Fitchean Ignorance and First-order Ignorance: A Neighborhood Look

Jie Fan

Institute of Philosophy, Chinese Academy of Sciences;

School of Humanities, University of Chinese Academy of Sciences

jiefan@ucas.ac.cn

Abstract

In a seminal work [20], Fine classifies several forms of ignorance, among which are Fitchean ignorance, first-order ignorance, Rumsfeld ignorance, and second-order ignorance. It is shown that there are some relationships among them, including that in **S4**, all higher-order ignorance are reducible to second-order ignorance. This is thought of as a disadvantage by some researchers. It is then natural to ask how to avoid this consequence. We deal with this issue in a much more general framework. In detail, we treat the forms of Fitchean ignorance and first-order ignorance as primitive modalities and study them as first-class citizens under neighborhood semantics, in which Rumsfeld ignorance and second-order ignorance are definable. We show that the aforementioned reducibility issue does not arise in our neighborhood setting. The main contributions include model-theoretical results such as expressivity and frame definability, and axiomatizations. Last but not least, by updating the neighborhood models via the intersection semantics, we extend the results to the dynamic case of public announcements, which gives us some applications to successful formulas.

Keywords: Fitchean ignorance, first-order ignorance, contingency, accident, unknown truths, expressivity, frame definability, axiomatizations, intersection semantics, successful formulas

1   Introduction

Ignorance has been a hotly discussed theme in epistemology and many other fields since Socrates, who professed ignorance in e.g. the *Apology* [41] (see also [2, 40]). Just as there has been no consensus on the definition of knowledge, there has been no consensus on the definition of ignorance. Instead, there has been at least three views in the literature: the standard view, the new view, and the logical view.[1] The standard view says that ignorance is merely the absence or lack of knowledge, the new view states that ignorance is the lack of true belief,[2] whereas the logical view holds that

---

[1] The terminology 'the standard view' is introduced in [35], 'the new view' is from [39], whereas the term 'the logical view' comes from [9].

[2] For the discussion on the standard view and the new view, see [29] and references therein.

ignorance means neither knowing nor knowing not [17, 18, 37, 46, 48, 49].[3]

Recently, various forms of ignorance have been proposed in the literature, such as pluralistic ignorance [3, 36, 42], circumscriptive ignorance [26], chronological ignorance [44], factive ignorance [27], relative ignorance [23], disjunctive ignorance [14]. In a seminal paper [20], instead of discussing the definition of ignorance, Fine classifies several forms of ignorance, among which are 'ignorance of (the fact that)' (also called 'Fitchean ignorance' there), 'first-order ignorance (whether)', 'Rumsfeld ignorance' and 'second-order ignorance'. One is *ignorant of* (the fact that) φ, if φ is the case but one does not know it. One is *(first-order) ignorant whether* φ, if one neither knows φ nor knows its negation. One is *Rumsfeld ignorant of* φ, if one is ignorant of the fact that one is ignorant whether φ. One is *second-order ignorant whether* φ, if one is ignorant whether one is ignorant whether φ.

Fine formulates the epistemological claim that knowledge of one's second-order ignorance is impossible, once knowledge is understood in the sense of system **S4**, and provides a formal proof for that claim. This is the main result of Fine's article. Moreover, to provide a proof in favor of his claim, Fine presents some relationships among some of the above forms. For instance, second-order ignorance implies first-order ignorance; second-order ignorance implies Rumsfeld ignorance, and vice versa (that is, Rumsfeld ignorance implies second-order ignorance); one does not know one is Rumsfeld ignorant; one does not know one is second-order ignorant.

However, all these results rely on the assumption that knowledge is understood in the sense of system **S4**. But the assumption itself is controversial. For instance, Williamson [53] rejects the KK principle (namely, $K\varphi \rightarrow KK\varphi$, an axiom in **S4**, also see [19]), and some researchers rejects the factivity of knowledge (namely, $K\varphi \rightarrow \varphi$, another axiom in **S4**), see e.g. [6, 24]. Moreover, as Fine shows, in **S4**, all higher-order ignorance are reducible to second-order ignorance, which is thought of as a disadvantage, called a *black hole* of ignorance in [20, p. 4031] and a *quite problematic phenomenon* in [5, p. 1060], because it makes it impossible for one to exit from second-order ignorance. Furthermore, one may easily check that none of the above forms of ignorance is a normal operator, since for instance, they are not monotone: ignorance of a conjunction does not entail ignorance of each conjunct.[4] This indicates that it may be interesting to explore the relationships among these forms in a more general setting.

In the current paper, we will focus on Fitchean ignorance and first-order ignorance. This is partly because they can define the aforementioned Rumsfeld ignorance and second-order ignorance, thereby enabling us to explore the relationships among these forms. More importantly, Fitchean ignorance and first-order ignorance correspond to accident (or 'accidental truths') and contingency, respectively, which in turn are important metaphysical concepts. For an overview of the importance of the two metaphysical concepts, we refer to [11] and the references therein. Despite being definable with knowledge, studying the two forms of ignorance as first-class citizens has some advantages. First, the logical properties of the notions can be seen more clearly. Second,

---

[3]To the best of our knowledge, the first to evidently investigate ignorance from the logical view is [48] — also see its extended journal version [49], although it includes an *unsound* transitive axiomatization, as shown in [18, pp. 102–103].

[4]An operator/a modality O is normal, if $\vDash \varphi_1 \wedge \cdots \wedge \varphi_n \rightarrow \psi$ implies $\vDash O\varphi_1 \wedge \cdots \wedge O\varphi_n \rightarrow O\psi$; O is monotone, if $\vDash \varphi \rightarrow \psi$ implies $\vDash O\varphi \rightarrow O\psi$.

the bundled operators are more succinct than their knowledge counterparts (see for instance [51], where it is shown that contingency logic (that is, the logic which has contingency or non-contingency as a sole primitive modality, see for instance [34]) is exponentially more succinct than the standard modal logic on the class of all Kripke models).[5]

It is important to distinguish these two forms. For instance, the Fitchean ignorance satisfies the so-called *Factivity Principle* (that is, if an agent is ignorant of φ then φ is true), but the first-order ignorance does not. Moreover, as mentioned above, the operators of the two forms are not normal. As is well known, neighborhood semantics has been a standard semantics tool for non-normal operators since its introduction in 1970 [33, 43], as [7, 38] shows. In the current paper, we will propose a bimodal logic with Fitchean ignorance and first-order ignorance, axiomatize the logic over various neighborhood frames, and explore the model-theoretical properties. As a side product, we also investigate the relationships among various forms of ignorance under the neighborhood semantics. As we will show, even under the neighborhood semantics, there are also some interesting relationships among first-order ignorance, second-order ignorance, and Rumsfeld ignorance, some of which are unclear from the literature. For example, under any neighborhood condition, Rumsfeld ignorance implies first-order ignorance, and second-order ignorance plus first-order ignorance implies Rumsfeld ignorance, whereas under a simple condition ((c), to be exact, defined below), Rumsfeld ignorance implies second-order ignorance, and thus Rumsfeld ignorance amounts to second-order ignorance plus first-order ignorance.

However, similar to the case for relational semantics [11], the situation may become quite involved if we study the two notions in a unified framework under the neighborhood semantics. For instance, we will be confronted with a difficulty in axiomatizing the bimodal logic, since we have only one neighborhood function to deal with two modal operators uniformly, which makes it hard to find suitable interaction axioms.

Since contingency and accident are metaphysical correspondents of first-order ignorance and Fitchean ignorance respectively, our technical results also apply to these correspondents. This also partly answers open questions posed in [13, 15]. Note that there is also an interesting comparison between our work and [14]: although the latter investigates a weak combination (disjunctive, that is) of the two forms of ignorance, we here treat both forms as primitive modalities and study them as first-class citizens.

Moreover, along with the line of thought in [11], we consider the dynamic extensions of our logics. The main motivation is that one's state of ignorance may change under some action. We consider the simplest case: public announcement. In the literature, there are mainly two ways of updating neighborhood models: intersection semantics and subset semantics [30, 31]. Here we only adopt the intersection semantics, partly because it can be seen a natural generalization of public announcement logic over Kripke semantics [30, p. 217], and partly because we have nice reduction axioms under this semantics.

The table below is an overview of the known results of which forms of ignorance

---

[5]Since for contingency logic, the class of all Kripke models corresponds to the class of augmented neighborhood models (see [16, Sec. 5]), contingency logic is also exponentially more succinct than the standard modal logic on the class of augmented neighborhood models and thus on the class of larger neighborhood models.

have been dealt with in which semantics in the literature, where 'Both' means both
Fitchean Ignorance and First-order Ignorance.

Forms of Ignorance	Kripke Semantics	Neighborhood Semantics
Only Fitchean Ignorance	[8, 22, 32, 45, 47]	[15, 21]
Only First-order Ignorance	[17, 18, 25, 28, 34, 49, 50, 54]	[10, 12, 16]
Both	[11, 13]	——

In this paper, we deal with both forms of Fitchean ignorance and first-order ignorance in neighborhood semantics, which is lacking in the literature. The remainder of the paper is organized as follows. After briefly reviewing the syntax and the neighborhood semantics of the bimodal logic of Fitchean ignorance and first-order ignorance and also some related logics (Sec. 2), we compare the relative expressivity (Sec. 3) and investigate the frame definability of the bimodal logic (Sec. 4). We axiomatize the bimodal logic over various classes of neighborhood frames (Sec. 5). By updating the neighborhood models via the intersection semantics, we find suitable reduction axioms and thus reduce the public announcements operators to the bimodal logic, which gives us good applications to successful formulas (defined in Sec. 6 below), where we obtain two fragments of successful formulas in the bimodal logic. Finally, we conclude with some future work in Sec. 7.

2   Syntax and Neighborhood Semantics

First, we introduce the languages and their neighborhood semantics involved in this paper.

Fix a nonempty set $\mathbf{P}$ of propositional variables, and let $p \in \mathbf{P}$. In what follows, $\mathcal{L}(\Box)$ is the language of standard epistemic logic, $\mathcal{L}(\nabla)$ is the language of the logic of (first-order) ignorance, $\mathcal{L}(\bullet)$ is the language of the logic of Fitchean ignorance[6], and $\mathcal{L}(\nabla, \bullet)$ is the language of the bimodal logic of Fitchean ignorance and first-order ignorance. We will mainly focus on $\mathcal{L}(\nabla, \bullet)$. For the sake of simplicity, we only exhibit the single-agent languages, but all our results also apply to multi-agent cases.

Definition 1 (Languages)**.**

$$
\begin{array}{llll}
\mathcal{L}(\Box) & \varphi & ::= & p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi & [4] \\
\mathcal{L}(\nabla) & \varphi & ::= & p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \nabla\varphi & [34] \\
\mathcal{L}(\bullet) & \varphi & ::= & p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \bullet\varphi & [32] \\
\mathcal{L}(\nabla, \bullet) & \varphi & ::= & p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \nabla\varphi \mid \bullet\varphi & [11]
\end{array}
$$

$\Box\varphi$ is read "one knows that φ", $\nabla\varphi$ is read "one is *(first-order) ignorant whether* φ", and $\bullet\varphi$ is read "one is (Fitchean) *ignorant of* (the fact that) φ", or "φ is an unknown truth". In the metaphysical setting, $\nabla\varphi$ and $\bullet\varphi$ are read, respectively, "it is contingent that φ" and "it is accidental that φ".[7] Among other connectives, $\top$, $\Diamond\varphi$, $\Delta\varphi$, and

---

[6]$\mathcal{L}(\bullet)$ is also called 'the logic of essence and accident' or 'the logic of unknown truths', see e.g. [32, 46].

[7]See [11]. There, it is contingent that φ, if φ is neither necessarily true nor necessarily false; it is accidental that φ, if φ is true but not necessarily true. Just as necessity is the metaphysical counterpart of knowledge, contingency and accident are the metaphysical counterparts of first-order ignorance and Fitchean ignorance, respectively.

$\circ\varphi$ abbreviate, respectively, $\varphi \vee \neg\varphi$, $\neg\Box\neg\varphi$, $\neg\nabla\varphi$, and $\neg\bullet\varphi$, read "it is epistemically possible that φ", " one knows whether φ", and "one is non-ignorant of φ".

Note that the forms of 'Rumsfeld ignorance (of φ)' and 'second-ignorance (whether φ)' can be defined as, respectively, $\bullet\nabla\varphi$ and $\nabla\nabla\varphi$.

The above languages are interpreted over neighborhood models.

Definition 2 (Neighborhood structures)**.** A *(neighborhood) model* is a triple $\mathcal{M} = \langle S, N, V \rangle$, where S is a nonempty set of states (also called 'points' or 'possible worlds', N is a neighborhood function from S to $\mathcal{P}(\mathcal{P}(S))$, and V is a valuation function. A *(neighborhood) frame* is a model without a valuation; in this case, we say that the model is based on the frame. A *pointed model* is a pair of a model with a point in it. Given an $s \in S$, an element of $N(s)$ is called 'a neighborhood of s'.

Definition 3 (Neighborhood properties)**.** Let $\mathcal{F} = \langle S, N \rangle$ be a frame, and $\mathcal{M}$ be a model based on $\mathcal{F}$. Let $s \in S$ and $X, Y \subseteq S$. We define various neighborhood properties as follows.

- (n): $N(s)$ *contains the unit*, if $S \in N(s)$.

- (r): $N(s)$ *contains its core*, if $\bigcap N(s) \in N(s)$.

- (i): $N(s)$ *is closed under intersections*, if $X, Y \in N(s)$ implies $X \cap Y \in N(s)$.

- (s): $N(s)$ is *supplemented*, or *closed under supersets*, if $X \in N(s)$ and $X \subseteq Y \subseteq S$ implies $Y \in N(s)$.

- (c): $N(s)$ is *closed under complements*, if $X \in N(s)$ implies $S \backslash X \in N(s)$.[8]

- (d): $X \in N(s)$ implies $S \backslash X \notin N(s)$.

- (t): $X \in N(s)$ implies $s \in X$.

- (b): $s \in X$ implies $\{u \in S \mid S \backslash X \notin N(u)\} \in N(s)$.

- (4): $X \in N(s)$ implies $\{u \in S \mid X \in N(u)\} \in N(s)$.

- (5): $X \notin N(s)$ implies $\{u \in S \mid X \notin N(u)\} \in N(s)$.

The function N possesses such a property, if for all $s \in S$, $N(s)$ has the property. $\mathcal{F}$ (and $\mathcal{M}$) has a property, if N has. In particular, we say that $\mathcal{F}$ (and $\mathcal{M}$) is *monotone*, if N has (s). $\mathcal{F}$ (and $\mathcal{M}$) is a *quasi-filter*, if N has (i) and (s); $\mathcal{F}$ (and $\mathcal{M}$) is a *filter*, if N also has (n).

Also, in what follows, we will use $\mathbb{C}_n$ to denote the class of (n)-models, and similarly for $\mathbb{C}_r$, etc. We use $\mathbb{C}_{\text{all}}$ for the class of all neighborhood models.

---

[8]The property (c) provides a new perspective for $\mathcal{L}(\nabla)$. In more detail, by restricting models to the ones with the property (c), we can simplify the neighborhood semantics of ∇ to that of $\Diamond$, and keep the logic (valid formulas) the same. This can weaken the too strong models so as to balance the syntax and models for $\mathcal{L}(\nabla)$. Under this new perspective, we can gain a lot of things, for example, bisimulation notions and their corresponding Hennessy-Milner Theorems, frame definability and simple axiomatizations. Moreover, it is shown that one of the bisimulation notions is equivalent to the notion of nbh-Δ-bisimulation, which helps us understand the essence of nbh-Δ-bisimulation proposed in [1]. See [10] for details.

Definition 4 (Semantics). Let $\mathcal{M} = \langle S, N, V \rangle$ be a model. Given a pointed model $(\mathcal{M}, s)$, the truth condition of formulas is defined recursively as follows:

$$
\begin{aligned}
\mathcal{M}, s \vDash p &\iff s \in V(p) \\
\mathcal{M}, s \vDash \neg\varphi &\iff \mathcal{M}, s \nvDash \varphi \\
\mathcal{M}, s \vDash \varphi \wedge \psi &\iff \mathcal{M}, s \vDash \varphi \text{ and } \mathcal{M}, s \vDash \psi \\
\mathcal{M}, s \vDash \Box\varphi &\iff \varphi^{\mathcal{M}} \in N(s) \\
\mathcal{M}, s \vDash \nabla\varphi &\iff \varphi^{\mathcal{M}} \notin N(s) \text{ and } S\backslash\varphi^{\mathcal{M}} \notin N(s) \\
\mathcal{M}, s \vDash \bullet\varphi &\iff \mathcal{M}, s \vDash \varphi \text{ and } \varphi^{\mathcal{M}} \notin N(s)
\end{aligned}
$$

where $\varphi^{\mathcal{M}}$ denotes the *truth set* of φ in $\mathcal{M}$, in symbols, $\varphi^{\mathcal{M}} = \{s \in S \mid \mathcal{M}, s \vDash \varphi\}$; given a set $X \subseteq S$, $S\backslash X$ denotes the complement of X with respect to S.

We say that φ is *true* in $(\mathcal{M}, s)$, if $\mathcal{M}, s \vDash \varphi$; we say that φ is valid on a model $\mathcal{M}$, notation: $\mathcal{M} \vDash \varphi$, if for all s in $\mathcal{M}$, we have $\mathcal{M}, s \vDash \varphi$; we say that φ is valid on a frame $\mathcal{F}$, notation: $\mathcal{F} \vDash \varphi$, if for all $\mathcal{M}$ based on $\mathcal{F}$, we have $\mathcal{M} \vDash \varphi$; we say that φ is valid over a class $\mathbb{F}$ of frames, notation: $\mathbb{F} \vDash \varphi$, if for all $\mathcal{F}$ in $\mathbb{F}$, we have $\mathcal{F} \vDash \varphi$; we say that φ is satisfiable over the class $\mathbb{F}$, if $\mathbb{F} \nvDash \neg\varphi$. Similar notions go to a set of formulas.

For the sake of reference, we also list the semantics of the aforementioned defined modalities as follows:

$$
\begin{aligned}
\mathcal{M}, s \vDash \Diamond\varphi &\iff S\backslash\varphi^{\mathcal{M}} \notin N(s) \\
\mathcal{M}, s \vDash \Delta\varphi &\iff \varphi^{\mathcal{M}} \in N(s) \text{ or } S\backslash\varphi^{\mathcal{M}} \in N(s) \\
\mathcal{M}, s \vDash \circ\varphi &\iff \mathcal{M}, s \vDash \varphi \text{ implies } \varphi^{\mathcal{M}} \in N(s).
\end{aligned}
$$

To illustrate the semantics, we show some validities. The following result is immediate by semantics.

Proposition 5. $\vDash \nabla\varphi \leftrightarrow (\neg\Box\varphi \wedge \neg\Box\neg\varphi), \vDash \bullet\varphi \leftrightarrow (\varphi \wedge \neg\Box\varphi)$.

This indicates that both forms of ignorance are definable with the knowledge operator $\Box$. In spite this, as we mentioned in the introduction, studying the two forms of ignorance as first-class citizens has some advantages.

Proposition 6. $\vDash \nabla\varphi \to \bullet\varphi \vee \bullet\neg\varphi$.

Proof. Let $\mathcal{M} = \langle S, N, V \rangle$ be any model and $s \in S$. Suppose that $\mathcal{M}, s \vDash \nabla\varphi$. Then $\varphi^{\mathcal{M}} \notin N(s)$ and $S\backslash\varphi^{\mathcal{M}} \notin N(s)$, that is, $(\neg\varphi)^{\mathcal{M}} \notin N(s)$. We have either $\mathcal{M}, s \vDash \varphi$ or $\mathcal{M}, s \vDash \neg\varphi$. If $\mathcal{M}, s \vDash \varphi$, since $\varphi^{\mathcal{M}} \notin N(s)$, we infer that $\mathcal{M}, s \vDash \bullet\varphi$; if $\mathcal{M}, s \vDash \neg\varphi$, since $(\neg\varphi)^{\mathcal{M}} \notin N(s)$, we derive that $\mathcal{M}, s \vDash \bullet\neg\varphi$. Therefore, $\mathcal{M}, s \vDash \bullet\varphi \vee \bullet\neg\varphi$. Since $(\mathcal{M}, s)$ is arbitrary, this establishes that $\vDash \nabla\varphi \to \bullet\varphi \vee \bullet\neg\varphi$. $\square$

In [20] it is shown that in **S4**, all higher-order ignorance are reducible to second-order ignorance (in symbols, $\mathbf{S4} \vdash \nabla^n\varphi \leftrightarrow \nabla\nabla\varphi$ for all $n > 1$), which makes it impossible for one to escape from second-order ignorance. Here we show that this problem does not arise in our neighborhood setting. Since the smallest neighborhood class mentioned above is the class of filters, it remains only to show that the formula is invalid over the class of filters.

Proposition 7. $\nabla\nabla\nabla\varphi \leftrightarrow \nabla\nabla\varphi$ is not valid over the class of filters.

Proof. Consider the following model $\mathcal{M}$:
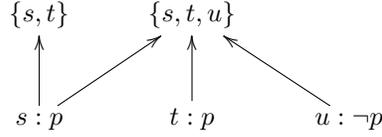
One may verify that $\mathcal{M}$ satisfies (n), (i) and (s), thus $\mathcal{M}$ is a filter.

Moreover, $\mathcal{M}, s \nVDash \nabla\nabla\nabla p \leftrightarrow \nabla\nabla p$. To see this, first notice that $p^{\mathcal{M}} = \{s, t\}$ and $S \backslash p^{\mathcal{M}} = \{u\}$. As $p^{\mathcal{M}} \in N(s)$, we have $\mathcal{M}, s \nVDash \nabla p$; as $p^{\mathcal{M}} \notin N(t)$, $S \backslash p^{\mathcal{M}} \notin N(t)$, we infer that $\mathcal{M}, t \vDash \nabla p$; similarly, $\mathcal{M}, u \vDash \nabla p$. Thus $(\nabla p)^{\mathcal{M}} = \{t, u\}$ and $S \backslash (\nabla p)^{\mathcal{M}} = \{s\}$, both of which are not neighborhoods of s, thus $\mathcal{M}, s \vDash \nabla\nabla p$. A similar argument gives us that $\mathcal{M}, t \vDash \nabla\nabla p$ and $\mathcal{M}, u \vDash \nabla\nabla p$. This shows that $(\nabla\nabla p)^{\mathcal{M}} = \{s, t, u\}$. Now using $\{s, t, u\} \in N(s)$, we conclude that $\mathcal{M}, s \nVDash \nabla\nabla\nabla p$, as desired. $\qquad\square$

3 Expressivity

In this section, we compare the relative expressivity of $\mathcal{L}(\nabla, \bullet)$ and other languages introduced before, over various classes of neighborhood models. Some expressivity results over the class of relational models have been obtained in [11] and [13].

To make our presentation self-contained, we introduce some necessary technical terms.

Definition 8. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages that are interpreted on the same class of models $\mathbb{C}$, where $\mathbb{C}$ ranges over classes of models which are models for $\mathcal{L}_1$ and for $\mathcal{L}_2$.

- $\mathcal{L}_2$ is *at least as expressive as* $\mathcal{L}_1$ *over* $\mathbb{C}$, notation: $\mathcal{L}_1 \preceq \mathcal{L}_2[\mathbb{C}]$, if for all $\varphi \in \mathcal{L}_1$, there exists $\psi \in \mathcal{L}_2$ such that for all $\mathcal{M} \in \mathbb{C}$ and all s in $\mathcal{M}$, we have that $\mathcal{M}, s \vDash \varphi$ iff $\mathcal{M}, s \vDash \psi$.

- $\mathcal{L}_1$ and $\mathcal{L}_2$ are *equally expressive over* $\mathbb{C}$, notation: $\mathcal{L}_1 \equiv \mathcal{L}_2[\mathbb{C}]$, if $\mathcal{L}_1 \preceq \mathcal{L}_2[\mathbb{C}]$ and $\mathcal{L}_2 \preceq \mathcal{L}_1[\mathbb{C}]$.

- $\mathcal{L}_1$ is *less expressive than* $\mathcal{L}_2$ *over* $\mathbb{C}$, notation: $\mathcal{L}_1 \prec \mathcal{L}_2[\mathbb{C}]$, if $\mathcal{L}_1 \preceq \mathcal{L}_2[\mathbb{C}]$ but $\mathcal{L}_2 \npreceq \mathcal{L}_1[\mathbb{C}]$.

- $\mathcal{L}_1$ and $\mathcal{L}_2$ are *incomparable in expressivity over* $\mathbb{C}$, notation: $\mathcal{L}_1 \asymp \mathcal{L}_2[\mathbb{C}]$, if $\mathcal{L}_1 \npreceq \mathcal{L}_2[\mathbb{C}]$ and $\mathcal{L}_2 \npreceq \mathcal{L}_1[\mathbb{C}]$.

It turns out that over the class of (c)-models and the class of (t)-models, $\mathcal{L}(\nabla)$ is at least as expressive as $\mathcal{L}(\bullet)$ (Prop. 13 and Prop. 14), whereas $\mathcal{L}(\nabla)$ is *not* at least as expressive as $\mathcal{L}(\bullet)$ over the class of models possessing either of the other eight neighborhood properties (Prop. 9-Prop. 11). The tricky thing is the construction of the desired models.

Proposition 9. $\mathcal{L}(\bullet) \not\preceq \mathcal{L}(\nabla)[\mathbb{C}]$, where $\mathbb{C} \in \{\mathbb{C}_{\text{all}}, \mathbb{C}_r, \mathbb{C}_i, \mathbb{C}_s, \mathbb{C}_d\}$.

Proof. Consider the following models, which come from [16, Prop. 2]. An arrow from a state x to a set X means that X is a neighborhood of x (Idem for other arrows).
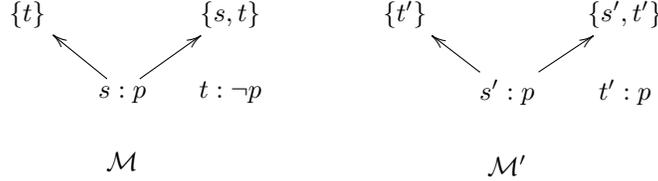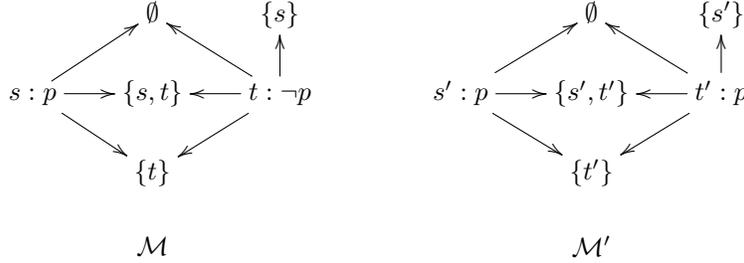

It has been shown in [16, Prop. 2] that both $\mathcal{M}$ and $\mathcal{M}'$ satisfy (r), (i), (s) and (d), and $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ cannot be distinguished by $\mathcal{L}(\nabla)$.

However, both pointed models can be distinguished by $\mathcal{L}(\bullet)$. To see this, note that $p^{\mathcal{M}} = \{s\}$ and $\{s\} \notin N(s)$, and thus $\mathcal{M}, s \vDash \bullet p$, whereas $\mathcal{M}', s' \nvDash \bullet p$, as $p^{\mathcal{M}'} = \{s', t'\} \in N'(s')$. $\qquad\square$

Proposition 10. $\mathcal{L}(\bullet) \not\preceq \mathcal{L}(\nabla)[\mathbb{C}]$, where $\mathbb{C} \in \{\mathbb{C}_n, \mathbb{C}_b\}$.

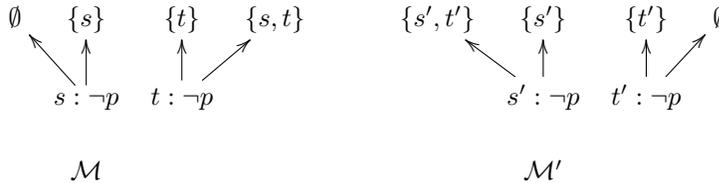*Proof.* Consider the following models, which come from [16, Prop. 3]:

It has been shown in [16, Prop. 3] that both $\mathcal{M}$ and $\mathcal{M}'$ satisfy (n) and (b), and $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ cannot be distinguished by $\mathcal{L}(\nabla)$.

However, both pointed models can be distinguished by $\mathcal{L}(\bullet)$. To see this, note that $p^{\mathcal{M}} = \{s\}$ and $\{s\} \notin N(s)$, and thus $\mathcal{M}, s \vDash \bullet p$, whereas $\mathcal{M}', s' \nvDash \bullet p$, as $p^{\mathcal{M}'} = \{s', t'\} \in N'(s')$. $\qquad\square$

Proposition 11. $\mathcal{L}(\bullet) \not\preceq \mathcal{L}(\nabla)[\mathbb{C}]$, where $\mathbb{C} \in \{\mathbb{C}_4, \mathbb{C}_5\}$.

Proof. Consider the following models:

Firstly, $\mathcal{M}$ and $\mathcal{M}'$ satisfy (4) and (5). In what follows we only show the claim for $\mathcal{M}$; the proof for the case $\mathcal{M}'$ is analogous.

- For (4): Suppose that $X \in N(s)$. Then $X = \emptyset$ or $X = \{s\}$. Notice that $\{u \mid X \in N(u)\} = \{s\} \in N(s)$. Similarly, we can demonstrate that (4) holds for $N(t)$.

- For (5): Assume that $X \notin N(s)$. Then $X = \{t\}$ or $X = \{s,t\}$. Notice that $\{u \mid X \notin N(u)\} = \{s\} \in N(s)$. A similar argument goes for $N(t)$.

Secondly, $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ cannot be distinguished by $\mathcal{L}(\nabla)$, that is to say, for all $\varphi \in \mathcal{L}(\nabla)$, we have that $\mathcal{M}, s \vDash \varphi$ iff $\mathcal{M}', s' \vDash \varphi$. The proof goes by induction on φ, where the only nontrivial case is $\nabla\varphi$. By semantics, we have the following equivalences:

$$
\begin{aligned}
& \mathcal{M}, s \vDash \nabla\varphi \\
\iff & \varphi^{\mathcal{M}} \notin N(s) \text{ and } (\neg\varphi)^{\mathcal{M}} \notin N(s) \\
\iff & \varphi^{\mathcal{M}} \notin \{\emptyset, \{s\}\} \text{ and } (\neg\varphi)^{\mathcal{M}} \notin \{\emptyset, \{s\}\} \\
\iff & \varphi^{\mathcal{M}} \neq \emptyset \text{ and } \varphi^{\mathcal{M}} \neq \{s\} \text{ and } (\neg\varphi)^{\mathcal{M}} \neq \emptyset \text{ and } (\neg\varphi)^{\mathcal{M}} \neq \{s\} \\
\iff & \varphi^{\mathcal{M}} \neq \emptyset \text{ and } \varphi^{\mathcal{M}} \neq \{s\} \text{ and } \varphi^{\mathcal{M}} \neq \{s,t\} \text{ and } \varphi^{\mathcal{M}} \neq \{t\} \\
\iff & \text{false}
\end{aligned}
$$

$$
\begin{aligned}
& \mathcal{M}', s' \vDash \nabla\varphi \\
\iff & \varphi^{\mathcal{M}'} \notin N'(s') \text{ and } (\neg\varphi)^{\mathcal{M}'} \notin N'(s') \\
\iff & \varphi^{\mathcal{M}'} \notin \{\{s', t'\}, \{s'\}\} \text{ and } (\neg\varphi)^{\mathcal{M}'} \notin \{\{s', t'\}, \{s'\}\} \\
\iff & \varphi^{\mathcal{M}'} \neq \{s', t'\} \text{ and } \varphi^{\mathcal{M}'} \neq \{s'\} \text{ and } (\neg\varphi)^{\mathcal{M}'} \neq \{s', t'\} \text{ and } (\neg\varphi)^{\mathcal{M}'} \neq \{s'\} \\
\iff & \varphi^{\mathcal{M}'} \neq \{s', t'\} \text{ and } \varphi^{\mathcal{M}'} \neq \{s'\} \text{ and } \varphi^{\mathcal{M}'} \neq \emptyset \text{ and } \varphi^{\mathcal{M}'} \neq \{t'\} \\
\iff & \text{false}
\end{aligned}
$$

In either case, the penultimate line of the proof merely states that φ cannot be interpreted on the related model: its denotation is *not* one of all possible subsets of the domain. We conclude that $\mathcal{M}, s \vDash \nabla\varphi$ iff $\mathcal{M}', s' \vDash \nabla\varphi$.

Finally, we show that $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ can be distinguished by $\mathcal{L}(\bullet)$. To see this, note that $\mathcal{M}, s \vDash \neg p$ and $(\neg p)^{\mathcal{M}} = \{s, t\} \notin N(s)$, and thus $\mathcal{M}, s \vDash \bullet\neg p$. However, since $(\neg p)^{\mathcal{M}'} = \{s', t'\} \in N'(s')$, we have $\mathcal{M}, s \nvDash \bullet\neg p$. $\qquad\square$

Remark 12. The reader may ask whether the figure in [16, Prop. 4] (as below), which is used to demonstrate that $\mathcal{L}(\nabla)$ is less expressive than $\mathcal{L}(\square)$ over $\mathbb{C}$ for $\mathbb{C} \in \{\mathbb{C}_4, \mathbb{C}_5\}$, applies to the above proposition.
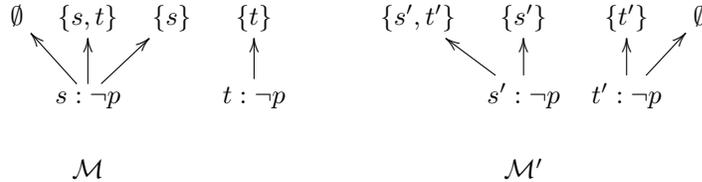
The answer is negative. This is because the pointed models $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ in this figure cannot be distinguished by $\mathcal{L}(\bullet)$ either. To see this, note that $\mathcal{M}, s \vDash \bullet\varphi$ iff $\mathcal{M}, s \vDash \varphi$ and $\varphi^{\mathcal{M}} \notin N(s)$, which by the construction of $N(s)$ implies that $s \in \varphi^{\mathcal{M}}$ and $\varphi^{\mathcal{M}} \neq \{s\}$ and $\varphi^{\mathcal{M}} \neq \{s, t\}$, which is impossible. It then follows that $\mathcal{M}, s \nvDash \bullet\varphi$. A similar argument can show that $\mathcal{M}', s' \nvDash \bullet\varphi$. Therefore, $\mathcal{M}, s \vDash \bullet\varphi$ iff $\mathcal{M}', s' \vDash \bullet\varphi$.

Proposition 13. $\mathcal{L}(\bullet) \preceq \mathcal{L}(\nabla)[\mathbb{C}_c]$.

Proof. It suffices to show that $\bullet\varphi \leftrightarrow (\varphi \wedge \nabla\varphi)$ is valid over the class of (c)-models. Let $\mathcal{M} = \langle S, N, V\rangle$ be a (c)-model and $s \in S$. Suppose that $\mathcal{M}, s \vDash \bullet\varphi$, it remains only to prove that $\mathcal{M}, s \vDash \varphi \wedge \nabla\varphi$. By supposition, we have $\mathcal{M}, s \vDash \varphi$ and $\varphi^{\mathcal{M}} \notin N(s)$. We have also $S\backslash\varphi^{\mathcal{M}} \notin N(s)$: otherwise, by (c), $S\backslash(S\backslash\varphi^{\mathcal{M}}) \in N(s)$, that is, $\varphi^{\mathcal{M}} \in N(s)$: a contradiction. Thus $\mathcal{M}, s \vDash \nabla\varphi$, and therefore $\mathcal{M}, s \vDash \varphi \wedge \nabla\varphi$. The converse is clear from the semantics. $\square$

Proposition 14. $\mathcal{L}(\bullet) \preceq \mathcal{L}(\nabla)[\mathbb{C}_t]$.

Proof. It suffices to show that $\bullet\varphi \leftrightarrow (\varphi \wedge \nabla\varphi)$ over the class of (t)-models. The proof is almost the same as that in Prop. 13, except that $S\backslash\varphi^{\mathcal{M}} \notin N(s)$ (that is, $(\neg\varphi)^{\mathcal{M}} \notin N(s)$) is obtained from $\mathcal{M}, s \vDash \varphi$ and the property (t). $\square$

Conversely, on the class of (c)-models and the class of (t)-models, $\mathcal{L}(\bullet)$ is at least as expressive as $\mathcal{L}(\nabla)$ (Prop. 18 and Prop. 19), whereas on the class of models possessing either of the other eight neighborhood properties, $\mathcal{L}(\bullet)$ is *not* at least as expressive as $\mathcal{L}(\nabla)$ (Prop. 15-Prop. 17). As a corollary, on the class of (c)-models and the class of (t)-models, $\mathcal{L}(\nabla)$, $\mathcal{L}(\bullet)$, and $\mathcal{L}(\nabla, \bullet)$ are equally expressive, whereas over the class of models possessing the eight neighborhood properties in question, $\mathcal{L}(\nabla)$ and $\mathcal{L}(\bullet)$ are both less expressive than $\mathcal{L}(\nabla, \bullet)$ (Coro. 20).

Proposition 15. $\mathcal{L}(\nabla) \not\preceq \mathcal{L}(\bullet)[\mathbb{C}]$, where $\mathbb{C} \in \{\mathbb{C}_{\text{all}}, \mathbb{C}_n, \mathbb{C}_r, \mathbb{C}_i, \mathbb{C}_s, \mathbb{C}_d, \mathbb{C}_b\}$.

Proof. Consider the following models:

It is straightforward to check that both $\mathcal{M}$ and $\mathcal{M}'$ satisfy (n), (r), (i), (s), and (d). In what follows, we show that $\mathcal{M}$ and $\mathcal{M}'$ both have the property (b).

- For $\mathcal{M}$: suppose that $s \in X$. Then $X = \{s\}$ or $X = \{s, t\}$. This implies that $\{u \mid S\backslash X \notin N(u)\} = \{s, t\} \in N(s)$. Similarly, we can show that (b) holds for $N(t)$.

- For $\mathcal{M}'$: assume that $s' \in X$. Then $X = \{s'\}$ or $X = \{s', t'\}$. If $X = \{s'\}$, then $\{u \mid S'\backslash X \notin N'(u)\} = \{t'\} \in N'(s')$; if $X = \{s', t'\}$, then $\{u \mid S'\backslash X \notin N'(u)\} = \{s', t'\} \in N'(s')$. Now assume that $t' \in X$. Then $X = \{t'\}$ or $X = \{s', t'\}$. If $X = \{t'\}$, then $\{u \mid S'\backslash X \notin N'(u)\} = \{s', t'\} \in N'(t')$; if $X = \{s', t'\}$, we can also show that $\{u \mid S'\backslash X \notin N'(u)\} = \{s', t'\} \in N'(t')$.
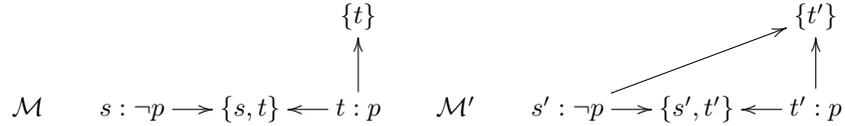
Moreover, $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ cannot be distinguished by $\mathcal{L}(\bullet)$. Here we use the

notion of •-morphisms introduced in [15, Def. 4.1].[9] Define a function $f : S \to S'$ such that $f(s) = s'$ and $f(t) = t'$. We prove that f is a •-morphism from $\mathcal{M}$ to $\mathcal{M}'$. The condition (Var) follows directly from the valuations. For the condition (•-Mor), we first prove that it holds for s: assume that $s \in f^{-1}[X']$ and $f^{-1}[X'] \notin N(s)$, then it must be that $X' = \{s'\}$. Then we have $f(s) = s' \in X'$ and $X' \notin N'(f(s))$. The converse is similar. In a similar way, we can show that (•-Mor) also holds for t. Then by [15, Prop. 4.1] (see also Footnote 9), we have $\mathcal{M}, s \vDash \varphi$ iff $\mathcal{M}', s' \vDash \varphi$ for all $\varphi \in \mathcal{L}(\bullet)$.

However, these pointed models can be distinguished by $\mathcal{L}(\nabla)$. This is because $\mathcal{M}, s \vDash \nabla p$ (as $p^{\mathcal{M}} = \{t\} \notin N(s)$ and $(\neg p)^{\mathcal{M}} = \{s\} \notin N(s)$) and $\mathcal{M}', s' \nvDash \nabla p$ (as $p^{\mathcal{M}'} = \{t'\} \in N'(s')$). $\qquad\square$

Proposition 16. $\mathcal{L}(\nabla) \npreceq \mathcal{L}(\bullet)[\mathbb{C}_4]$.

Proof. Consider the following models:

$$
\begin{array}{cccc}
& \{t\} & & \{t'\} \\
& \uparrow & \nearrow & \uparrow \\
\mathcal{M} \quad s : \neg p \longrightarrow \{s,t\} \longleftarrow t : p & & \mathcal{M}' \quad s' : \neg p \longrightarrow \{s',t'\} \longleftarrow t' : p
\end{array}
$$

Firstly, both $\mathcal{M}$ and $\mathcal{M}'$ have (4).

- For $\mathcal{M}$: Suppose that $X \in N(s)$. Then $X = \{s,t\}$, and so $\{u \mid X \in N(u)\} = \{s,t\} \in N(s)$. Now assume that $X \in N(t)$. Then $X = \{t\}$ or $X = \{s,t\}$. If $X = \{t\}$, then $\{u \mid X \in N(u)\} = \{t\} \in N(t)$; if $X = \{s,t\}$, then $\{u \mid X \in N(u)\} = \{s,t\} \in N(t)$.

- For $\mathcal{M}'$: Suppose that $X \in N'(s')$. Then $X = \{t'\}$ or $X = \{s',t'\}$. Either case implies that $\{u \mid X \in N'(u)\} = \{s',t'\} \in N'(s')$. Now assume that $X \in N'(t')$. Then $X = \{t'\}$ or $X = \{s',t'\}$. Again, either case implies that $\{u \mid X \in N'(u)\} = \{s',t'\} \in N'(t')$.

Secondly, similar to the proof of the corresponding part in Prop. 15, we can show that $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ cannot be distinguished by $\mathcal{L}(\bullet)$.

It remains only to show that $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ can be distinguished by $\mathcal{L}(\nabla)$. The proof for this is analogous to that in Prop. 15. $\qquad\square$

Proposition 17. $\mathcal{L}(\nabla) \npreceq \mathcal{L}(\bullet)[\mathbb{C}_5]$.

---

[9]The notion of •-morphisms is defined as follows. Let $\mathcal{M} = \langle S, N, V \rangle$ and $\mathcal{M}' = \langle S', N', V' \rangle$ be neighborhood models. A function $f : S \to S'$ is a •-morphism from $\mathcal{M}$ to $\mathcal{M}'$, if for all $s \in S$,

(Var) $s \in V(p)$ iff $f(s) \in V'(p)$ for all $p \in \mathbf{P}$,

(•-Mor) for all $X' \subseteq S'$, $[s \in f^{-1}[X']$ and $f^{-1}[X'] \notin N(s)] \Longleftrightarrow [f(s) \in X'$ and $X' \notin N'(f(s))]$.

It is then demonstrated in [15, Prop. 4.1] that the formulas of $\mathcal{L}(\bullet)$ are invariant under •-morphisms. In detail, let $\mathcal{M}$ and $\mathcal{M}'$ be neighborhood models, and let f be a •-morphism from $\mathcal{M}$ to $\mathcal{M}'$. Then for all $s \in S$, for all $\varphi \in \mathcal{L}(\bullet)$, we have that $\mathcal{M}, s \vDash \varphi$ iff $\mathcal{M}', f(s) \vDash \varphi$.

Proof. Consider the following models:

Firstly, both $\mathcal{M}$ and $\mathcal{M}'$ possess the property (5). Since for all $X \subseteq S = \{s,t\}$, $X \in N(t)$, the property (5) is possessed vacuously by $N(t)$. Similarly, (5) is also possessed vacuously by $N'(t')$. It remains only to show that both $N(s)$ and $N'(s')$ also have (5).

- For $N(s)$: suppose that $X \notin N(s)$, then $X = \emptyset$ or $X = \{s,t\}$. Either case implies that $\{u \in S \mid X \notin N(u)\} = \{s\} \in N(s)$.

- For $N'(s')$: assume that $X \notin N'(s')$, then $X = \{s',t'\}$. This follows that $\{u \in S' \mid X \notin N'(u)\} = \{s'\} \in N'(s')$.

Secondly, $(\mathcal{M}, s)$ and $(\mathcal{M}', s)$ cannot be distinguished by $\mathcal{L}(\bullet)$. Again, this can be shown as the corresponding part in Prop. 15.

Finally, $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ can be distinguished by $\mathcal{L}(\nabla)$. On one hand, $p^{\mathcal{M}} = \{s,t\} \notin N(s)$ and $S \backslash p^{\mathcal{M}} = \emptyset \notin N(s)$, thus $\mathcal{M}, s \vDash \nabla p$. On the other hand, $S' \backslash p^{\mathcal{M}'} = \emptyset \in N'(s')$, thus $\mathcal{M}', s' \nvDash \nabla p$. $\qquad\square$

Proposition 18. $\mathcal{L}(\nabla) \preceq \mathcal{L}(\bullet)[\mathbb{C}_c]$.

Proof. We show that over the class of (c)-models, $\vDash \nabla\varphi \leftrightarrow \bullet\varphi \vee \bullet\neg\varphi$. By Prop. 6, $\vDash \nabla\varphi \rightarrow \bullet\varphi \vee \bullet\neg\varphi$. It suffices to prove that $\mathbb{C}_c \vDash \bullet\varphi \vee \bullet\neg\varphi \rightarrow \nabla\varphi$.

Suppose that $\mathcal{M} = \langle S, N, V \rangle$ be a (c)-model and $s \in S$. Assume that $\mathcal{M}, s \vDash \bullet\varphi \vee \bullet\neg\varphi$. Then $\mathcal{M}, s \vDash \bullet\varphi$ or $\mathcal{M}, s \vDash \bullet\neg\varphi$. If $\mathcal{M}, s \vDash \bullet\varphi$, then $\varphi^{\mathcal{M}} \notin N(s)$. By (c), we have $S \backslash \varphi^{\mathcal{M}} \notin N(s)$, so $\mathcal{M}, s \vDash \nabla\varphi$. If $\mathcal{M}, s \vDash \bullet\neg\varphi$, then $(\neg\varphi)^{\mathcal{M}} \notin N(s)$, namely $S \backslash \varphi^{\mathcal{M}} \notin N(s)$. By (c) again, we obtain $\varphi^{\mathcal{M}} \notin N(s)$, and thus $\mathcal{M}, s \vDash \nabla\varphi$. Therefore, $\mathcal{M}, s \vDash \nabla\varphi$. $\qquad\square$

Proposition 19. $\mathcal{L}(\nabla) \preceq \mathcal{L}(\bullet)[\mathbb{C}_t]$.

Proof. We claim that over the class of (t)-models, $\vDash \nabla\varphi \leftrightarrow \bullet\varphi \vee \bullet\neg\varphi$. The proof is almost the same as that in Prop. 18, except that $\mathbb{C}_t \vDash \bullet\varphi \vee \bullet\neg\varphi \rightarrow \nabla\varphi$ is obtained as follows: if $\mathcal{M}, s \vDash \bullet\varphi$, then $\mathcal{M}, s \vDash \varphi$ and $\varphi^{\mathcal{M}} \notin N(s)$, thus $\mathcal{M}, s \nvDash \neg\varphi$, namely $s \notin (\neg\varphi)^{\mathcal{M}}$, and then by (t), we infer that $(\neg\varphi)^{\mathcal{M}} \notin N(s)$, namely $S \backslash \varphi^{\mathcal{M}} \notin N(s)$, so $\mathcal{M}, s \vDash \nabla\varphi$; similarly, we can show that if $\mathcal{M}, s \vDash \bullet\neg\varphi$ then $\mathcal{M}, s \vDash \nabla\varphi$. $\qquad\square$

With the above results in mind, we have the following result, which extends the expressivity results over Kripke models in [11].

Corollary 20. Where $\mathbb{C} \in \{\mathbb{C}_{\text{all}}, \mathbb{C}_n, \mathbb{C}_r, \mathbb{C}_i, \mathbb{C}_s, \mathbb{C}_d, \mathbb{C}_b, \mathbb{C}_4, \mathbb{C}_5\}$, $\mathcal{L}(\nabla) \asymp \mathcal{L}(\bullet)[\mathbb{C}]$, and $\mathcal{L}_1 \prec \mathcal{L}(\nabla, \bullet)[\mathbb{C}]$, where $\mathcal{L}_1 \in \{\mathcal{L}(\nabla), \mathcal{L}(\bullet)\}$. Where $\mathbb{C} \in \{\mathbb{C}_c, \mathbb{C}_t\}$, $\mathcal{L}_1 \equiv \mathcal{L}_2[\mathbb{C}]$, where $\mathcal{L}_1, \mathcal{L}_2 \in \{\mathcal{L}(\nabla), \mathcal{L}(\bullet), \mathcal{L}(\nabla, \bullet)\}$.

Moreover, over the class of (c)-models and the class of (t)-models, $\mathcal{L}(\nabla, \bullet)$ and $\mathcal{L}(\square)$ are equally expressive (Prop. 23), whereas over the class of models possessing either of the other eight neighborhood properties except for (d), $\mathcal{L}(\nabla, \bullet)$ is less expressive than $\mathcal{L}(\square)$ (Prop. 21 and Prop. 22).

Proposition 21. $\mathcal{L}(\nabla, \bullet) \prec \mathcal{L}(\square)[\mathbb{C}]$, where $\mathbb{C} \in \{\mathbb{C}_{\text{all}}, \mathbb{C}_r, \mathbb{C}_i, \mathbb{C}_4, \mathbb{C}_5\}$.

Proof. Use Remark 12 and [16, Prop. 4]. First, [16, Prop. 4] has shown that $\mathcal{M}$ and $\mathcal{M}'$ in Remark 12 satisfy (4) and (5). Also, one may easily verify that these two models possess (r) and (i). Moreover, Remark 12 shows that $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ cannot be distinguished by $\mathcal{L}(\bullet)$, whereas [16, Prop. 4] shows that $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ cannot be distinguished by $\mathcal{L}(\nabla)$ but can be distinguished by $\mathcal{L}(\square)$. □

Proposition 22. $\mathcal{L}(\nabla, \bullet) \prec \mathcal{L}(\square)[\mathbb{C}]$, where $\mathbb{C} \in \{\mathbb{C}_n, \mathbb{C}_s, \mathbb{C}_b\}$.

Proof. Consider the following models $\mathcal{M} = \langle S, N, V \rangle$ and $\mathcal{M}' = \langle S', N', V' \rangle$, where $S = \{s, t\}$ and $S' = \{s', t'\}$.
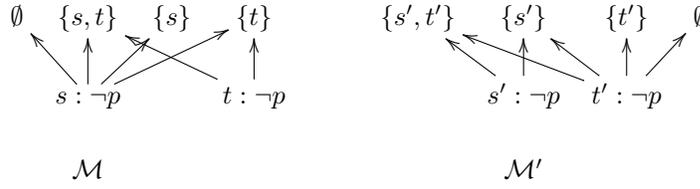
It should be straightforward to check that both $\mathcal{M}$ and $\mathcal{M}'$ possess (n) and (s). Moreover, both models also possess (b), shown as follows. Since for all $X \subseteq S$, we have $X \in N(s)$, one may easily see that $N(s)$ has (b). Similarly, we can show that $N'(t')$ has (b). Besides,

- $N(t)$ has (b): suppose that $t \in X$, then $X = \{t\}$ or $X = \{s, t\}$. The first case implies $\{u \in S \mid S \backslash X \notin N(u)\} = \{u \in S \mid \{s\} \notin N(u)\} = \{t\} \in N(t)$, and the second case implies that $\{u \in S \mid S \backslash X \notin N(u)\} = \{u \in S \mid \emptyset \notin N(u)\} = \{t\} \in N(t)$, as desired.

- $N'(s')$ has (b): assume that $s' \in X$, then $X = \{s'\}$ or $X = \{s', t'\}$. The first case entails that $\{u \in S' \mid S' \backslash X \notin N'(u)\} = \{u \in S' \mid \{t'\} \notin N'(u)\} = \{s'\} \in N'(s')$, and the second case entails that $\{u \in S' \mid S' \backslash X \notin N'(u)\} = \{u \in S' \mid \emptyset \notin N'(u)\} = \{s'\} \in N'(s')$, as desired.

Next, we show $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ cannot be distinguished by $\mathcal{L}(\nabla, \bullet)$. That is, for all $\varphi \in \mathcal{L}(\nabla, \bullet)$, we have $\mathcal{M}, s \vDash \varphi$ iff $\mathcal{M}', s' \vDash \varphi$. The proof proceeds by induction on φ, where the nontrivial cases are $\nabla\varphi$ and $\bullet\varphi$. The proof for the case $\bullet\varphi$ is shown as in Remark 12. For the case $\nabla\varphi$, we have the following equivalences.

$$
\begin{aligned}
& \mathcal{M}, s \vDash \nabla\varphi \\
\Longleftrightarrow \quad & \varphi^{\mathcal{M}} \notin N(s) \text{ and } S \backslash \varphi^{\mathcal{M}} \notin N(s) \\
\Longleftrightarrow \quad & \varphi^{\mathcal{M}} \neq \emptyset \text{ and } \varphi^{\mathcal{M}} \neq \{s, t\} \text{ and } \varphi^{\mathcal{M}} \neq \{s\} \text{ and } \varphi^{\mathcal{M}} \neq \{t\} \\
\Longleftrightarrow \quad & \text{false}
\end{aligned}
$$

$$\mathcal{M}', s' \vDash \nabla \varphi$$
$$\iff \quad \varphi^{\mathcal{M}'} \notin N'(s') \text{ and } S' \backslash \varphi^{\mathcal{M}'} \notin N'(s')$$
$$\iff \quad \varphi^{\mathcal{M}'} \notin \{\{s', t'\}, \{s'\}\} \text{ and } S \backslash \varphi^{\mathcal{M}'} \notin \{\{s', t'\}, \{s'\}\}$$
$$\iff \quad \varphi^{\mathcal{M}'} \neq \{s', t'\} \text{ and } \varphi^{\mathcal{M}'} \neq \{s'\} \text{ and } \varphi^{\mathcal{M}'} \neq \emptyset \text{ and } \varphi^{\mathcal{M}'} \neq \{t'\}$$
$$\iff \quad \text{false}$$

In either case, the penultimate line of the equivalences states that φ cannot be interpreted on the related model: its denotation is not one of all possible subsets of the domain. We therefore conclude that $\mathcal{M}, s \vDash \nabla \varphi$ iff $\mathcal{M}', s' \vDash \nabla \varphi$.

Finally, $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ can be distinguished by $\mathcal{L}(\Diamond)$. To see this, note that $p^{\mathcal{M}} = \emptyset \in N(s)$, thus $\mathcal{M}, s \vDash \Box p$; however, $p^{\mathcal{M}'} = \emptyset \notin N'(s')$, and thus $\mathcal{M}', s' \nvDash \Box p$. $\qquad \square$

Proposition 23. $\mathcal{L}(\nabla, \bullet) \equiv \mathcal{L}(\Box)[\mathbb{C}]$, where $\mathbb{C} \in \{\mathbb{C}_c, \mathbb{C}_t\}$.

Proof. Straightforward from $\mathcal{L}(\nabla, \bullet) \equiv \mathcal{L}(\nabla)[\mathbb{C}]$ (see Coro. 20) and $\mathcal{L}(\nabla) \equiv \mathcal{L}(\Box)[\mathbb{C}]$ (see [16, Prop. 5, Prop. 6]), where $\mathbb{C} \in \{\mathbb{C}_c, \mathbb{C}_t\}$. $\qquad \square$

We do not know whether $\mathcal{L}(\nabla, \bullet)$ is less expressive than $\mathcal{L}(\Box)$ over the class of (d)-models. We conjecture the answer is positive. We leave it for future work.

We summarize the results in this section as follows.

$\mathcal{L}(\nabla) \asymp \mathcal{L}(\bullet)[\mathbb{C}]$, where $\mathbb{C} \in \{\mathbb{C}_{\text{all}}, \mathbb{C}_n, \mathbb{C}_r, \mathbb{C}_i, \mathbb{C}_s, \mathbb{C}_d, \mathbb{C}_b, \mathbb{C}_4, \mathbb{C}_5\}$	(Coro. 20)
$\mathcal{L}(\nabla) \equiv \mathcal{L}(\bullet)[\mathbb{C}]$, where $\mathbb{C} \in \{\mathbb{C}_c, \mathbb{C}_t\}$	(Coro. 20)
$\mathcal{L}(\nabla) \prec \mathcal{L}(\nabla, \bullet)[\mathbb{C}]$, where $\mathbb{C} \in \{\mathbb{C}_{\text{all}}, \mathbb{C}_n, \mathbb{C}_r, \mathbb{C}_i, \mathbb{C}_s, \mathbb{C}_d, \mathbb{C}_b, \mathbb{C}_4, \mathbb{C}_5\}$	(Coro. 20)
$\mathcal{L}(\bullet) \prec \mathcal{L}(\nabla, \bullet)[\mathbb{C}]$, where $\mathbb{C} \in \{\mathbb{C}_{\text{all}}, \mathbb{C}_n, \mathbb{C}_r, \mathbb{C}_i, \mathbb{C}_s, \mathbb{C}_d, \mathbb{C}_b, \mathbb{C}_4, \mathbb{C}_5\}$	(Coro. 20)
$\mathcal{L}(\nabla) \equiv \mathcal{L}(\nabla, \bullet)[\mathbb{C}]$, where $\mathbb{C} \in \{\mathbb{C}_c, \mathbb{C}_t\}$	(Coro. 20)
$\mathcal{L}(\bullet) \equiv \mathcal{L}(\nabla, \bullet)[\mathbb{C}]$, where $\mathbb{C} \in \{\mathbb{C}_c, \mathbb{C}_t\}$	(Coro. 20)
$\mathcal{L}(\nabla, \bullet) \prec \mathcal{L}(\Box)[\mathbb{C}]$, where $\mathbb{C} \in \{\mathbb{C}_{\text{all}}, \mathbb{C}_n, \mathbb{C}_r, \mathbb{C}_i, \mathbb{C}_s, \mathbb{C}_b, \mathbb{C}_4, \mathbb{C}_5\}$	(Props. 21, 22)
$\mathcal{L}(\nabla, \bullet) \equiv \mathcal{L}(\Box)[\mathbb{C}]$, where $\mathbb{C} \in \{\mathbb{C}_c, \mathbb{C}_t\}$	(Coro. 23)

4 Frame Definability

We have shown in the previous section that $\mathcal{L}(\nabla, \bullet)$ is more expressive than $\mathcal{L}(\nabla)$ and $\mathcal{L}(\bullet)$ (at the level of models). It may then be natural to ask whether a similar situation holds at the level of frames. It is shown in [16, Prop. 7] that all frame properties in Def. 3, in particular (n), are undefinable in $\mathcal{L}(\nabla)$. In what follows, we shall show that all frame properties in question except for (n) are undefinable in $\mathcal{L}(\nabla, \bullet)$, thus $\mathcal{L}(\nabla, \bullet)$ is also more expressive than $\mathcal{L}(\nabla)$ and $\mathcal{L}(\bullet)$ at the level of frames. First, we need some related notion.

Definition 24. Let Γ be a set of $\mathcal{L}(\nabla, \bullet)$-formulas, and P a neighborhood property. We say that Γ defines P, if for all frames $\mathcal{F}$, $\mathcal{F}$ has P if and only if $\mathcal{F} \vDash \Gamma$. If Γ is a singleton, say $\{\varphi\}$, we will write $\mathcal{F} \vDash \varphi$ rather than $\mathcal{F} \vDash \{\varphi\}$. We say that P is definable in $\mathcal{L}(\nabla, \bullet)$, if there exists a set of $\mathcal{L}(\nabla, \bullet)$-formulas that defines it.

Proposition 25. The frame property (n) is definable in $\mathcal{L}(\nabla, \bullet)$.

Proof. [15] has shown that (n) is defined in $\mathcal{L}(\bullet)$, by $\circ\top$. Therefore, (n) is also definable in $\mathcal{L}(\nabla, \bullet)$, by $\circ\top$. $\square$

Proposition 26. The frame properties (r), (i), (c), (d), (t) and (b) are undefinable in $\mathcal{L}(\nabla, \bullet)$.

Proof. Consider the following frames $\mathcal{F}_1 = \langle S_1, N_1 \rangle$, $\mathcal{F}_2 = \langle S_2, N_2 \rangle$, and $\mathcal{F}_3 = \langle S_3, N_3 \rangle$[10]:
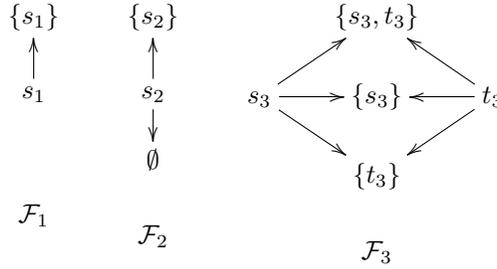
It has been observed in [16, Prop. 7] that $\mathcal{F}_1$ satisfies (d) and (t) but $\mathcal{F}_2$ does not. Also, it is straightforward to check that $\mathcal{F}_2$ satisfies (c) but $\mathcal{F}_1$ does not. Moreover, $\mathcal{F}_2$ satisfies (r), (i) and (b), whereas $\mathcal{F}_3$ does not. To see $\mathcal{F}_3$ does not satisfy (b), note that $s_3 \in \{s_3\}$ but $\{u \in S_3 \mid \{t_3\} \notin N_3(u)\} = \emptyset \notin N_3(s_3)$. In what follows, we show that for all $\varphi \in \mathcal{L}(\nabla, \bullet)$, $\mathcal{F}_1 \vDash \varphi$ iff $\mathcal{F}_2 \vDash \varphi$ iff $\mathcal{F}_3 \vDash \varphi$.

Suppose that $\mathcal{F}_1 \nvDash \varphi$. Then there exists $\mathcal{M}_1 = \langle \mathcal{F}_1, V_1 \rangle$ such that $\mathcal{M}_1, s_1 \nvDash \varphi$. Define a valuation V_2 on $\mathcal{F}_2$ as $s_2 \in V_2(p)$ iff $s_1 \in V_1(p)$ for all $p \in \mathbf{P}$. By induction on φ, we show that $(*)$: $\mathcal{M}_1, s_1 \vDash \varphi$ iff $\mathcal{M}_2, s_2 \vDash \varphi$, where $\mathcal{M}_2 = \langle \mathcal{F}_2, V_2 \rangle$. The nontrivial cases are $\nabla\varphi$ and $\bullet\varphi$. The case $\nabla\varphi$ can be shown as in [16, Prop. 7]. For the case $\bullet\varphi$, notice that $\mathcal{M}_1, s_1 \vDash \bullet\varphi$ iff ($\mathcal{M}_1, s_1 \vDash \varphi$ and $\varphi^{\mathcal{M}_1} \notin N_1(s_1)$) iff ($\mathcal{M}_1, s_1 \vDash \varphi$ and $\varphi^{\mathcal{M}_1} \neq \{s_1\}$), where the last one is a contradiction, and thus $\mathcal{M}_1, s_1 \nvDash \bullet\varphi$; a similar argument gives us $\mathcal{M}_2, s_2 \nvDash \bullet\varphi$. We have thus proved $(*)$. This entails that $\mathcal{M}_2, s_2 \nvDash \varphi$, and thus $\mathcal{F}_2 \nvDash \varphi$. The converse is similar. Therefore, $\mathcal{F}_1 \vDash \varphi$ iff $\mathcal{F}_2 \vDash \varphi$.

It remains only to show that $\mathcal{F}_2 \vDash \varphi$ iff $\mathcal{F}_3 \vDash \varphi$. Assume that $\mathcal{F}_2 \nvDash \varphi$. Then there exists $\mathcal{M}_2 = \langle \mathcal{F}_2, V_2 \rangle$ such that $\mathcal{M}_2, s_2 \nvDash \varphi$. Define a valuation V_3 on $\mathcal{F}_3$ such that $s_3 \in V_3(p)$ iff $s_2 \in V_2(p)$ for all $p \in \mathbf{P}$. By induction on $\varphi \in \mathcal{L}(\nabla, \bullet)$, we show that $(**)$: $\mathcal{M}_2, s_2 \vDash \varphi$ iff $\mathcal{M}_3, s_3 \vDash \varphi$, where $\mathcal{M}_3 = \langle \mathcal{F}_3, V_3 \rangle$. The nontrivial cases are $\nabla\varphi$ and $\bullet\varphi$. Again, the case $\nabla\varphi$ can be shown as in [16, Prop. 7]. For the case $\bullet\varphi$, just note that $\mathcal{M}_3, s_3 \vDash \bullet\varphi$ iff ($\mathcal{M}_3, s_3 \vDash \varphi$ and $\varphi^{\mathcal{M}_3} \notin N_3(s_3)$) iff ($\mathcal{M}_3, s_3 \vDash \varphi$ and $\varphi^{\mathcal{M}_3} \neq \{s_3\}$ and $\varphi^{\mathcal{M}_3} \neq \{t_3\}$ and $\varphi^{\mathcal{M}_3} \neq \{s_3, t_3\}$) iff false. Thus $(**)$ holds. This implies that $\mathcal{M}_3, s_3 \nvDash \varphi$, and then $\mathcal{F}_3 \nvDash \varphi$. The converse is analogous. Therefore, $\mathcal{F}_2 \vDash \varphi$ iff $\mathcal{F}_3 \vDash \varphi$.

If (r) were to be defined by a set of $\mathcal{L}(\nabla, \bullet)$-formulas, say Σ, then as $\mathcal{F}_2$ satisfies (r), we have $\mathcal{F}_2 \vDash \Sigma$. Then we should also have $\mathcal{F}_3 \vDash \Sigma$, which means that $\mathcal{F}_3$ has (r): a contradiction. Therefore, (r) is undefinable in $\mathcal{L}(\nabla, \bullet)$. Similarly, we can show other frame properties in question are undefinable in $\mathcal{L}(\nabla, \bullet)$. $\square$

---

[10]These frames come from [16, Prop. 7], where with an extra frame they are together used to show that all ten neighborhood properties described in Def. 3 are undefinable in $\mathcal{L}(\nabla)$.

Proposition 27. The frame properties (s) and (4) are undefinable in $\mathcal{L}(\nabla, \bullet)$.

Proof. Consider the following frames $\mathcal{F} = \langle S, N \rangle$ and $\mathcal{F}' = \langle S', N' \rangle$, where $S = \{s, t\}$ and $S' = \{s', t'\}$:
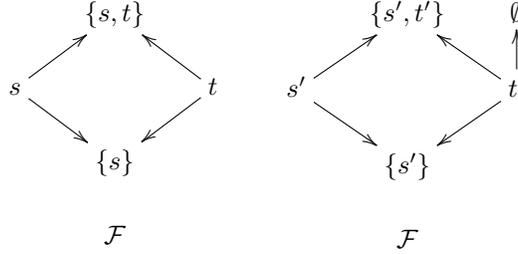
Firstly, one may easily see that $\mathcal{F}$ has (s). Also, $\mathcal{F}$ has (4). Suppose that $X \in N(s)$, to show that $\{u \in S \mid X \in N(u)\} \in N(s)$. By supposition, $X = \{s\}$ or $X = \{s, t\}$. Either case implies that $\{u \in S \mid X \in N(u)\} = \{s, t\} \in N(s)$. Thus $N(s)$ has (4). A similar argument applies to showing that $N(t)$ has (4).

Secondly, $\mathcal{F}'$ does not have (s), since $\emptyset \in N'(t')$ and $\emptyset \subseteq \{t'\}$ but $\{t'\} \notin N'(t')$. Moreover, $\mathcal{F}'$ does not have (4). This is because, for instance, $\emptyset \in N'(t')$ but $\{u \in S' \mid \emptyset \in N'(u)\} = \{t'\} \notin N'(t')$.

Thirdly, for all $\psi \in \mathcal{L}(\nabla, \bullet)$, we have that $\mathcal{F} \vDash \psi$ iff $\mathcal{F}' \vDash \psi$. Suppose that $\mathcal{F} \nvDash \psi$. Then there exists $\mathcal{M} = \langle \mathcal{F}, V \rangle$ and $x \in S$ such that $\mathcal{M}, x \nvDash \psi$. Define V' to be a valuation on $\mathcal{F}'$ such that for all $p \in \mathbf{P}$, $s \in V(p)$ iff $s' \in V'(p)$, and $t \in V(p)$ iff $t' \in V'(p)$. In what follows, we show $(*)$: for all $\varphi \in \mathcal{L}(\nabla, \bullet)$, $\mathcal{M}, s \vDash \varphi$ iff $\mathcal{M}', s' \vDash \varphi$, and $\mathcal{M}, t \vDash \varphi$ iff $\mathcal{M}', t' \vDash \varphi$, where $\mathcal{M}' = \langle \mathcal{F}', V' \rangle$. We proceed by induction on φ, where the nontrivial cases are $\nabla\varphi$ and $\bullet\varphi$.

For the case $\nabla\varphi$, we have the following equivalences.

$$
\begin{aligned}
&\mathcal{M}, s \vDash \nabla\varphi \\
\iff\quad & \varphi^{\mathcal{M}} \notin N(s) \text{ and } S \backslash \varphi^{\mathcal{M}} \notin N(s) \\
\iff\quad & \varphi^{\mathcal{M}} \notin \{\{s\}, \{s, t\}\} \text{ and } S \backslash \varphi^{\mathcal{M}} \notin \{\{s\}, \{s, t\}\} \\
\iff\quad & \varphi^{\mathcal{M}} \neq \{s\} \text{ and } \varphi^{\mathcal{M}} \neq \{s, t\} \text{ and } \varphi^{\mathcal{M}} \neq \{t\} \text{ and } \varphi^{\mathcal{M}} \neq \emptyset
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{M}', s' \vDash \nabla\varphi \\
\iff\quad & \varphi^{\mathcal{M}'} \notin N'(s') \text{ and } S' \backslash \varphi^{\mathcal{M}'} \notin N'(s') \\
\iff\quad & \varphi^{\mathcal{M}'} \notin \{\{s'\}, \{s', t'\}\} \text{ and } S' \backslash \varphi^{\mathcal{M}'} \notin \{\{s'\}, \{s', t'\}\} \\
\iff\quad & \varphi^{\mathcal{M}} \neq \{s'\} \text{ and } \varphi^{\mathcal{M}'} \neq \{s', t'\} \text{ and } \varphi^{\mathcal{M}'} \neq \{t'\} \text{ and } \varphi^{\mathcal{M}'} \neq \emptyset
\end{aligned}
$$

In each case, the last line of the above proofs states that φ cannot be interpreted on the related models, which is impossible. Thus $\mathcal{M}, s \nvDash \nabla\varphi$ and $\mathcal{M}', s' \nvDash \nabla\varphi$. Analogously, we can show that $\mathcal{M}, t \nvDash \nabla\varphi$ and $\mathcal{M}', t' \nvDash \nabla\varphi$.

For the case $\bullet\varphi$, we have the following equivalences.

$$
\begin{aligned}
&\mathcal{M}, s \vDash \bullet\varphi \\
\iff\quad & \mathcal{M}, s \vDash \varphi \text{ and } \varphi^{\mathcal{M}} \notin N(s) \\
\iff\quad & \mathcal{M}, s \vDash \varphi \text{ and } \varphi^{\mathcal{M}} \neq \{s\} \text{ and } \varphi^{\mathcal{M}} \neq \{s, t\} \\
\iff\quad & \text{false}
\end{aligned}
$$

$$\mathcal{M}', s' \vDash \bullet\varphi$$
$$\Longleftrightarrow \quad \mathcal{M}', s' \vDash \varphi \text{ and } \varphi^{\mathcal{M}'} \notin N'(s')$$
$$\Longleftrightarrow \quad \mathcal{M}', s' \vDash \varphi \text{ and } \varphi^{\mathcal{M}'} \neq \{s'\} \text{ and } \varphi^{\mathcal{M}'} \neq \{s', t'\}$$
$$\Longleftrightarrow \quad \text{false}$$

This shows that $\mathcal{M}, s \vDash \bullet\varphi$ iff $\mathcal{M}', s' \vDash \bullet\varphi$. Therefore, $\mathcal{M}, s \vDash \varphi$ iff $\mathcal{M}', s' \vDash \varphi$ for all $\varphi \in \mathcal{L}(\nabla, \bullet)$.

$$\mathcal{M}, t \vDash \bullet\varphi$$
$$\Longleftrightarrow \quad \mathcal{M}, t \vDash \varphi \text{ and } \varphi^{\mathcal{M}} \notin N(t)$$
$$\Longleftrightarrow \quad \mathcal{M}, t \vDash \varphi \text{ and } \varphi^{\mathcal{M}} \neq \{s\} \text{ and } \varphi^{\mathcal{M}} \neq \{s, t\}$$
$$\Longleftrightarrow \quad \mathcal{M}, t \vDash \varphi \text{ and } \mathcal{M}, s \nvDash \varphi$$
$$\stackrel{\text{IH}}{\Longleftrightarrow} \quad \mathcal{M}', t' \vDash \varphi \text{ and } \mathcal{M}', s' \nvDash \varphi$$
$$\Longleftrightarrow \quad \mathcal{M}', t' \vDash \varphi \text{ and } \varphi^{\mathcal{M}'} \neq \{s'\} \text{ and } \varphi^{\mathcal{M}'} \neq \{s', t'\} \text{ and } \varphi^{\mathcal{M}'} \neq \emptyset$$
$$\Longleftrightarrow \quad \mathcal{M}', t' \vDash \varphi \text{ and } \varphi^{\mathcal{M}'} \notin N'(t')$$
$$\Longleftrightarrow \quad \mathcal{M}', t' \vDash \bullet\varphi$$

This gives us that $\mathcal{M}, t \vDash \varphi$ iff $\mathcal{M}', t' \vDash \varphi$ for all $\varphi \in \mathcal{L}(\nabla, \bullet)$. We have now completed the proof of $(*)$.

If (s) were to be defined by a set of $\mathcal{L}(\nabla, \bullet)$-formulas, say Γ, then as $\mathcal{F}$ has (s), we would have $\mathcal{F} \vDash \Gamma$, thus we should also have $\mathcal{F}' \vDash \Gamma$, that is, $\mathcal{F}'$ has (s): a contradiction. Therefore, (s) is not definable in $\mathcal{L}(\nabla, \bullet)$. Similarly, we can obtain the undefinability of (4) in $\mathcal{L}(\nabla, \bullet)$. $\square$

Proposition 28. The frame property (5) is undefinable in $\mathcal{L}(\nabla, \bullet)$.

Proof. Consider the following frames $\mathcal{F} = \langle S, N \rangle$ and $\mathcal{F}' = \langle S', N' \rangle$, where $S = \{s, t\}$ and $S' = \{s', t'\}$:
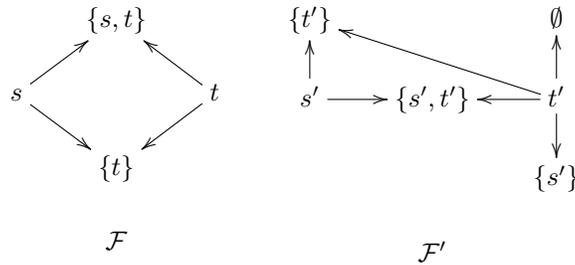
Firstly, $\mathcal{F}$ has (5). Suppose that $X \notin N(s)$, to prove that $\{u \in S \mid X \notin N(u)\} \in N(s)$. By supposition, $X = \emptyset$ or $X = \{s\}$. Either case implies that $\{u \in S \mid X \notin N(u)\} = \{s, t\} \in N(s)$. Thus $N(s)$ has (5). A similar argument shows that $N(t)$ has (5).

Secondly, $\mathcal{F}'$ does not satisfy (5). For instance, $\emptyset \notin N'(s')$ and $\{u \in S' \mid \emptyset \notin N'(u)\} = \{s'\} \notin N'(s')$.

Thirdly, for all $\psi \in \mathcal{L}(\nabla, \bullet)$, we have that $\mathcal{F} \vDash \psi$ iff $\mathcal{F}' \vDash \psi$. Suppose that $\mathcal{F} \nvDash \psi$. Then there exists $\mathcal{M} = \langle \mathcal{F}, V \rangle$ and $x \in S$ such that $\mathcal{M}, x \nvDash \psi$. Define V' to be a valuation on $\mathcal{F}'$ such that for all $p \in \mathbf{P}$, $s \in V(p)$ iff $s' \in V'(p)$, and $t \in V(p)$

iff $t' \in V'(p)$. In what follows, we show $(**)$: for all $\varphi \in \mathcal{L}(\nabla, \bullet)$, $\mathcal{M}, s \vDash \varphi$ iff $\mathcal{M}', s' \vDash \varphi$, and $\mathcal{M}, t \vDash \varphi$ iff $\mathcal{M}', t' \vDash \varphi$, where $\mathcal{M}' = \langle \mathcal{F}', V' \rangle$. We proceed by induction on φ, where the nontrivial cases are $\nabla\varphi$ and $\bullet\varphi$.

For the case $\nabla\varphi$, we have the following equivalences.

$$
\begin{aligned}
&\quad \mathcal{M}, s \vDash \nabla\varphi \\
\Longleftrightarrow\quad & \varphi^{\mathcal{M}} \notin N(s) \text{ and } S \backslash \varphi^{\mathcal{M}} \notin N(s) \\
\Longleftrightarrow\quad & \varphi^{\mathcal{M}} \notin \{\{t\}, \{s,t\}\} \text{ and } S \backslash \varphi^{\mathcal{M}} \notin \{\{t\}, \{s,t\}\} \\
\Longleftrightarrow\quad & \varphi^{\mathcal{M}} \neq \{t\} \text{ and } \varphi^{\mathcal{M}} \neq \{s,t\} \text{ and } \varphi^{\mathcal{M}} \neq \{s\} \text{ and } \varphi^{\mathcal{M}} \neq \emptyset
\end{aligned}
$$

$$
\begin{aligned}
&\quad \mathcal{M}', s' \vDash \nabla\varphi \\
\Longleftrightarrow\quad & \varphi^{\mathcal{M}'} \notin N'(s') \text{ and } S' \backslash \varphi^{\mathcal{M}'} \notin N'(s') \\
\Longleftrightarrow\quad & \varphi^{\mathcal{M}'} \notin \{\{t'\}, \{s',t'\}\} \text{ and } S' \backslash \varphi^{\mathcal{M}'} \notin \{\{t'\}, \{s',t'\}\} \\
\Longleftrightarrow\quad & \varphi^{\mathcal{M}'} \neq \{t'\} \text{ and } \varphi^{\mathcal{M}'} \neq \{s',t'\} \text{ and } \varphi^{\mathcal{M}'} \neq \{s'\} \text{ and } \varphi^{\mathcal{M}'} \neq \emptyset
\end{aligned}
$$

In each case, the last line of the above proofs states that φ cannot be interpreted on the related models, which is impossible. Thus $\mathcal{M}, s \nvDash \nabla\varphi$ and $\mathcal{M}', s' \nvDash \nabla\varphi$. Analogously, we can show that $\mathcal{M}, t \nvDash \nabla\varphi$ and $\mathcal{M}', t' \nvDash \nabla\varphi$.

For the case $\bullet\varphi$, we have the following equivalences.

$$
\begin{aligned}
&\quad \mathcal{M}, t \vDash \bullet\varphi \\
\Longleftrightarrow\quad & \mathcal{M}, t \vDash \varphi \text{ and } \varphi^{\mathcal{M}} \notin N(t) \\
\Longleftrightarrow\quad & \mathcal{M}, t \vDash \varphi \text{ and } \varphi^{\mathcal{M}} \neq \{t\} \text{ and } \varphi^{\mathcal{M}} \neq \{s,t\} \\
\Longleftrightarrow\quad & \text{false}
\end{aligned}
$$

$$
\begin{aligned}
&\quad \mathcal{M}', t' \vDash \bullet\varphi \\
\Longleftrightarrow\quad & \mathcal{M}', t' \vDash \varphi \text{ and } \varphi^{\mathcal{M}'} \notin N'(t') \\
\Longleftrightarrow\quad & \mathcal{M}', t' \vDash \varphi \text{ and } \varphi^{\mathcal{M}'} \neq \emptyset \text{ and } \varphi^{\mathcal{M}'} \neq \{s'\} \text{ and } \varphi^{\mathcal{M}'} \neq \{t'\} \text{ and } \varphi^{\mathcal{M}'} \neq \{s',t'\} \\
\Longleftrightarrow\quad & \text{false}
\end{aligned}
$$

Thus $\mathcal{M}, t \vDash \bullet\varphi$ iff $\mathcal{M}', t' \vDash \bullet\varphi$. Therefore, $\mathcal{M}, t \vDash \varphi$ iff $\mathcal{M}', t' \vDash \varphi$ for all $\varphi \in \mathcal{L}(\nabla, \bullet)$.

$$
\begin{aligned}
&\quad \mathcal{M}, s \vDash \bullet\varphi \\
\Longleftrightarrow\quad & \mathcal{M}, s \vDash \varphi \text{ and } \varphi^{\mathcal{M}} \notin N(s) \\
\Longleftrightarrow\quad & \mathcal{M}, s \vDash \varphi \text{ and } \varphi^{\mathcal{M}} \neq \{t\} \text{ and } \varphi^{\mathcal{M}} \neq \{s,t\} \\
\Longleftrightarrow\quad & \mathcal{M}, s \vDash \varphi \text{ and } \mathcal{M}, t \nvDash \varphi \\
\overset{\text{IH}}{\Longleftrightarrow}\quad & \mathcal{M}', s' \vDash \varphi \text{ and } \mathcal{M}', t' \nvDash \varphi \\
\Longleftrightarrow\quad & \mathcal{M}', s' \vDash \varphi \text{ and } \varphi^{\mathcal{M}'} \neq \{t'\} \text{ and } \varphi^{\mathcal{M}'} \neq \{s',t'\} \\
\Longleftrightarrow\quad & \mathcal{M}', s' \vDash \varphi \text{ and } \varphi^{\mathcal{M}'} \notin N'(s') \\
\Longleftrightarrow\quad & \mathcal{M}', s' \vDash \bullet\varphi
\end{aligned}
$$

Therefore, $\mathcal{M}, s \vDash \varphi$ iff $\mathcal{M}', s' \vDash \varphi$ for all $\varphi \in \mathcal{L}(\nabla, \bullet)$. This completes the proof of $(**)$. Thus there exists $x' \in S'$ such that $\mathcal{M}', x' \nvDash \psi$, and hence $\mathcal{F}' \nvDash \psi$. The converse is similar.

If (5) were to be defined by a set of $\mathcal{L}(\nabla, \bullet)$-formulas, say Γ, then as $\mathcal{F}$ has (5), we would have $\mathcal{F} \vDash \Gamma$, thus we should also have $\mathcal{F}' \vDash \Gamma$, that is, $\mathcal{F}'$ has (5): a contradiction. Therefore, (5) is not definable in $\mathcal{L}(\nabla, \bullet)$. $\qquad\square$

5 Axiomatizations

In this section, we axiomatize $\mathcal{L}(\nabla, \bullet)$ over various classes of neighborhood frames.

5.1 Classical Modal Logic

5.1.1 Proof System and Soundness

Definition 29. The classical modal logic of $\mathcal{L}(\nabla, \bullet)$, denoted $\mathbf{E}^{\nabla\bullet}$, consists of the following axioms and inference rules:

TAUT	all instances of tautologies
E1	$\nabla\varphi \leftrightarrow \nabla\neg\varphi$
E2	$\bullet\varphi \to \varphi$
E3	$\nabla\varphi \to \bullet\varphi \vee \bullet\neg\varphi$
MP	$\dfrac{\varphi,\ \varphi \to \psi}{\psi}$
RE∇	$\dfrac{\varphi \leftrightarrow \psi}{\nabla\varphi \leftrightarrow \nabla\psi}$
RE$\bullet$	$\dfrac{\varphi \leftrightarrow \psi}{\bullet\varphi \leftrightarrow \bullet\psi}$

Intuitively, E1 says that one is (first-order) ignorant whether a proposition holds if and only if one is ignorant whether its negation holds; E2 says that one is (Fitchean) ignorant of the fact that φ only if it is the case that φ; E3 is the bridge axiom in $\mathbf{E}^{\nabla\bullet}$, which describes the relationships between Fitchean ignorance and first-order ignorance: if one is ignorant whether φ, then either one is ignorant of the fact that φ or one is ignorant of the fact that φ is not the case; RE∇ and RE$\bullet$ concerns the replacement of equivalences for first-order ignorance and Fitchean ignorance, respectively.

Notions of derivations and provability are defined as usual. We use $\vdash_{\mathbb{S}} \varphi$ to denote that φ is provable in a system $\mathbb{S}$. We always omit the system when it is clear from the context.

It is straightforward by axiom E2 that $\bullet\nabla\varphi \to \nabla\varphi$ is provable in $\mathbf{E}^{\nabla\bullet}$, which says that under any neighborhood condition, Rumsfeld ignorance implies first-order ignorance.

The following result states how to derive Fitchean ignorance, which means that if one is *ignorant whether* a *true* proposition holds, then one is *ignorant of* the proposition. It will be used in several places below (for instance, Lemma 34, Lemma 37, and Prop. 44.

Proposition 30. $\vdash \nabla\varphi \wedge \varphi \to \bullet\varphi$. Equivalently, $\vdash \circ\varphi \wedge \varphi \to \Delta\varphi$.

Proof. We have the following proof sequence.

(i)	$\nabla\varphi \to \bullet\varphi \vee \bullet\neg\varphi$	E3
(ii)	$\bullet\neg\varphi \to \neg\varphi$	E2
(iii)	$\nabla\varphi \to \bullet\varphi \vee \neg\varphi$	$(i), (ii)$
(iv)	$\nabla\varphi \wedge \varphi \to \bullet\varphi$	(iii)

$\square$

As a corollary, we have $\vdash \nabla\nabla\varphi \wedge \nabla\varphi \rightarrow \bullet\nabla\varphi$. This means, in terms of Fine [20], second-order ignorance plus first-order ignorance implies Rumsfeld ignorance. On one hand, this is not noticed in Fine [20]; on the other hand, this plus the transitivity entails that second-order ignorance implies Rumsfeld ignorance, which is stronger (since we do not need **S**4) than a result in the paper in question.

The following result indicates how to derive a proposition from Fitchean ignorance.

Proposition 31. $\vdash \bullet(\circ\varphi \vee \psi \rightarrow \varphi) \rightarrow \varphi$

Proof. We have the following proof sequence.

$$
\begin{array}{lll}
(i) & \bullet(\circ\varphi \vee \psi \rightarrow \varphi) \rightarrow (\circ\varphi \vee \psi \rightarrow \varphi) & \text{E2} \\
(ii) & (\circ\varphi \vee \psi \rightarrow \varphi) \rightarrow (\circ\varphi \rightarrow \varphi) & \text{TAUT} \\
(iii) & \bullet(\circ\varphi \vee \psi \rightarrow \varphi) \rightarrow (\circ\varphi \rightarrow \varphi) & (i), (ii) \\
(iv) & \bullet\varphi \rightarrow \varphi & \text{E2} \\
(v) & \bullet(\circ\varphi \vee \psi \rightarrow \varphi) \rightarrow \varphi & (iii), (iv)
\end{array}
$$

$\square$

Proposition 32. $\mathbf{E}^{\nabla\bullet}$ is sound with respect to the class of all (neighborhood) frames.

Proof. The nontrivial part is the validity of axiom E3. This has been shown in Prop. 6.
$\square$

5.1.2 Completeness

The completeness of $\mathbf{E}^{\nabla\bullet}$ is demonstrated via a standard construction of a canonical model. Define the *proof set* of φ, notation: $|\varphi|$, as $\{s \in S^c \mid \varphi \in s\}$.

Definition 33. The canonical model for $\mathbf{E}^{\nabla\bullet}$ is $\mathcal{M}^c = \langle S^c, N^c, V^c \rangle$, where

- $S^c = \{s \mid s \text{ is a maximal consistent set for } \mathbf{E}^{\nabla\bullet}\}$,

- $N^c(s) = \{|\varphi| \mid \circ\varphi \wedge \Delta\varphi \in s\}$,

- $V^c(p) = \{s \in S^c \mid p \in s\}$.

Lemma 34. For all $\varphi \in \mathcal{L}(\nabla, \bullet)$, for all $s \in S^c$, we have

$$\mathcal{M}^c, s \vDash \varphi \iff \varphi \in s.$$

That is, $|\varphi| = \varphi^{\mathcal{M}^c}$.

Proof. By induction on φ. The nontrivial cases are $\nabla\varphi$ and $\bullet\varphi$.

For case $\nabla\varphi$:

First, suppose that $\nabla\varphi \in s$, to show that $\mathcal{M}^c, s \vDash \nabla\varphi$. By supposition, we have $\Delta\varphi \notin s$. Then by definition of N^c, we infer that $|\varphi| \notin N^c(s)$. By supposition again and axiom E1, we have $\nabla\neg\varphi \in s$, and thus $\Delta\neg\varphi \notin s$, and hence $|\neg\varphi| \notin N^c(s)$, that is, $S^c \backslash |\varphi| \notin N^c(s)$. By the induction hypothesis, we have $\varphi^{\mathcal{M}^c} \notin N^c(s)$ and $S^c \backslash \varphi^{\mathcal{M}^c} \notin N^c(s)$. Therefore, $\mathcal{M}^c, s \vDash \nabla\varphi$.

Conversely, assume that $\nabla\varphi \notin s$ (that is, $\nabla\neg\varphi \notin s$), to show that $\mathcal{M}^c, s \nvDash \nabla\varphi$. By assumption, $\Delta\varphi \in s$ and $\Delta\neg\varphi \in s$. Since $s \in S^c$, we have either $\varphi \in s$ or $\neg\varphi \in s$. If $\varphi \in s$, then by axiom E2, $\circ\neg\varphi \in s$, and then $|\neg\varphi| \in N^c(s)$, viz. $S^c\backslash|\varphi| \in N^c(s)$, which by the induction hypothesis implies that $S^c\backslash\varphi^{\mathcal{M}^c} \in N^c(s)$. If $\neg\varphi \in s$, then again by axiom E2, $\circ\varphi \in s$, thus $|\varphi| \in N^c(s)$, which by the induction hypothesis entails that $\varphi^{\mathcal{M}^c} \in N^c(s)$. We have now shown that either $\varphi^{\mathcal{M}^c} \in N^c(s)$ or $S^c\backslash\varphi^{\mathcal{M}^c} \in N^c(s)$, and we therefore conclude that $\mathcal{M}^c, s \nvDash \nabla\varphi$.

For case $\bullet\varphi$:

First, suppose that $\bullet\varphi \in s$, to show that $\mathcal{M}^c, s \vDash \bullet\varphi$. By supposition and axiom E2, we have $\varphi \in s$. By the induction hypothesis, $\mathcal{M}^c, s \vDash \varphi$. By supposition, $\circ\varphi \notin s$, then using the definition of N^c, we infer that $|\varphi| \notin N^c(s)$, which by induction means that $\varphi^{\mathcal{M}^c} \notin N^c(s)$. Therefore, $\mathcal{M}^c, s \vDash \bullet\varphi$.

Conversely, assume that $\bullet\varphi \notin s$, to demonstrate that $\mathcal{M}^c, s \nvDash \bullet\varphi$. By assumption, $\circ\varphi \in s$. If $\mathcal{M}^c, s \nvDash \varphi$, it is obvious that $\mathcal{M}^c, s \nvDash \bullet\varphi$. Otherwise, by the induction hypothesis, we have $\varphi \in s$, then $\circ\varphi \wedge \varphi \in s$. By Prop. 30, $\Delta\varphi \in s$, and thus $|\varphi| \in N^c(s)$, by induction we obtain $\varphi^{\mathcal{M}^c} \in N^c(s)$, and therefore we have also $\mathcal{M}^c, s \nvDash \bullet\varphi$. $\square$

It is then a standard exercise to show the following.

Theorem 35. $\mathbf{E}^{\nabla\bullet}$ is sound and strongly complete with respect to the class of all neighborhood frames.

5.2 Extensions

In this part, we consider some extensions of $\mathbf{E}^{\nabla\bullet}$.

5.2.1 $\mathbf{E}_\mathbf{c}^{\nabla\bullet}$

Let $\mathbf{E}_\mathbf{c}^{\nabla\bullet}$ be the smallest extension of $\mathbf{E}^{\nabla\bullet}$ with the following axiom, denoted E4:

$$\bullet\varphi \to \nabla\varphi.$$

Intuitively, E4 says that Fitchean ignorance implies first-order ignorance.

From E4 we can easily prove $\Delta\varphi \to \circ\varphi$. This turns the canonical model for $\mathbf{E}^{\nabla\bullet}$ (Def. 33) into the following simpler one.

Definition 36. The canonical model for $\mathbf{E}_\mathbf{c}^{\nabla\bullet}$ is a triple $\mathcal{M}^c = \langle S^c, N^c, V^c \rangle$, where

- $S^c = \{s \mid s \text{ is a maximal consistent set for } \mathbf{E}_\mathbf{c}^{\nabla\bullet}\}$

- $N^c(s) = \{|\varphi| \mid \Delta\varphi \in s\}$

- $V^c(p) = \{s \in S^c \mid p \in s\}$.

Lemma 37. For all $\varphi \in \mathcal{L}(\nabla, \bullet)$, for all $s \in S^c$, we have

$$\mathcal{M}^c, s \vDash \varphi \iff \varphi \in s.$$

That is, $|\varphi| = \varphi^{\mathcal{M}^c}$.

Proof. By induction on φ. The nontrivial cases are $\nabla\varphi$ and $\bullet\varphi$. The case $\nabla\varphi$ has been shown in [16, Lemma 1]. It suffices to show the case $\bullet\varphi$.

Suppose that $\bullet\varphi \in s$. Then by axiom E2, we have $\varphi \in s$; by axiom E4, we derive that $\nabla\varphi \in s$, and thus $\Delta\varphi \notin s$. This follows that $|\varphi| \notin N^c(s)$. By the induction hypothesis, $\mathcal{M}^c, s \vDash \varphi$ and $\varphi^{\mathcal{M}^c} \notin N^c(s)$. Therefore, $\mathcal{M}^c, s \vDash \bullet\varphi$.

Conversely, assume that $\bullet\varphi \notin s$, to show that $\mathcal{M}^c, s \nvDash \bullet\varphi$, which by the induction hypothesis amounts to showing that $\varphi \notin s$ or $|\varphi| \in N^c(s)$. For this, suppose that $\varphi \in s$, this plus the assumption implies that $\circ\varphi \wedge \varphi \in s$. Then by Prop. 30, $\Delta\varphi \in s$, and therefore $|\varphi| \in N^c(s)$. $\square$

Proposition 38. $\mathcal{M}^c$ possesses the property (c).

Proof. Refer to [16, Thm. 2]. $\square$

Now it is a standard exercise to show the following.

Theorem 39. $\mathbf{E}_{\mathbf{c}}^{\nabla\bullet}$ is sound and strongly complete with respect to the class of (c)-frames.

In the neighborhood context (c), there are some relationships between Rumsfeld ignorance, second-order ignorance and first-order ignorance. The following is immediate from the axiom E4.

Proposition 40. $\bullet\nabla\varphi \to \nabla\nabla\varphi$ is provable in $\mathbf{E}_{\mathbf{c}}^{\nabla\bullet}$.

This says that under the condition (c), Rumsfeld ignorance implies second-order ignorance. This contrasts the result of Fine's paper [20] that in $\mathbf{S4}$, Rumsfeld ignorance implies second-order ignorance.

Combined with an instance of the axiom E2 ($\bullet\nabla\varphi \to \nabla\varphi$) and $\vdash \nabla\nabla\varphi \wedge \nabla\varphi \to \bullet\nabla\varphi$ (see the remark after Prop. 30), it follows that within the neighborhood context (c), Rumsfeld ignorance amounts to second-order ignorance plus first-order ignorance, and thus Rumsfeld ignorance is definable in terms of first-order ignorance. This is absent in the literature.

5.2.2 $\mathbf{EN}^{\nabla\bullet}$

Let $\mathbf{EN}^{\nabla\bullet} = \mathbf{E}^{\nabla\bullet} + \circ\top$. From $\circ\top$ and Prop. 30 it follows that $\Delta\top$ is provable in $\mathbf{EN}^{\nabla\bullet}$.

Theorem 41. $\mathbf{EN}^{\nabla\bullet}$ is sound and strongly complete with respect to the class of all (n)-frames.

Proof. For soundness, by Prop. 32, it remains only to show the validity of $\circ\top$ over the class of (n)-frames. The validity of $\circ\top$ can be found in Prop. 25.

For completeness, define the canonical model $\mathcal{M}^c$ w.r.t. $\mathbf{EN}^{\nabla\bullet}$ as in Def. 33. By Thm. 35, it suffices to show that $\mathcal{M}^c$ possesses (n). By the construction of $\mathbf{EN}^{\nabla\bullet}$, for all $s \in S^c$, we have $\circ\top \wedge \Delta\top \in s$, and thus $|\top| \in N^c(s)$, that is, $S^c \in N^c(s)$, as desired. $\square$

5.2.3 Monotone Logic

Let $\mathbf{M}^{\nabla\bullet}$ be the extension of $\mathbf{E}^{\nabla\bullet}$ plus the following extra axioms:

$$
\begin{array}{ll}
\text{M1} & \nabla(\varphi \vee \psi) \wedge \nabla(\neg\varphi \vee \chi) \to \nabla\varphi \\
\text{M2} & \bullet(\varphi \vee \psi) \wedge \bullet(\neg\varphi \vee \chi) \to \nabla\varphi \\
\text{M3} & \bullet(\varphi \vee \psi) \wedge \nabla(\neg\varphi \vee \chi) \to \nabla\varphi \\
\text{M4} & \circ\varphi \wedge \varphi \to \circ(\varphi \vee \psi)
\end{array}
$$

Compared to the aforementioned systems, $\mathbf{M}^{\nabla\bullet}$ is much more complex.

Prop. 42 and Prop. 43 tell us how to derive $\Delta\top$ and $\circ\top$ in $\mathbf{M}^{\nabla\bullet}$, respectively. They will be used in Section 6.

Proposition 42. $\Delta\varphi \to \Delta\top$, equivalently, $\nabla\top \to \nabla\varphi$, is provable in $\mathbf{M}^{\nabla\bullet}$.

Proof. We have the following proof sequence in $\mathbf{M}^{\nabla\bullet}$.

$$
\begin{array}{lll}
(1) & \nabla(\varphi \vee \top) \wedge \nabla(\neg\varphi \vee \top) \to \nabla\varphi & \text{M1} \\
(2) & \top \leftrightarrow \varphi \vee \top & \text{TAUT} \\
(3) & \nabla\top \leftrightarrow \nabla(\varphi \vee \top) & (2), \text{RE}\nabla \\
(4) & \top \leftrightarrow \neg\varphi \vee \top & \text{TAUT} \\
(5) & \nabla\top \leftrightarrow \nabla(\neg\varphi \vee \top) & (4), \text{RE}\nabla \\
(6) & \nabla\top \to \nabla\varphi & (1), (3), (5) \\
(7) & \Delta\varphi \to \Delta\top & (6), \text{Def. } \Delta
\end{array}
$$

$\square$

Intuitively, $\nabla\top \to \nabla\varphi$ says that if one is ignorant about whether $\top$ holds, then one is ignorant about whether everything holds.

Proposition 43. $\circ\varphi \wedge \varphi \to \circ\top$, equivalently, $\bullet\top \to \bullet\varphi \vee \neg\varphi$, is provable in $\mathbf{M}^{\nabla\bullet}$.

Proof. We have the following proof sequence in $\mathbf{M}^{\nabla\bullet}$.

$$
\begin{array}{lll}
(1) & \varphi \wedge \circ\varphi \to \circ(\varphi \vee \top) & \text{M4} \\
(2) & \varphi \vee \top \leftrightarrow \top & \text{TAUT} \\
(3) & \bullet(\varphi \vee \top) \leftrightarrow \bullet\top & (2), \text{RE}\bullet \\
(4) & \neg\bullet(\varphi \vee \top) \leftrightarrow \neg\bullet\top & (3) \\
(5) & \circ(\varphi \vee \top) \leftrightarrow \circ\top & (4), \text{Def. } \circ \\
(6) & \varphi \wedge \circ\varphi \to \circ\top & (1), (5)
\end{array}
$$

$\square$

Intuitively, $\bullet\top \to \bullet\varphi \vee \neg\varphi$ says that if one is ignorant of $\top$, then one is ignorant of every fact, unless the fact is false.

Proposition 44. $\vdash \circ\varphi \wedge \varphi \to \Delta(\varphi \vee \psi)$.

Proof. We have the following proof sequence in $\mathbf{M}^{\nabla\bullet}$.

$$
\begin{array}{lll}
(1) & \circ\varphi \wedge \varphi \rightarrow \circ(\varphi \vee \psi) & \text{M4} \\
(2) & \circ(\varphi \vee \psi) \wedge (\varphi \vee \psi) \rightarrow \Delta(\varphi \vee \psi) & \text{Prop. } 30 \\
(3) & \varphi \rightarrow \varphi \vee \psi & \text{TAUT} \\
(4) & \circ\varphi \wedge \varphi \rightarrow \Delta(\varphi \vee \psi) & (1)-(3)
\end{array}
$$

$\square$

Proposition 45. $\mathbf{M}^{\nabla\bullet}$ is sound with respect to the class of all (s)-frames.

Proof. By soundness of $\mathbf{E}^{\nabla\bullet}$ (Prop. 32), it suffices to show the validity of the extra axioms. Let $\mathcal{M} = \langle S, N, V \rangle$ be an arbitrary (s)-model and $s \in S$.

For M1: suppose that $\mathcal{M}, s \vDash \nabla(\varphi \vee \psi) \wedge \nabla(\neg\varphi \vee \chi)$. Then $(\varphi \vee \psi)^{\mathcal{M}} \notin N(s)$ and $(\neg\varphi \vee \chi)^{\mathcal{M}} \notin N(s)$, that is, $\varphi^{\mathcal{M}} \cup \psi^{\mathcal{M}} \notin N(s)$ and $(\neg\varphi)^{\mathcal{M}} \cup \chi^{\mathcal{M}} \notin N(s)$. Since $\varphi^{\mathcal{M}} \subseteq \varphi^{\mathcal{M}} \cup \psi^{\mathcal{M}}$ and $N(s)$ is closed under supersets, we must have $\varphi^{\mathcal{M}} \notin N(s)$. Similarly, we can show that $(\neg\varphi)^{\mathcal{M}} \notin N(s)$. Therefore, $\mathcal{M}, s \vDash \nabla\varphi$, as desired. Similarly, we can show the validity of M2 and M3.

For M4: assume that $\mathcal{M}, s \vDash \circ\varphi \wedge \varphi$. Then $\varphi^{\mathcal{M}} \in N(s)$. Since $\varphi^{\mathcal{M}} \subseteq \varphi^{\mathcal{M}} \cup \psi^{\mathcal{M}} = (\varphi \vee \psi)^{\mathcal{M}}$, by the property (s), we have $(\varphi \vee \psi)^{\mathcal{M}} \in N(s)$, and therefore $\mathcal{M}, s \vDash \circ(\varphi \vee \psi)$. $\square$

Definition 46. Let Λ be an extension of $\mathbf{M}^{\nabla\bullet}$. A triple $\mathcal{M}^{\Lambda} = \langle S^{\Lambda}, N^{\Lambda}, V^{\Lambda} \rangle$ is a *canonical neighborhood model* for Λ if

- $S^{\Lambda} = \{s \mid s \text{ is a maximal consistent set for } \Lambda\}$,

- $|\varphi| \in N^{\Lambda}(s)$ iff $\Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \psi) \in s$ for all ψ,

- $V^{\Lambda}(p) = |p| = \{s \in S^{\Lambda} \mid p \in s\}$.

We need to show that N^{Λ} is well defined.

Proposition 47. Let $s \in S^{\Lambda}$ as defined in Def. 46. If $|\varphi| = |\varphi'|$, then $\Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \psi) \in s$ for all ψ iff $\Delta(\varphi' \vee \psi) \wedge \circ(\varphi' \vee \psi) \in s$ for all ψ.

Proof. Suppose that $|\varphi| = |\varphi'|$, then $\vdash \varphi \leftrightarrow \varphi'$, and thus $\vdash \varphi \vee \psi \leftrightarrow \varphi' \vee \psi$. By RE$\nabla$, RE$\bullet$, Def. Δ and Def. $\circ$, we infer that $\vdash \Delta(\varphi \vee \psi) \leftrightarrow \Delta(\varphi' \vee \psi)$ and $\vdash \circ(\varphi \vee \psi) \leftrightarrow \circ(\varphi' \vee \psi)$, and hence $\vdash \Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \psi) \leftrightarrow \Delta(\varphi' \vee \psi) \wedge \circ(\varphi' \vee \psi)$. Therefore, $\Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \psi) \in s$ for all ψ iff $\Delta(\varphi' \vee \psi) \wedge \circ(\varphi' \vee \psi) \in s$ for all ψ. $\square$

Lemma 48. Let $\mathcal{M}^{\Lambda} = \langle S^{\Lambda}, N^{\Lambda}, V^{\Lambda} \rangle$ be an arbitrary canonical neighborhood model for any system Λ extending $\mathbf{M}^{\nabla\bullet}$. Then for all $s \in S^{\Lambda}$, for all $\varphi \in \mathcal{L}(\nabla, \bullet)$, we have

$$
\mathcal{M}^{\Lambda}, s \vDash \varphi \iff \varphi \in s.
$$

That is, $\varphi^{\mathcal{M}^{\Lambda}} = |\varphi|$.

Proof. By induction on φ. The nontrivial cases are $\nabla\varphi$ and $\bullet\varphi$.

For case $\nabla\varphi$:

Suppose that $\mathcal{M}^\Lambda, s \nvDash \nabla\varphi$, to show that $\nabla\varphi \notin s$. By supposition and induction hypothesis, $|\varphi| \in N^\Lambda(s)$ or $S\backslash|\varphi| \in N^\Lambda(s)$ (that is, $|\neg\varphi| \in N^\Lambda(s)$). If $|\varphi| \in N^\Lambda(s)$, then $\Delta(\varphi \vee \psi) \in s$ for all ψ. By letting $\psi = \bot$, we infer that $\Delta\varphi \in s$, and thus $\nabla\varphi \notin s$. If $|\neg\varphi| \in N^\Lambda(s)$, with a similar argument we can show that $\Delta\neg\varphi \in s$, that is, $\Delta\varphi \in s$, and we also have $\nabla\varphi \notin s$.

Conversely, assume that $\mathcal{M}^\Lambda, s \vDash \nabla\varphi$, to prove that $\nabla\varphi \in s$. By assumption and induction hypothesis, $|\varphi| \notin N^\Lambda(s)$ and $S\backslash|\varphi| \notin N^\Lambda(s)$, that is, $|\neg\varphi| \notin N^\Lambda(s)$. Then $\Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \psi) \notin s$ for some ψ, and $\Delta(\neg\varphi \vee \chi) \wedge \circ(\neg\varphi \vee \chi) \notin s$ for some χ. We consider the following cases.

- $\Delta(\varphi \vee \psi) \notin s$ and $\Delta(\neg\varphi \vee \chi) \notin s$. That is, $\nabla(\varphi \vee \psi) \in s$ and $\nabla(\neg\varphi \vee \chi) \in s$. Then by axiom M1, we infer that $\nabla\varphi \in s$.

- $\Delta(\varphi \vee \psi) \notin s$ and $\circ(\neg\varphi \vee \chi) \notin s$. That is, $\nabla(\varphi \vee \psi) \in s$ and $\bullet(\neg\varphi \vee \chi) \in s$. By axiom M3, $\nabla\neg\varphi \in s$, that is, $\nabla\varphi \in s$.

- $\circ(\varphi \vee \psi) \notin s$ and $\Delta(\neg\varphi \vee \chi) \notin s$. That is, $\bullet(\varphi \vee \psi) \in s$ and $\nabla(\neg\varphi \vee \chi) \in s$. Then by axiom M3, we derive that $\nabla\varphi \in s$.

- $\circ(\varphi \vee \psi) \notin s$ and $\circ(\neg\varphi \vee \chi) \notin s$. That is, $\bullet(\varphi \vee \psi) \in s$ and $\bullet(\neg\varphi \vee \chi) \in s$. By axiom M2, we obtain that $\nabla\varphi \in s$.

Either case implies that $\nabla\varphi \in s$, as desired.

For case $\bullet\varphi$.

Suppose that $\bullet\varphi \in s$, to show that $\mathcal{M}^\Lambda, s \vDash \bullet\varphi$. By supposition and axiom E2, we obtain $\varphi \in s$, which by the induction hypothesis means that $\mathcal{M}^\Lambda, s \vDash \varphi$. We have also $|\varphi| \notin N^\Lambda(s)$: otherwise, by definition of N^Λ, we should have $\circ(\varphi \vee \psi) \in s$ for all ψ, which then implies that $\circ\varphi \in s$ (by letting $\psi = \bot$), a contradiction. Then by the induction hypothesis, $\varphi^{\mathcal{M}^\Lambda} \notin N^\Lambda(s)$. Therefore, $\mathcal{M}^\Lambda, s \vDash \bullet\varphi$.

Conversely, assume that $\bullet\varphi \notin s$ (that is, $\circ\varphi \in s$), to prove that $\mathcal{M}^\Lambda, s \nvDash \bullet\varphi$. For this, suppose that $\mathcal{M}^\Lambda, s \vDash \varphi$, by the induction hypothesis, we have $\varphi \in s$, and then $\circ\varphi \wedge \varphi \in s$. By axiom M4 and Prop. 44, $\Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \psi) \in s$ for all ψ. By definition of N^Λ, we derive that $|\varphi| \in N^\Lambda(s)$. Then by the induction hypothesis again, we conclude that $\varphi^{\mathcal{M}^\Lambda} \in N^\Lambda(s)$. Therefore, $\mathcal{M}^\Lambda, s \nvDash \bullet\varphi$, as desired. $\square$

Given an extension Λ of $\mathbf{M}^{\nabla\bullet}$, the minimal canonical neighborhood model for Λ, denoted $\mathcal{M}_0^\Lambda = \langle S^\Lambda, N_0^\Lambda, V^\Lambda \rangle$, is defined such that $N_0^\Lambda(s) = \{|\varphi| \mid \Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \psi) \in s$ for all $\psi\}$. Note that $\mathcal{M}_0^\Lambda$ is not necessarily supplemented. Therefore, we define a notion of supplementation, which comes from [7].

Definition 49. Let $\mathcal{M} = \langle S, N, V \rangle$ be a neighborhood model. The *supplementation* of $\mathcal{M}$, denoted $\mathcal{M}^+$, is a triple $\langle S, N^+, V \rangle$, in which for every $s \in S$, $N^+(s)$ is the superset closure of $N(s)$; namely, for each $s \in S$,

$$N^+(s) = \{X \subseteq S \mid Y \subseteq X \text{ for some } Y \in N(s)\}.$$

One may easily show that $\mathcal{M}^+$ is supplemented, that is, $\mathcal{M}^+$ possesses (s). Also, $N(s) \subseteq N^+(s)$. Moreover, the properties of being closed under intersections and containing the unit are closed under the supplementation. The proof is routine.

Proposition 50. Let $\mathcal{M} = \langle S, N, V \rangle$ be a neighborhood model and $\mathcal{M}^+$ be its supplementation. If $\mathcal{M}$ possesses (i), then so does $\mathcal{M}^+$; if $\mathcal{M}$ possesses (n), then so does $\mathcal{M}^+$.

In what follows, we will use $(\mathcal{M}_0^\Lambda)^+$ to denote the supplementation of $\mathcal{M}_0^\Lambda$, namely $(\mathcal{M}_0^\Lambda)^+ = \langle S^\Lambda, (N_0^\Lambda)^+, V^\Lambda \rangle$, where Λ extends $\mathbf{M}^{\nabla \bullet}$. By the definition of supplementation, $(\mathcal{M}_0^\Lambda)^+$ is an (s)-model. To show the completeness of $\mathbf{M}^{\nabla \bullet}$ over the class of (s)-frames, by Lemma 48, it remains only to show that $(\mathcal{M}_0^\Lambda)^+$ is a canonical neighborhood model for Λ.

Lemma 51. Let Λ extend $\mathbf{M}^{\nabla \bullet}$. For every $s \in S^\Lambda$, we have

$$|\varphi| \in (N_0^\Lambda)^+(s) \iff \Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \psi) \in s \text{ for all } \psi.$$

Proof. Right-to-Left: Immediate by the definition of N_0^Λ and the fact that $N_0^\Lambda(s) \subseteq (N_0^\Lambda)^+(s)$.

Left-to-Right: Suppose that $|\varphi| \in (N_0^\Lambda)^+(s)$, to prove that $\Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \psi) \in s$ for all ψ. By supposition, $X \subseteq |\varphi|$ for some $X \in N_0^\Lambda(s)$. Then there must be a χ such that $X = |\chi|$, and thus $\Delta(\chi \vee \psi) \wedge \circ(\chi \vee \psi) \in s$ for all ψ, and hence $\Delta(\chi \vee \varphi \vee \psi) \wedge \circ(\chi \vee \varphi \vee \psi) \in s$. From $|\chi| \subseteq |\varphi|$, it follows that $\vdash \chi \to \varphi$, and then $\vdash \chi \vee \varphi \vee \psi \leftrightarrow \varphi \vee \psi$, and thus $\vdash \Delta(\chi \vee \varphi \vee \psi) \leftrightarrow \Delta(\varphi \vee \psi)$ and $\vdash \circ(\chi \vee \varphi \vee \psi) \leftrightarrow \circ(\varphi \vee \psi)$ by RE∇, RE$\bullet$, Def. Δ and Def. $\circ$. Therefore, $\Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \psi) \in s$ for all ψ. $\square$

Based on the previous analysis, we have the following.

Theorem 52. $\mathbf{M}^{\nabla \bullet}$ is sound and strongly complete with respect to the class of all (s)-frames.

We conclude this part with some results which will be used in Section 6. The following result states that if one is ignorant of the fact that either φ holds or one is ignorant whether φ holds, then one is either ignorant of the fact that φ or ignorant whether φ holds.

Proposition 53. $\bullet(\varphi \vee \nabla \varphi) \to (\bullet \varphi \vee \nabla \varphi)$ is provable in $\mathbf{M}^{\nabla \bullet}$.[11]

Proof. By Thm. 52, it suffices to show the formula is valid over the class of (s)-frames. Let $\mathcal{M} = \langle S, N, V \rangle$ be an (s)-model and $s \in S$.

Suppose, for reductio, that $\mathcal{M}, s \vDash \bullet(\varphi \vee \nabla \varphi)$ and $\mathcal{M}, s \nvDash \bullet \varphi \vee \nabla \varphi$. From the former, it follows that $\mathcal{M}, s \vDash \varphi \vee \nabla \varphi$ and $(\varphi \vee \nabla \varphi)^{\mathcal{M}} \notin N(s)$; from the latter, it follows that $\mathcal{M}, s \nvDash \bullet \varphi$ and $\mathcal{M}, s \nvDash \nabla \varphi$. This implies that $\mathcal{M}, s \vDash \varphi$, which plus $\mathcal{M}, s \nvDash \bullet \varphi$ gives us $\varphi^{\mathcal{M}} \in N(s)$. Since $\varphi^{\mathcal{M}} \subseteq (\varphi \vee \nabla \varphi)^{\mathcal{M}}$, by (s), we conclude that $(\varphi \vee \nabla \varphi)^{\mathcal{M}} \in N(s)$: a contradiction, as desired. $\square$

---

[11]In fact, we can show a stronger result: $\bullet(\varphi \vee \nabla \psi) \to \bullet \varphi \vee \nabla \psi$ is provable in $\mathbf{M}^{\nabla \bullet}$. But we do not need such a strong result below.

The following result says that if one is ignorant of the fact that either non-ignorance of φ or non-ignorance whether φ holds implies that φ, then one is ignorant of the fact that φ.

Proposition 54. $\bullet(\circ\varphi \vee \Delta\varphi \to \varphi) \to \bullet\varphi$ is provable in $\mathbf{M}^{\nabla\bullet}$.

Proof. By Thm. 52, it remains only to prove that the formula is valid over the class of (s)-frames. Let $\mathcal{M} = \langle S, N, V \rangle$ be an (s)-model and $s \in S$.

Assume, for reductio, that $\mathcal{M}, s \vDash \bullet(\circ\varphi \vee \Delta\varphi \to \varphi)$ and $\mathcal{M}, s \nvDash \bullet\varphi$. The former implies $\mathcal{M}, s \vDash \circ\varphi \vee \Delta\varphi \to \varphi$ and $(\circ\varphi \vee \Delta\varphi \to \varphi)^{\mathcal{M}} \notin N(s)$; the latter entails that $\mathcal{M}, s \vDash \circ\varphi$. Then $\mathcal{M}, s \vDash \varphi$, and thus $\varphi^{\mathcal{M}} \in N(s)$. One may easily verify that $\varphi^{\mathcal{M}} \subseteq (\circ\varphi \vee \Delta\varphi \to \varphi)^{\mathcal{M}}$. Then by (s), we conclude that $(\circ\varphi \vee \Delta\varphi \to \varphi)^{\mathcal{M}} \in N(s)$: a contradiction. $\square$

5.2.4 EMN$^{\nabla\bullet}$

Define $\mathbf{EMN}^{\nabla\bullet} := \mathbf{M}^{\nabla\bullet} + \circ\top$. This system will be useful in Sec. 6.

Theorem 55. $\mathbf{EMN}^{\nabla\bullet}$ is sound and strongly complete with respect to the class of all (sn)-frames.

Proof. The soundness follows directly from Thm. 41 and Prop. 45.

For completeness, define $\mathcal{M}_0^\Lambda$ and $(\mathcal{M}_0^\Lambda)^+$ as before w.r.t. $\mathbf{EMN}^{\nabla\bullet}$. By Prop. 50, it remains only to show that $N_0^\Lambda(s)$ possesses (n). By axiom $\circ\top$ and the derivable formula $\Delta\top$, we have $\vdash \circ\top$ and $\vdash \Delta\top$. Then by $\vdash (\top \vee \psi) \leftrightarrow \top$, RE$\nabla$, RE$\bullet$, Def. Δ and Def. $\circ$, we infer that for all $s \in S^\Lambda$, $\Delta(\top \vee \psi) \wedge \circ(\top \vee \psi) \in s$ for all ψ, and thus $|\top| \in N^\Lambda(s)$, that is, $S \in N_0^\Lambda(s)$, as desired. $\square$

5.2.5 Regular Logic

Define $\mathbf{R}^{\nabla\bullet} := \mathbf{M}^{\nabla\bullet} + \text{R1} + \text{R2}$, where

$$\text{R1} \quad \Delta\varphi \wedge \Delta\psi \to \Delta(\varphi \wedge \psi)$$
$$\text{R2} \quad \circ\varphi \wedge \circ\psi \to \circ(\varphi \wedge \psi)$$

Proposition 56. $\mathbf{R}^{\nabla\bullet}$ is sound with respect to the class of quasi-filters.

Proof. By soundness of $\mathbf{M}^{\nabla\bullet}$, it remains to prove the validity of R1 and R2. The validity of R1 has been shown in [12, Prop. 3(iv)], and the validity of R2 has been shown in [15, Thm. 5.2]. $\square$

Proposition 57. Let Λ extend $\mathbf{R}^{\nabla\bullet}$. Then the minimal canonical model $\mathcal{M}_0^\Lambda$ has the property (i). As a corollary, its supplementation is a quasi-filter.

Proof. Suppose that $X, Y \in N_0^\Lambda(s)$, to show that $X \cap Y \in N_0^\Lambda(s)$. By supposition, there exist φ and χ such that $X = |\varphi|$ and $Y = |\chi|$, and then $\Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \psi) \in s$ for all ψ, and $\Delta(\chi \vee \psi) \wedge \circ(\chi \vee \psi) \in s$ for all ψ. By axioms R1 and R2, we can obtain that $\Delta((\varphi \wedge \chi) \vee \psi) \wedge \circ((\varphi \wedge \chi) \vee \psi) \in s$ for all ψ, which implies that $|\varphi \wedge \chi| \in N_0^\Lambda(s)$, that is, $X \cap Y \in N_0^\Lambda(s)$. $\square$

Theorem 58. $\mathbf{R}^{\nabla\bullet}$ is sound and strongly complete with respect to the class of quasi-filters.

5.2.6 $\mathbf{K}^{\nabla\bullet}$

Define $\mathbf{K}^{\nabla\bullet} := \mathbf{R}^{\nabla\bullet} + \circ\top$.

Again, like the case of $\mathbf{EN}^{\nabla\bullet}$(Sec. 5.2.2), $\Delta\top$ is derivable from $\circ\top$ and Prop. 30. This hints us that the inference rule R1 in [11, Def. 12] is actually dispensable. (Fact 13 therein is derivable from axiom A1 and axiom A6 thereof. Then by R2, we have $\vdash \varphi$ implies $\vdash \circ\varphi \wedge \varphi$, and then $\vdash \Delta\varphi$. Thus we derive R1 there.)

Theorem 59. $\mathbf{K}^{\nabla\bullet}$ is sound and strongly complete with respect to the class of filters.

Proof. For soundness, by Prop. 56, it suffices to show the validity of $\circ\top$ over the class of filters. This follows immediately from Prop. 25.

For completeness, define $\mathcal{M}_0^\Lambda$ and $(\mathcal{M}_0^\Lambda)^+$ as before w.r.t. $\mathbf{K}^{\nabla\bullet}$. By Prop. 57 and Prop. 50, it remains only to show that $N_0^\Lambda(s)$ possesses (n). This is similar to the corresponding part in Thm. 55. $\qquad\square$

Inspired by the definition of N^Λ, one may define the canonical relation for the extensions of $\mathbf{K}^{\nabla\bullet}$ as follows:

$sR^N t$ iff for all φ, if $\Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \psi) \in s$ for all ψ, then $\varphi \in t$.

Recall that the original definition of canonical relation given in [11, Def. 18], which is inspired by a schema therein, is as follows:

$sR^K t$ iff there exists δ such that (a) $\bullet\delta \in s$, and (b) for all φ, if $\Delta\varphi \wedge \circ(\neg\delta \to \varphi) \in s$, then $\varphi \in t$.

One may ask what the relationships between R^N and R^K is. As we shall see, they are equal to each other. Before this, we need some preparation.

Proposition 60. $\vdash \bullet\delta \wedge \Delta\varphi \wedge \circ(\neg\delta \to \varphi) \to \Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \chi)$

Proof. By Thm. 59, it remains only to show that this formula is valid over the class of filters.

Let $\mathcal{M} = \langle S, N, V \rangle$ be a filter and $s \in S$. Suppose that $\mathcal{M}, s \vDash \bullet\delta \wedge \Delta\varphi \wedge \circ(\neg\delta \to \varphi)$, to show $\mathcal{M}, s \vDash \Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \chi)$. By $\mathcal{M}, s \vDash \bullet\delta$, we have $\mathcal{M}, s \vDash \delta$ and $\delta^{\mathcal{M}} \notin N(s)$. By $\mathcal{M}, s \vDash \Delta\varphi$, we infer that $\varphi^{\mathcal{M}} \in N(s)$ or $S\backslash\varphi^{\mathcal{M}} \in N(s)$. Since $\mathcal{M}, s \vDash \delta$, we derive that $\mathcal{M}, s \vdash \neg\delta \to \varphi$. Then by $\mathcal{M}, s \vDash \circ(\neg\delta \to \varphi)$, we get $(\neg\delta \to \varphi)^{\mathcal{M}} \in N(s)$, that is, $\delta^{\mathcal{M}} \cup \varphi^{\mathcal{M}} \in N(s)$. If $S\backslash\varphi^{\mathcal{M}} \in N(s)$, then as $N(s)$ has the property (i), $(\delta^{\mathcal{M}} \cup \varphi^{\mathcal{M}}) \cap (S\backslash\varphi^{\mathcal{M}}) \in N(s)$, viz. $\delta^{\mathcal{M}} \cap (S\backslash\varphi^{\mathcal{M}}) \in N(s)$. Since $N(s)$ possesses the property (s) and $\delta^{\mathcal{M}} \cap (S\backslash\varphi^{\mathcal{M}}) \subseteq \delta^{\mathcal{M}}$, it follows that $\delta^{\mathcal{M}} \in N(s)$: a contradiction. This entails that $S\backslash\varphi^{\mathcal{M}} \notin N(s)$, and thus $\varphi^{\mathcal{M}} \in N(s)$. Note that $\varphi^{\mathcal{M}} \subseteq \varphi^{\mathcal{M}} \cup \psi^{\mathcal{M}} = (\varphi \vee \psi)^{\mathcal{M}}$ and $\varphi^{\mathcal{M}} \subseteq \varphi^{\mathcal{M}} \cup \chi^{\mathcal{M}} = (\varphi \vee \chi)^{\mathcal{M}}$. Using (s) again, we conclude that $(\varphi \vee \psi)^{\mathcal{M}} \in N(s)$ and $(\varphi \vee \chi)^{\mathcal{M}} \in N(s)$, and therefore $\mathcal{M}, s \vDash \Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \chi)$, as desired. $\qquad\square$

Proposition 61. Let Λ be an extension of $\mathbf{K}^{\nabla\bullet}$. Then for all $s, t \in S^\Lambda$, $sR^N t$ iff $sR^K t$.

Proof. Suppose that $sR^N t$, to show that $sR^K t$. By supposition, for all φ, if $\Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \psi) \in s$ for all ψ, then $\varphi \in t$. Letting $\varphi = \bot$, we can infer that $\Delta\psi \wedge \circ\psi \notin s$ for some ψ. If $\Delta\psi \notin s$, that is, $\nabla\psi \in s$, then by axiom E3, we derive that $\bullet\psi \in s$ or $\bullet\neg\psi \in s$. If $\circ\psi \notin s$, then we have $\bullet\psi \in s$. Either case implies that $\bullet\delta \in s$ for some

δ. Now suppose for any φ' that $\Delta\varphi' \wedge \circ(\neg\delta \to \varphi') \in s$. By Prop. 60, we infer that $\Delta(\varphi' \vee \chi) \wedge \circ(\varphi' \vee \chi) \in s$ for all χ. Then by supposition again, we conclude that $\varphi' \in t$. Therefore, $sR^K t$.

Conversely, assume that $sR^K t$, then there exists δ such that $(a) \bullet\delta \in s$, and (b) for all φ, if $\Delta\varphi \wedge \circ(\neg\delta \to \varphi) \in s$, then $\varphi \in t$. It remains to prove that $sR^N t$. For this, suppose for any φ that $\Delta(\varphi \vee \psi) \wedge \circ(\varphi \vee \psi) \in s$ for all ψ. By letting $\psi = \bot$, we obtain that $\Delta\varphi \in s$; by letting $\psi = \delta$, we infer that $\circ(\neg\delta \to \varphi) \in s$. Thus $\Delta\varphi \wedge \circ(\neg\delta \to \varphi) \in s$. Then by (b), we conclude that $\varphi \in t$, and therefore $sR^N t$, as desired. $\square$

The above proposition gives us another definition of the canonical relation (thus canonical model) for $\mathbf{K}^{\nabla\bullet}$ in [11].

6 Updating Neighborhood Models

6.1 The Intersection Semantics

Like knowledge, ignorance can be changed by actions. For instance, it was raining in Melbourne (p) but I was (first-order) ignorant whether p holds (thus I was also (Fitchean) ignorant of p). After Lloyd told me about p by correspondence, my ignorance about p disappeared.[12] In this section, we extend the previous results to the dynamic case of public announcements.[13] Syntactically, we add the construct $[\varphi]\varphi$ into the previous languages $\mathcal{L}(\nabla)$, $\mathcal{L}(\bullet)$ and $\mathcal{L}(\nabla, \bullet)$, and denote the obtained extensions by $\mathcal{L}(\nabla, [\cdot])$, $\mathcal{L}(\bullet, [\cdot])$, $\mathcal{L}(\nabla, \bullet, [\cdot])$, respectively. $[\psi]\varphi$ is read "after every truthful public announcement of ψ, φ holds". Also, as usual, $\langle\psi\rangle\varphi$ abbreviates $\neg[\psi]\neg\varphi$. Semantically, we adopt the intersection semantics in the literature (e.g. [30, 31, 55]). To define the semantics, we first introduce the notion of intersection models.

Definition 62. [30, Def. 3] Let $\mathcal{M} = \langle S, N, V \rangle$ be a monotone model, and X a nonempty subset of S. Define *the intersection model* $\mathcal{M}^{\cap X} = \langle X, N^{\cap X}, V^X \rangle$ induced from X in the following.

- for every $s \in X$, $N^{\cap X} = \{Y \mid Y = P \cap X \text{ for some } P \in N(s)\}$,

- $V^X(p) = V(p) \cap X$.

Proposition 63. [30, Prop. 2] The neighborhood property (s) is preserved under taking the intersection submodel. That is, if $\mathcal{M}$ is a monotone neighborhood model with the

---

[12]As we will show in Prop. 91 and Prop. 92, $\nabla p \to [p]\neg\nabla p$ and $\bullet p \to [p]\neg\bullet p$ are both provable in $\mathbf{EMN}^{\nabla\bullet[\cdot]}$.

[13]One may say that, since both forms of ignorance can be defined in terms of the knowledge operator (see Prop. 5), one can analyze the dynamic of ignorance with public announcement operators in dynamic epistemic logic. Despite this, we think that when it comes to the succinctness and simplicity, studying the dynamics of ignorance with a logic that treats ignorance with its own operators is better than the treatment with dynamic epistemic logic. Take Prop. 77 as an example. Compared to the proof of the provability of $[p \wedge \neg\bullet p \wedge \neg\nabla p](p \wedge \neg\bullet p \wedge \neg\nabla p)$ in $\mathbf{M}^{\nabla\bullet[\cdot]}$, showing the provability of the corresponding DEL-formulas (more precisely, $[p \wedge \neg(p \wedge \neg\Box p) \wedge \neg(\neg\Box p \wedge \neg\Box\neg p)](p \wedge \neg(p \wedge \neg\Box p) \wedge \neg(\neg\Box p \wedge \neg\Box\neg p))$) is a much more cumbersome work. Moreover, the dynamic properties of the forms of ignorance can be seen more clearly, as e.g. the reduction axioms $\mathrm{A}\nabla$ and $\mathrm{A}\bullet$ shows below.

domain S, then for any nonempty subset X of S, the intersection submodel $\mathcal{M}^{\cap X}$ is also monotone.

Now, given a monotone neighborhood model $\mathcal{M} = \langle S, N, V \rangle$ and a state $s \in S$,

$$\mathcal{M}, s \vDash [\psi]\varphi \iff \mathcal{M}, s \vDash \psi \text{ implies } \mathcal{M}^{\cap\psi}, s \vDash \varphi$$

where $\mathcal{M}^{\cap\psi}$ is the intersection submodel $\mathcal{M}^{\cap\psi^{\mathcal{M}}}$, and the notion of intersection submodels is defined as above.

The following reduction axioms are from [11, Sec. 7]. We will show that all of them are valid under intersection semantics.

AP	$[\psi]p \leftrightarrow (\psi \to p)$
AN	$[\psi]\neg\varphi \leftrightarrow (\psi \to \neg[\psi]\varphi)$
AC	$[\psi](\varphi \wedge \chi) \leftrightarrow ([\psi]\varphi \wedge [\psi]\chi)$
AA	$[\psi][\chi]\varphi \leftrightarrow [\psi \wedge [\psi]\chi]\varphi$
A∇	$[\psi]\nabla\varphi \leftrightarrow (\psi \to \nabla[\psi]\varphi \wedge \nabla[\psi]\neg\varphi)$
A$\bullet$	$[\psi]\bullet\varphi \leftrightarrow (\psi \to \bullet[\psi]\varphi)$

The following reduction axioms are derivable from the above reduction axioms.

AΔ	$[\psi]\Delta\varphi \leftrightarrow (\psi \to \Delta[\psi]\varphi \vee \Delta[\psi]\neg\varphi)$
A$\circ$	$[\psi]\circ\varphi \leftrightarrow (\psi \to \circ[\psi]\varphi)$

Theorem 64. Let Λ be a system of $\mathcal{L}(\nabla)$ (resp. $\mathcal{L}(\bullet)$, $\mathcal{L}(\nabla, \bullet)$). If Λ is sound and strongly complete with respect to the class of monotone neighborhood frames, then so is Λ plus AP, AN, AC, AA and A∇ (resp. plus AP, AN, AC, AA and A$\bullet$, plus AP, AN, AC, AA, A∇ and A$\bullet$) under intersection semantics.

Proof. We only need to show the validity of A∇. The proof for the validity of A$\bullet$ has been shown in [15, Thm. 6.1], where the axiom is named A$\bullet$Int, and the validity of other reduction axioms can be found in [30, Thm. 1], [31, Thm. 2, Thm. 3] and [55, Prop. 3.1]. This then gives us the soundness. Moreover, the completeness can be shown via a standard reduction method, see e.g. [52]. Let $\mathcal{M} = \langle S, N, V \rangle$ be a monotone model and $s \in S$.

To begin with, suppose that $\mathcal{M}, s \vDash [\psi]\nabla\varphi$ and $\mathcal{M}, s \vDash \psi$, to show that $\mathcal{M}, s \vDash \nabla[\psi]\varphi \wedge \nabla[\psi]\neg\varphi$. By supposition, we have $\mathcal{M}^{\cap\psi}, s \vDash \nabla\varphi$, which implies $\varphi^{\mathcal{M}^{\cap\psi}} \notin N^{\cap\psi}(s)$ and $\psi^{\mathcal{M}} \backslash \varphi^{\mathcal{M}^{\cap\psi}} \notin N^{\cap\psi}(s)$.

We claim that $\mathcal{M}, s \vDash \nabla[\psi]\varphi$, that is, $([\psi]\varphi)^{\mathcal{M}} \notin N(s)$ and $S \backslash ([\psi]\varphi)^{\mathcal{M}} \notin N(s)$. If $([\psi]\varphi)^{\mathcal{M}} \in N(s)$, then $([\psi]\varphi)^{\mathcal{M}} \cap \psi^{\mathcal{M}} \in N^{\cap\psi}(s)$. As $([\psi]\varphi)^{\mathcal{M}} \cap \psi^{\mathcal{M}} \subseteq \varphi^{\mathcal{M}^{\cap\psi}}$, by (s), we have $\varphi^{\mathcal{M}^{\cap\psi}} \in N^{\cap\psi}(s)$: a contradiction. If $S \backslash ([\psi]\varphi)^{\mathcal{M}} \in N(s)$, then $(S \backslash ([\psi]\varphi)^{\mathcal{M}}) \cap \psi^{\mathcal{M}} \in N^{\cap\psi}(s)$. Note that $(S \backslash ([\psi]\varphi)^{\mathcal{M}}) \cap \psi^{\mathcal{M}} \subseteq \psi^{\mathcal{M}} \backslash \varphi^{\mathcal{M}^{\cap\psi}}$: for any $x \in (S \backslash ([\psi]\varphi)^{\mathcal{M}}) \cap \psi^{\mathcal{M}}$, $x \notin ([\psi]\varphi)^{\mathcal{M}}$, thus $x \in \psi^{\mathcal{M}}$ and $x \notin \varphi^{\mathcal{M}^{\cap\psi}}$, and hence $x \in \psi^{\mathcal{M}} \backslash \varphi^{\mathcal{M}^{\cap\psi}}$. By (s) again, $\psi^{\mathcal{M}} \backslash \varphi^{\mathcal{M}^{\cap\psi}} \in N^{\cap\psi}(s)$: a contradiction again.

We also claim that $\mathcal{M}, s \vDash \nabla[\psi]\neg\varphi$, that is, $([\psi]\neg\varphi)^{\mathcal{M}} \notin N(s)$ and $S \backslash ([\psi]\neg\varphi)^{\mathcal{M}} \notin N(s)$. If $([\psi]\neg\varphi)^{\mathcal{M}} \in N(s)$, then $([\psi]\neg\varphi)^{\mathcal{M}} \cap \psi^{\mathcal{M}} \in N^{\cap\psi}(s)$. As $([\psi]\neg\varphi)^{\mathcal{M}} \cap \psi^{\mathcal{M}} \subseteq \psi^{\mathcal{M}} \backslash \varphi^{\mathcal{M}^{\cap\psi}}$, we infer by (s) that $\psi^{\mathcal{M}} \backslash \varphi^{\mathcal{M}^{\cap\psi}} \in N^{\cap\psi}(s)$: a contradiction. If

$S\backslash([\psi]\neg\varphi)^{\mathcal{M}} \in N(s)$, then $(S\backslash([\psi]\neg\varphi)^{\mathcal{M}})\cap\psi^{\mathcal{M}} \in N^{\cap\psi}(s)$. Since $(S\backslash([\psi]\neg\varphi)^{\mathcal{M}})\cap$ $\psi^{\mathcal{M}} \subseteq \varphi^{\mathcal{M}^{\cap\psi}}$, by (s) again, we derive that $\varphi^{\mathcal{M}^{\cap\psi}} \in N^{\cap\psi}(s)$: a contradiction again.

Conversely, assume that $\mathcal{M}, s \vDash \psi \to \nabla[\psi]\varphi \wedge \nabla[\psi]\neg\varphi$, to prove that $\mathcal{M}, s \vDash [\psi]\nabla\varphi$. For this, we suppose that $\mathcal{M}, s \vDash \psi$, it remains only to show that $\mathcal{M}^{\cap\psi}, s \vDash \nabla\varphi$, that is, $\varphi^{\mathcal{M}^{\cap\psi}} \notin N^{\cap\psi}(s)$ and $\psi^{\mathcal{M}}\backslash\varphi^{\mathcal{M}^{\cap\psi}} \notin N^{\cap\psi}(s)$. By assumption and supposition, we obtain $\mathcal{M}, s \vDash \nabla[\psi]\varphi \wedge \nabla[\psi]\neg\varphi$. This follows that $([\psi]\varphi)^{\mathcal{M}} \notin N(s)$ and $S\backslash([\psi]\varphi)^{\mathcal{M}} \notin N(s)$, and $([\psi]\neg\varphi)^{\mathcal{M}} \notin N(s)$ and $S\backslash([\psi]\neg\varphi)^{\mathcal{M}} \notin N(s)$.

We claim that $\varphi^{\mathcal{M}^{\cap\psi}} \notin N^{\cap\psi}(s)$. Otherwise, that is, $\varphi^{\mathcal{M}^{\cap\psi}} \in N^{\cap\psi}(s)$, we have $\varphi^{\mathcal{M}^{\cap\psi}} = P \cap \psi^{\mathcal{M}}$ for some $P \in N(s)$. This implies that $P \subseteq ([\psi]\varphi)^{\mathcal{M}}$: for any $x \in P$, we would have $x \in ([\psi]\varphi)^{\mathcal{M}}$, since if $x \in \psi^{\mathcal{M}}$, then $x \in P \cap \psi^{\mathcal{M}} = \varphi^{\mathcal{M}^{\cap\psi}}$. By (s), $([\psi]\varphi)^{\mathcal{M}} \in N(s)$: a contradiction.

We also claim that $\psi^{\mathcal{M}}\backslash\varphi^{\mathcal{M}^{\cap\psi}} \notin N^{\cap\psi}(s)$. Otherwise, that is, $\psi^{\mathcal{M}}\backslash\varphi^{\mathcal{M}^{\cap\psi}} \in N^{\cap\psi}(s)$, we infer that $\psi^{\mathcal{M}}\backslash\varphi^{\mathcal{M}^{\cap\psi}} = P \cap \psi^{\mathcal{M}}$ for some $P \in N(s)$. It then follows that $P \subseteq ([\psi]\neg\varphi)^{\mathcal{M}}$: for any $x \in P$, we have $x \in ([\psi]\neg\varphi)^{\mathcal{M}}$, since if $x \in \psi^{\mathcal{M}}$, then $x = P\cap\psi^{\mathcal{M}} = \psi^{\mathcal{M}}\backslash\varphi^{\mathcal{M}^{\cap\psi}}$, and so $x \in (\neg\varphi)^{\mathcal{M}^{\cap\psi}}$. By (s) again, $([\psi]\neg\varphi)^{\mathcal{M}} \in N(s)$: a contradiction, as desired. $\qquad\square$

For the sake of reference, we use $\mathbf{M}^{\nabla\bullet[\cdot]}$ (resp. $\mathbf{EMN}^{\nabla\bullet[\cdot]}$) to denote the extension of $\mathbf{M}^{\nabla\bullet}$ (resp. $\mathbf{EMN}^{\nabla\bullet}$) with all the above reduction axioms. By dropping all axioms and inference rules involving $\bullet$ from $\mathbf{M}^{\nabla\bullet[\cdot]}$, we obtain the system $\mathbf{M}^{\nabla[\cdot]}$; by dropping all axioms and inference rules involving ∇ from $\mathbf{M}^{\nabla\bullet[\cdot]}$ (resp. $\mathbf{EMN}^{\nabla\bullet[\cdot]}$), we obtain the system $\mathbf{M}^{\bullet[\cdot]}$ (resp. $\mathbf{EMN}^{\bullet[\cdot]}$).

6.2 Application: Successful Formulas

In what follows, mainly as technical applications of the above reduction axioms, we will focus on some successful formulas in our languages. A formula is said to be *successful*, if it still holds after being announced; in symbols, $\vDash [\varphi]\varphi$. Recall that $\neg\bullet p$ is shown to be successful under the relational semantics in [11, Prop. 39] and under the intersection semantics in [15, Prop. 6.5]. We will follow this line of research and say much more. As we shall show in a syntactic way, any combination of p, $\neg p$, $\neg\bullet p$, and $\neg\nabla p$ via conjunction (or, via disjunction) is successful under the intersection semantics.[14] This gives us the syntactic characterization of two fragments of successful formulas in $\mathcal{L}(\nabla, \bullet)$.

$$
\begin{aligned}
\varphi &::= p \mid \neg p \mid \neg\bullet p \mid \neg\nabla p \mid \varphi \wedge \varphi \\
\varphi &::= p \mid \neg p \mid \neg\bullet p \mid \neg\nabla p \mid \varphi \vee \varphi.
\end{aligned}
$$

To begin with, we show that, provably, any combination of p, $\neg p$, $\neg\bullet p$, and $\neg\nabla p$ via *conjunction* is successful under the intersection semantics.

Proposition 65. p is successful under the intersection semantics. That is, $[p]p$ is provable in $\mathbf{M}^{\bullet[\cdot]}$.

---

[14]It is natural to ask whether any combination of the four constructs via conjunction or disjunction is successful under the intersection semantics. We will investigate this more complex situation in another scenario.

Proof. Straightforward by AP. ☐

Proposition 66. $\neg p$ is successful under the intersection semantics. That is, $[\neg p]\neg p$ is provable in $\mathbf{M}^{\bullet[\cdot]}$.

Proof. Straightforward by AN and AP. ☐

Proposition 67. $\neg\bullet p$ is successful under the intersection semantics. That is, $[\neg\bullet p]\neg\bullet p$ is provable in $\mathbf{M}^{\bullet[\cdot]}$.

Proof. Refer to [15, Prop. 6.5]. ☐

Proposition 68. $\neg\nabla p$ is successful under the intersection semantics. That is, $[\neg\nabla p]\neg\nabla p$ is provable in $\mathbf{M}^{\nabla[\cdot]}$.

Proof. We have the following proof sequence in $\mathbf{M}^{\nabla[\cdot]}$.

$$
\begin{array}{lll}
& [\neg\nabla p]\neg\nabla p & \\
\leftrightarrow & (\neg\nabla p \rightarrow \neg[\neg\nabla p]\nabla p) & \text{AN} \\
\leftrightarrow & (\neg\nabla p \rightarrow \neg(\neg\nabla p \rightarrow \nabla[\neg\nabla p]p \wedge \nabla[\neg\nabla p]\neg p) & \text{A}\nabla \\
\leftrightarrow & (\neg\nabla p \rightarrow \neg(\nabla(\neg\nabla p \rightarrow p) \wedge \nabla(\neg\nabla p \rightarrow \neg(\neg\nabla p \rightarrow p)))) & \text{AP, AN} \\
\leftrightarrow & (\nabla(\neg\nabla p \rightarrow p) \wedge \nabla(\neg\nabla p \rightarrow \neg p) \rightarrow \nabla p) & \text{TAUT, RE}\nabla \\
\leftrightarrow & (\nabla(p \vee \nabla p) \wedge \nabla(\neg p \vee \nabla p) \rightarrow \nabla p) & \text{TAUT, RE}\nabla \\
\leftrightarrow & \top & \text{M1}
\end{array}
$$

Therefore, $[\neg\nabla p]\neg\nabla p$ is provable in $\mathbf{M}^{\nabla[\cdot]}$. ☐

Intuitively, $[\neg\nabla p]\neg\nabla p$ means that "after being told that one is not ignorant whether p, one is still not ignorant whether p." In other words, one's non-ignorance about a fact cannot be altered by being announced.

Proposition 69. $p \wedge \neg p$ is successful under the intersection semantics.

Proof. Note that $p \wedge \neg p$ is equivalent to $\bot$, and $\bot$ is successful. ☐

Proposition 70. $p \wedge \neg\bullet p$ is successful under the intersection semantics. That is, $[p \wedge \neg\bullet p](p \wedge \neg\bullet p)$ is provable in $\mathbf{M}^{\bullet[\cdot]}$.

Proof. We have the following proof sequence in $\mathbf{M}^{\bullet[\cdot]}$.

$$
\begin{array}{lll}
& [p \wedge \neg\bullet p](p \wedge \neg\bullet p) & \\
\leftrightarrow & ([p \wedge \neg\bullet p]p \wedge [p \wedge \neg\bullet p]\neg\bullet p) & \text{AC} \\
\leftrightarrow & (p \wedge \neg\bullet p \rightarrow p) \wedge (p \wedge \neg\bullet p \rightarrow \neg[p \wedge \neg\bullet p]\bullet p) & \text{AP, AN} \\
\leftrightarrow & (p \wedge \neg\bullet p \rightarrow \neg(p \wedge \neg\bullet p \rightarrow \bullet[p \wedge \neg\bullet p]p)) & \text{A}\bullet \\
\leftrightarrow & (p \wedge \neg\bullet p \rightarrow \neg\bullet[p \wedge \neg\bullet p]p) & \text{TAUT} \\
\leftrightarrow & (p \wedge \neg\bullet p \rightarrow \neg\bullet(p \wedge \neg\bullet p \rightarrow p)) & \text{AP} \\
\leftrightarrow & (p \wedge \neg\bullet p \rightarrow \neg\bullet\top) & \text{TAUT, RE}\bullet \\
\leftrightarrow & (p \wedge \circ p \rightarrow \circ\top) & \text{Def. } \circ \\
\leftrightarrow & \top & \text{Prop. 43}
\end{array}
$$

Therefore, $[p \wedge \neg\bullet p](p \wedge \neg\bullet p)$ is provable in $\mathbf{M}^{\bullet[\cdot]}$. ☐

Proposition 71. $p \wedge \neg \nabla p$ is successful under the intersection semantics. That is, $[p \wedge \neg \nabla p](p \wedge \neg \nabla p)$ is provable in $\mathbf{M}^{\nabla[\cdot]}$.

Proof. We have the following proof sequence in $\mathbf{M}^{\nabla[\cdot]}$.

$$
\begin{array}{lll}
& [p \wedge \neg \nabla p](p \wedge \neg \nabla p) & \\
\leftrightarrow & ([p \wedge \neg \nabla p]p \wedge [p \wedge \neg \nabla p]\neg \nabla p) & \text{AC} \\
\leftrightarrow & (p \wedge \neg \nabla p \to p) \wedge (p \wedge \neg \nabla p \to \neg[p \wedge \neg \nabla p]\nabla p) & \text{AP, AN} \\
\leftrightarrow & (p \wedge \neg \nabla p \to \neg(p \wedge \neg \nabla p \to \nabla[p \wedge \neg \nabla p]p \wedge \nabla[p \wedge \neg \nabla p]\neg p)) & \text{A}\nabla \\
\leftrightarrow & (p \wedge \neg \nabla p \to \neg(\nabla[p \wedge \neg \nabla p]p \wedge \nabla[p \wedge \neg \nabla p]\neg p)) & \text{TAUT} \\
\leftrightarrow & (p \wedge \neg \nabla p \to \neg(\nabla \top \wedge \nabla[p \wedge \neg \nabla p]\neg p)) & \text{AP, RE}\nabla \\
\leftrightarrow & (p \wedge \neg \nabla p \to \neg \nabla \top \vee \neg \nabla(p \wedge \neg \nabla p \to \neg p)) & \text{AN, AP, RE}\nabla \\
\leftrightarrow & (p \wedge \Delta p \to \Delta \top \vee \Delta(p \wedge \Delta p \to \neg p)) & \text{Def. } \Delta \\
\end{array}
$$

By Prop. 42, $\Delta p \to \Delta \top$ is provable in $\mathbf{M}^{\nabla}$, so is the last formula in the above proof sequence, and thus $[p \wedge \neg \nabla p](p \wedge \neg \nabla p)$ is provable in $\mathbf{M}^{\nabla[\cdot]}$. $\qquad \square$

Proposition 72. $\neg p \wedge \neg \bullet p$ is successful under the intersection semantics.

Proof. By E2, $\neg p \wedge \neg \bullet p$ is equivalent to $\neg p$. And we have already known from Prop. 66 that $\neg p$ is successful. $\qquad \square$

Proposition 73. $\neg p \wedge \neg \nabla p$ is successful under the intersection semantics. That is, $[\neg p \wedge \neg \nabla p](\neg p \wedge \neg \nabla p)$ is provable in $\mathbf{M}^{\nabla[\cdot]}$.

Proof. We have the following proof sequence in $\mathbf{M}^{\nabla[\cdot]}$.

$$
\begin{array}{lll}
& [\neg p \wedge \neg \nabla p](\neg p \wedge \neg \nabla p) & \\
\leftrightarrow & [\neg p \wedge \neg \nabla p]\neg p \wedge [\neg p \wedge \neg \nabla p]\neg \nabla p & \text{AC} \\
\leftrightarrow & (\neg p \wedge \neg \nabla p \to \neg p) \wedge (\neg p \wedge \neg \nabla p \to \neg[\neg p \wedge \neg \nabla p]\nabla p) & \text{AN, AP} \\
\leftrightarrow & (\neg p \wedge \neg \nabla p \to \neg[\neg p \wedge \neg \nabla p]\nabla p) & \text{TAUT} \\
\leftrightarrow & (\neg p \wedge \neg \nabla p \to \neg(\nabla[\neg p \wedge \neg \nabla p]p \wedge \nabla[\neg p \wedge \neg \nabla p]\neg p) & \text{A}\nabla \\
\leftrightarrow & (\neg p \wedge \Delta p \to \Delta(\neg p \wedge \Delta p \to p) \vee \Delta \top) & \text{AP, AN, RE}\nabla \\
\leftrightarrow & \top & \text{Prop. 42} \\
\end{array}
$$

Therefore, $[\neg p \wedge \neg \nabla p](\neg p \wedge \neg \nabla p)$ is provable in $\mathbf{M}^{\nabla[\cdot]}$. $\qquad \square$

We have seen that both $\neg \bullet p$ and $\neg \nabla p$ are successful. One natural question would be whether their conjunction, viz. $\neg \bullet p \wedge \neg \nabla p$, is successful. Note that this does not obviously hold, since for instance, both p and $\neg K p$ are successful, whereas $p \wedge \neg K p$ is not, see e.g. [52, Example 4.34].

Proposition 74. $\neg \bullet p \wedge \neg \nabla p$ is successful under the intersection semantics. That is, $[\neg \bullet p \wedge \neg \nabla p](\neg \bullet p \wedge \neg \nabla p)$ is provable in $\mathbf{M}^{\nabla \bullet[\cdot]}$.

Proof. By AC, $[\neg \bullet p \wedge \neg \nabla p](\neg \bullet p \wedge \neg \nabla p) \leftrightarrow ([\neg \bullet p \wedge \neg \nabla p]\neg \bullet p \wedge [\neg \bullet p \wedge \neg \nabla p]\neg \nabla p)$. We show that both $[\neg \bullet p \wedge \neg \nabla p]\neg \bullet p$ and $[\neg \bullet p \wedge \neg \nabla p]\neg \nabla p$ are provable in $\mathbf{M}^{\nabla \bullet[\cdot]}$.

We have the following proof sequence in $\mathbf{M}^{\nabla\bullet[\cdot]}$.

$$
\begin{aligned}
& [\neg\bullet p \wedge \neg\nabla p]\neg\bullet p \\
\leftrightarrow\ & (\neg\bullet p \wedge \neg\nabla p \rightarrow \neg[\neg\bullet p \wedge \neg\nabla p]\bullet p) && \text{AN} \\
\leftrightarrow\ & (\neg\bullet p \wedge \neg\nabla p \rightarrow \neg(\neg\bullet p \wedge \neg\nabla p \rightarrow \bullet[\neg\bullet p \wedge \neg\nabla p]p)) && \text{A}\bullet \\
\leftrightarrow\ & (\neg\bullet p \wedge \neg\nabla p \rightarrow \neg\bullet[\neg\bullet p \wedge \neg\nabla p]p) && \text{TAUT} \\
\leftrightarrow\ & (\neg\bullet p \wedge \neg\nabla p \rightarrow \neg\bullet(\neg\bullet p \wedge \neg\nabla p \rightarrow p)) && \text{AP, RE}\bullet \\
\leftrightarrow\ & (\bullet(\neg\bullet p \wedge \neg\nabla p \rightarrow p) \rightarrow (\bullet p \vee \nabla p)) && \text{TAUT} \\
\leftrightarrow\ & (\bullet(\bullet p \vee \nabla p \vee p) \rightarrow (\bullet p \vee \nabla p)) && \text{TAUT, RE}\bullet \\
\leftrightarrow\ & (\bullet(p \vee \nabla p) \rightarrow (\bullet p \vee \nabla p)) && \text{E2, RE}\bullet \\
\leftrightarrow\ & \top && \text{Prop. 53}
\end{aligned}
$$

Note that the penultimate equivalence holds, because by axiom E2, $\vdash \bullet p \rightarrow p$, thus $\vdash (\bullet p \vee \nabla p \vee p) \leftrightarrow (p \vee \nabla p)$; then using RE$\bullet$, we infer that $\vdash \bullet(\bullet p \vee \nabla p \vee p) \leftrightarrow \bullet(p \vee \nabla p)$.

$$
\begin{aligned}
& [\neg\bullet p \wedge \neg\nabla p]\neg\nabla p \\
\leftrightarrow\ & (\neg\bullet p \wedge \neg\nabla p \rightarrow \neg[\neg\bullet p \wedge \neg\nabla p]\nabla p) && \text{AN} \\
\leftrightarrow\ & (\neg\bullet p \wedge \neg\nabla p \rightarrow \neg(\neg\bullet p \wedge \neg\nabla p \rightarrow \nabla[\neg\bullet p \wedge \neg\nabla p]p \wedge \\
& \quad \nabla[\neg\bullet p \wedge \neg\nabla p]\neg p)) && \text{A}\nabla \\
\leftrightarrow\ & (\neg\bullet p \wedge \neg\nabla p \rightarrow \neg(\nabla[\neg\bullet p \wedge \neg\nabla p]p \wedge \nabla[\neg\bullet p \wedge \neg\nabla p]\neg p)) && \text{TAUT} \\
\leftrightarrow\ & (\neg\bullet p \wedge \neg\nabla p \rightarrow \neg(\nabla(\neg\bullet p \wedge \neg\nabla p \rightarrow p) \wedge \nabla(\neg\bullet p \wedge \neg\nabla p \rightarrow \neg p)) && \text{AP, AN} \\
\leftrightarrow\ & (\nabla(p \vee \bullet p \vee \nabla p) \wedge \nabla(\neg p \vee \bullet p \vee \nabla p) \rightarrow (\nabla p \vee \bullet p)) && \text{TAUT, RE}\bullet \\
\leftrightarrow\ & \top && \text{M1}
\end{aligned}
$$

Thus both $[\neg\bullet p \wedge \neg\nabla p]\neg\bullet p$ and $[\neg\bullet p \wedge \neg\nabla p]\neg\nabla p$ are provable in $\mathbf{M}^{\nabla\bullet[\cdot]}$. Therefore, $[\neg\bullet p \wedge \neg\nabla p](\neg\bullet p \wedge \neg\nabla p)$ is provable in $\mathbf{M}^{\nabla\bullet[\cdot]}$. $\square$

Intuitively, $[\neg\bullet p \wedge \neg\nabla p](\neg\bullet p \wedge \neg\nabla p)$ says that after being told that one is neither ignorant whether nor ignorant of p, one is still neither ignorant whether nor ignorant of p. In short, one's non-ignorance whether and non-ignorance of a fact cannot be altered by being announced.

The following two propositions can be shown as in Prop. 69.

Proposition 75. $p \wedge \neg p \wedge \neg\bullet p$ is successful under the intersection semantics.

Proposition 76. $p \wedge \neg p \wedge \neg\nabla p$ is successful under the intersection semantics.

Proposition 77. $p \wedge \neg\bullet p \wedge \neg\nabla p$ is successful under the intersection semantics. That is, $[p \wedge \neg\bullet p \wedge \neg\nabla p](p \wedge \neg\bullet p \wedge \neg\nabla p)$ is provable in $\mathbf{M}^{\nabla\bullet[\cdot]}$.

Proof. By AC, $[p \wedge \neg\bullet p \wedge \neg\nabla p](p \wedge \neg\bullet p \wedge \neg\nabla p) \leftrightarrow ([p \wedge \neg\bullet p \wedge \neg\nabla p]p \wedge [p \wedge \neg\bullet p \wedge \neg\nabla p]\neg\bullet p \wedge [p \wedge \neg\bullet p \wedge \neg\nabla p]\neg\nabla p)$. One may easily verify that $[p \wedge \neg\bullet p \wedge \neg\nabla p]p$ is provable in $\mathbf{M}^{\nabla\bullet[\cdot]}$. It remains only to show that both $[p \wedge \neg\bullet p \wedge \neg\nabla p]\neg\bullet p$ and $[p \wedge \neg\bullet p \wedge \neg\nabla p]\neg\nabla p$ are provable in the system in question.

We have the following proof sequence in $\mathbf{M}^{\nabla\bullet[\cdot]}$.

$$
\begin{aligned}
& [p \wedge \neg\bullet p \wedge \neg\nabla p]\neg\bullet p \\
\leftrightarrow\ & (p \wedge \neg\bullet p \wedge \neg\nabla p \rightarrow \neg[p \wedge \neg\bullet p \wedge \neg\nabla p]\bullet p) && \text{AN} \\
\leftrightarrow\ & (p \wedge \neg\bullet p \wedge \neg\nabla p \rightarrow \neg\bullet[p \wedge \neg\bullet p \wedge \neg\nabla p]p) && \text{A}\bullet \\
\leftrightarrow\ & (p \wedge \neg\bullet p \wedge \neg\nabla p \rightarrow \neg\bullet\top) && \text{AP, RE}\bullet \\
\leftrightarrow\ & (p \wedge \circ p \wedge \Delta p \rightarrow \circ\top) && \text{Def. } \circ, \text{Def. } \Delta
\end{aligned}
$$

By Prop. 43, $p \wedge \circ p \to \circ \top$ is provable in $\mathbf{M}^\bullet$, so is the last formula in the above proof sequence, and thus $[p \wedge \neg \bullet p \wedge \neg \nabla p] \neg \bullet p$ is provable in $\mathbf{M}^{\nabla \bullet [\cdot]}$.

Also, we have the following proof sequence in $\mathbf{M}^{\nabla \bullet [\cdot]}$.

$$
\begin{array}{lll}
& [p \wedge \neg \bullet p \wedge \neg \nabla p] \neg \nabla p & \\
\leftrightarrow & (p \wedge \neg \bullet p \wedge \neg \nabla p \to \neg [p \wedge \neg \bullet p \wedge \neg \nabla p] \nabla p) & \text{AN} \\
\leftrightarrow & (p \wedge \neg \bullet p \wedge \neg \nabla p \to \neg (\nabla [p \wedge \neg \bullet p \wedge \neg \nabla p] p \wedge \nabla [p \wedge \neg \bullet p \wedge \neg \nabla p] \neg p)) & \text{A} \nabla \\
\leftrightarrow & (p \wedge \circ p \wedge \Delta p \to \Delta [p \wedge \neg \bullet p \wedge \neg \nabla p] p \vee \Delta [p \wedge \neg \bullet p \wedge \neg \nabla p] \neg p) & \text{Def. } \circ, \text{Def. } \Delta \\
\leftrightarrow & (p \wedge \circ p \wedge \Delta p \to \Delta \top \vee \Delta [p \wedge \neg \bullet p \wedge \neg \nabla p] \neg p) & \text{AP}, \text{RE} \nabla
\end{array}
$$

By Prop. 42, $\Delta p \to \Delta \top$ is provable in $\mathbf{M}^\nabla$, thus the last formula in the above proof sequence is provable in $\mathbf{M}^{\nabla \bullet}$. Therefore, $[p \wedge \neg \bullet p \wedge \neg \nabla p] \neg \nabla p$ is provable in $\mathbf{M}^{\nabla \bullet [\cdot]}$.

According to the previous analysis, $[p \wedge \neg \bullet p \wedge \neg \nabla p](p \wedge \neg \bullet p \wedge \neg \nabla p)$ is provable in $\mathbf{M}^{\nabla \bullet [\cdot]}$. □

Proposition 78. $\neg p \wedge \neg \bullet p \wedge \neg \nabla p$ is successful under the intersection semantics. That is, $[\neg p \wedge \neg \bullet p \wedge \neg \nabla p](\neg p \wedge \neg \bullet p \wedge \neg \nabla p)$ is provable in $\mathbf{M}^{\nabla \bullet [\cdot]}$.

Proof. By axiom E2, $\neg p \wedge \neg \bullet p \wedge \neg \nabla p$ is equivalent to $\neg p \wedge \neg \nabla p$. And we have already shown in Prop. 73 that $\neg p \wedge \neg \nabla p$ is successful under the intersection semantics. □

Proposition 79. $p \wedge \neg p \wedge \neg \bullet p \wedge \neg \nabla p$ is successful under the intersection semantics.

Proof. The proof is similar to that of Prop. 69. □

Now we demonstrate that any combination of p, $\neg p$, $\neg \bullet p$, and $\neg \nabla p$ via *disjunction* is successful under the intersection semantics. First, one may show that $[\psi](\varphi \vee \chi) \leftrightarrow ([\psi]\varphi \vee [\psi]\chi)$ is provable from the above reduction axioms. For the sake of reference, we denote it AD.

Proposition 80. $p \vee \neg p$ is successful under the intersection semantics.

Proof. Note that $p \vee \neg p$ is equivalent to $\top$, and $\top$ is successful. □

Proposition 81. $p \vee \neg \bullet p$ is successful under the intersection semantics. That is, $[p \vee \neg \bullet p](p \vee \neg \bullet p)$ is provable in $\mathbf{M}^{\bullet [\cdot]}$.

Proof. Just note that $p \vee \neg \bullet p$ is equivalent to $\bullet p \to p$, which by E2 is equivalent to $\top$. And $\top$ is successful. □

Proposition 82. $p \vee \neg \nabla p$ is successful under the intersection semantics. That is, $[p \vee \neg \nabla p](p \vee \neg \nabla p)$ is provable in $\mathbf{M}^{\nabla [\cdot]}$.

Proof. We have the following proof sequence in $\mathbf{M}^{\nabla [\cdot]}$.

$$
\begin{array}{lll}
& [p \vee \neg \nabla p](p \vee \neg \nabla p) & \\
\leftrightarrow & ([p \vee \neg \nabla p] p \vee [p \vee \neg \nabla p] \neg \nabla p) & \text{AD} \\
\leftrightarrow & (p \vee \neg \nabla p \to p) \vee (p \vee \neg \nabla p \to \neg [p \vee \neg \nabla p] \nabla p) & \text{AP, AN} \\
\leftrightarrow & (p \vee \neg \nabla p \to p) \vee (p \vee \neg \nabla p \to \neg (\nabla [p \vee \neg \nabla p] p \wedge \nabla [p \vee \neg \nabla p] \neg p)) & \text{A} \nabla \\
\leftrightarrow & (p \vee \neg \nabla p \to p) \vee (p \vee \neg \nabla p \to \neg (\nabla (p \vee \neg \nabla p \to p) \wedge \nabla (p \vee \neg \nabla p \to \neg p))) & \text{AP, AN} \\
\leftrightarrow & (p \vee \neg \nabla p \to p \vee \neg (\nabla (p \vee \neg \nabla p \to p) \wedge \nabla (p \vee \neg \nabla p \to \neg p))) & \text{TAUT} \\
\leftrightarrow & (\neg p \wedge \nabla (p \vee \neg (p \vee \neg \nabla p)) \wedge \nabla (\neg p \vee \neg (p \vee \neg \nabla p)) \to \neg p \wedge \nabla p) & \text{TAUT, RE} \nabla \\
\leftrightarrow & \top & \text{M1}
\end{array}
$$

Therefore, $[p \vee \neg \nabla p](p \vee \neg \nabla p)$ is provable in $\mathbf{M}^{\nabla[\cdot]}$. $\qquad\square$

Proposition 83. $\neg p \vee \neg \bullet p$ is successful under the intersection semantics.

Proof. By E2, $\neg p \vee \neg \bullet p$ is equivalent to $\neg \bullet p$, and Prop. 67 has shown that $\neg \bullet p$ is successful under the intersection semantics. $\qquad\square$

Proposition 84. $\neg p \vee \neg \nabla p$ is successful under the intersection semantics. That is, $[\neg p \vee \neg \nabla p](\neg p \vee \neg \nabla p)$ is provable in $\mathbf{M}^{\nabla[\cdot]}$.

Proof. We have the following proof sequence in $\mathbf{M}^{\nabla[\cdot]}$.

$$
\begin{array}{lll}
& [\neg p \vee \neg \nabla p](\neg p \vee \neg \nabla p) & \\
\leftrightarrow & [\neg p \vee \neg \nabla p]\neg p \vee [\neg p \vee \neg \nabla p]\neg \nabla p & \text{AD} \\
\leftrightarrow & (\neg p \vee \neg \nabla p \to \neg p) \vee (\neg p \vee \neg \nabla p \to \neg[\neg p \vee \neg \nabla p]\nabla p) & \text{AN, AP} \\
\leftrightarrow & (\neg p \vee \neg \nabla p \to \neg p) \vee (\neg p \vee \neg \nabla p \to \neg(\nabla[\neg p \vee \neg \nabla p]p \wedge \nabla[\neg p \vee \neg \nabla p]\neg p)) & \text{A}\nabla \\
\leftrightarrow & (\neg p \vee \neg \nabla p \to \neg p) \vee (\neg p \vee \neg \nabla p \to \neg(\nabla(\neg p \vee \neg \nabla p \to p) \wedge \nabla(\neg p \vee \neg \nabla p \to \neg p))) & \text{AP, AN} \\
\leftrightarrow & (\neg p \vee \neg \nabla p \to \neg p \vee \neg(\nabla(\neg p \vee \neg \nabla p \to p) \wedge \nabla(\neg p \vee \neg \nabla p \to \neg p))) & \text{TAUT} \\
\leftrightarrow & p \wedge \nabla(p \vee \neg(\neg p \vee \neg \nabla p)) \wedge \nabla(\neg p \vee \neg(\neg p \vee \neg \nabla p)) \to p \wedge \nabla p & \text{TAUT} \\
\leftrightarrow & \top & \text{M1}
\end{array}
$$

Therefore, $[\neg p \vee \neg \nabla p](\neg p \vee \neg \nabla p)$ is provable in $\mathbf{M}^{\nabla[\cdot]}$. $\qquad\square$

Proposition 85. $\neg \bullet p \vee \neg \nabla p$ is successful under the intersection semantics. That is, $[\neg \bullet p \vee \neg \nabla p](\neg \bullet p \vee \neg \nabla p)$ is provable in $\mathbf{M}^{\nabla \bullet[\cdot]}$.

Proof. We have the following proof sequence in $\mathbf{M}^{\nabla \bullet[\cdot]}$.

$$
\begin{array}{lll}
& [\neg \bullet p \vee \neg \nabla p](\neg \bullet p \vee \neg \nabla p) & \\
\leftrightarrow & ([\neg \bullet p \vee \neg \nabla p]\neg \bullet p \vee [\neg \bullet p \vee \neg \nabla p]\neg \nabla p) & \text{AD} \\
\leftrightarrow & (\neg \bullet p \vee \neg \nabla p \to \neg[\neg \bullet p \vee \neg \nabla p]\bullet p) \vee & \\
& (\neg \bullet p \vee \neg \nabla p \to \neg[\neg \bullet p \vee \neg \nabla p]\nabla p) & \text{AN} \\
\leftrightarrow & (\neg \bullet p \vee \neg \nabla p \to \neg \bullet[\neg \bullet p \vee \neg \nabla p]p) \vee & \\
& (\neg \bullet p \vee \neg \nabla p \to \neg(\nabla[\neg \bullet p \vee \neg \nabla p]p \wedge & \\
& \nabla[\neg \bullet p \vee \neg \nabla p]\neg p)) & \text{A}\bullet, \text{A}\nabla \\
\leftrightarrow & (\neg \bullet p \vee \neg \nabla p \to \neg \bullet(\neg \bullet p \vee \neg \nabla p \to p)) \vee & \\
& (\neg \bullet p \vee \neg \nabla p \to \neg(\nabla(\neg \bullet p \vee \neg \nabla p \to p) \wedge & \\
& \nabla(\neg \bullet p \vee \neg \nabla p \to \neg p))) & \text{AP, AN} \\
\leftrightarrow & (\neg \bullet p \vee \neg \nabla p \to \neg \bullet(\neg \bullet p \vee \neg \nabla p \to p) \vee & \\
& \neg(\nabla(\neg \bullet p \vee \neg \nabla p \to p) \wedge \nabla(\neg \bullet p \vee \neg \nabla p \to \neg p))) & \text{TAUT} \\
\leftrightarrow & \bullet(\neg \bullet p \vee \neg \nabla p \to p) \wedge \nabla(\neg \bullet p \vee \neg \nabla p \to p) \wedge & \\
& \nabla(\neg \bullet p \vee \neg \nabla p \to \neg p) \to \bullet p \wedge \nabla p & \text{TAUT} \\
\leftrightarrow & \bullet(\circ p \vee \Delta p \to p) \wedge \nabla(\circ p \vee \Delta p \to p) \wedge & \\
& \nabla(\circ p \vee \Delta p \to \neg p) \to \bullet p \wedge \nabla p & \text{Def. } \circ, \text{Def. } \Delta
\end{array}
$$

By Prop. 54, $\bullet(\circ p \vee \Delta p \to p) \to \bullet p$ is provable in $\mathbf{M}^{\nabla \bullet}$; by axiom M1 and RE$\nabla$, we can show the provability of $\nabla(\circ p \vee \Delta p \to p) \wedge \nabla(\circ p \vee \Delta p \to \neg p) \to \nabla p$ in $\mathbf{M}^{\nabla \bullet}$. Therefore, $[\neg \bullet p \vee \neg \nabla p](\neg \bullet p \vee \neg \nabla p)$ is provable in $\mathbf{M}^{\nabla \bullet[\cdot]}$. $\qquad\square$

Intuitively, $[\neg\bullet p \vee \neg\nabla p](\neg\bullet p \vee \neg\nabla p)$ says that one's either non-ignorance of or non-ignorance whether a fact cannot be altered by being announced: after being told that one is either not ignorant of or not ignorant whether p, one is still either not ignorant of or not ignorant whether p.

Next two propositions are shown as in Prop. 80.

Proposition 86. $p \vee \neg p \vee \neg\bullet p$ is successful under the intersection semantics.

Proposition 87. $p \vee \neg p \vee \neg\nabla p$ is successful under the intersection semantics.

Proposition 88. $p \vee \neg\bullet p \vee \neg\nabla p$ is successful under the intersection semantics. That is, $[p \vee \neg\bullet p \vee \neg\nabla p](p \vee \neg\bullet p \vee \neg\nabla p)$ is provable in $\mathbf{M}^{\nabla\bullet[\cdot]}$.

Proof. By axiom E2, $p \vee \neg\bullet p$ is equivalent to $\top$, so is $p \vee \neg\bullet p \vee \neg\nabla p$. And $\top$ is successful. $\qquad\square$

Proposition 89. $\neg p \vee \neg\bullet p \vee \neg\nabla p$ is successful under the intersection semantics.

Proof. By E2, $\neg p \vee \neg\bullet p \vee \neg\nabla p$ is equivalent to $\neg\bullet p \vee \neg\nabla p$, and we have shown in Prop. 85 that $\neg\bullet p \vee \neg\nabla p$ is successful under the intersection semantics. $\qquad\square$

Proposition 90. $p \vee \neg p \vee \neg\bullet p \vee \neg\nabla p$ is successful under the intersection semantics.

Proof. The proof is similar to that of Prop. 80. $\qquad\square$

Now we come back to the example in the opening paragraph of this section. We will show that $\nabla p \to [p]\neg\nabla p$ and $\bullet p \to [p]\neg\bullet p$ are both provable in $\mathbf{EMN}^{\nabla\bullet[\cdot]}$.[15]

Proposition 91. $\nabla p \to [p]\neg\nabla p$ is provable in $\mathbf{EMN}^{\nabla\bullet[\cdot]}$.

Proof. We show a stronger result: $[p]\neg\nabla p$ is provable in $\mathbf{EMN}^{\nabla\bullet[\cdot]}$. For this, we have the following proof sequence in $\mathbf{EMN}^{\nabla\bullet[\cdot]}$.

$$
\begin{array}{lll}
 & [p]\neg\nabla p & \\
\leftrightarrow & p \to \neg[p]\nabla p & \text{AN} \\
\leftrightarrow & p \to \neg(p \to \nabla[p]p \wedge \nabla[p]\neg p) & \text{A}\nabla \\
\leftrightarrow & p \to \neg(p \to \nabla[p]p \wedge \nabla(p \to \neg[p]p)) & \text{AN} \\
\leftrightarrow & p \to \neg\nabla(p \to p) \vee \neg\nabla(p \to \neg(p \to p)) & \text{AP} \\
\leftrightarrow & p \to \neg\nabla\top \vee \neg\nabla\neg p & \text{TAUT}, \text{RE}\nabla \\
\leftrightarrow & p \to \Delta\top \vee \Delta\neg p & \text{Def. } \Delta
\end{array}
$$

Since $\Delta\top$ is provable in $\mathbf{EN}^{\nabla\bullet}$, it is also provable in $\mathbf{EMN}^{\nabla\bullet[\cdot]}$. Therefore, $[p]\neg\nabla p$ is provable in $\mathbf{EMN}^{\nabla\bullet[\cdot]}$. $\qquad\square$

Proposition 92. $\bullet p \to [p]\neg\bullet p$ is provable in $\mathbf{EMN}^{\nabla\bullet[\cdot]}$.

---

[15]Note that the system $\mathbf{M}^{\nabla\bullet[\cdot]}$ is too weak to derive these two formulas, for which the countermodels are not hard to find.

Proof. We show a stronger result: $[p]\neg\bullet p$ is provable in $\mathbf{EMN}^{\nabla\bullet[\cdot]}$. For this, we have the following proof sequence in $\mathbf{EMN}^{\nabla\bullet[\cdot]}$.

$$
\begin{aligned}
& [p]\neg\bullet p \\
\leftrightarrow\ & p \to \neg[p]\bullet p && \text{AN} \\
\leftrightarrow\ & p \to \neg(p \to \bullet[p]p) && \text{A}\bullet \\
\leftrightarrow\ & p \to \neg\bullet[p]p && \text{TAUT} \\
\leftrightarrow\ & p \to \neg\bullet(p \to p) && \text{AP}, \text{RE}\bullet \\
\leftrightarrow\ & p \to \neg\bullet\top && \text{TAUT}, \text{RE}\bullet \\
\leftrightarrow\ & p \to \circ\top && \text{Def.}\ \circ
\end{aligned}
$$

Since $\circ\top$ is an axiom of $\mathbf{EMN}^{\nabla\bullet}$, $p \to \circ\top$ is provable in $\mathbf{EMN}^{\nabla\bullet[\cdot]}$. Therefore, $[p]\neg\bullet p$ is provable in $\mathbf{EMN}^{\nabla\bullet[\cdot]}$. $\qquad\square$

Recall from [11, Sec. 7.2] that it is shown in a proof-theoretical perspective that Moore sentences are self-refuting and the negation of Moore sentences are all successful in the minimal Kripke logic; in symbol, $[\bullet p]\neg\bullet p$ and $[\neg\bullet p]\neg\bullet p$ are both provable in $\mathbf{K}^{\bullet[\cdot]}$. We now check if they are provable in the weaker systems $\mathbf{M}^{\bullet[\cdot]}$ and $\mathbf{EMN}^{\bullet[\cdot]}$. It is shown that $[\neg\bullet p]\neg\bullet p$ is provable in $\mathbf{M}^{\bullet[\cdot]}$ (thus also provable in $\mathbf{EMN}^{\bullet[\cdot]}$) (see [15, Prop. 6.5], also see Prop. 67). In contrast, $[\bullet p]\neg\bullet p$ is not provable in $\mathbf{M}^{\bullet[\cdot]}$, but provable in $\mathbf{EMN}^{\bullet[\cdot]}$.

Proposition 93. $[\bullet p]\neg\bullet p$ is *not* provable in $\mathbf{M}^{\bullet[\cdot]}$, but provable in $\mathbf{EMN}^{\bullet[\cdot]}$.

Proof. We have the following proof sequence in $\mathbf{M}^{\bullet[\cdot]}$.

$$
\begin{aligned}
[\bullet p]\neg\bullet p \quad \leftrightarrow\ & (\bullet p \to \neg[\bullet p]\bullet p) && \text{AN} \\
\leftrightarrow\ & (\bullet p \to \neg(\bullet p \to \bullet[\bullet p]p)) && \text{A}\bullet \\
\leftrightarrow\ & (\bullet p \to \neg\bullet[\bullet p]p) && \text{TAUT} \\
\leftrightarrow\ & (\bullet p \to \neg\bullet(\bullet p \to p)) && \text{AP}, \text{RE}\bullet \\
\leftrightarrow\ & (\bullet p \to \neg\bullet\top) && \text{E2}, \text{RE}\bullet
\end{aligned}
$$

Note that $\bullet p \to \neg\bullet\top$ is not valid over the class of monotone neighborhood frames $\mathbb{F}_s$, neither is $[\bullet p]\neg\bullet p$. To see this, consider a monotone neighborhood model $\mathcal{M} = \langle S, N, V\rangle$ where $S = \{s\}$, $N(s) = \emptyset$ and $V(p) = \{s\}$. One may verify that $\mathcal{M}, s \vDash \bullet p$ and $\mathcal{M}, s \vDash \bullet\top$, thus $\mathbb{F}_s \nvDash \bullet p \to \neg\bullet\top$.

On the other hand, as $\neg\bullet\top$, that is, $\circ\top$ is an axiom in $\mathbf{EMN}^{\bullet[\cdot]}$, thus $[\bullet p]\neg\bullet p$ is provable in $\mathbf{EMN}^{\bullet[\cdot]}$. $\qquad\square$

7 Conclusion and Future Work

The primary aim of this paper is to investigate the bimodal logic of Fitchean ignorance and first-order ignorance under the neighborhood semantics. The main contributions include model-theoretical results such as expressivity and frame definability, and axiomatizations. We showed that the reducibility issue in [20] stating that all higher-order ignorance are reducible to second-order ignorance does not arise in our neighborhood setting. Besides, by updating the neighborhood models via the intersection semantics,

we found suitable reduction axioms and thus reduced the public announcement operators to the bimodal logic. This gives us good applications to successful formulas, and we obtain two fragments of successful formulas in the bimodal logic. As a side product, we also explored the relationships among various forms of ignorance. Given that Fitchean ignorance and first-order ignorance are epistemic counterparts of accident and contingency, we also answered an open question raised in [13, 15], where it is asked how to explore the neighborhood semantics of bimodal logic with contingency and accident.

For future work, we hope to know whether $\mathcal{L}(\nabla, \bullet)$ is less expressive than $\mathcal{L}(\Diamond)$ over the class of (d)-models. We conjecture the answer is positive, but the model constructions seems hard, where the desired models both needs at least three points. Moreover, as we have seen, the proofs of the expressivity and frame definability results involve nontrivial (if not highly nontrivial) constructions of neighborhood models and frames, we thus also hope to find the bisimulation notion for $\mathcal{L}(\nabla, \bullet)$ under the neighborhood semantics.

Acknowledgements

References

[1] Z. Bakhtiari, H. van Ditmarsch, and H. H. Hansen. Neighbourhood contingency bisimulation. In *Indian Conference on Logic and Its Applications*, pages 48–63. Springer, Berlin, Heidelberg, 2017, January.

[2] R. Bett. Socratic ignorance. In Donald R. Morrison, editor, *The Cambridge Companion to Socrates*, pages 215–236. Cambridge University Press, 2011.

[3] J. C. Bjerring, J. U. Hansen, and N. J. L. L. Pedersen. On the rationality of pluralistic ignorance. *Synthese*, 191(11):2445–2470, 2014.

[4] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, Cambridge, 2001. Cambridge Tracts in Theoretical Computer Science 53.

[5] S. Bonzio, V. Fano, P. Graziani, and M. Pra Baldi. A logical modeling of severe ignorance. *Journal of Philosophical Logic*, 52:1053–1080, 2023.

[6] A. M. Bricker. Knowledge as a (non-factive) mental state. *Erkenntnis*, 2023. https://doi.org/10.1007/s10670-023-00697-0.

[7] B. F. Chellas. *Modal Logic: An Introduction*. Cambridge University Press, 1980.

[8] J. Fan. Logics of essence and accident. arXiv preprint arXiv:1506.01872, unpublished manuscript, 2015.

[9] J. Fan. Removing your ignorance by announcing group ignorance: A group announcement logic for ignorance. *Studies in Logic*, 9(4):4–33, 2016.

[10] J. Fan. Neighborhood contingency logic: A new perspective. *Studies in Logic*, 11(4):37–55, 2018.

[11] J. Fan. Bimodal logics with contingency and accident. *Journal of Philosophical Logic*, 48:425–445, 2019.

[12] J. Fan. A family of neighborhood contingency logics. *Notre Dame Journal of Formal Logic*, 60(4):683–699, 2019.

[13] J. Fan. Bimodal logic with contingency and accident: Bisimulation and axiomatizations. *Logica Universalis*, 15:123–147, 2021. https://doi.org/10.1007/s11787-021-00270-9.

[14] J. Fan. A logic for disjunctive ignorance. *Journal of Philosophical Logic*, 50(6):1293–1312, 2021. https://doi.org/10.1007/s10992-021-09599-4.

[15] J. Fan. Unknown truths and false beliefs: Completeness and expressivity results for the neighborhood semantics. *Studia Logica*, 110:1–45, 2022. https://doi.org/10.1007/s11225-021-09950-5.

[16] J. Fan and H. van Ditmarsch. Neighborhood contingency logic. In M. Banerjee and S. Krishna, editors, *Logic and Its Application*, volume 8923 of *Lecture Notes in Computer Science*, pages 88–99. Springer, 2015.

[17] J. Fan, Y. Wang, and H. van Ditmarsch. Almost necessary. In *Advances in Modal Logic*, volume 10, pages 178–196, 2014.

[18] J. Fan, Y. Wang, and H. van Ditmarsch. Contingency and knowing whether. *The Review of Symbolic Logic*, 8(1):75–107, 2015.

[19] C. Fawell. Interrogating the linguistic argument for KK. *Journal of Philosophy*, forthcoming.

[20] K. Fine. Ignorance of ignorance. *Synthese*, 195:4031–4045, 2018. https://doi.org/10.1007/s11229-017-1406-z.

[21] D. Gilbert and G. Venturi. Neighborhood semantics for logics of unknown truths and false beliefs. *The Australasian Journal of Logic*, 14(1):246–267, 2017.

[22] D. Gilbert and G. Venturi. A note on logics of essence and accident. *Logic Journal of the IGPL*, 28(5):881–891, 2020.

[23] V. Goranko. On relative ignorance. *Filosofiska Notiser*, 8(1):119–140, 2021.

[24] A. Hazlett. The myth of factive verbs. *Philosophy and phenomenological research*, 80(3):497–522, 2010.

[25] L. Humberstone. The logic of non-contingency. *Notre Dame Journal of Formal Logic*, 36(2):214–229, 1995.

[26] K. Konolige. Circumscriptive ignorance. In *AAAI*, pages 202–204, 1982.

[27] E. Kubyshkina and M. Petrolo. A logic for factive ignorance. *Synthese*, pages 1–12, 2019.

[28] S. Kuhn. Minimal non-contingency logic. *Notre Dame Journal of Formal Logic*, 36(2):230–234, 1995.

[29] P. Le Morvan and R. Peels. The nature of ignorance: Two views. In R. Peels and M. Blaauw, editors, *The epistemic dimensions of ignorance*. Cambridge: Cambridge University Press, 2016.

[30] M. Ma and K. Sano. How to update neighborhood models. In *International Workshop on Logic, Rationality and Interaction*, pages 204–217. Springer, 2013.

[31] M. Ma and K. Sano. How to update neighbourhood models. *Journal of Logic and Computation*, 28(8):1781–1804, December 2018. https://doi.org/10.1093/logcom/exv026.

[32] J. Marcos. Logics of essence and accident. *Bulletin of the Section of Logic*, 34(1):43–56, 2005.

[33] R. Montague. Universal grammar. *Theoria*, 36:373–398, 1970.

[34] H. Montgomery and R. Routley. Contingency and non-contingency bases for normal modal logics. *Logique et Analyse*, 9:318–328, 1966.

[35] P. Le Morvan. On ignorance: A reply to Peels. *Philosophia*, 39(2):335–344, 2011.

[36] Hubert J O'Gorman and Stephen L Garry. Pluralistic ignorance — a replication and extension. *Public Opinion Quarterly*, 40(4):449–458, 1976.

[37] E. J. Olsson and C. Proietti. Explicating ignorance and doubt: A possible worlds approach. *The Epistemic Dimensions of Ignorance*, 2015.

[38] E. Pacuit. *Neighborhood semantics for modal logic*. Springer, 2017.

[39] R. Peels. Ignorance is Lack of True Belief: A Rejoinder to Le Morvan. *Philosophia*, 39(2):344–355, 2011.

[40] B. Plato. *The Apology of Socrates*. Edited by Adam, A. M., Cambridge University Press, 1964.

[41] B. Plato. *Apology*. BookRix, 2019.

[42] C. Proietti and E. J. Olsson. A ddl approach to pluralistic ignorance and collective belief. *Journal of Philosophical Logic*, 43(2):499–515, 2014.

[43] D. Scott. Advice on modal logic. *Philosophical Problems in Logic: Some Recent Developments*, pages 143–173, 1970.

[44] Y. Shoham. Chronological ignorance. In *Proceedings of the Fifth National Conference on Artificial Intelligence*, pages 389–393, 1986.

[45] C. Steinsvold. Completeness for various logics of essence and accident. *Bulletin of the Section of Logic*, 37(2):93–101, 2008.

[46] C. Steinsvold. A note on logics of ignorance and borders. *Notre Dame Journal of Formal Logic*, 49(4):385–392, 2008.

[47] C. Steinsvold. The boxdot conjecture and the language of essence and accident. *Australasian Journal of Logic*, 10:18–35, 2011.

[48] W. van der Hoek and A. Lomuscio. Ignore at your peril - towards a logic for ignorance. In *Proc. of 2nd AAMAS*, pages 1148–1149. ACM, 2003.

[49] W. van der Hoek and A. Lomuscio. A logic for ignorance. *Electronic Notes in Theoretical Computer Science*, 85(2):117–133, 2004.

[50] W. van der Hoek and A. Lomuscio. A logic for ignorance. In *Declarative agent languages and technologies. DALT 2003. Lecture Notes in Computer Science*, volume 2990, pages 97–108. Springer, Berlin, Heidelberg, 2004.

[51] H. van Ditmarsch, J. Fan, W. van der Hoek, and P. Iliev. Some exponential lower bounds on formula-size in modal logic. In *Advances in Modal Logic*, volume 10, pages 139–157, 2014.

[52] H. van Ditmarsch, W. van der Hoek, and B. Kooi. *Dynamic Epistemic Logic*, volume 337 of *Synthese Library*. Springer, 2007.

[53] T. Williamson. *Knowledge and its Limits*. Oxford University Press, 2000.

[54] E. Zolin. Completeness and definability in the logic of noncontingency. *Notre Dame Journal of Formal Logic*, 40(4):533–547, 1999.

[55] J. Zvesper. *Playing with information*. PhD thesis, Institute for Logic, Language and Computation, University of Amsterdam, 2010.