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An Unconstructivisable Paradox: A Counterexample to Tennant's Conjecture

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Abstract

In 'A New Unified Account of Truth and Paradox', Neil Tennant makes the following conjecture: Paradoxes are never strictly classical; all of the classical paradoxes are constructivisable. In this paper, we will present a counterexample to Tennant's conjecture; there are classical paradoxes that cannot be constructivized.

Keywords— Intuitionistic Logic, Paradox, Normalization, Constructivisability.

In [8], Neil Tennant makes the following conjecture:

> Paradoxes are never strictly classical. The kind of conceptual trouble that a paradox reveals will afflict the intuitionist just as seriously as it does the classicist. Therefore, attempted solutions to the paradoxes, if they are to be genuine solutions, must be available to the intuitionist. Nothing about an attempted solution to a paradox should imply that the trouble it reveals lies with strictly classical moves of reasoning ([8], p.589).

This conjecture comes as a response to a possible objection that suggests that maybe there are strictly classical paradoxes:

> But how can you be so sure that there are not any *strictly classical* paradoxes? Perhaps there are paradoxes whose associated proofs of $\bot$ *have* to make use of strictly classical rules and which *cannot* be emulated by the constructivist? ([8], p.589, emphasis in original).

To which Tennant raises a challenge:

> *Give an example of a semantic paradox whose associated deductive reasoning cannot be constructivized.* At present, all the known paradoxes, and their constructive regimentations, indicate that an example of this kind would be remarkable indeed, and would occasion much re-thinking ([8], p.590, emphasis in original).

In this paper, we aim to do just that; we will show that there are paradoxes that fit Tennant's theory of paradox yet they are not constructivisable—they can be constructed in a classical setting but not in an intuitionistic setting.

1 Constructing an Unconstructivisable Paradox

In [7, 8], Tennant proposed a proof-theoretic criterion for logico-semantic paradoxes. The criterion simply states that the natural deduction proofs of logico-semantic paradoxes are proofs that cannot be put in normal form. In order to show that there are unconstructivisable paradoxes, we must first show that our paradox fits Tennant's criterion for logico-semantic paradoxes.

First, we will be working with what I called in a previous work [1] a partially transparent predicate. A *partially transparent predicate* P is a predicate that has the following rules ([1], p. 267-268):

(a) If $\vdash \varphi$ then $\vdash P(\ulcorner \varphi \urcorner)$
(b) $P(\ulcorner \varphi \urcorner) \vdash \varphi$

Some possible candidates for partially transparent predicates are the knowability predicate and the provability predicate. Without loss of generality, we will be working with the provability predicate (Bew). The aforementioned rules are in sequent calculus. However, since Tennant's criterion uses natural deduction, we want to represent the rules for the provability predicate (Bew) in natural deduction. Note that the restriction is not on Bew elimination, but rather, on Bew introduction. The rule simply states that in order to introduce the provability predicate on a sentence, that sentence must first be provable. Thus, Bew-I and Bew-E would look as follows. Where "$\ulcorner\ \urcorner$" is a naming device:

$$\frac{Bew(\ulcorner \varphi \urcorner)}{\varphi}\ {\scriptstyle Bew\text{-E}} \qquad \frac{\varphi}{Bew(\ulcorner \varphi \urcorner)}\ {\scriptstyle Bew\text{-I}^{\oslash}}$$

$\oslash$ here indicates a restriction on Bew-I. The restriction is simple: in order to introduce Bew on a sentence φ, there must be no undischarged assumptions leading to φ.

We will also assume that we have a method that allows sentences to refer to themselves. From there, let ε be $\neg Bew(\ulcorner \varepsilon \urcorner) \vee \psi$. Notice, since ε is a self-referential sentence, it observes the following *id est* rules:

$$\frac{\neg Bew(\ulcorner \varepsilon \urcorner) \vee \psi}{\varepsilon}\ {\scriptstyle \text{Def } \varepsilon} \qquad \frac{\varepsilon}{\neg Bew(\ulcorner \varepsilon \urcorner) \vee \psi}\ {\scriptstyle \text{Def } \varepsilon}$$

Since we added Bew rules and id est rules, the standard definition of a normal proof must be adjusted to accommodate these additional rules.[1] We will adopt the following definitions from ([4], p.157) with the appropriate adjustments:

Definition 1. A *detour (cut)* in a deduction is an occurrence of a formula A which is

[1] Thanks are due to the anonymous reviewer for raising this question on how the id est rules should be treated with regard to normalization.

1. the conclusion of an I-rule, EFQ, IP, $\vee$E, $\exists$E, or id est rule and the major premise of an E-rule, *or*
2. the conclusion of an id est rule and the premise of an id est rule.

The first condition would, for example, flag instances of Bew-I immediately followed by Bew-E as detours, while the second condition flags instances where an id est rule is applied only to switch back via another id est rule.

Some detours, however, are not direct, but rather dragged out over the course of several steps in a derivation. So, to account for such detours, we need a further definition:[2]

Definition 2. A *detour segment (cut segment)* in a deduction is a sequence $A_1, ..., A_n$ of formula occurrences in the deduction, such that

1. A_1 is not the conclusion of IP, $\vee$E, $\exists$E,
2. A_i where $1 \leq i < n$ is
   (a) a minor premise of a $\vee$E or $\exists$E inference, and A_{i+1} is its conclusion, *or*
   (b) the minor premise of a $\neg$E inference where the major premise of that $\neg$E inference is an IP-assumption, and A_{i+1} is the conclusion of the IP discharging the premise, *or*
   (c) an application of an id est rule
3. A_n is a major premise of an E-rule (i.e., A_n is a detour).

We are now in a position to define a normal deduction:

Definition 3. A deduction (proof) is *normal* if it has no detours (this includes detour segments).[3]

So, the only difference from the standard definition of a detour segment is the addition of (c). Since detour segments flag "sequence of intermediate identical formula occurrences" ([4], p.157), the addition of (c) ensures that we are treating ε and $\neg Bew(\ulcorner \varepsilon \urcorner) \vee \psi$ as identical.

We now turn to show that there is a paradoxical proof of ψ using ε in **NK**, and that that paradoxical proof cannot be put in normal form. We later show that there is no proof of ψ using ε in **NJ**.

The following is the paradoxical proof generated by ε in **NK**. Throughout the rest of the paper, we will omit the naming device "$\ulcorner\ \urcorner$" for readability.

Π:

[2] Again, this definition is adopted from ([4], p.157) with slight modifications.

[3] Note that these definitions are tailored for the current specific id est rules and the Bew-rules. Further conditions might be required for different id est rules and different rules for semantic predicates. For instance, if we have a truth-teller τ that is identical to $Tr(\ulcorner \tau \urcorner)$, then Tr-I on τ followed by an id est rule would count as a detour (similarly for other combinations between Tr rules on τ and τ-id-est rules).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[Bew(\varepsilon)]^2}{Bew(\varepsilon)\lor\neg Bew(\varepsilon)}\,\lor\text{I}\qquad[\neg(Bew(\varepsilon)\lor\neg Bew(\varepsilon))]^1}{\bot}\,\neg\text{E}}{\neg Bew(\varepsilon)}\,\neg\text{I}^{\,2}}{Bew(\varepsilon)\lor\neg Bew(\varepsilon)}\,\lor\text{I}\qquad[\neg(Bew(\varepsilon)\lor\neg Bew(\varepsilon))]^1}{\dfrac{\bot}{Bew(\varepsilon)\lor\neg Bew(\varepsilon)}\,IP^{\,1}}\,\neg\text{E}$$

Σ:

$$\dfrac{\dfrac{\Pi}{Bew(\varepsilon)\lor\neg Bew(\varepsilon)}\,IP^{\,1}\qquad\dfrac{[Bew(\varepsilon)]^3}{\varepsilon}\,Bew\text{ E}\qquad\dfrac{\dfrac{[\neg Bew(\varepsilon)]^3}{\neg Bew(\varepsilon)\lor\psi}\,\lor\text{I}}{\varepsilon}\,\text{Def }\varepsilon}{\varepsilon}\,\lor\text{E}^{\,3}$$

Ξ:

$$\dfrac{\dfrac{\dfrac{\Sigma}{\varepsilon}\,\lor\text{E}^{\,3}}{\neg Bew(\varepsilon)\lor\psi}\,\text{Def }\varepsilon\qquad\dfrac{\dfrac{\dfrac{\dfrac{\Sigma}{\varepsilon}\,\lor\text{E}^{\,3}}{Bew(\varepsilon)}\,Bew\text{-I}\qquad[\neg Bew(\varepsilon)]^4}{\bot}\,\neg\text{ E}}{\psi}\,EFQ\qquad[\psi]^4}{\psi}\,\lor\text{E}^{\,4}$$

2 The Non-normalizability of the Paradoxical Proof Generated by ε

In the previous proof, Π is in normal form. Σ, on the other hand, is not in normal form; the conclusion of IP is a major premise of a ∨E, and so there is a direct detour. There is also a detour segment running from Σ beginning with ∨I and ending in the major premise of ∨E in Ξ (i.e., the detour segment is of length 4, where the *length of a segment* $A_1,\ldots,A_n$ is n)[4]. We will focus on attempting to normalize the direct detour in Σ, but the reader can check that if we, instead, start with the attempt of normalizing the aforementioned detour segment, then the attempt would be futile as well.

In order to normalize the direct detour in Σ[5], we need to use Yuuki Andou's method [2], namely:[6]

[4] Thanks to the anonymous reviewer for pointing out this detour segment.

[5] We are calling it a *direct detour* because it can be seen as a detour segment of length 1.

[6] See also [4], for more on Andou's conversions.

$$\dfrac{\dfrac{\begin{matrix}\dfrac{[\neg(\mathcal{A}\vee\mathcal{B})]^i \qquad \begin{matrix}\delta_0\\ \vdots\\ \mathcal{A}\vee\mathcal{B}\end{matrix}}{\bot}\,\neg\text{E}\\ \vdots\\ \delta_1\\ \vdots\\ \bot\end{matrix}}{\mathcal{A}\vee\mathcal{B}}\,{}^{i}\,IP \qquad \begin{matrix}[\mathcal{A}]^j\\ \vdots\\ \delta_2\\ \vdots\\ \mathcal{C}\end{matrix} \qquad \begin{matrix}[\mathcal{B}]^j\\ \vdots\\ \delta_3\\ \vdots\\ \mathcal{C}\end{matrix}}{\mathcal{C}}\,{}^{j}\,\vee\text{E} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\dfrac{[\neg\mathcal{C}]^i \qquad \dfrac{\begin{matrix}\delta_0\\ \vdots\\ \mathcal{A}\vee\mathcal{B}\end{matrix} \quad \begin{matrix}[\mathcal{A}]^j\\ \vdots\\ \delta_2\\ \vdots\\ \mathcal{C}\end{matrix} \quad \begin{matrix}[\mathcal{B}]^j\\ \vdots\\ \delta_3\\ \vdots\\ \mathcal{C}\end{matrix}}{\mathcal{C}}\,{}^{j}\,\vee\text{E}}{\bot}\,\neg\text{E}\\ \vdots\\ \delta_1\\ \vdots\\ \bot\end{matrix}}{\mathcal{C}}\,{}^{i}\,IP$$

Applying Andou's normalization method, we would get the following:

$$\dfrac{\dfrac{[\neg\varepsilon]^1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\neg\varepsilon]^1 \qquad \dfrac{\dfrac{[Bew(\varepsilon)]^2}{Bew(\varepsilon)\vee\neg Bew(\varepsilon)}\,\vee\text{I} \qquad \dfrac{[Bew(\varepsilon)]^3}{\varepsilon}\,Bew\ \text{E} \qquad \dfrac{\dfrac{[\neg Bew(\varepsilon)]^3}{\neg Bew(\varepsilon)\vee\psi}\,\vee\text{I}}{\varepsilon}\,\text{Def }\varepsilon}{\varepsilon}\,{}^{3}\,\vee\text{E}}{\bot}\,\neg\text{E}}{\neg Bew(\varepsilon)}\,{}^{2}\,\neg\text{I}}{\neg Bew(\varepsilon)\vee\psi}\,\vee\text{I}}{\varepsilon}\,\text{Def }\varepsilon}{\bot}\,\neg\text{E}}{\varepsilon}\,{}^{1}\,IP$$

This is not in normal form since the conclusion of $\vee$I is a major premise of an $\vee$E. Let us attempt to normalize it:

Σ^*:

$$\dfrac{\dfrac{[\neg\varepsilon]^1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\neg\varepsilon]^1 \qquad \dfrac{[Bew(\varepsilon)]^2}{\varepsilon}\,Bew\ \text{E}}{\bot}\,\neg\text{E}}{\neg Bew(\varepsilon)}\,{}^{2}\,\neg\text{I}}{\neg Bew(\varepsilon)\vee\psi}\,\vee\text{I}}{\varepsilon}\,\text{Def }\varepsilon}{\bot}\,\neg\text{E}}{\varepsilon}\,{}^{1}\,IP$$

It looks like we normalized the proof, but alas, we did not. This proof of ε uses IP as its last step, which then is used as a major premise of $\vee$E in Ξ. In other words, Ξ^* is now the following:

$$\dfrac{\dfrac{\dfrac{\Sigma^*}{\varepsilon}\,{}^{1}\,IP}{\neg Bew(\varepsilon)\vee\psi}\,\text{Def }\varepsilon \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Sigma^*}{\varepsilon}\,{}^{1}\,IP}{Bew(\varepsilon)}\,Bew\text{-I} \qquad [\neg Bew(\varepsilon)]^4}{\bot}\,\neg\ \text{E}}{\psi}\,EFQ \qquad [\psi]^4}{\psi}\,{}^{4}\,\vee\ \text{E}$$

That is, in $\Xi*$, there is an id est rule applied on the ε that was the result from IP, and that formula is a major premise of a $\vee$-E. In fact, from our definition of a detour segment, this constitutes as a detour segment of length 5 starting from the conclusion of $\vee$I in $\Sigma*$ to the major premise of $\vee$E in $\Xi*$.[7] Because we have proven a disjunction via IP, only to eliminate it using $\vee$-E, Andou's conversion is called for. If we apply Andou's method to normalize this detour, we will get another detour:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Part of }\Sigma*}{\vdots}}{\neg Bew(\varepsilon)}}{\neg Bew(\varepsilon)\vee\psi}\,\vee\text{I}\qquad \dfrac{\dfrac{[\neg Bew(\varepsilon)]^4}{\vdots}}{\psi}\qquad [\psi]^4}{\dfrac{\psi}{\vdots}}\,{}^4\vee\text{E}$$

The proof continues because there is an undischarged assumption in that part of $\Sigma*$. To normalize this detour, it *cannot* be converted to:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Part of }\Sigma*}{\vdots}}{\neg Bew(\varepsilon)}}{\neg Bew(\varepsilon)\vee\psi}\,\vee\text{I}}{\varepsilon}\,\text{Def }\varepsilon}{Bew(\varepsilon)}\,Bew\text{ I}\qquad \dfrac{\dfrac{\text{Part of }\Sigma*}{\vdots}}{\neg Bew(\varepsilon)}}{\dfrac{\bot}{\psi}}$$

The reason why this is an illegitimate conversion is that Bew introduction cannot be applied if there is an undischarged assumption, and indeed, there is an undischarged $[\neg\varepsilon]$. To discharge it, we do the whole derivation $\Sigma*$ ending with an IP move. But the problem persists because now we do not have a proof of $\neg Bew(\varepsilon)$ on the right-hand side because there is an undischarged assumption. To discharge the assumption, IP is required, but that only establishes ε not ψ (keep in mind that EFQ cannot discharge an assumption). But now, to prove ψ, we need to go through $\Xi*$ again. Hence, we are stuck in a loop.

The non-normalizability of the paradoxical proof using ε is unsurprising. If it were normalizable, then that suggests that in a classical sequent calculus setting, there is a proof of ψ using the sentence ε without employing Cut. This would entail that the non-transitive logic ST is trivial since Bew is weaker than the truth predicate, and so, there must be a Cut-free proof of ψ using a disjunctive

[7] If one is not convinced that our definition of detour segment should include the id est moves as in (c), then it suffices to point out that id est moves can be pushed upwards (i.e., a permutative reduction), resulting in a direct detour (i.e., we can let IP establish $\neg Bew(\varepsilon)\vee\psi$ and the id est move is used on ε rather than on $\neg Bew(\varepsilon)\vee\psi$).

Curry sentence. However, we know for a fact that ST (which includes a fully transparent truth predicate) is not trivial [3, 5, 6]. Therefore, there cannot be a normal proof of ψ using the sentence ε. Therefore, the paradoxical proof generated by ε fits Tennant's criterion for logico-semantic paradoxes.

3 The Paradoxical Proof Generated by ε is Unconstructivisable

In the paradoxical proof above, we used IP to establish ε. In order for the proof to go through, we must be able to prove ε to both establish the disjunction $\neg Bew(\varepsilon) \vee \psi$ and detach the disjunction via $Bew(\varepsilon)$ to get ψ. So, for a constructive paradoxical proof to get through, we must establish ε without using IP. This, we claim, is not possible.

The use of IP must be replaced with other assumption-discharging rules. The only candidates are $\vee$E, $\to$I, and $\neg$I. To use $\vee$E, we must have a disjunction to eliminate. The only possible disjunctions are $Bew(\varepsilon) \vee \neg Bew(\varepsilon)$ or $\neg Bew(\varepsilon) \vee \psi$ (i.e., ε itself).

The former cannot be proven without IP. That is, to prove $Bew(\varepsilon) \vee \neg Bew(\varepsilon)$, we can either assume $\neg(Bew(\varepsilon) \vee \neg Bew(\varepsilon))$ and employ IP or derive one of the disjuncts. However, to derive either disjunct, it must be done without IP because each disjunct entails ε—the sentence which we are trying to prove without IP. But then, we are back to our initial options: $\vee$E, $\to$I, and $\neg$I. If we choose $\vee$E via $Bew(\varepsilon) \vee \neg Bew(\varepsilon)$, then we would be stuck in an infinite regress. So, it must be one of the other options, but as we will see, none would work.

The other possible $\vee$E is by establishing $\neg Bew(\varepsilon) \vee \psi$ (without IP) and using it as the major premise. But this is ε itself. In other words, we must prove ε without IP in order to prove ε without IP. Thus, this option is not viable either.

As for $\to$I, it can get us only insofar as $Bew(\varepsilon) \to \psi$. However, the arrow would not detach without $Bew(\varepsilon)$ which we do not have because $Bew(\varepsilon) \to \psi$ is not ε. Thus, to detach the conditional, we must be able to derive ε, once again, without IP. Hence, we are back to the same options. Alternatively, we prove $\neg Bew(\varepsilon) \vee \psi$ from $Bew(\varepsilon) \to \psi$, alas, this is not possible without IP.

Similarly, $\neg$I can get us $\neg\neg\varepsilon$ but that would not help us to detach ε to get ψ unless we can prove double negation elimination for $\neg\neg\varepsilon$. However, a double negation elimination would require none other than IP. Of course, there are instances of double negation elimination that do not use IP (e.g., $\neg\neg\neg\varphi \to \neg\varphi$)[8], but those instances would not help us. What we are after is ε without any negations in order to detach the disjunction. So what we are after is $\neg\neg\varepsilon \to \varepsilon$. This requires IP on the assumption $\neg\varepsilon$ unless ε is derivable by other means that use IP. Those other means are the ones we started with $\vee$E, $\to$I, and $\neg$I. Thus, for each option, in order to prove ε without using IP, one must already be able to prove ε without IP.

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[8] Many thanks to the anonymous reviewer for pressing this point.

Another way to frame this, which is due to the anonymous reviewer, is to assume that we have a constructive proof of ε without any undischarged assumptions. Since we cannot use IP on $[\neg\varepsilon]$, ε must have been reached via an id est move on $\neg Bew(\varepsilon) \vee \psi$. Now, we cannot have reached $\neg Bew(\varepsilon) \vee \psi$ via IP, so we either proved ψ or proved $\neg Bew(\varepsilon)$ and applied $\vee$I. If ψ was proved, then since it is an arbitrary sentence, it was proved by $\vee$E via a proof of ε without undischarged assumptions. So, there is a shorter proof of ε. If the proof of $\neg Bew(\varepsilon) \vee \psi$ is via an $\vee$I on $\neg Bew(\varepsilon)$, then $\neg Bew(\varepsilon)$ must have been reached either via establishing $\neg\psi$ and ε without undischarged assumptions (thus, there is a shorter proof of ε), or from $\neg$I on the assumption $[Bew(\varepsilon)]$. If it was established via $\neg$I, then that requires a contradiction either from ψ and $\neg\psi$ (which again comes from a proof of ε, and so, there is a shorter proof of ε), or from a proof of $\neg Bew(\varepsilon)$ (and so, there is already a shorter proof of ε). Alternatively, the contradiction is via proving $\neg\varepsilon$ (with no undischarged assumptions), but that can only come by via $\neg$I on $[\varepsilon]$ and so, there must be another contradiction, and the candidates are the same as above. Moreover, if there is a contradiction (with no undischarged assumptions) that can be reached from the assumption $[\varepsilon]$, then since ε follows from our initial assumption, $[Bew(\varepsilon)]$, there is a shorter proof of $\neg Bew(\varepsilon)$, and so, a shorter proof of ε.

Thus, if we assume that there is a constructive proof of ε with no undischarged assumptions, then from that proof, there is a shorter proof of ε with no undischarged assumptions. Hence, by the least number principle, there can be no such constructive proof of ε without undischarged assumptions.

We showed that we cannot reach ψ by proving ε because ε cannot be proven constructively without undischarged assumptions. Are there other ways to prove ψ? The answer is negative. Recall that ψ is an arbitrary sentence, and so, it cannot be proven (intuitionistically or classically) without relying on a paradoxical sentence. In our setup, we only have one candidate for that job, namely, ε. Though ε is able to get us ψ classically because it is provable classically, it cannot do its job constructively because it is not provable constructively.

4 A Possible Objection

One might reformulate Tennant's conjecture as follows: Not every single paradox has to be constructivisable; rather, the logico-conceptual resources must admit of constructive paradoxes, and those paradoxes are not normalizable.[9] They might point out that our choice of ε is a classical equivalent of the provability Curry sentence $Bew(\ulcorner\varepsilon\urcorner) \to \psi$,[10] and the paradoxical proof of provability Curry is indeed constructivisable. Thus, one might conclude that the logico-conceptual resources used in our unconstructivisable paradox above admit paradoxes that are constructivisable as shown with the provability Curry.[11]

This, however, is undoubtedly an ad hoc! Relying on the fact that $\neg Bew(\ulcorner\varepsilon\urcorner)\vee$

[9] Tennant suggested this reformulation in a private conversation (5/25/2022).
[10] See [1] for more on provability Curry.
[11] Tennant via a private conversation (5/25/2022).

ψ and $Bew(\ulcorner\varepsilon\urcorner) \to \psi$ are classically equivalent as proof that they use the same logico-conceptual resources is self-defeating. The fact that these two sentences are not constructively equivalent shows that they do not share the same logico-conceptual resources. What counts as the same logico-conceptual resources would depend on the background logic that is being used. In a classical setting, the provability Curry is equivalent to our initial ε, but in an intuitionistic setting, the provability Curry and our initial ε are not equivalent. One is paradoxical, while the other one is not.

In other words, just because $\neg Bew(\ulcorner\varepsilon\urcorner) \vee \psi$ intuitionistically implies $Bew(\ulcorner\varepsilon\urcorner) \to \psi$, and the latter is paradoxical in Tennant's account, we cannot conclude that $\neg Bew(\ulcorner\varepsilon\urcorner) \vee \psi$ is likewise paradoxical in conformity with Tennant's conjecture. As shown earlier, we absolutely need the other direction of this implication. Namely, we need $Bew(\ulcorner\varepsilon\urcorner) \to \psi$ to intuitionistically imply $\neg Bew(\ulcorner\varepsilon\urcorner) \vee \psi$. This implication, sadly, does not hold intuitionistically.

Perhaps what Tennant meant was that the paradoxical proof above relies on two basic logico-conceptual resources, namely, the partially transparent provability predicate and a means of self-reference.[12] The conjecture, then, simply implies that there are constructive paradoxes that rely on these two basic resources. If that is the case, then surely that is a much weaker and less interesting conjecture. It would amount to the claim that intuitionistic logic is afflicted by self-referential semantic paradoxes once those semantic notions are introduced to the system; no one would deny this. Moreover, a question remains: why is IP (or its equivalence) not counted as one of the logical resources that the paradoxical proof above relies on? If the rules of logic are not part of the logico-conceptual resources, then that would already assume that there is no way to set the classical paradoxes apart from constructive ones. Nonetheless, there are reasons to think that this is not the initial conjecture; Tennant's challenge was to "[g]ive an example of a semantic paradox whose associated deductive reasoning cannot be constructivized"([8], p.590), and we met this challenge. Altering this challenge to "show that there are no constructive paradoxes that make use of the basic resources needed for a paradox" would amount to moving the goalpost.

In conclusion, we showed a counterexample to Tennant's conjecture that every classical logico-semantic paradox can be constructivized. We showed that the paradoxical proof generated by ε in **NK** fits Tennant's proof-theoretic criterion for logico-semantic paradoxes, yet there are no paradoxical proofs generated by ε in **NJ**. Therefore, pace Tennant's conjecture, there are unconstructivisable paradoxes.

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[12]Thanks to the anonymous reviewer for this possible interpretation

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