = 1$ for each value $a_1, \dots, a_n$ in $\mathcal{V}$, then it is essentially a matrix. Set-theoretically, these are distinct structures, but they essentially represent the same structure and also characterize the same logics.

In the following, we make claims that hold for both matrices and Nmatrices. In these cases, we use the expression “(N)matrix” instead of explicitly expressing the structure is either matrix or Nmatrix.

Definition 2.3 (valuation). A *valuation* of an (N)matrix $\mathcal{M}$ is a function from the set of all formulas to the domain of the matrix, i.e., $v : \mathcal{W}(\mathcal{L}) \rightarrow \mathcal{V}_{\mathcal{M}}$.

The set of all valuations of a matrix $\mathcal{M}$ is denoted by $\mathcal{V}_{\mathcal{M}}^{\mathcal{W}(\mathcal{L})}$.

Definition 2.4 (normal valuation). Given (N)matrix an $\mathcal{M}$, a valuation v when $\mathcal{M}$ is a matrix is called *normal* if, for any formula $\diamond(\alpha_1, \dots, \alpha_n) \in \mathcal{W}(\mathcal{L})$ (where $\diamond$ is an n -place operator), it satisfies the following condition:

$$v(\diamond(\alpha_1, \dots, \alpha_n)) = \bar{\diamond}(v(\alpha_1), \dots, v(\alpha_n))$$

If $\mathcal{M}$ is an Nmatrix, then v is normal if it satisfies:

$$v(\diamond(\alpha_1, \dots, \alpha_n)) \in \bar{\diamond}(v(\alpha_1), \dots, v(\alpha_n))$$

Therefore, a *normal* valuation is simply a valuation that *respects* every connective interpreted in $\mathcal{M}$. A *non-normal* valuation is a valuation for (N)matrix $\mathcal{M}$ that does not satisfy the corresponding condition for at least one connective of its language.

The set of all normal valuations of $\mathcal{M}$ is denoted by $\Lambda_{\mathcal{M}}$. For an (N)matrix $\mathcal{M}_X$, we simply write Λ_X .

Definition 2.5 (satisfaction). Given an (N)matrix $\mathcal{M}$ and a valuation v of $\mathcal{M}$, we say that v *satisfies* an inference $\Gamma \vdash \alpha$ if $v(\beta) \notin \mathcal{D}$ for some $\beta \in \Gamma$ or $v(\alpha) \in \mathcal{D}$. We also say that v satisfies a sequent $\Gamma \Rightarrow \Delta$ if $v(\beta) \notin \mathcal{D}$ for some $\beta \in \Gamma$ or $v(\alpha) \in \mathcal{D}$ for some $\alpha \in \Delta$.

We sometimes use the word “model” for a valuation that satisfies a certain set of inferences or sequents.

Definition 2.6 (validity). An inference $\Gamma \vdash \alpha$ is valid in a matrix $\mathcal{M}$ if it is satisfied by every normal valuation of the matrix. In other words, for every $v \in \Lambda_{\mathcal{M}}$, v satisfies $\Gamma \vdash \alpha$.

Definition 2.7 (induced logic). The logic induced by a matrix $\mathcal{M}$ is a pair $\langle \vdash_{\mathcal{M}}, \mathcal{L} \rangle$, where $\mathcal{L}$ is the language of $\mathcal{M}$ and $\vdash_{\mathcal{M}}$ is a consequence relation containing all and only valid inferences of $\mathcal{M}$. We denote a valid inference of this logic by $\Gamma \vdash_{\mathcal{M}} \alpha$ or write $\Gamma \vdash \alpha \in \vdash_{\mathcal{M}}$.

Definition 2.8 (LP). The logic **LP** is induced by the 3-valued matrix $\mathcal{M}_{\text{LP}} = \langle t, \top, f, \bar{\top}, \bar{f}, \bar{\top}, \bar{f}, \bar{\top}, \bar{f} \rangle$, where the connectives are interpreted as follows:

$\bar{\wedge}$	t	f	$\top$	$\bar{\vee}$	t	f	$\top$	$\bar{\neg}$	t	f	$\top$
t	t	f	$\top$	t	t	t	t	t	t	f	$\top$
f	f	f	f	f	t	f	$\top$	f	f	t	$\top$
$\top$	$\top$	f	$\top$	$\top$	t	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

The truth table for implication is defined by $a \supset b =_{def} \bar{\neg} a \bar{\vee} b$ for every pair of values a and b in $\{t, \top, f\}$.

By a simple induction it can be easily seen that the valuation $v^{\top}$, defined as $v^{\top}(\phi) = \top$ for every sentence ϕ of its language, is indeed a *normal* valuation of $\mathcal{M}_{\text{LP}}$. Since $\top$ is a designated value, this valuation is a *trivial* valuation, as it sees every sentence as true.

Definition 2.9 ($\mathbf{J}_3$). The logic $\mathbf{J}_3$ in the language $\mathcal{L}_{\mathbf{F}}^c$ is induced by the 3-valued matrix $\mathcal{M}_{\mathbf{J}_3} = \langle \{t, \top, f\}, \{t, \top\}, \{\bar{\wedge}, \bar{\vee}, \bar{\neg}, \bar{\supset}, \bar{\mathbf{F}}\} \rangle$, where the connectives $\bar{\wedge}, \bar{\vee}$ and $\bar{\neg}$ are interpreted as in $\mathbf{LP}$, but $\bar{\mathbf{F}} = f$, and implication is interpreted by the following function:

$$a \bar{\supset} b = \begin{cases} b & \text{if } a \neq f, \\ t & \text{if } a = f. \end{cases}$$

The logic $\mathbf{LP}$ was introduced by Priest [13] in 1979 and, before that, by Asenjo [23] in 1966. The interpretation for conjunction, negation, and disjunction is known as the *Kleene strong connectives* (see [24]). A difference from Kleene’s logic is that the third value, $\top$, is designated here. This value was added to deal with paradoxes and antinomies; it represents the “both true and false” or “paradoxical” value, while the other values, t and f , behave classically. An important feature of this logic is that it has the same theorems as classical logic.

The logic $\mathbf{J}_3$ has been introduced by several authors with different signatures and motivations. It was initially introduced by D’Ottaviano and da Costa [11] in 1970 with the signature of a modal language to address Jaskowski’s problem. It was subsequently rediscovered by Carnielli and Marcos [20] in 2002 to deal with inconsistent databases. However, our presentation, which includes a primitive falsum, is based on Avron’s presentation [2, p.112]. This logic has a well-behaved implication, and the deduction theorem holds for it, while it does not hold for the implication of $\mathbf{LP}$. Another important feature of this logic is its expressive power: a wide range of 3-valued paraconsistent logics can be embedded in this logic (see [2, Corollary 4.67(2)]). It is interesting to see whether the semantics of such a rich logic can be read off from the syntax alone.

Another logic we are going to discuss is $\mathbf{mbC}$. This logic belongs to the Brazilian school of paraconsistent logic, mainly introduced by Carnielli *et al.* [6]. The main idea of this school is that there is a unary *consistency* operator $\circ$ that divides propositions into two types: “normal” and “abnormal.” The principle that allows deducing everything from contradictions applies only to normal propositions. Therefore, in these logics, $\alpha, \neg\alpha \vdash \beta$ is not a valid inference in general, but if we add $\circ\alpha$ to the premises, we obtain a valid inference $\circ\alpha, \alpha, \neg\alpha \vdash \beta$. The logic $\mathbf{mbC}$ serves as the base for a large family of logics called LFIs. It has been proven that this logic cannot be semantically characterized by finite matrices (see [6, p.122]). Hence, to introduce this logic, we need to use an alternative semantics: the *valuational semantics*.

Definition 2.10 (mbC-valuation). A function $v : \mathcal{W}(\mathcal{L}_{\mathbf{F}}^c) \rightarrow \{0, 1\}$ is a *mbC-valuation* if it satisfies the following conditions:

1. $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$
2. $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$
3. $v(\alpha \supset \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$
4. if $v(\neg\alpha) = 0$ then $v(\alpha) = 1$
5. if $v(\circ\alpha) = 1$ then $v(\alpha) = 0$ or $v(\neg\alpha) = 0$.

The set of all mbC-valuations is denoted by $\Lambda_{\mathbf{mbC}}$. We call any valuation that is not an mbC-valuation a *non-normal valuation* for $\mathbf{mbC}$. An inference $\Gamma \vdash \alpha$ is valid in $\mathbf{mbC}$ iff, for every valuation $v \in \Lambda_{\mathbf{mbC}}$, either $v(\gamma) = 0$ for some $\gamma \in \Gamma$ or $v(\alpha) = 1$. We show valid inferences as $\Gamma \vdash_{\mathbf{mbC}} \alpha$.

The definition of the consequence relation induced by bi-valuational semantics is also similar to the one in Definition 2.7; however, there is no finite matrix $\mathcal{M}$. Thus, the consequence relation $\vdash_{\text{mbC}}$ is the set of all valid inferences for any valuation in Λ_{mbC} .

Since the conditions (4) and (5) are not biconditional statements, they violate the principle of *compositionality*: the truth value of a formula is not uniquely determined by the truth values of its constituents. For example, consider an mbC-valuation v such that $v(\alpha) = 1$; the value of $v(\neg\alpha)$ is not determined from this and can be either 1 or 0. This logic can be axiomatized by adding the axioms $\alpha \vee \neg\alpha$ and $\alpha \supset (\neg\alpha \supset \beta)$ to the usual Hilbert-style system of the positive fragment of classical logic. The soundness and completeness of this system are proved in [6, p.36].

One might think that the above logics are essentially semantical and not a good fit for inferentialists at all. But as we will see, these logics all have a good proof system, and as long as there is a nice syntactical characterization of a logic, there should be a way to obtain these semantics if they are their “real” semantics. This must be achieved by tracking the idea of *what follows from what*, with minimal assumptions on semantics. Garson explains this point with a nice metaphor:

Would it be possible, at least in principle, for alien anthropologists from a planet circling the star Alpha Centauri (who initially know nothing about our language) to learn what our sentences mean by simply investigating the way we reason from one to another in different circumstances?
[9, p.1]

Here, we want to know what happens when a user of the language reasons in a paraconsistent way. Similar questions regarding paraconsistency in the inferentialist framework have been raised by Trafford [19] and implicitly by Hjortland [10].

Given a consequence relation, we want to identify the set of all valuations that satisfy every inference of it. Thus, we have the following definition:

Definition 2.11 (models of a consequence). Given a consequence relation $\vdash$ of a logic in the language $\mathcal{L}$, and an (N)matrix $\mathcal{M}$ with a domain of $2 \leq n$ members, the set $\text{mod}_{\mathcal{M}}(\vdash)$ is defined as follows:

$$\text{mod}_{\mathcal{M}}(\vdash) = \{v \in \mathcal{V}_{\mathcal{M}}^{\mathcal{W}(\mathcal{L})} : v \text{ satisfies every } \Gamma \vdash \alpha \text{ such that } \Gamma \vdash \alpha \in \vdash\}$$

For a fixed finite matrix, we omit the index and write the number of its values instead as $\text{mod}_n(\vdash)$. Particularly, for the two-valued matrix $\mathcal{M}_2^{\mathcal{L}}$ of a fixed language, instead of $\text{mod}_{\mathcal{M}_2^{\mathcal{L}}}(\vdash)$, we simply write $\text{mod}_2(\vdash)$. Or for the consequence relation of **LP**, since the related matrix is 3-valued, instead of $\text{mod}_{\mathcal{M}_{\text{LP}}}(\vdash)$, we write $\text{mod}_3(\vdash)$. This notation can be confusing when several such n-valued (N)matrices exist for a given consequence relation. However, since this is not the case in this paper, no confusion should arise.

We call these *n-valued models of the consequence relation* $\vdash$. Notice that this definition also applies to multiple-conclusion consequences, where the definition of satisfaction is then the satisfaction of sequents. Therefore, we denote the set of all 2-valued models of a logic **L** by $\text{mod}_2(\Rightarrow_{\text{L}})$ and the 3-valued ones by $\text{mod}_3(\Rightarrow_{\text{L}})$.

The set $\text{mod}_2(\vdash)$, when $\vdash$ is the consequence relation of classical logic, is referred to as the “deductive model” in Garson [9, p.13]. Here, we have generalized the idea for any n-valued matrices.

When there are some non-normal valuations in $\text{mod}_n(\vdash)$ (valuations that are not *normal* by Definition 2.4), we say that this consequence relation *admits* non-normal

models. To determine whether a consequence relation admits a non-normal model in a given matrix $\mathcal{M}$ with n values, we must check whether the following holds:

$$\text{mod}_{\mathcal{M}}(\vdash) = \Lambda_{\mathcal{M}}$$

If this is the case, it means that there is no non-normal model.

2.2 Proof-Theory

The following Gentzen-type system of positive classical logic provides a common basis for the subsequent logics introduced in this paper. Note that the rules for contraction and commutation are omitted because Γ and Δ are sets.

Definition 2.12. LK^+ is a Gentzen-type proof system for the positive fragment of classical logic and is defined by the following rules:

Axioms: $\alpha \Rightarrow \alpha$

Operational rules:

$$\begin{array}{ll} [\wedge \Rightarrow] \frac{\Gamma, \alpha, \beta \Rightarrow \Delta}{\Gamma \alpha \wedge \beta \Rightarrow \Delta} & [\Rightarrow \wedge] \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta} \\ [\Rightarrow \vee] \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} & [\vee \Rightarrow] \frac{\Gamma, \alpha \Rightarrow \Delta \quad \Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow \Delta} \\ [\Rightarrow \supset] \frac{\Gamma, \alpha \Rightarrow \beta, \Delta}{\Gamma \Rightarrow \alpha \supset \beta, \Delta} & [\supset \Rightarrow] \frac{\Gamma \Rightarrow \alpha, \Delta \quad \Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \supset \beta \Rightarrow \Delta} \end{array}$$

Structural rules:

$$\begin{array}{ll} [W] \frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'} & [\text{Cut}] \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma', \alpha \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \end{array}$$

3 Non-Normal models

3.1 Single-conclusion

Although the logic **mbC** cannot be characterized by any finite matrix, we would like to know whether all the two-valued models of this consequence relation are exactly those valuations that satisfy the conditions given in Definition 2.10. In fact, we want to see whether $\text{mod}_2(\vdash_{\text{mbC}})$ contains exactly the elements of Λ_{mbC} . We show that this is not the case: this single-conclusion consequence relation, just like classical logic, admits non-normal models that do not respect the intended truth conditions of its intended semantics. One non-normal model is the trivial valuation mentioned above, which assigns 1 to every sentence. This valuation does not respect the truth condition of $\circ\alpha$ (item 5 of Definition 2.10) but satisfies the remaining conditions.

The next proposition shows that another non-normal model, called *provability valuation*⁴ persists for this consequence relation.

⁴See Garson [9, p. 29].

Proposition 3.1. The consequence relation $\vdash_{\mathbf{mbC}}$ admits non-normal models, i.e.:

$$\text{mod}_2(\vdash_{\mathbf{mbC}}) \neq \Lambda_{\mathbf{mbC}}$$

Proof. Let $v : \mathcal{W}(\mathcal{L}_0^c) \rightarrow \{0, 1\}$ be a valuation such that, for every α , if $\vdash_{\mathbf{mbC}} \alpha$ then $v(\alpha) = 1$, otherwise $v(\alpha) = 0$.

Now we show that v satisfies every inference of $\vdash_{\mathbf{mbC}}$. Consider an inference $\Gamma \vdash_{\mathbf{mbC}} \beta$. If $\nVdash_{\mathbf{mbC}} \alpha$ for some $\alpha \in \Gamma$ or $\vdash_{\mathbf{mbC}} \beta$, by definition it satisfies the inference, and we are done. Suppose that this is not the case; then, for every $\alpha \in \Gamma$, we have $\vdash_{\mathbf{mbC}} \alpha$ and $\nVdash_{\mathbf{mbC}} \beta$. Since $\nVdash_{\mathbf{mbC}} \beta$, there exists some mbC-valuation $u \in \Lambda_{\mathbf{mbC}}$ such that $u(\beta) = 0$. But u also satisfies $\Gamma \vdash_{\mathbf{mbC}} \beta$, which means that for some $\alpha \in \Gamma$, $v(\alpha) = 0$, thus $\nVdash_{\mathbf{mbC}} \alpha$, contradicting our assumption about Γ . Clearly, v is not an mbC-valuation, since it violates clause 4 of Definition 2.10 for any propositional variable. $\square$

The above valuation violates the *semantical law of excluded middle*: it can assign 0 to a sentence and its negation. For example, consider an atomic proposition p ; neither p nor its negation $\neg p$ is a theorem, and therefore both have the value 0.

Suppose that v is a non-normal valuation. The previous proposition means that $\vdash_{\mathbf{mbC}}$ can be alternatively characterized by $\Lambda_{\mathbf{mbC}} \cup \{v\}$. This means that v cannot be excluded by this consequence relation. This situation is reminiscent of the existence of non-standard models for Peano Arithmetic (PA), which cannot be excluded by the first-order language.

We will see that, just as in the case of classical logic, formulating a multiple-conclusion sequent system will exclude the non-normal models of $\mathbf{mbC}$ (just as moving from first-order PA to second-order PA excludes the non-standard models). However, as we will see, this is not the case for $\mathbf{LP}$ and $\mathbf{J}_3$.

3.2 Going multiple-conclusion

The following multiple-conclusion sequent system, introduced by Avron in [2, p. 244], provides a sequent-style characterization of $\mathbf{mbC}$.

Definition 3.1 ($\mathbf{G}_{\mathbf{mbC}}$). $\mathbf{G}_{\mathbf{mbC}}$ is obtained from LK^+ and the following rules:

$$[\Rightarrow \neg] \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \alpha} \quad [\circ \Rightarrow] \frac{\Gamma \Rightarrow \alpha, \Delta \quad \Gamma \Rightarrow \neg \alpha, \Delta}{\Gamma, \circ \alpha \Rightarrow \Delta}$$

Let us denote the consequence relation of $\mathbf{G}_{\mathbf{mbC}}$ by $\Rightarrow_{\mathbf{mbC}}$, that is, $\Rightarrow_{\mathbf{mbC}}$ is the set of all sequents provable in $\mathbf{G}_{\mathbf{mbC}}$. The following proposition shows that $\Rightarrow_{\mathbf{mbC}}$ uniquely determines mbC-valuations.

Proposition 3.2. The multiple-conclusion consequence relation $\Rightarrow_{\mathbf{mbC}}$ does not admit non-normal models (i.e., $\text{mod}_2(\Rightarrow_{\mathbf{mbC}}) = \Lambda_{\mathbf{mbC}}$).

Proof. $\Lambda_{\mathbf{mbC}} \subseteq \text{mod}_2(\Rightarrow_{\mathbf{mbC}})$ is trivial (since every valuation in $\Lambda_{\mathbf{mbC}}$ is a model of $\Rightarrow_{\mathbf{mbC}}$ by soundness); we only show that every valuation in $\text{mod}_2(\Rightarrow_{\mathbf{mbC}})$ is an mbC-valuation, so it must satisfy all clauses in Definition 2.10. Note that the clauses 1-3 are identical to the classical cases. Let $v \in \text{mod}_2(\Rightarrow_{\mathbf{mbC}})$:

- Clause 1: Suppose that $v(\alpha \wedge \beta) = 1$. Since both $\alpha \wedge \beta \Rightarrow_{\mathbf{mbC}} \alpha$ and $\alpha \wedge \beta \Rightarrow_{\mathbf{mbC}} \beta$ are provable, we get $v(\alpha) = v(\beta) = 1$. Now suppose that $v(\alpha) = v(\beta) = 1$; by $\alpha, \beta \Rightarrow_{\mathbf{mbC}} \alpha \wedge \beta$ we conclude that $v(\alpha \wedge \beta) = 1$.

- Clause 2: Suppose that $v(\alpha \vee \beta) = 1$. Since $\alpha \vee \beta \Rightarrow_{\mathbf{mbC}} \alpha, \beta$, we have either $v(\alpha) = 1$ or $v(\beta) = 1$. Conversely, suppose $v(\alpha) = 1$ or $v(\beta) = 1$; in either case, by $\alpha \Rightarrow_{\mathbf{mbC}} \alpha \vee \beta$ and $\beta \Rightarrow_{\mathbf{mbC}} \alpha \vee \beta$, we get $v(\alpha \vee \beta) = 1$.
- Clause 3: Suppose that $v(\alpha \supset \beta) = 1$ and $v(\alpha) \neq 0$. From $\alpha, \alpha \supset \beta \Rightarrow_{\mathbf{mbC}} \beta$, we get $v(\beta) = 1$. Conversely, suppose that $v(\alpha) = 0$ or $v(\beta) = 1$. If $v(\alpha) = 0$, then by $\Rightarrow_{\mathbf{mbC}} \alpha \supset \beta, \alpha$, we get $v(\alpha \supset \beta) = 1$. If $v(\beta) = 1$, then it follows from $\beta \Rightarrow_{\mathbf{mbC}} \alpha \supset \beta$ that $v(\alpha \supset \beta) = 1$. So in either case, $v(\alpha \supset \beta) = 1$.
- Clause 4: By an application of $[\Rightarrow \neg]$, we can obtain the sequent $\Rightarrow_{\mathbf{mbC}} \alpha, \neg\alpha$, which means that either $v(\alpha) = 1$ or $v(\neg\alpha) = 1$.
- Clause 5: By an application of $[\circ \Rightarrow]$, the sequent $\circ\alpha, \neg\alpha, \alpha \Rightarrow_{\mathbf{mbC}}$ is provable. Now assume that $v(\circ\alpha) = 1$; since the other side of the sequent is empty, we must have either $v(\neg\alpha) = 0$ or $v(\alpha) = 0$.

□

A little inspection makes it clear that the non-normal valuation constructed in Proposition 3.1 is excluded by the rule $[\Rightarrow \neg]$, as it allows us to prove $\Rightarrow \alpha, \neg\alpha$, which enforces the semantical law of excluded middle. The trivial valuation is also excluded by $[\circ \Rightarrow]$, which ensures the derivability of $\alpha, \circ\alpha, \neg\alpha \Rightarrow$, as it can be easily checked that the trivial valuation cannot satisfy this sequent.

So far, multiple-conclusion proof systems seem promising, but we will see that they fail to determine the intended semantics of **LP** and **J₃**.

Definition 3.2 (G_{LP}). **G_{LP}** is defined by all the rules and axioms of LK^+ except $[\Rightarrow \supset]$, $[\supset \Rightarrow]$, plus the following rules and axioms:

Axioms: $\Rightarrow \alpha, \neg\alpha$

Rules:

$$\begin{array}{ll}
 [\neg \neg \Rightarrow] \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma, \neg\neg\alpha \Rightarrow \Delta} & [\Rightarrow \neg] \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \neg\alpha} \\
 [\Rightarrow \neg\wedge] \frac{\Gamma \Rightarrow \Delta, \neg\alpha, \neg\beta}{\Gamma \Rightarrow \Delta, \neg(\alpha \wedge \beta)} & [\neg\wedge \Rightarrow] \frac{\Gamma, \neg\alpha \Rightarrow \Delta \quad \Gamma, \neg\beta \Rightarrow \Delta}{\Gamma, \neg(\alpha \wedge \beta) \Rightarrow \Delta} \\
 [\neg\vee \Rightarrow] \frac{\Gamma, \neg\alpha, \neg\beta \Rightarrow \Delta}{\Gamma, \neg(\alpha \vee \beta) \Rightarrow \Delta} & [\Rightarrow \neg\vee] \frac{\Gamma \Rightarrow \Delta, \neg\alpha \quad \Gamma \Rightarrow \Delta, \neg\beta}{\Gamma \Rightarrow \Delta, \neg(\alpha \vee \beta)}
 \end{array}$$

This system has been proven to be sound and complete with respect to the matrix $\mathcal{M}_{\mathbf{LP}}$ presented in Definition 2.8 (see [2, Theorem 4.95]). Now, we examine $\text{mod}_3(\Rightarrow_{\mathbf{LP}})$ with respect to the 3-valued matrix $\mathcal{M}_{\mathbf{LP}}$.

Next, we introduce a proof system for **J₃**.

Definition 3.3 (G_{J₃}). **G_{J₃}** is defined by all the rules and axioms of **G_{LP}**, plus $[\Rightarrow \supset]$, $[\supset \Rightarrow]$ of LK^+ and the following rules and axioms:

$$\begin{array}{ll}
 [\neg \supset \Rightarrow] \frac{\Gamma, \alpha, \neg\beta \Rightarrow \Delta}{\Gamma, \neg(\alpha \supset \beta) \Rightarrow \Delta} & [\Rightarrow \neg \supset] \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \neg\beta}{\Gamma \Rightarrow \Delta, \neg(\alpha \supset \beta)} \\
 [\mathbf{F} \Rightarrow] \Gamma, \mathbf{F} \Rightarrow \Delta & [\Rightarrow \neg \mathbf{F}] \Gamma \Rightarrow \Delta, \neg\mathbf{F}
 \end{array}$$

Similar to the previous proof system, the consequence relation of $\mathbf{G}_{\mathbf{J}_3}$ is denoted by $\Rightarrow_{\mathbf{J}_3}$. Soundness and completeness of this system were proved by Avron [2, p. 127].

A little reflection leads us to a more general and rather obvious fact about matrices and multiple-conclusion proof systems. The fact is that multiple-conclusion systems *cannot* uniquely characterize the behavior of n -valued matrices when $3 \leq n$ (with a minor constraint on their connectives), regardless of whether they are paraconsistent or not. Therefore, instead of proving the existence of non-normal models separately for $\mathbf{LP}$ and $\mathbf{J}_3$, we prove a stronger statement and then derive the existence of such models as a corollary.

We need a simple definition to state the next proposition. When a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ has an n -place connective $\diamond$ such that $\bar{\diamond}(a_1, a_2, \dots, a_k) \in \mathcal{D}$ for some $a_1, a_2, \dots, a_k$ in $\mathcal{V}$, we say that $\diamond$ is a *satisfiable connective*. When $\bar{\diamond}(b_1, b_2, \dots, b_k) \in (\mathcal{V} - \mathcal{D})$ for some $b_1, b_2, \dots, b_k$, it is called a *falsifiable connective*. For example, conjunction, disjunction, and implication are both satisfiable and falsifiable in the matrices $\mathcal{M}_{\mathbf{J}_3}$ and $\mathcal{M}_{\mathbf{LP}}$, and $\mathbf{F}$ in $\mathcal{M}_{\mathbf{J}_3}$ is not a satisfiable connective. Thus, having satisfiable and falsifiable connectives is a rather fair condition, and most useful matrices have such connectives. In the matrices we have presented so far, there is no connective that is not falsifiable. However, if we add a constant for one of the designated values to the language, we will have such a connective as well.

The following proposition holds for matrices with both satisfiable and falsifiable connectives:

Proposition 3.3. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix with $3 \leq |\mathcal{V}|$ for a language $\mathcal{L}$, with a satisfiable connective $\diamond$ and a falsifiable connective $\blacklozenge$, then any sound and complete multiple-conclusion consequence $\Rightarrow$ for this matrix admits non-normal models, i.e.:

$$\text{mod}_n(\Rightarrow) \neq \Lambda_{\mathcal{M}}$$

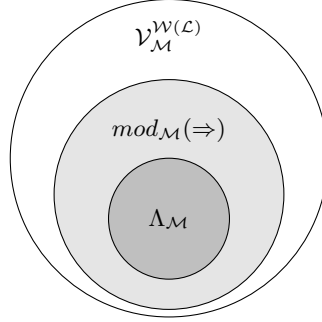
for $3 \leq |\mathcal{V}| = n$.

Proof. Suppose a valid sequent $\Gamma \Rightarrow \Delta$. We construct a non-normal valuation that satisfies this sequent. Since $\mathcal{V}$ has at least 3 elements, either $2 \leq |\mathcal{D}|$ or $2 \leq |\mathcal{V} - \mathcal{D}|$. We consider each case separately:

- $2 \leq |\mathcal{D}|$: Consider two distinct values t_1 and t_2 in $\mathcal{D}$ such that $\bar{\diamond}(a_1, a_2, \dots, a_k) = t_1$ for some $a_1, a_2, \dots, a_k$ in $\mathcal{V}$. Let u be a normal valuation of $\mathcal{M}$ such that $u(p_i) = a_i$ for all $1 \leq i \leq n$. Define the valuation v as follows: for every atomic formula, let $v(p_i) = u(p_i)$ ($i \in N$); for any complex formula α , if $u(\alpha) = t_1$, let $v(\alpha) = t_2$, otherwise let $v(\alpha) = u(\alpha)$. Now, it is easy to see that for any formula α , $v(\alpha) \in \mathcal{D}$ iff $u(\alpha) \in \mathcal{D}$. Hence, v satisfies $\Gamma \Rightarrow \Delta$ because u satisfies it (by soundness). However, v is non-normal because $v(\diamond(p_1, p_2, \dots, p_k)) = t_2 \neq \bar{\diamond}(v(p_1), v(p_2), \dots, v(p_k)) = t_1$, while $v(p_i) = a_i$ for all $1 \leq i \leq k$.
- $2 \leq |\mathcal{V} - \mathcal{D}|$: This case can be handled similarly. Consider two distinct values f_1 and f_2 in $\mathcal{V} - \mathcal{D}$ such that $\bar{\blacklozenge}(b_1, b_2, \dots, b_m) = f_1$ for some $b_1, b_2, \dots, b_m$ in $\mathcal{V}$. Let u be a normal valuation of $\mathcal{M}$ such that $u(p_i) = b_i$ for all $1 \leq i \leq m$. Construct the valuation v such that for every atomic variable $v(p_i) = u(p_i)$ ($i \in N$), and for any complex formula α , $v(\alpha) = f_2$ provided that $u(\alpha) = f_1$, otherwise $v(\alpha) = u(\alpha)$. Again, it is easy to see that for any formula α , $v(\alpha) \in \mathcal{D}$ iff $u(\alpha) \in \mathcal{D}$. Thus, v satisfies the sequent as u does, but it is also non-normal since $v(\blacklozenge(p_1, p_2, \dots, p_m)) = f_2 \neq \bar{\blacklozenge}(v(p_1), v(p_2), \dots, v(p_m)) = f_1$, because $v(p_i) = b_i$ for all $1 \leq i \leq m$.

In both cases, we showed that the system admits a non-normal valuation: the first one does not respect the truth condition of $\Diamond$, while the second one violates the falsity condition of $\Diamond$. $\square$

The above theorem states that a multi-conclusion sequent is not capable of expressing a matrix-based semantics with more than 2 values. Therefore, $\Lambda_{\mathcal{M}}$ is a proper subset of $mod_{\mathcal{M}}(\Rightarrow)$. The following diagram shows this situation:



One point to avoid possible confusion: Shoesmith and Smiley [16] have a theorem stating that any multiple-conclusion logic characterized by a finite matrix is categorical (see Theorem 17.5 in [16]). This claim might appear to be in conflict with the theorem above. However, the reader should note that the meaning of categoricity used there is different from the one we have in mind here. Shoesmith and Smiley introduce a relation for the *equivalence* of matrices — a relation that even allows matrices with different numbers of truth values to be considered equivalent (see [16, p. 264]). According to their definition, a logic is *categorical* if all matrices that characterize it are equivalent under this relation. So, according to their definition, a logic is not categorical if it is characterized by a non-equivalent matrix. That is, even in the non-categorical case, a non-normal valuation (in respect of one matrix) must also be the normal valuations of another matrix. Our notion of *categoricity*, however, is more liberal; we allow non-normal valuations that are not normal with respect to any possible matrices, not only in respect of the intended matrix. It is closer to what Carnap had in mind:⁵ a logical calculus is categorical if and only if the only valuations that satisfy its consequence relations are the intended (or standard) ones. In fact, our definition is more inferentialistic in spirit and depends more on valuations than on matrices. The set of all valuations must satisfy certain conditions that we expect it. In the case where we have a matrix which represents the intended semantics, the valuations in model of consequence must be normal with respect to that matrix. In other words, in this case, there must be a homomorphism between the language and the algebra corresponding to the matrix, or simply being *compositional* with respect to connectives of the matrix. However, this is not strictly necessary. The important point is that valuations must satisfy the conditions that we regard as *intended*. For

⁵For example, Carnap's non-normal model of single-conclusion classical logic, which assigns 1 to theorems and 0 to non-theorems, is not a homomorphism between the language and any possible two-valued matrix. It is a non-normal valuation, while does not belong to any two-valued matrix, because this valuation is not *compositional* for $\vee$, i.e., the value of $\phi \vee \psi$ is not uniquely determined by the values of ϕ and ψ , for some case (take p and q) it has the value 1 and for others (take p and $\neg p$) it has the value 0, while all of its components have the value 0.

example, in the case of **mbC**, as we have seen, it cannot, in principle, be characterized by a finite matrix. So, expecting its valuations to be a homomorphism to a matrix is an unreasonable expectation. There, the normality of a valuation is defined by satisfying the semantic conditions that were listed. Or, in logics whose semantics are based on Nmatrices, the valuations' conditions for normality are completely different from being normal in matrices.

Now, as an application, we can see that the above theorem guarantees the existence of non-normal models for the two logics **LP** and **J₃**.

Proposition 3.4. The consequence relations $\Rightarrow_{\mathbf{LP}}$ and $\Rightarrow_{\mathbf{J}_3}$ admit non-normal models, i.e.:

$$\begin{aligned} \text{mod}_3(\Rightarrow_{\mathbf{LP}}) &\neq \Lambda_{\mathbf{LP}} \\ \text{mod}_3(\Rightarrow_{\mathbf{J}_3}) &\neq \Lambda_{\mathbf{J}_3} \end{aligned}$$

Proof. The conjunction $\bar{\wedge}$ in both $\mathcal{M}_{\mathbf{LP}}$ and $\mathcal{M}_{\mathbf{J}_3}$ is satisfiable and falsifiable. Thus, the result follows from Proposition 3.3. $\square$

It is interesting that conjunction is the only connective that remains untouched by non-normal models of classical logic. In other words, even single-conclusion proof systems can fix the meaning of conjunction (see [9, p.14]). However, as we saw above, there is a non-normal valuation of these three-valued logics that even destroys the meaning of conjunction.

All of the above propositions indicate that multiple-conclusion systems are not strong enough to provide a categorical axiomatization of the intended matrices, even though they perform well for the bi-valuational semantics of **mbC** or the semantics of classical logic. Borrowing the terminology of model theory, we can say that multiple-conclusion systems *can* be 2-categorical⁶ for certain logics (such as classical logic or **mbC**), but are not n-categorical for $3 \leq n$. Therefore, we may say that Dicher's [7] Hopeful Monsters are not that hopeful!⁷

4 Reading off Semantics

4.1 Motivations

If multiple-conclusion systems are insufficient to provide a categorical (or a full formalization) of certain paraconsistent logics, what should a model-theoretic inferentialist do? Is it a dead end? We believe that having a categorical axiomatization is desirable but not essential for model-theoretic inferentialism, as long as there is a *uniform* method to *read off* the underlying semantics. In fact, if there exists a proof system that offers a categorical axiomatization but is overly complex or semantically loaded, an alternative way of deriving the semantics from syntax might be preferable for an inferentialist. Of course, this method should not violate the general constraints of inferentialism. Here, we claim that such a way indeed exists.

⁶By "n-categorical" we mean a consequence relation that does not admit n-valued non-normal models.

⁷Dicher's defense of multiple-conclusion systems is not directly related to our discussion. We merely point out that multiple conclusions *alone* are not particularly effective for our present purpose, although they are significant from other perspectives.

Our source of inspiration is Garson [8]. In that paper, Garson considers the set of all bi-valuations V that satisfy the usual rule of implication in natural deduction. He observes that some non-normal valuations in V do not respect the classical truth table of implication, meaning that the natural deduction rule for implication does not enforce the classical interpretation. He then asks: “Could the explanation for this be that the semantics expressed by implication [in the natural deduction system] is intuitionistic?” The answer is affirmative.⁸ Garson shows that V is essentially a Kripke-style intuitionistic semantics for implication in disguise. He treats each v in V as a possible world by defining a binary accessibility relation R over it. This relation turns out to be reflexive and transitive (similar to the frame of Kripke-style intuitionistic semantics), and he shows that the following statement holds for V :

- For all formulas α and β and any v and v' in V , $v(\alpha \supset \beta) = 1$ iff vRv' and $v'(\alpha) = 1$ then $v'(\beta) = 1$.

The proposition above precisely states the truth condition of implication in Kripke-style intuitionistic semantics. The important lesson for us here is that we can view the old bi-valuations from a completely different perspective, and it may be possible to extract a non-classical meaning from them. We shift the models from 3-valued valuations to 2-valued ones.

Another source of inspiration for this work is what is known as *Suszko’s Thesis* [28] — the idea that there are only two logical values: *true* and *false*.⁹ To support this claim, using a technique known as Suszko’s Reduction, he shows that every structural Tarskian many-valued propositional logic can be characterized by a bi-valuational semantics (see [29] for a detailed discussion of Suszko’s Thesis). We may hope to establish the converse of Suszko’s Reduction for our logics — that is, to find a method by which all normal valuations can be constructed from two-valued valuations. In fact, we have a rather straightforward method to convert two-valued valuations into 3-valued ones. This method was used by one of the authors in [30] to study the class of non-normal models of classical logic. In the next section, we use a slightly modified version of it.

After accepting that the models should be changed to two-valued ones, the question arises: which consequence relation should we consider — single-conclusion or multiple-conclusion consequence? The important fact we should not ignore is the partial success of multiple-conclusion systems. As we saw, they can capture the intended bi-valuational semantics of **mbC** and classical logic. This shows that multiple-conclusion sequents can impose more structure on 2-valued valuations than on 3-valued ones. Multiple-conclusion systems have also been defended within several inferentialist frameworks by various authors (e.g., Restall [14], Dicher [7]). Therefore, their adoption should not pose a significant problem for inferentialists in general. In what follows, we aim to recover the intended semantics from 2-valued models of multiple-conclusion consequence relations, i.e., from $mod_2(\Rightarrow)$. This can be viewed as something like the converse of Suszko’s Reduction for two-valued models of multiple-conclusion consequence relations.

⁸Even though Garson later admits that this view is not entirely convincing — see, for instance, [25].

⁹We would like to thank the reviewer for bringing Suszko’s Thesis to our attention.

4.2 Changing The Models

It seems natural for an inferentialist to start with a 2-valued semantics, since she might not know much about the semantics of her system. However, after some reflection, she might find that certain sequents are not derivable. For example, in this case, she may notice that $\alpha, \neg\alpha \Rightarrow \beta$ is not generally derivable. This means that there exists some valuation v in which both α and $\neg\alpha$ are true. Such a valuation can be seen as an *inconsistent* valuation with respect to α . It is natural to classify the space of all valuations into three disjoint categories: (1) valuations that see α inconsistently (as a *glutty* state), (2) those that see α as classically true, and (3) those that see α as classically false. This classification is also motivated by Priest's (1979) view of paradoxical sentences, namely that there are sentences that are true and their negations are also true. Now, we investigate the relationships between these valuations with respect to different connectives. In the next definition, we attempt to make this idea precise.

Definition 4.1. For a language $\mathcal{L} \in \{\mathcal{L}_f^c, \mathcal{L}_o^c, \mathcal{L}^c\}$ and any $U \subseteq \{0, 1\}^{\mathcal{W}(\mathcal{L})}$ and $\alpha \in \mathcal{W}(\mathcal{L})$ the sets t_α, f_α and $\top_\alpha$ are defined over U as follows:

$$\begin{aligned} t_\alpha &= \{v \in U : v(\alpha) = 1 \text{ and } v(\neg\alpha) = 0\} \\ f_\alpha &= \{v \in U : v(\alpha) = 0 \text{ and } v(\neg\alpha) = 1\} \\ \top_\alpha &= \{v \in U : v(\alpha) = 1 \text{ and } v(\neg\alpha) = 1\} \end{aligned}$$

Then we can construct a 3-valued valuation from the old 2-valued valuation. For any valuation $v \in U$, the valuation $\hat{v} : \mathcal{W}(\mathcal{L}) \rightarrow \{t, f, \top\}$ is defined for any α as follows¹⁰:

$$\hat{v}(\alpha) = \begin{cases} t & \text{if } v \in t_\alpha, \\ f & \text{if } v \in f_\alpha, \\ \top & \text{if } v \in \top_\alpha. \end{cases}$$

This function might be partial because v might satisfy none of the three conditions. We denote the set of all $\hat{v}$ constructed over U , by $U/3_\top$ ¹¹, i.e. $U/3_\top = \{\hat{v} : v \in U\}$.

Before constructing 3-valued valuations from $\text{mod}_2(\Rightarrow_{\mathbf{LP}})$, we briefly explore the relations of these different categories of valuation to each other, and show the following useful facts about t_α, f_α and $\top_\alpha$ defined over it:

Lemma 4.1. For each α and β , the following conditions hold for $t_\alpha, f_\alpha, \top_\alpha$ and $t_\beta, f_\beta, \top_\beta$ defined over $\text{mod}_2(\Rightarrow_{\mathbf{LP}})$:

1. $t_\alpha \cup f_\alpha \cup \top_\alpha = \text{mod}_2(\Rightarrow_{\mathbf{LP}})$.
2. $t_\alpha = f_{\neg\alpha}$.
3. $f_\alpha = t_{\neg\alpha}$.
4. $\top_\alpha = \top_{\neg\alpha}$.
5. $t_\alpha \cap t_\beta = t_{\alpha \wedge \beta}$.

¹⁰A similar definition has previously been used by the author in [30], with the difference that there the third value is a gappy value.

¹¹Avron [2] denotes the class of all 3-valued matrices with two of designated values by $3_\top$. We adopt this notation to emphasize that the constructed valuations corresponds to a matrix within this class.

6. $f_\alpha \cup f_\beta = f_{\alpha \wedge \beta}$.
7. $(\top_\alpha - f_\beta) \cup (\top_\beta - f_\alpha) = \top_{\alpha \wedge \beta}$.
8. $f_\alpha \cap f_\beta = f_{\alpha \vee \beta}$.
9. $t_\alpha \cup t_\beta = t_{\alpha \vee \beta}$.
10. $(\top_\alpha - t_\beta) \cup (\top_\beta - t_\alpha) = \top_{\alpha \vee \beta}$.

Proof. 1. First of all, it should be noted that $t_\alpha, \top_\alpha$ and f_α are defined on the set $\text{mod}_2(\Rightarrow_{\mathbf{LP}})$. Now assume that there is some $v \in \text{mod}_2(\Rightarrow_{\mathbf{LP}})$ but $v \notin (t_\alpha \cup f_\alpha \cup \top_\alpha)$, then we must have $v(\alpha) = v(\neg\alpha) = 0$, but it contradicts the fact that v satisfies $\Rightarrow_{\mathbf{LP}} \alpha, \neg\alpha$.

This item simplifies other cases. For example, to show that $v \in f_\alpha$, it is sufficient to show that $v(\alpha) = 0$, and we do not need to show $v(\neg\alpha) = 1$.

2. Let $v \in t_\alpha$, then $v(\alpha) = 1$ and $v(\neg\alpha) = 0$, from $\alpha \Rightarrow_{\mathbf{LP}} \neg\neg\alpha$ we get $v(\neg\neg\alpha) = 1$, from this and $v(\neg\alpha) = 0$ we get $v \in f_{\neg\alpha}$. Conversely, let $v \in f_{\neg\alpha}$, then $v(\neg\neg\alpha) = 1$, from $\neg\neg\alpha \Rightarrow_{\mathbf{LP}} \alpha$ we get $v(\alpha) = 1$, from this and $v(\neg\alpha) = 0$ it follows that $v \in t_\alpha$.
3. $t_{\neg\alpha} = f_{\neg\neg\alpha}$ follows from the previous item, since both $\neg\neg\alpha \Rightarrow_{\mathbf{LP}} \alpha$ and $\alpha \Rightarrow_{\mathbf{LP}} \neg\neg\alpha$ are valid in $\mathbf{LP}$, we can substitute $\neg\neg\alpha$ by α , thus $t_{\neg\alpha} = f_\alpha$.
4. Let $v \in \top_\alpha$, then $v(\alpha) = v(\neg\alpha) = 1$, from $\alpha \Rightarrow_{\mathbf{LP}} \neg\neg\alpha$ it follows $v(\neg\neg\alpha) = 1$, thus we have $v(\neg\alpha) = 1$ and $v(\neg\neg\alpha) = 1$, which means that $v \in \top_{\neg\alpha}$. Conversely, let $v \in \top_{\neg\alpha}$, then $v(\neg\alpha) = v(\neg\neg\alpha) = 1$, from $\neg\neg\alpha \Rightarrow_{\mathbf{LP}} \alpha$ it follows $v(\alpha) = 1 = v(\neg\alpha)$, thus $v \in \top_\alpha$.
5. Suppose $v \in t_\alpha \cap t_\beta$, then $v(\alpha) = v(\beta) = 1$ and $v(\neg\alpha) = v(\neg\beta) = 0$. It follows from $\alpha, \beta \Rightarrow_{\mathbf{LP}} \alpha \wedge \beta$ that $v(\alpha \wedge \beta) = 1$, also from $\neg(\alpha \wedge \beta) \Rightarrow_{\mathbf{LP}} \neg\alpha, \neg\beta$ it follows that $v(\neg(\alpha \wedge \beta)) = 0$, thus $v \in t_{\alpha \wedge \beta}$. Conversely, suppose $v \in t_{\alpha \wedge \beta}$, then $v(\alpha \wedge \beta) = 1$ and $v(\neg(\alpha \wedge \beta)) = 0$, from $\alpha \wedge \beta \Rightarrow_{\mathbf{LP}} \alpha$ and $\alpha \wedge \beta \Rightarrow_{\mathbf{LP}} \beta$ we get $v(\alpha) = v(\beta) = 1$. From $\neg\alpha \Rightarrow_{\mathbf{LP}} \neg(\alpha \wedge \beta)$ and $\neg\beta \Rightarrow_{\mathbf{LP}} \neg(\alpha \wedge \beta)$ we get $v(\neg\beta) = v(\neg\alpha) = 0$, thus $v \in t_\alpha$ and $v \in t_\beta$.
6. Let $v \in f_\beta$. Then $v(\beta) = 0$ and $v(\neg\beta) = 1$. From $\alpha \wedge \beta \Rightarrow_{\mathbf{LP}} \beta$, we have $v(\alpha \wedge \beta) = 0$, so $v \in f_{\alpha \wedge \beta}$ (note that we do not need to check $v(\neg(\alpha \wedge \beta)) = 1$, it follows from item 1). The other case, $v \in f_\alpha$, can be handled similarly. Conversely, suppose $v \in f_{\alpha \wedge \beta}$. Then $v(\alpha \wedge \beta) = 0$. Assume, to the contrary, that $v \notin f_\alpha$ and $v \notin f_\beta$. Then $v(\alpha) = v(\beta) = 1$ (since, from item 1, we have $v \in (\top_\alpha \cup t_\alpha)$ and $v \in (\top_\beta \cup t_\beta)$). Then $\alpha, \beta \Rightarrow_{\mathbf{LP}} \alpha \wedge \beta$ implies $v(\alpha \wedge \beta) = 1$, a contradiction. Therefore, we must have either $v \in f_\alpha$ or $v \in f_\beta$.
7. Let $v \in \top_\alpha - f_\beta$, then $v(\alpha) = v(\neg\alpha) = 1$ and $v(\beta) = 1$ (since $v \notin f_\beta$ implies $v \in \top_\beta \cup t_\beta$). Now, from $\alpha, \beta \Rightarrow_{\mathbf{LP}} \alpha \wedge \beta$ we get $v(\alpha \wedge \beta) = 1$, and from $\neg\alpha \Rightarrow_{\mathbf{LP}} \neg(\alpha \wedge \beta)$ we get $v(\neg(\alpha \wedge \beta)) = 1$, thus $v \in \top_{\alpha \wedge \beta}$. The other case $v \in \top_\beta - f_\alpha$ can be handled similarly. Now suppose that $v \in \top_{\alpha \wedge \beta}$, then $v(\alpha \wedge \beta) = v(\neg(\alpha \wedge \beta)) = 1$, from $\alpha \wedge \beta \Rightarrow_{\mathbf{LP}} \alpha$ and $\alpha \wedge \beta \Rightarrow_{\mathbf{LP}} \beta$ we get $v(\alpha) = v(\beta) = 1$, so $v \notin f_\beta$ and $v \notin f_\alpha$. From $\neg(\alpha \wedge \beta) \Rightarrow_{\mathbf{LP}} \neg\alpha, \neg\beta$ therefore either $v(\neg\alpha) = 1$ or $v(\neg\beta) = 1$, in the first case we get $v \in \top_\alpha - f_\beta$, in the second one $v \in \top_\beta - f_\alpha$, thus $v \in (\top_\alpha - f_\beta) \cup (\top_\beta - f_\alpha)$.
8. Since $\alpha \vee \beta \Rightarrow_{\mathbf{LP}} \neg(\neg\alpha \wedge \neg\beta)$ and $\neg(\neg\alpha \wedge \neg\beta) \Rightarrow_{\mathbf{LP}} \alpha \vee \beta$ we have $f_{\alpha \vee \beta} = f_{\neg(\neg\alpha \wedge \neg\beta)}$, then by item (2) $f_{\neg(\neg\alpha \wedge \neg\beta)} = t_{\neg\alpha \wedge \neg\beta}$. From (5) we get $t_{\neg\alpha \wedge \neg\beta} = t_{\neg\alpha} \cap t_{\neg\beta}$, and from (3) $t_{\neg\alpha} \cap t_{\neg\beta} = f_\alpha \cap f_\beta$, thus $f_\alpha \cap f_\beta = f_{\alpha \vee \beta}$.

9. This case follows by respective application of (3), (6) and (2): $t_{\alpha \vee \beta} = t_{\neg(\neg\alpha \wedge \neg\beta)} = f_{(\neg\alpha \wedge \neg\beta)} = f_{\neg\alpha} \cup f_{\neg\beta} = t_\alpha \cup t_\beta$.
10. By $\alpha \vee \beta \Rightarrow_{\mathbf{LP}} \neg(\neg\alpha \wedge \neg\beta)$ and $\neg(\neg\alpha \wedge \neg\beta) \Rightarrow_{\mathbf{LP}} \alpha \vee \beta$ and also $\neg(\alpha \vee \beta) \Rightarrow_{\mathbf{LP}} \neg\neg(\neg\alpha \wedge \neg\beta)$ and $\neg\neg(\neg\alpha \wedge \neg\beta) \Rightarrow_{\mathbf{LP}} \neg(\alpha \vee \beta)$ we get $\top_{\alpha \vee \beta} = \top_{\neg(\neg\alpha \wedge \neg\beta)}$. Now, by applications of (4), (7) and (2),(4) we get: $\top_{\alpha \vee \beta} = \top_{\neg(\neg\alpha \wedge \neg\beta)} = \top_{\neg\alpha \wedge \neg\beta} = (\top_{\neg\alpha} - f_{\neg\beta}) \cup (\top_{\neg\beta} - f_{\neg\alpha}) = (\top_\alpha - t_\beta) \cup (\top_\beta - t_\alpha)$.

□

Lemma 4.2. All conditions in Lemma 4.1 hold for any $t_\alpha, f_\alpha, \top_\alpha$ and $t_\beta, f_\beta, \top_\beta$ defined over $\text{mod}_2(\Rightarrow_{\mathbf{J}_3})$, and the following:

1. $f_\alpha \cup t_\beta = t_{\alpha \supset \beta}$.
2. $\top_\beta - f_\alpha = \top_{\alpha \supset \beta}$.
3. $f_\beta - f_\alpha = f_{\alpha \supset \beta}$.
4. $f_F = \text{mod}_2(\Rightarrow_{\mathbf{J}_3})$.

Proof. Because $\mathbf{J}_3$ and $\mathbf{LP}$ share the same fragments for negation, conjunction and disjunction, all items of Lemma 4.1 hold for $\mathbf{J}_3$. We just show the above facts about F and implication:

1. Let $v \in f_\alpha \cup t_\beta$, if $v \in f_\alpha$ then $v(\alpha) = 0$ and $v(\neg\alpha) = 1$, but then $\Rightarrow_{\mathbf{J}_3} \alpha \supset \beta, \alpha$ implies $v(\alpha \supset \beta) = 1$, and $\neg(\alpha \supset \beta) \Rightarrow_{\mathbf{J}_3} \alpha$ implies $v(\neg(\alpha \supset \beta)) = 0$, thus $v \in t_{\alpha \supset \beta}$. if $v \in t_\beta$, then $v(\beta) = 1$ and $v(\neg\beta) = 0$, from $\beta \Rightarrow_{\mathbf{J}_3} \alpha \supset \beta$ we have $v(\alpha \supset \beta) = 1$ and from $\neg(\alpha \supset \beta) \Rightarrow_{\mathbf{J}_3} \neg\beta$, $v(\neg(\alpha \supset \beta)) = 0$, thus in both case we have $v \in t_{\alpha \supset \beta}$. Conversely, let $v \in t_{\alpha \supset \beta}$, then from $\neg\beta, \alpha \Rightarrow_{\mathbf{J}_3} \neg(\alpha \supset \beta)$ we get either $v(\neg\beta) = 0$ or $v(\alpha) = 0$, but then from $\Rightarrow_{\mathbf{J}_3} \beta, \neg\beta$ and $\Rightarrow_{\mathbf{J}_3} \alpha, \neg\alpha$ we get either $v(\beta) = 1$ or $v(\neg\alpha) = 1$, thus $v \in f_\alpha \cup t_\beta$.
2. Suppose $v \in \top_\beta - f_\alpha$, then $v(\beta) = v(\neg\beta) = 1$ and $v(\alpha) = 1$. But then $v(\alpha \supset \beta) = 1$ follows from $\beta \Rightarrow_{\mathbf{J}_3} \alpha \supset \beta$. By $\alpha, \neg\beta \Rightarrow_{\mathbf{J}_3} \neg(\alpha \supset \beta)$ we get $v(\neg(\alpha \supset \beta)) = 1$, therefore $v \in \top_{\alpha \supset \beta}$. For the converse, suppose $v \in \top_{\alpha \supset \beta}$, then from $\neg(\alpha \supset \beta) \Rightarrow_{\mathbf{J}_3} \alpha$ we get $v(\alpha) = 1$ which means $v \notin f_\alpha$. Then $\neg(\alpha \supset \beta) \Rightarrow_{\mathbf{J}_3} \neg\beta$ implies $v(\neg\beta) = 1$, and by $\alpha, \alpha \supset \beta \Rightarrow_{\mathbf{J}_3} \beta$ we have $v(\beta) = 1$, thus $v \in \top_\beta$ and $v \in \top_\beta - f_\alpha$.
3. Let $v \in f_\beta - f_\alpha$, then $v(\alpha) = 1$ and $v(\beta) = 0$, from $\alpha, \alpha \supset \beta \Rightarrow_{\mathbf{J}_3} \beta$ we get $v(\alpha \supset \beta) = 0$. Since v cannot be in $t_{\alpha \supset \beta}$ or $\top_{\alpha \supset \beta}$, we get $v \in f_{\alpha \supset \beta}$. For the converse, suppose $v \in f_{\alpha \supset \beta}$, then $v(\alpha \supset \beta) = 0$, but $\Rightarrow_{\mathbf{J}_3} \alpha \supset \beta, \alpha$, which implies $v(\alpha) = 1$, thus $v \notin f_\alpha$. $v(\beta) = 0$ follows from $\beta \Rightarrow_{\mathbf{J}_3} (\alpha \supset \beta)$, therefore $v \in f_\beta - f_\alpha$.
4. Suppose $v \in \text{mod}_2(\Rightarrow_{\mathbf{J}_3})$, then v must satisfy $F \Rightarrow_{\mathbf{J}_3}$, thus $v(F) = 0$. It also must satisfy $\Rightarrow_{\mathbf{J}_3} \neg F$, therefore $v(\neg F) = 1$, which means $v \in f_F$. The other side is trivial.

□

The next step is to see that the valuations defined over the above categories of valuations are indeed the normal valuations of the intended matrices, the following natural definition helps us to express this fact:

Definition 4.2 (determination). Suppose an (N)matrix $\mathcal{M}$ has the language $\mathcal{L} \in \{\mathcal{L}_F^c, \mathcal{L}_\circ^c, \mathcal{L}^c\}$ and let $U \subseteq \{0, 1\}^{\mathcal{W}(\mathcal{L})}$, we say $U/3_\top$ *determines* the normal valuations of $\mathcal{M}$ if the following holds:

$$U/3_\top = \Lambda_{\mathcal{M}}$$

The definition above tells us that if the set of three-valued valuations defined over the set of two-valued models of a consequence relation are precisely the normal three-valued models, then we can say that this consequence relation somehow determines the the intended semantics.

Two following propositions show that $mod_2(\Rightarrow_{\mathbf{LP}})/3_\top$ and $mod_2(\Rightarrow_{\mathbf{J}_3})/3_\top$ can determine their corresponding normal valuations.

Proposition 4.1. $mod_2(\Rightarrow_{\mathbf{LP}})/3_\top$ determines the normal valuation of the matrix $\mathcal{M}_{\mathbf{LP}}$, i.e.:

$$mod_2(\Rightarrow_{\mathbf{LP}})/3_\top = \Lambda_{\mathbf{LP}}$$

Proof. To show that $\Lambda_{\mathbf{LP}} \subseteq mod_2(\Rightarrow_{\mathbf{LP}})/3_\top$, let $u \in \Lambda_{\mathbf{LP}}$, define $v : \mathcal{L}^c \rightarrow \{0, 1\}$ as follows:

$$v(\alpha) = \begin{cases} 1 & \text{if } u(\alpha) = t \text{ or } u(\alpha) = \top \\ 0 & \text{if } u(\alpha) = f. \end{cases}$$

Now, for every formula α , $v(\alpha) = 1$ iff $u(\alpha) \in \mathcal{D}$ ($\mathcal{D}$ is the set of designated values of $\mathcal{M}_{\mathbf{LP}}$). But v satisfies every valid inference in $\mathbf{LP}$ because u is a normal model, therefore $v \in mod_2(\Rightarrow_{\mathbf{LP}})$. So $\hat{v} \in mod_2(\Rightarrow_{\mathbf{LP}})/3_\top$. It is easy to see that for every formula $\hat{v}(\alpha) = u(\alpha)$, therefore $\hat{v} = u$ and $u \in mod_2(\Rightarrow_{\mathbf{LP}})/3_\top$.

To prove the other direction, let $u \in mod_2(\Rightarrow_{\mathbf{LP}})/3_\top$, then $u = \hat{v}$ for some $v \in mod_2(\Rightarrow_{\mathbf{LP}})$, clearly $\hat{v}$ is total, this follows from Lemma 4.1(1), so it is enough to show it respects the truth tables of $\mathbf{LP}$.

- *Negation* ($\neg$) :

1. $\neg t = f$: if $\hat{v}(\alpha) = t$, then Lemma 4.1(2) implies $\hat{v}(\neg\alpha) = f$.
2. $\neg f = t$: if $\hat{v}(\alpha) = f$, then Lemma 4.1(3) implies $\hat{v}(\neg\alpha) = t$.
3. $\neg \top = \top$: if $\hat{v}(\alpha) = \top$, then Lemma 4.1(4) implies $\hat{v}(\neg\alpha) = \top$.

Therefore, in all cases we have $\hat{v}(\neg\alpha) = \neg\hat{v}(\alpha)$.

- *Conjunction* ($\wedge$):

1. $t \wedge t = t$: It follows from Lemma 4.1(5) that $\hat{v}(\alpha \wedge \beta) = t$ iff $\hat{v}(\alpha) = \hat{v}(\beta) = t$.
2. $t \wedge f = f \wedge t = \top \wedge f = f \wedge \top = f \wedge f = f$: It means that at least one conjunct is f , but by Lemma 4.1(6) we have $\hat{v}(\alpha) = f$ or $\hat{v}(\beta) = f$ iff $\hat{v}(\alpha \wedge \beta) = f$.
3. $\top \wedge t = t \wedge \top = \top \wedge \top = \top$: One conjunct is $\top$ and the other is not f , this case follows from Lemma 4.1(7) : $\hat{v}(\alpha \wedge \beta) = \top$ iff $\hat{v}(\alpha) \neq f$ and $\hat{v}(\beta) \neq f$ and either $\hat{v}(\alpha) = \top$ or $\hat{v}(\beta) = \top$.

- *Disjunction* can be handled similarly, using items (8),(9) and (10) of Lemma 4.1.

□

Proposition 4.2. $mod_2(\Rightarrow_{\mathbf{J}_3})/3_{\top}$ determines the normal valuations of the matrix $\mathcal{M}_{\mathbf{J}_3}$.

Proof. $\Lambda_{\mathbf{J}_3} \subseteq mod_2(\Rightarrow_{\mathbf{J}_3})/3_{\top}$ can be proved exactly like the previous proof. So we only show the other direction.

Negation, disjunction and conjunction can be handled like the previous proposition. We show it for implication and falsum. Let $u \in mod_2(\Rightarrow_{\mathbf{J}_3})/3_{\top}$, then $u = \hat{v}$ for some $v \in mod_2(\Rightarrow_{\mathbf{J}_3})$.

- *Implication* ($\supset$): if $\hat{v}(\alpha) = f$ then lemma 4.2(1) implies $\hat{v}(\alpha \supset \beta) = t$. Now assume $\hat{v}(\alpha) \neq f$, there are three possible value for $\hat{v}(\beta)$: if $\hat{v}(\beta) = t$ then $\hat{v}(\alpha \supset \beta) = t$ by Lemma 4.2(1). if $\hat{v}(\beta) = \top$ then $\hat{v}(\alpha \supset \beta) = \top$ by Lemma 4.2(2) and if $\hat{v}(\beta) = f$ then $\hat{v}(\alpha \supset \beta) = f$ by Lemma 4.2(3). So we have the truth condition of implication in Definition 2.9.
- *Falsum* ($\mathbf{F}$): Lemma 4.2(4) simply implies $\hat{v}(\mathbf{F}) = f$.

□

In Proposition 3.2, we saw that the multiple-conclusion consequence relation of **mbC** can fix the mbC-valuations we introduced in Definition 2.10. This means that there is no non-normal valuation in $mod_2(\Rightarrow_{\mathbf{mbC}})$ in respect to this semantics. But what happens if we apply the above construction to $mod_2(\Rightarrow_{\mathbf{mbC}})$? Interestingly, this construction determines an alternative semantics for **mbC**: a sound and complete 3-valued Nmatrix introduced by Avron [3]. This shows that the method is working uniformly. Even in the case of **mbC** which is not characterizable by finite deterministic matrices, the method can give us a sound and complete non-deterministic matrix. One advantage of this type of semantics over mbC-valuations is that this non-deterministic matrix has a separate truth table for each connective (see the truth table in Proposition 4.3 below), contrary to the mbC-valuations in which truth conditions of consistency and negation operators were interdefined. In the next two propositions, we show that the normal valuation of the Nmatrix of **mbC** can be determined with our method.

Lemma 4.3. For each α and β , the following conditions hold for any $t_\alpha, f_\alpha, \top_\alpha$ and $t_\beta, f_\beta, \top_\beta$ defined over $mod_2(\Rightarrow_{\mathbf{mbC}})$:

1. $t_\alpha \cup f_\alpha \cup \top_\alpha = mod_2(\Rightarrow_{\mathbf{mbC}})$.
2. $t_\alpha = f_{\neg\alpha}$.
3. $f_\alpha \cup \top_\alpha = \top_{\neg\alpha} \cup t_{\neg\alpha}$.
4. $\top_\alpha \subset f_{\circ\alpha}$.
5. $f_\alpha \cup t_\alpha \subset t_{\circ\alpha} \cup f_{\circ\alpha} \cup \top_{\circ\alpha}$.
6. $f_\alpha \cup f_\beta = f_{\alpha\wedge\beta}$.
7. $(t_\alpha \cup \top_\alpha) - f_\beta = t_{\alpha\wedge\beta} \cup \top_{\alpha\wedge\beta}$.
8. $f_\alpha \cap f_\beta = f_{\alpha\vee\beta}$.
9. $t_\alpha \cup t_\beta \cup \top_\alpha \cup \top_\beta = t_{\alpha\vee\beta} \cup \top_{\alpha\vee\beta}$.
10. $f_\beta - f_\alpha = f_{\alpha\supset\beta}$.
11. $t_\beta \cup \top_\beta \cup f_\alpha = t_{\alpha\supset\beta} \cup \top_{\alpha\supset\beta}$.

Proof. 1. The proof is the same as that of Lemma 4.1 (1).

2. Suppose $v \in t_\alpha$, then $v(\neg\alpha) = 0$ and $v(\neg\neg\alpha) = 1$ follows from $\Rightarrow_{\mathbf{mbC}} \neg\alpha, \neg\neg\alpha$, thus $v \in f_{\neg\alpha}$. For the converse, suppose $v \in f_{\neg\alpha}$, then $v(\neg\alpha) = 0$, but $\Rightarrow_{\mathbf{mbC}} \alpha, \neg\alpha$ implies $v(\alpha) = 1$, thus $v \in t_\alpha$.
3. Let $v \in f_\alpha \cup \top_\alpha$, then either $v \in f_\alpha$ or $v \in \top_\alpha$, in both case we get $v(\neg\alpha) = 1$, which means $v \notin f_{\neg\alpha}$, thus $v \in \top_{\neg\alpha} \cup t_{\neg\alpha}$. Conversely, suppose $v \in \top_{\neg\alpha} \cup t_{\neg\alpha}$, in both case we have $v(\neg\alpha) = 1$, then either $v(\alpha) = 1$ or $v(\alpha) = 0$, in the first case we get $v \in \top_\alpha$, in the second one $v \in f_\alpha$, thus $v \in f_\alpha \cup \top_\alpha$.
4. Suppose $v \in \top_\alpha$, so $v(\alpha) = v(\neg\alpha) = 1$, but then $v(\circ\alpha) = 0$ follows from $\alpha, \neg\alpha, \circ\alpha \Rightarrow_{\mathbf{mbC}}$, thus $v \in f_{\circ\alpha}$.
5. Just notice that $t_{\circ\alpha} \cup f_{\circ\alpha} \cup \top_{\circ\alpha} = \text{mod}_2(\Rightarrow_{\mathbf{mbC}})$.
6. The proof is the same as that of Lemma 4.1 (6).
7. This item easily follows from the previous item, just notice that $\text{mod}_2(\Rightarrow_{\mathbf{mbC}}) - (f_\alpha \cup f_\beta) = (t_\alpha \cup \top_\alpha) - f_\beta$ and $\text{mod}_2(\Rightarrow_{\mathbf{mbC}}) - f_{\alpha\wedge\beta} = t_{\alpha\wedge\beta} \cup \top_{\alpha\wedge\beta}$.
8. Let $v \in f_\alpha \cap f_\beta$, then $v(\alpha) = v(\beta) = 0$, from $\alpha \vee \beta \Rightarrow_{\mathbf{mbC}} \alpha, \beta$ it follows $v(\alpha \vee \beta) = 0$, thus $v \in f_{\alpha\vee\beta}$. Conversely, if $v \in f_{\alpha\vee\beta}$ then from $\alpha \Rightarrow_{\mathbf{mbC}} \alpha \vee \beta$ and $\beta \Rightarrow_{\mathbf{mbC}} \alpha \vee \beta$ it follows $v \in f_\alpha \cap f_\beta$.
9. It follows from the previous case, since $\text{mod}_2(\Rightarrow_{\mathbf{mbC}}) - (f_\alpha \cap f_\beta) = t_\alpha \cup t_\beta \cup \top_\alpha \cup \top_\beta$ and $\text{mod}_2(\Rightarrow_{\mathbf{mbC}}) - f_{\alpha\vee\beta} = t_{\alpha\vee\beta} \cup \top_{\alpha\vee\beta}$.
10. Suppose $v \in f_\beta - f_\alpha$, then $v(\alpha) = 1$ and $v(\beta) = 0$, from $\alpha, \alpha \supset \beta \Rightarrow_{\mathbf{mbC}} \beta$ we get $v(\alpha \supset \beta) = 0$, thus $v \in f_{\alpha\supset\beta}$. For the converse, let $v \in f_{\alpha\supset\beta}$ then from $\Rightarrow_{\mathbf{mbC}} \alpha \supset \beta, \alpha$ and $\beta \Rightarrow_{\mathbf{mbC}} \alpha \supset \beta$ it follows that $v(\alpha) = 1$ and $v(\beta) = 0$, which means $v \in f_\beta$ and $v \notin f_\alpha$.
11. This case is also the equality between complements of the previous case.

□

Proposition 4.3. $\text{mod}_2(\Rightarrow_{\mathbf{mbC}})/3_\top$ determines the normal valuations of the Nmatrix $\mathcal{M}_B = \langle \{t, f, \top\}, \{t, \top\}, \mathcal{O} \rangle$, in which $\mathcal{O}$ assigns the following functions to each connective:

$\bar{\wedge}$	t	f	$\top$	$\bar{\vee}$	t	f	$\top$
t	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$	t	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
f	$\{f\}$	$\{f\}$	$\{f\}$	f	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$	$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$

$\bar{\supset}$	t	f	$\top$
t	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$
f	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$

$\bar{\circ}$	t	$\bar{\neg}$	t
t	$\{t, \top, f\}$	t	$\{f\}$
f	$\{t, \top, f\}$	f	$\{t, \top\}$
$\top$	$\{f\}$	$\top$	$\{t, \top\}$

Proof. Again, $\Lambda_B \subseteq \text{mod}_2(\Rightarrow_{\mathbf{mbC}})/3_\top$ can be proved exactly like the proof for **LP**. So we only show the other direction. Let $\hat{v} \in \text{mod}_2(\Rightarrow_{\mathbf{mbC}})/3_\top$:

Negation ($\neg$): Consider $\hat{v}(\neg\alpha)$, if $\hat{v}(\alpha) = \top$ or $\hat{v}(\alpha) = f$ then by Lemma 4.3(3) $\hat{v}(\neg\alpha) \in \{t, \top\}$, if $\hat{v}(\alpha) = t$ then by Lemma 4.3(2) we get $\hat{v}(\neg\alpha) = f$, so in all cases $\hat{v}(\neg\alpha) \in \bar{\neg}\hat{v}(\alpha)$ according to the above table.

- *Consistency* ($\circ$): Consider $\hat{v}(\circ\alpha)$, if $\hat{v}(\alpha) = t$ or $\hat{v}(\alpha) = f$ then by Lemma 4.3(5) we get $\hat{v}(\circ\alpha) \in \{t, f, \top\}$, if $\hat{v}(\alpha) = \top$ then from Lemma 4.3(4) it follows $\hat{v}(\circ\alpha) = f$, thus in all cases we get $\hat{v}(\circ\alpha) \in \circ\hat{v}(\alpha)$.
- *Conjunction* ($\wedge$): if $\hat{v}(\alpha) = f$ or $\hat{v}(\beta) = f$, then Lemma 4.3(6) implies $\hat{v}(\alpha \wedge \beta) = f$. Lemma 4.3(7) implies in other cases we get $\hat{v}(\alpha \wedge \beta) \in \{t, \top\}$, so according to the above table, in all cases we have $\hat{v}(\alpha \wedge \beta) \in \hat{v}(\alpha) \wedge \hat{v}(\beta)$.
- *Disjunction* ($\vee$): if $\hat{v}(\alpha) = f$ and $\hat{v}(\beta) = f$, then by Lemma 4.3(8) we get $\hat{v}(\alpha \vee \beta) = f$. For other cases $\hat{v}(\alpha \vee \beta) \in \{t, \top\}$ follows from Lemma 4.3(9). Again, in all cases we have $\hat{v}(\alpha \vee \beta) \in \hat{v}(\alpha) \vee \hat{v}(\beta)$.
- *Implication* ($\supset$): if $\hat{v}(\alpha) \neq f$ and $\hat{v}(\beta) = f$, then by Lemma 4.3(10) we get $\hat{v}(\alpha \supset \beta) = f$. For other cases $\hat{v}(\alpha \supset \beta) \in \{t, \top\}$ follows from Lemma 4.3(11). $\square$

Note that from the proposition 3.2 we have $\text{mod}_2(\Rightarrow_{\mathbf{mbC}}) = \Lambda_{\mathbf{mbC}}$, by the above theorem we get:

$$\Lambda_{\mathbf{mbC}}/3_{\top} = \Lambda_{\mathcal{B}}$$

This means that the normal valuations of the 3-valued Nmatrix for $\mathbf{mbC}$ (i.e., $\Lambda_{\mathcal{B}}$) can be constructed from its bi-valuational semantics (which is its original semantics introduced by the Brazilian school). The idea of the non-deterministic matrix was not known when the LFIs were introduced. It was formally introduced by Avron, and then he found it useful in providing effective semantics for this large family of paraconsistent logics (see [1, p.xxvi]). Many logics that cannot be characterized by finite matrices can be characterized using finite non-deterministic matrices. But as we saw above, our construction over models of the consequence relation of $\mathbf{mbC}$ naturally leads to this kind of structure. The situation is similar to what Garson [9, p.69] describes regarding the meaning of natural deduction rules and their relation to the idea of forcing in set theory¹². He says, “Alpha Centaurian anthropologists are puzzled that we think of forcing as a discovery. They know it was under our noses all along, for it is what predicate logic really means.” Similarly, we can say that the concept of the non-deterministic matrix was under our noses all along.

5 Closing Remarks

Let us take a closer look at what we have accomplished so far. First, we saw that multiple-conclusion consequence could not provide a categorical axiomatization of the 3-valued logics mentioned earlier. This might lead us to abandon multiple-conclusion systems in favor of alternatives such as n-sided sequents, as in Hjortland [10]. However, we realized that it is better to be patient. By employing 2-valued valuations within the multiple-conclusion framework, we can retrieve the intended 3-valued semantics with ease. The process is simple: we change the models back to 2-valued ones and then reveal that they are essentially the intended 3-valued valuations in a different form. We showed that even when we abandon the idea of categorical axiomatization, it is still possible to extract semantics from syntax and remain under the umbrella of model-theoretic inferentialism.

¹²Forcing is the key concept in Cohen’s proof of the independence of the Continuum Hypothesis from ZFC.

Both multiple-conclusion and n-sided sequents diverge from standard practices and are not found in everyday usage. But multiple-conclusion approaches are more justifiable, having been supported by several authors with varying philosophical motives (see [14] and [7]). These multiple-conclusion systems are less complicated, less artificial, and less semantically loaded than n-sided calculus. More importantly, we think they remain closer to the traditional concept of inference, which is about what follows from what. Moreover, although n-sided sequent calculus works well for categorical axiomatization of finite-valued matrices, its applicability to paraconsistent logics, such as **mbC** and its extensions, which cannot be characterized by finite matrices, remains unclear. However, we know that there are well-behaved multiple-conclusion proof systems for these logics.

We also conjecture that this method can be applied to logics that are not paraconsistent. For instance, if in Definition 4.1 we consider the third value as those assigning 0 to both α and $\neg\alpha$ and apply it to the two-valued models of a multiple-conclusion system for **K3** (see Tabakci [18]), we would obtain the semantics of this logic. We can have more than three values. For example, we guess that if we allow both gappy and glutty values in our definition and apply it to the multiple-conclusion system for the logic **Belpap – Dunn**¹³ we would get a sound and complete matrix for this logic.

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References

- [1] Arieli, Ofer & Zamansky, Anna (eds.) (2021). *Arnon Avron on Semantics and Proof Theory of Non-Classical Logics*. Springer Verlag
- [2] Arnon Avron, Ofer Arieli & Anna Zamansky (2018): *Theory of Effective Propositional Paraconsistent Logics*. Studies in Logic, College Publications.
- [3] Avron, Arnon. 2007. Non-deterministic semantics for logics with a consistency operator. *Journal of Approximate Reasoning* 45: 271–287.
- [4] Bonnay, Denis & Westerståhl, Dag (2016). Compositionality Solves Carnap’s Problem. *Erkenntnis* 81 (4):721–739.
- [5] Carnap, Rudolf (1943). *Formalization of logic*. Cambridge, Mass.,: Harvard university press.
- [6] Carnielli, Walter & Coniglio, Marcelo Esteban (2016). *Paraconsistent Logic: Consistency, Contradiction and Negation*. Basel, Switzerland: Springer International Publishing. Edited by Marcelo Esteban Coniglio.
- [7] Dicher, Bogdan (2020). Hopeful Monsters: A Note on Multiple Conclusions. *Erkenntnis* 85 (1):77–98.
- [8] Garson, James W. (2001). Natural Semantics: Why Natural Deduction is Intuitionistic. *Theoria* 67 (2):114–139.

¹³See Chapter 5 of Avron *et al.* [2].

- [9] Garson, James W. (2013). *What Logics Mean: From Proof Theory to Model-Theoretic Semantics*. New York: Cambridge University Press.
- [10] Hjortland, Ole Thomassen (2014). Speech Acts, Categoricity, and the Meanings of Logical Connectives. *Notre Dame Journal of Formal Logic* 55 (4):445-467.
- [11] I. D'Ottaviano and N. da Costa. Sur un Problème de Jankowski. *Comptes Rendus Hebdomadaires de l'Academie des Sciences, Paris*, 270A, 1349–1353, 1970.
- [12] Peregrin, Jaroslav (2014). *Inferentialism: Why Rules Matter*. London and New York: Palgrave-Macmillan.
- [13] Priest, Graham (1979). Logic of Paradox. *Journal of Philosophical Logic* 8 (1):219-241.
- [14] Restall, Greg (2005). Multiple Conclusions. In Petr Hájek, Luis Valdés-Villanueva & Dag Westerstaahl (eds.), *Logic, Methodology and Philosophy of Science*. College Publications.
- [15] Brandom, Robert (2000). *Articulating reasons: an introduction to inferentialism*. Cambridge: Harvard University Press.
- [16] Shoesmith, D. J. & Smiley, Timothy (1978). *Multiple Conclusion Logic*. Cambridge, England / New York London Melbourne: Cambridge University Press.
- [17] Smiley, Timothy (1996). Rejection. *Analysis* 56 (1):1–9.
- [18] Tabakci, Selcuk Kaan (2024). Categoricity Problem for LP and K3. *Studia Logica*:1-35. doi:10.1007/s11225-024-10098-1
- [19] Trafford, James (2014). An Inferentialist Approach to Paraconsistency. *Abstracta* 8 (1):55-73.
- [20] W. A. Carnielli and J. Marcos. A taxonomy of C-systems. In: W. A. Carnielli, M. E. Coniglio, and I. M. L. D'Ottaviano, editors, *Paraconsistency: The Logical Way to the Inconsistent*, volume 228 of *Lecture Notes in Pure and Applied Mathematics*, pages 1–94. Marcel Dekker, 2002.
- [21] Tennant, Neil, 'Inferentialism, Logicism, Harmony, and a Counterpoint', in Alexander Miller (ed.), *Logic, Language, and Mathematics: Themes from the Philosophy of Crispin Wright* (Oxford, 2020; online edn, Oxford Academic, 20 Aug. 2020), <https://doi.org/10.1093/oso/9780199278343.003.0009>, accessed 28 Sept. 2025.
- [22] Speitel, S. G. (2020). *Logical Constants between Inference and Reference - An Essay in the Philosophy of Logic*. UC San Diego.
- [23] Asenjo, F. G. (1966). A Calculus for Antinomies. *Notre Dame Journal of Formal Logic* 16 (1):103-105.
- [24] Kleene, S. C. (1952). *Introduction to Metamathematics*. Amsterdam: North-Holland Publishing Company.
- [25] Garson, J. W.: "Expressive Power and Incompleteness of Propositional Logics." *Journal of Philosophical Logic* 39 (2):159-171. 2010.
- [26] Rumfitt, Ian (1997). The categoricity problem and truth-value gaps. *Analysis* 57 (4):223-235.
- [27] Steinberger, Florian (2011). Why Conclusions Should Remain Single. *Journal of Philosophical Logic* 40 (3):333-355.

- [28] Suszko, R., ‘The Fregean axiom and Polish mathematical logic in the 1920’s’, *Studia Logica* 36 (1977), 373–380.
- [29] Wansing, Heinrich & Shramko, Yaroslav (2008). Suszko’s Thesis, Inferential Many-valuedness, and the Notion of a Logical System. *Studia Logica* 88 (3):405–429.
- [30] Jafari, Javid (2025). Non-Classical Beasts Under the Carpet. *Topoi* 44:1-8.