Formulas of one variable in **KR**

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Abstract

This note corrects an error in an earlier paper [8] claiming that the algebra of three-variable formulas in the logic **KR** is finite. Here we show that the algebra of one-variable formulas in this logic is infinite.

1 Semantics of the logic KR

The logic **KR** is the result of adding to the relevant logic **R** the axiom *ex falso quodlibet*, that is to say, $(A \wedge \neg A) \to B$. We formulate the language of **KR** with the Boolean connectives $\wedge, \vee, \neg$ together with the fusion operator $\circ$ and the propositional constants $t, \top, \bot$; implication $A \to B$ can be defined as $\neg(A \circ \neg B)$.

A **KR** model structure (KRms) $\mathcal{K} = \langle S, R, 0 \rangle$ is a 3-place relation R on a set containing a distinguished element 0, satisfying the postulates:

1. $R0ab \Leftrightarrow a = b$;

2. $Raaa$;

3. $Rabc \Rightarrow (Rbac \,\&\, Racb)$ (total symmetry);

4. $(Rabc \,\&\, Rcde) \Rightarrow \exists f (Radf \,\&\, Rfbe)$ (Pasch's postulate).

Given a **KR** model structure $\mathcal{K} = \langle S, R, 0 \rangle$, we can define an algebra $\mathfrak{A}(\mathcal{K})$ as follows:

Definition 1.1 *The algebra $\mathfrak{A}(\mathcal{K}) = \langle \mathcal{P}(S), \cap, \cup, \neg, \top, \bot, t, \circ \rangle$ is defined on the Boolean algebra $\langle \mathcal{P}(S), \cap, \cup, \neg, \top, \bot \rangle$ of all subsets of S, where $\top = S, \bot = \emptyset, t = \{0\}$, and the operator $A \circ B$ is defined by*

$$A \circ B = \{c \mid \exists a \in A, b \in B \ (Rabc)\}.$$

The algebra $\mathfrak{A}(\mathcal{K})$ is a De Morgan monoid [1],[3] in which $a \wedge \neg a = \bot$, where $\bot$ is the least element of the monoid; we shall call any such algebra a **KR**-*algebra*. Hence the fusion operator $A \circ B$ is associative, commutative, and monotone. In addition, it distributes over union, it satisfies the square-increasing property, and t is the monoid identity:

$$A \circ (B \circ C) = (A \circ B) \circ C,$$

$$A \circ B = B \circ A,$$
$$A \circ (B \cup C) = (A \circ B) \cup (A \circ C),$$
$$(A \subseteq B \wedge C \subseteq D) \Rightarrow A \circ C \subseteq B \circ D,$$
$$A \subseteq A \circ A,$$
$$A \circ t = A.$$

In what follows, we shall assume basic results from the theory of De Morgan monoids, referring the reader to the expositions in Anderson and Belnap [1] and Dunn and Restall [3] for more background.

Definition 1.2 *If $\mathfrak{A}$ is a* **KR***-algebra then an* interpretation *of the language L of* **KR** *in $\mathfrak{A}$ is an assignment $I(A)$ mapping formulas A into elements of $\mathfrak{A}$ that satisfies the conditions:*

1. *$I(t) = t$, $I(\top) = \top$, $I(\bot) = \bot$;*

2. *$I(A \wedge B) = I(A) \wedge I(B)$;*

3. *$I(A \vee B) = I(A) \vee I(B)$;*

4. *$I(\neg A) = \neg I(A)$;*

5. *$I(A \circ B) = I(A) \circ I(B)$.*

If I is an interpretation in a **KR***-algebra, then a formula A is* satisfied *by the interpretation if $t \leq I(A)$.*

Theorem 1.3 *If A is a theorem of* **KR***, then it is satisfied by any interpretation in a* **KR***-algebra.*

In a **KR**-algebra, we can single out a subset of the elements that form a lattice; this lattice plays a key role in the analysis of the logic **KR**.

Definition 1.4 *Let $\mathfrak{A}$ be a* **KR***-algebra. The family $\mathcal{L}(\mathfrak{A})$ is defined to be the elements of $\mathfrak{A}$ that are $\geq t$ and idempotent, that is to say, $a \in \mathcal{L}(\mathfrak{A})$ if and only if $a \circ a = a$ and $t \leq a$. If $\mathcal{K}$ is a* **KR** *model structure, then we define $\mathcal{L}(\mathcal{K})$ to be $\mathcal{L}(\mathfrak{A}(\mathcal{K}))$.*

If $\mathcal{K} = \langle S, R, 0 \rangle$ is a **KR** model structure, then a subset A of S is a *linear subspace* if it satisfies the condition

$$(a, b \in A \wedge Rabc) \Rightarrow c \in A.$$

A lattice is *modular* if it satisfies the implication

$$x \geq z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z.$$

For background on modular lattice theory, the reader can consult the texts of Birkhoff [2] or Grätzer [4].

Lemma 1.5 *If $\mathcal{K}$ is a* **KR** *model structure, then the elements of $\mathcal{L}(\mathcal{K})$ are exactly the non-empty linear subspaces of $\mathcal{K}$.*

Proof. The lemma follows from the definition of $A \circ B$ and the fact that $Raa0$ and $Raaa$ hold in any **KR** model structure. □

Theorem 1.6 *If $\mathfrak{A}$ is a* **KR***-algebra, then $\mathcal{L}(\mathfrak{A})$, ordered by containment, forms a modular lattice, with least element t, and the lattice operations of meet and join defined by $a \wedge b$ and $a \circ b$.*

Proof. See Theorem 2.5 of [7]. □

2 Constructing models from projective spaces

The main result is proved by constructing a model from a projective space; the method in fact works in a more general context. Let L be a lattice with 0. We define the *height* function: for $a \in L$, let $h(a)$ denote the length of a longest maximal chain in the interval lattice $[0, a]$ if there is a finite longest maximal chain; otherwise put $h(a) = \infty$. A lattice L has *finite length* if $0 \le h(a) < \infty$ for all $a \in L$. An *atom* a of L is an element with $h(a) = 1$; we also refer to atoms as *points*. If a, b, c are distinct points, then we say that they are *collinear* if they all belong to a common element d with $h(d) = 2$.

If a and b are lattice elements, then we say that b *covers* a, $a \prec b$, if $a < b$, but there is no element c so that $a < c < b$; we write $a \preceq b$ if $a \prec b$ or $a = b$. A lattice is *semimodular* if it satisfies the *Upper Covering Condition*:

$$a \prec b \Longrightarrow (a \vee c \preceq b \vee c).$$

In a semimodular lattice of finite length, any two maximal chains have the same length [4, p. 172]. A lattice L is *atomistic* if every element of L is a join of atoms. A *geometric lattice* is an atomistic semimodular lattice of finite length [2, Ch. IV][9, p. 53].

Definition 2.1 *Let L be a geometric lattice of length at least 2, and S the points in L. Define a ternary relation R on the elements of $S \cup \{0\}$ by:*

$$Rabc \Leftrightarrow a \vee b = b \vee c = a \vee c,$$

and let $\mathcal{K}(L)$ be $\langle S, R, 0 \rangle$.

Lemma 2.2 *The defined ternary relation $Rabc$ satisfies the conditions:*

1. *If $a = b$, then either $c = 0$ or $a = b = c$;*

2. *If $a \neq b$ then either*

    (a) *$a = 0$ and $R0bb$ or $b = 0$ and $Ra0a$, or*

    (b) *a, b, c are distinct collinear points.*

Proof. If $a = b$, then $a = a \vee c$, so that $c \leq a$. If $c \neq 0$, then since a and c are both points, it follows that $a = b = c$.

If $a \neq b$, and $a = 0$ then $b = b \vee c = 0 \vee c = c$; similarly if $b = 0$. If $a \neq b$ and $a \neq 0$, then $a \prec a \vee b$ and $h(a \vee b) = 2$, so that $h(b \vee c) = h(a \vee c) = 2$, showing that $a \neq c$ and $b \neq c$, so that a, b, c are distinct collinear points. $\qquad\square$

Lemma 2.3 *If L is a modular geometric lattice, then $\mathcal{K}(L)$ is a* **KR** *model structure.*

Proof. The Lemma follows by essentially the same proof as in the case of Theorem 2.7 of [7]. $\qquad\square$

Let $\mathbb{F}$ be a field, and $\mathbb{F}^{k+1}$ the vector space of dimension $k + 1$ over $\mathbb{F}$. The family of vector subspaces of $\mathbb{F}^{k+1}$, ordered by containment, forms a modular geometric lattice [4, p. 203], which we denote by L_k. It also forms a projective space $\mathbb{FP}^k$ of dimension k over $\mathbb{F}$. The points of this projective space are the one-dimensional subspaces, while the zero element of the lattice is $\{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector in $\mathbb{F}^{k+1}$. The projective dimension of a subspace is one less than the vector space dimension, so that the points have projective dimension 0 and the zero element has projective dimension -1. If U is a subspace in the lattice L_k, then $h(U)$ coincides with the dimension of U in the vector space $\mathbb{F}^{k+1}$. Applying Lemma 2.3 and Definition 1.1 to the modular geometric lattice L_k, we define $\mathcal{K}_k = \mathcal{K}(L_k)$ and $\mathfrak{A}_k = \mathfrak{A}(\mathcal{K}_k)$.

3 The main construction

In this section, we apply the results of the earlier sections to show that the propositions $p, p^2, p^3, \ldots, p^k, \ldots$ are pairwise distinct in **KR**, where $p^{j+1} = p^j \circ p$. This is the same sequence of formulas used by Bob Meyer in a paper [5] from 1970, showing that these propositions are pairwise distinct in **R**. We prove this by showing that for any k, we can find a set A of points in a projective space so that $A, A^2, A^3, \ldots, A^k$ are all distinct. If b is a point and A a set of points, then we abbreviate $A \circ \{b\}$ as $A \circ b$ and $\{a\} \circ \{b\}$ as $a \circ b$. The main construction is based on the real projective space $\mathbb{RP}^k$, for $k > 1$.

Lemma 3.1 *Let B be a finite set of points in $\mathbb{RP}^k$ that spans a linear subspace U with $h(U) = k$, and define $A = B \cup \{0\}$. For $1 \leq j \leq k$, let $\mathcal{F}_j$ be the family of subsets $C \subseteq A$ in $\mathbb{RP}^k$, where C is a linear subspace spanned by a finite subset of A, and $h(C) \leq j$. Then $A^j = \bigcup \mathcal{F}_j$.*

Proof. We argue by induction on j. For $j = 1$, $A^1 = \bigcup\{\{0, a\} | a \in A\}$, where $\{0, a\}$, for $a \in A$, is a linear subspace of height 1. Assume the lemma for j, so that $A^j = \bigcup \mathcal{F}_j$, Then, using the basic properties of the fusion operator,

$$A^{j+1} = A^j \circ A = \bigcup \mathcal{F}_j \circ A = \bigcup \{C \circ a \mid C \in \mathcal{F}_j \wedge a \in A\}.$$

We need to show that

$$\mathcal{F}_{j+1} = \{C \circ a \mid C \in \mathcal{F}_j \wedge a \in A\}.$$

If $C \in \mathcal{F}_j$ and $a \in A$, then $h(C \circ a) \leq h(C) + 1 \leq j + 1$, showing that $C \circ a \in \mathcal{F}_{j+1}$. Conversely, assume that $D \in \mathcal{F}_{j+1}$, so that D is a linear subspace spanned by a finite subset E of A and $h(D) \leq j + 1$. If $h(D) \leq j$, then $D \in \mathcal{F}_j$, so choosing $a \in E$, $D = D \circ a$. If $h(D) = j + 1$ we can assume that E is a minimal spanning set generating D. If $a \in E$, then $D = F \circ a$, where F is the linear subspace spanned by $E \setminus a$, so that $h(F) = j$ and $F \in \mathcal{F}_j$. Hence $D \in \{C \circ a \mid C \in \mathcal{F}_j \wedge a \in A\}$, completing the proof of the equality. $\square$

Theorem 3.2 *Let B be a finite set of points in $\mathbb{RP}^k$ spanning a linear subspace U with $h(U) = k$, and $A = B \cup \{0\}$. Then the sets of points in the family $A, A^2, A^3, \ldots, A^k$ are pairwise distinct.*

    Proof. Since $C \subseteq C \circ C$ for any set of points C, it follows that

$$A \subseteq A^2 \subseteq A^3 \subseteq \cdots \subseteq A^k,$$

so that it suffices to show that all of these containments are proper. By Lemma 3.1, A^{j+1} is a union of subspaces of height $j + 1$. Any such subspace covers a subspace of height j, so that there is an element of A in $A^{j+1} \setminus A^j$. $\square$

    The theorem can be illustrated geometrically. In three-dimensional real projective space, let B be a set of four points in general position, so that they form a regular tetrahedron, and define $A = B \cup \{0\}$. Then A^2 is the union of one-dimensional subspaces, that are determined by the six lines joining pairs of points in A. The set A^3 is the union of the four planes determined by a face of the tetrahedron, while A^4 is the whole three-dimensional space. Figure 1 illustrates the construction of A^2, with $A = \{0, a, b, c, d\}$.

Corollary 3.3 *The propositions $p, p^2, p^3, \ldots, p^k, \ldots$ are pairwise distinct in the logic **KR**.*

    Proof. If we interpret the variable p in the projective space $\mathbb{RP}^k$ so that $I(p) = A$, where B is a finite set of points in $\mathbb{RP}^k$ spanning a linear subspace U with $h(U) = k$, and $A = B \cup \{0\}$, then Theorem 3.2 shows that $I(p), I(p^2), I(p^3), \ldots, I(p^k)$ are pairwise distinct sets of points. Hence, for $i < j \leq k$, $p^i \leftrightarrow p^j$ is unprovable in **KR**, since for an interpretation I in a **KR** algebra, $t \leq I(C \leftrightarrow D)$ if and only if $I(C) = I(D)$. $\square$

4   Conclusion

In conclusion, we point out the error in the earlier paper [8]. The basic mistake is in the proof of Theorem 3.3, claiming that the free **KR**-algebra on three generators is finite. The proof correctly notes that the free modular lattice on
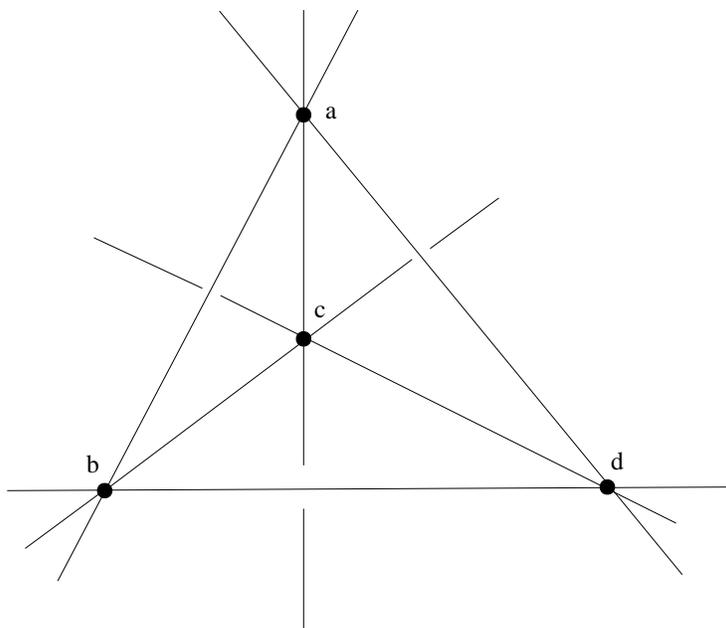
Figure 1: A tetrahedron in real projective 3-space

three generators is finite (a well known result of lattice theory, dating back to an article of Richard Dedekind in 1900). Furthermore, it also correctly claims that the **KR** model structure $\mathcal{K}_3$ constructed from this lattice, using the construction of Lemma 2.3, is also finite. The error lies in the last sentence, which claims that the free **KR**-algebra in three generators is also finite, based on the nonsensical claim that this algebra is isomorphic to $\mathcal{L}(\mathcal{K}_3)$. If at this point, the proof had claimed an isomorphism to $\mathfrak{A}[\mathcal{L}(\mathcal{K}_3)]$, then the finiteness result would follow. However, as we have just seen, the free **KR**-algebra in one generator is infinite, so the whole proof is worthless.

An intriguing open problem is this: what is the smallest number of variables for which we can prove undecidability in various relevant logics? For the logics between **T-W + A15** and **KR**, a paper from 2007 [6] shows undecidability of the four-variable fragment. So far, this is the best result in this area. A simplified proof of this result for the case of **KR** is included in the 2023 paper [8] cited above (this simplified proof is not affected by the error explained in the preceding paragraph).

The result of the current note re-opens the question for the logic **KR** that was supposedly laid to rest by the 2023 paper containing the mistake. Thus we have a new and intriguing open question.

Problem 4.1 *Is the one-variable fragment of* **KR** *decidable?*

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