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A multiplicative ingredient for ω-inconsistency

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Abstract

This paper presents a distinctively multiplicative quantificational principle that arguably captures the problematic aspects of Zardini's infinitary rules for a multiplicative quantifier within the context of the semantic paradoxes and the theoretical goal to obtain a (ω)-consistent theory of transparent truth. After showing that the principle is derivable with Zardini's rules and that one obtains through vacuous quantification an inconsistent theory of truth if truth is transparent, the paper presents two results regarding the principle and ω-inconsistency. First, the principle is used to obtain a non-classical variant of McGee's ω-inconsistency result for certain classical theories of truth. Second, it is demonstrated that the conditions for a truth-theoretic variant of Bacon's ω-inconsistency result for certain non-classical theories of transparent truth implies that the principle holds for the paradoxical formula. Finally, the paper argues that the paradoxical reasoning that the principle enables is structurally similar to the kind of infinitary reasoning popularised by Hilbert's Grand Hotel.

Keywords: Non-contractive logic, multiplicative quantifier, theories of truth, semantic paradoxes, omega-inconsistency, McGee's theorem

1 Introduction

Within substructural logics as an area of research, one of the central distinctions is that between additive and multiplicative connectives. For example, additive and multiplicative conjunction, here represented respectively by $\wedge$ and &, can be defined as follows in a single-succedent sequent calculus where the antecedent is a possibly empty finite multiset of formulas:[1][2]

[1]The labels "additive" and "multiplicative" come from the literature on linear logic, see e.g. Girard (1987). Other research traditions or perspectives use different labels. For example, the same distinction is expressed in terms of weak vs strong connectives by Gottwald (2022), as extensional vs intensional connectives by Humberstone (2020) and as lattice-theoretic vs group-theoretic connectives by Paoli (2002). In the literature on relevant logics, "conjunction" and "disjunction" are used for additive conjunction and disjunction, and "fusion" and "fission" for their multiplicative counterparts, see e.g. Dunn and Restall (2002).

[2]Sequents will throughout the paper be single-succedent with a multiset antecedent. The paper also assumes some familiarity with sequent calculi as proof-theoretic framework. See for example Negri and von Plato (2001) or Troelstra and Schwichtenberg (2000) for an introduction.

$$\frac{A, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ (\wedge\mathrm{L}^1) \qquad \frac{B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ (\wedge\mathrm{L}^2) \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ (\wedge\mathrm{R})$$

$$\frac{A, B, \Gamma \Rightarrow C}{A\&B, \Gamma \Rightarrow C}\ (\&\mathrm{L}) \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma' \Rightarrow B}{\Gamma, \Gamma' \Rightarrow A\&B}\ (\&\mathrm{R})$$

The differences between the rules for multiplicative and additive conjunction can be characterised as follows. In the additive case, the right introduction rule ($\wedge$R) has shared context and there are two left introduction rules ($\wedge\mathrm{L}^1$) and ($\wedge\mathrm{L}^2$), one for each immediate subformula where that formula occurs as active formula in the antecedent. In the multiplicative case, the right introduction rule (&R) has independent contexts and there is only one left introduction rule (&L) where every immediate subformula occurs as active formula in the antecedent.

Importantly, a distinction between additive and multiplicative conjunction requires that one of the following structural rules is not admissible:[3]

$$\text{Contraction: } \frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C} \qquad \text{Weakening: } \frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}$$

The admissibility of weakening implies that $A\&B \Rightarrow A \wedge B$ is derivable, and the admissibility of contraction implies that $A \wedge B \Rightarrow A\&B$ is derivable.

One can generalise the description of the distinction between additive and multiplicative conjunction to obtain additive and multiplicative universal quantifiers. The additive universal quantifier is defined as follows where y is an eigenvariable; it does not occur free in the conclusion-sequent of the right rule:

$$\frac{A(t/x), \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\ (\forall\mathrm{L}_A) \qquad \frac{\Gamma \Rightarrow A(y/x)}{\Gamma \Rightarrow \forall x A}\ (\forall\mathrm{R})$$

As with the left introduction rules ($\wedge\mathrm{L}^1$) and ($\wedge\mathrm{L}^2$) for additive conjunction, the rule ($\forall\mathrm{L}_A$) is weakening the antecedent by replacing a single instance with a formula implying every instance (the universally quantified formula itself). The right rule ($\forall$R) corresponds to the right rule ($\wedge$R) for additive conjunction since the premise-sequent is normally derivable for any term through substitution because y is an eigenvariable. It should thus hold for every instance of the universally quantified formula.

Whereas there is a clear consensus about the adequacy of the rules for the additive quantifiers, this is not the case with multiplicative quantification. That there should be such a notion of quantification has been argued by Paoli (2005) and an infinitary proposal has been developed by Zardini (2011) along the following lines. As in the case of multiplicative conjunction, it is natural to conclude that the left introduction rule for a multiplicative universal quantifier should include every instance as active formula in the premise-sequent and that the right introduction rule should have one premise-sequent with independent contexts

[3] A rule is admissible just in case there is a derivation ending with the conclusion-sequent whenever there is a derivation ending with the premise-sequent(s).

for every instance. This proposal amounts to the following rules in the case of single-succedent sequents:[4]

$$\frac{A(t_0/x), A(t_1/x), A(t_2/x), \ldots, \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\ \forall^Z\mathrm{L}$$

$$\frac{\Gamma_i \Rightarrow A(t_i/x) \quad \text{for each } i}{\Gamma_0, \Gamma_1, \Gamma_2, \ldots \Rightarrow \forall x A}\ \forall^Z\mathrm{R}$$

The rules assume an enumeration of the terms $t_0, t_1, t_2, \ldots$. If the corresponding set is denumerable, then the antecedent of the premise-sequent of ($\forall^Z$L) will be, and the antecedent of the conclusion-sequent of ($\forall^Z$R) may be, an infinite multiset. In addition, ($\forall^Z$R) will have infinitely many premise-sequents, one for each term.

The adequacy of these rules and the corresponding notion of a multiplicative universal quantifier depends on context of use for the quantifier. I will in this paper restrict my attention to the application proposed by Zardini (2011), namely as a notion of quantification compatible with naive semantic concepts and in particular transparent truth.[5]

Zardini (2011, p.503-504) observes that contraction is a crucial ingredient in the semantic paradoxes and suggests that we can tame them by restricting that rule. To show how this can be done, Zardini (2011) proceeds to present a theory of transparent truth based on a logic with multiplicative connectives and quantifiers, where the universal multiplicative quantifier is defined with the multiple-succedent variants of the rules ($\forall^Z$L) and ($\forall^Z$R) above. To show that the proposal is consistent, Zardini (2011) presents a cut-elimination proof for the sequent calculus defining the theory.

However, the results by Da Ré and Rosenblatt (2018), Fjellstad (2018), Fjellstad (2020), Fjellstad and Olsen (2021), Petersen (2023) and Nicolai et al. (2023) suggest that the above rules are inadequate for that purpose.

A formal theory of truth where truth is represented with a unary predicate requires a method that associates each formula with a unique closed term and allows for the generation of paradoxical formulas such as the liar sentence, a sentence which "says of itself" that it is not true. This can be achieved for example through meta-linguistic stipulation, where one simply assumes an appropriate mapping in the meta-theory which also ensures the existence of paradoxical formulas because one stipulates for example that the closed term λ is the name

[4]Corresponding rules for an infinitary multiplicative disjunction are presented and explored by Pereira and Haeusler (1999). The original rules by Zardini (2011) involve multiple-succedent sequents. That is not of importance for this paper, and I will thus stick to single-succedent sequents.

[5]A unary predicate *Tr* intended to represent "is a true sentence" is transparent if and only if a formula A is intersubstitutable within a valid inference with the formula $Tr(t)$ where t is a closed term that functions as a name for A. See e.g. Beall et al. (2023) for an introduction to the liar paradox and the complications arising from a transparent truth-predicate within for example classical logic.

For other applications of multiplicative quantification, see e.g. Paoli (2005) and Lanzinger et al. (2022).

for the formula $Tr(\lambda) \to \bot$ where then Tr is the designated truth-predicate.[6] Alternatively, one can use so-called Gödel-coding by adding a sufficiently strong theory of arithmetic, associate each formula with a unique numeral and use either the strong or the weak diagonal lemma to ensure the existence of paradoxical formulas, depending on whether the theory defines function-symbols for every primitive recursive function.[7] In the sprit of primitive recursive arithmetic, one can also skip the detour through the strong diagonal lemma by simply letting for example a nullary function-symbol $\dot{\lambda}$ represent the (primitive recursive) nullary function that returns the Gödel-code of $Tr(\dot{\lambda}) \to \bot$. While I will do something along the latter lines in this paper, Zardini (2011) uses meta-linguistic stipulation.

Now, Da Ré and Rosenblatt (2018) show that Zardini's theory of truth is trivial if defined for a language of arithmetic and expanded with equations for certain primitive recursive functions. They achieve this by proving that the ω-inconsistency result by Bacon (2013) holds for such a theory. Triviality follows because an ω-inconsistent theory closed under the ω-rule is inconsistent, and the rule ($\forall^Z$R) is in effect an ω-rule if formulated for a language of arithmetic. However, Da Ré and Rosenblatt (2018) did not take their result to imply that something is wrong with the consistency proof presented by Zardini (2011) since their result requires a theory of arithmetic together with Gödel-coding and Zardini (2011) used meta-linguistic stipulation to generate paradoxical formulas. In response to that, Fjellstad (2018) simplifies the argument presented by Da Ré and Rosenblatt (2018) with a sequent calculus for labelled formulas that defines a necessity-like predicate using rules that are structurally identical to Zardini's rules for multiplicative quantifiers and which uses meta-linguistic stipulation to generate the paradoxical formula. The simplified argument is employed to show that there must be an error in the consistency proof presented by Zardini (2011). That there is an error is demonstrated by Fjellstad (2020). Nicolai et al. (2023) explains how to amend the proof presented by Zardini (2011) to avoid that error, but they also proceed to present a more fundamental error with the consistency proof.

Finally, Fjellstad and Olsen (2021) and Petersen (2023) point out that the standard understanding of vacuous quantification suffices for triviality with Zardini's multiplicative quantifiers. Nicolai et al. (2023) builds on this observation to show that classical propositional logic can be faithfully interpreted through vacuous quantification with Zardini's rules for multiplicative quantifiers.

The purpose of this paper is to present a principle (MI) which is derivable with Zardini's rules and which seems to capture the problematic aspects of Zardini's rules: it allows for triviality through vacuous quantification with transparent truth and suffices for ω-inconsistency under the same truth-theoretic assumptions as the result by McGee (1985). I also show that (MI) holds for the paradoxical formula employed in a variant of the ω-inconsistency result by Ba-

[6]For a presentation of this method, sometimes referred to as meta-linguistic coding, see Ripley (2012).

[7]For the distinction between the weak and the strong diagonal lemma, see for example Heck (2007).

con (2013). Finally, the paper argues that the paradoxical reasoning that (MI) enables is structurally similar to the kind of infinitary reasoning popularised by Hilbert's Grand Hotel.

Section 2 presents the logical and arithmetical background assumptions together with the principle (MI). Section 3 presents some observations about (MI), among others a comparison with classical quantification and that it leads to triviality through vacuous quantification with transparent truth. Section 4 presents the observations regarding McGee's and Bacon's ω-inconsistency results. Section 5 presents the comparison with Hilbert's Grand Hotel.

2 Logical and arithmetical preliminaries

I will work with a first-order language for arithmetic based on $\rightarrow$, &, $\forall$ and $\bot$ as logical vocabulary, 0 and $\dot{s}$ as arithmetic vocabulary in addition to the one-place function-symbols $\dot{T}$, $\dot{\mu}$ and $\dot{\gamma}$, and one two-place function-symbol $\dot{\&}$. The language will also contain a unary predicate *Tr* and a nullary function-symbol $\dot{\lambda}$. I will use $\ulcorner A \urcorner$ to represent the numeral of the Gödel-code of the formula A. Polish notation will be used for complex terms to avoid that parentheses clutter the formulas. All substitutions of equals will be performed lazily in the meta-theory.

Importantly, by explicitly defining the required primitive recursive functions through equations, I am in effect working with a fragment of Primitive Recursive Arithmetic (PRA) as opposed to say Robinson Arithmetic with quantifiers relativised to the natural numbers (Q) or Peano Arithmetic (PA). I thus do not assume that something like the weak diagonal lemma involving a complex formula holds for the theory of arithmetic obtained by expanding $\mathcal{C}\text{BCK}_\bot$ as defined below with additive quantifiers and the axioms of Q or PA. This stands in contrast to for example McGee (1985) and research on axiomatic theories of truth in general as presented by Halbach (2010). The reader worrying about self-reference may observe that one still obtains the strong diagonal lemma along the lines of Burgess (1986) and Heck (2007) using a function-symbol for the substitution-function. I could then have used that lemma to define for example a complex term instead of using a nullary function-symbol $\dot{\lambda}$ to generate a liar-like sentence. However, I have chosen to bracket such technicalities. Instead, it suffices to underline that I work with function-symbols defined along the lines of PRA.

For the propositional vocabulary I will use the following rules:

$$\frac{}{\bot, \Gamma \Rightarrow C}\ \bot \qquad \frac{}{A, \Gamma \Rightarrow A}\ \text{id}$$

$$\frac{\Gamma \Rightarrow A \qquad B, \Gamma' \Rightarrow C}{A \rightarrow B, \Gamma, \Gamma' \Rightarrow C}\ {\rightarrow}\text{L} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\ {\rightarrow}\text{R}$$

$$\frac{A, B, \Gamma \Rightarrow C}{A \& B, \Gamma \Rightarrow C}\ \&\text{L} \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma' \Rightarrow B}{\Gamma, \Gamma' \Rightarrow A \& B}\ \&\text{R}$$

$$\frac{\Gamma \Rightarrow A \qquad A, \Gamma' \Rightarrow C}{\Gamma, \Gamma' \Rightarrow C}\text{cut}$$

Weakening is admissible through the arbitrary context in the antecedent of every rule and the initial sequents. Contraction is not admissible. Being a sequent calculus for the logic BCK with additive falsum, I will call this calculus $\mathcal{C}\text{BCK}_\bot$.

The following single-premise rule is admissible through the rules ($\bot$) and ($\to$L):

$$\frac{\Gamma \Rightarrow A}{A \to \bot, \Gamma \Rightarrow C}\to\text{L}/\bot$$

This is thus not a primitive rule in the system, but I will use it on occasion to reduce the number of leafs and thereby save some horizontal space.

The infinitary rules for a multiplicative universal quantifier become the following multiplicative ω-rules when formulated for a language of arithmetic:

$$\frac{A(0/x), A(\dot{s}0/x), A(\dot{s}\dot{s}0/x), \ldots, \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\forall^\omega\text{L}$$

$$\frac{\Gamma_0 \Rightarrow A(0/x) \qquad \Gamma_1 \Rightarrow A(\dot{s}0/x) \qquad \Gamma_2 \Rightarrow A(\dot{s}\dot{s}0/x) \qquad \ldots}{\Gamma_0, \Gamma_1, \Gamma_2, \ldots \Rightarrow \forall x A}\forall^\omega\text{R}$$

Let's call the sequent calculus obtained by expanding $\mathcal{C}\text{BCK}_\bot$ with these rules $\mathcal{C}\text{BCK}_\bot^{\forall\omega}$. Observe that whereas the antecedent in sequents of $\mathcal{C}\text{BCK}_\bot$ are possibly empty finite multisets of formulas, the antecedent in sequents of $\mathcal{C}\text{BCK}_\bot^{\forall\omega}$ are possibly empty countable multisets of formulas.

The principle that will take the centre stage of this paper is the following labelled (MI) for *The Multiplicative Ingredient*:[8]

$$\text{(MI)} \qquad \forall x A \Rightarrow \forall y A(\dot{s}y/x) \& A(0/x)$$

As announced in the introduction, the principle (MI) is derivable in $\mathcal{C}\text{BCK}_\bot^{\forall\omega}$:

$$\dfrac{(\&\text{R})\,\dfrac{\dfrac{A(\dot{s}0/x) \Rightarrow A(\dot{s}0/x) \quad A(\dot{s}\dot{s}0/x) \Rightarrow A(\dot{s}\dot{s}0/x) \quad \ldots}{A(\dot{s}0/x), A(\dot{s}\dot{s}0/x), \ldots \Rightarrow \forall x A(\dot{s}x/x)}(\forall^\omega\text{R}) \qquad A(0/x) \Rightarrow A(0/x)}{A(\dot{s}0/x), A(\dot{s}\dot{s}0/x), \ldots, A(0/x) \Rightarrow \forall x A(\dot{s}x/x) \& A(0/x)}}{\forall x A \Rightarrow \forall x A(\dot{s}x/x) \& A(0/x)}(\forall^\omega\text{L})$$

Being an infinitary derivation, what is presented here is of course merely the shape of the derivation. I will use $\mathcal{C}\text{BCK}_\bot^{\text{MI}}$ as label for the calculus obtained by expanding $\mathcal{C}\text{BCK}_\bot$ with (MI).

In addition to $\mathcal{C}\text{BCK}_\bot$, $\mathcal{C}\text{BCK}_\bot^{\forall\omega}$ and $\mathcal{C}\text{BCK}_\bot^{\text{MI}}$, I will also define in the next section the calculus $\mathcal{C}\text{BCK}_\bot^{\forall}$ by replacing the infinitary quantifier rules with rules corresponding to (MI) and its converse supplemented with a distributive rule

[8] This and other principles in this paper are presented in a sequent format with $\Rightarrow$ rather than with a conditional $\to$. This makes the use of the principle more intuitive in our proof-theoretic framework since $\mathcal{C}\text{BCK}_\bot$ has Cut rather than Conditional Detachment as a primitive rule. The sequent $\Rightarrow A \to B$ is anyway equiderivable with the sequent $A \Rightarrow B$.

for ∀. It is intended as a preliminary proposal for a finitary variant of $\mathcal{C}\text{BCK}_{\perp}^{\forall\omega}$, hence the omission of ω in the label. The following table summarises this for the reader:

$\mathcal{C}\text{BCK}_{\perp}$	Propositional rules
$\mathcal{C}\text{BCK}_{\perp}^{\forall\omega}$	$\mathcal{C}\text{BCK}_{\perp}$+ ($\forall^{\omega}$R) and ($\forall^{\omega}$R)
$\mathcal{C}\text{BCK}_{\perp}^{\text{MI}}$	$\mathcal{C}\text{BCK}_{\perp}$+ (MI)
$\mathcal{C}\text{BCK}_{\perp}^{\forall}$	$\mathcal{C}\text{BCK}_{\perp}$+ finitary rules based on (MI) + distribution

Principles for the function-symbols and *Tr* will be introduced when needed.

3 Classical, multiplicative and vacuous quantification

The purpose of this section is to provide the reader with a better understanding of the principle (MI). This is achieved as follows. Subsection 3.1 compares (MI) with the standard left introduction rule for the universal quantifier in first-order classical logic. Subsection 3.2 shows how one can obtain with rules corresponding to (MI) and its converse a preliminary proposal for a finitary variant $\mathcal{C}\text{BCK}_{\perp}^{\forall}$ of $\mathcal{C}\text{BCK}_{\perp}^{\forall\omega}$. Finally, subsection 3.3 demonstrates that $\mathcal{C}\text{BCK}_{\perp}^{\forall}$ expanded with transparent truth is trivial.

Before I embark on those subsections, however, the following general remark about (MI) is appropriate. As the reader may have noticed, I refer to (MI) as a multiplicative quantificational principle, but the principle involves the constant 0 and a function-symbol $\dot{s}$. The use of arithmetical vocabulary makes it very tempting to consider this as an arithmetical rather than a logical principle. Considering the topic of the paper being ω-inconsistency, I see no harm in working with a quantificational principle that explicitly includes arithmetical vocabulary. However, one can also understand the use of 0 and $\dot{s}$ as merely a convenient way to generate a denumerable list of terms for each element in the universe.

3.1 A quick comparison with classical quantification

The principle (MI) looks very similar to the standard rule for the universal quantifier in the sequent calculi g3c and g3i for classical and intuitionistic logic:[9]

$$\frac{\forall x A, A(t/x), \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\ (\forall\text{L})$$

The repetition of $\forall xA$ in the premise-sequent ensures that contraction is admissible as long as the sequent calculus satisfy certain other conditions.[10]

[9] See e.g. Negri and von Plato (2001) for a presentation of the calculi.

[10] Spelling out these conditions is beyond the scope of this paper. The curious reader is referred to Negri and von Plato (2001).

However, (MI) makes a different claim. (∀L) corresponds to the sequent $\forall xA \Rightarrow \forall xA \& A(t/x)$ which involves an explicit repetition in the succedent: $A(t/x)$ is one of the instances of $\forall xA$, so the succedent is claiming both that A holds for t, and that A holds for everything, including t. (MI) has instead a strong multiplicative flavour analogous to the infinitary rules presented in section 2: if everything is A, then 0 is A and every successor is A. There is no repetition here since 0 is not one of the successors. The rule corresponding to (MI) looks like this:

$$\frac{\forall y A(\dot{s}y/x), A(0/x), \Gamma \Rightarrow C}{\forall xA, \Gamma \Rightarrow C}\ (\forall \mathrm{L}_M)$$

The formula $\forall xA$ is not repeated in the premise-sequent. Instead, there are two formulas that jointly imply that A holds for every instance, assuming that every term equals a numeral for a natural number. The two active formulas in the premise-sequent represent respectively the *first* and the *rest* of the instances. Although the premise-sequent does not explicitly contain every instance as active formula as in ($\forall^\omega$L), the rest of the instances are implicitly present and may be obtained one by one through repeated inverted applications of the rule. The rule arguably captures within a finitary context to some extent the idea of a multiplicative rule as presented in section 1 for the case where there are infinitely many subformulas.

3.2 Finitary rules for a multiplicative quantifier

Now, the rule ($\forall \mathrm{L}_M$) is naturally paired with the following right introduction rule:

$$\frac{\Gamma \Rightarrow \forall y A(\dot{s}y/x) \qquad \Gamma' \Rightarrow A(0/x)}{\Gamma, \Gamma' \Rightarrow \forall xA}\ (\forall \mathrm{R}_M)$$

The corresponding sequent $\forall y A(\dot{s}y/x), A(0/x) \Rightarrow \forall xA$ is also derivable with ($\forall^\omega$L) and ($\forall^\omega$R), which makes ($\forall \mathrm{R}_M$) admissible in $\mathcal{C}\textsc{bck}_\perp^{\forall\omega}$.

However, interestingly enough, the pair ($\forall \mathrm{L}_M$) and ($\forall \mathrm{R}_M$) are insufficient to derive $\forall xA \Rightarrow \forall xA$ which means that they are insufficient to derive (MI):

$$\dfrac{\dfrac{\forall xA(\dot{s}x/x) \Rightarrow \forall xA(\dot{s}x/x) \qquad A(0/x) \Rightarrow A(0/x)}{\forall xA(\dot{s}x/x), A(0/x) \Rightarrow \forall xA(\dot{s}x/x) \& A(0/x)}\ (\&\mathrm{R})}{\forall xA \Rightarrow \forall xA(\dot{s}x/x) \& A(0/x)}\ (\forall \mathrm{L}_M)$$

Attempting to derive the left leaf by applying ($\forall \mathrm{L}_M$) will result in a regress.

One solution is to include $\forall xA \Rightarrow \forall xA$ as an axiom, i.e. include initial sequents for complex formulas. Another solution is to add the following rule familiar from proof theory for generalised quantifiers where y is an eigenvariable:[11]

$$\frac{B_0(y/z_0),\ldots B_n(y/z_0) \Rightarrow A(y/x)}{\Gamma', \forall z_0 B_0, \ldots, \forall z_n B_n \Rightarrow \forall x A}\ (\forall\text{D})$$

The premise-antecedent may be empty and the arbitrary context Γ' in the conclusion-antecedent ensures admissibility of weakening.

In both cases, the above leaf is derivable. The sequent calculus obtained by expanding $\mathcal{C}\text{BCK}_\perp$ with ($\forall\text{L}_M$), ($\forall\text{R}_M$) and ($\forall$D) is referred to as $\mathcal{C}\text{BCK}_\perp^\forall$.

The solution to include ($\forall$D) results in a stronger quantifier. For example, one can in $\mathcal{C}\text{BCK}_\perp^\forall$ derive $\forall x(A\&B) \Rightarrow \forall x A\&B$ if B is free from y in the formula $(A\&B)(y/x)$:

$$\dfrac{\dfrac{\dfrac{A(0/x) \Rightarrow A(0/x) \qquad \dfrac{\dfrac{(A\&B)(\dot{s}(y/x)) \Rightarrow A(\dot{s}(y/x))}{\forall x(A\&B)(\dot{s}x/x) \Rightarrow \forall x A(\dot{s}x/x)}\ (\forall\text{D})}{}}{\forall x(A\&B)(\dot{s}x/x), A(0/x) \Rightarrow \forall x A}\ (\forall\text{R}_M) \qquad B \Rightarrow B}{\dfrac{\dfrac{\forall x(A\&B)(\dot{s}x/x), A(0/x), B \Rightarrow \forall x A\&B}{\forall x(A\&B)(\dot{s}x/x), (A\&B)(0/x) \Rightarrow \forall x A\&B}\ (\&\text{L})}{\forall x(A\&B) \Rightarrow \forall x A\&B}\ (\forall\text{L}_M)}\ (\&\text{R})$$

Compare this with the following derivation in $\mathcal{C}\text{BCK}_\perp^{\forall\omega}$, that is, with the infinitary rules:

$$\dfrac{\dfrac{A(0/x) \Rightarrow A(0/x) \qquad (A\&B)(\dot{s}0/x) \Rightarrow A(\dot{s}0/x) \qquad \ldots}{A(0/x), (A\&B)(\dot{s}0/x), \ldots \Rightarrow \forall x A}\ (\forall^\omega\text{R}) \qquad B \Rightarrow B}{\dfrac{\dfrac{A(0/x), B, (A\&B)(\dot{s}0/x), \ldots \Rightarrow \forall x A\&B}{(A\&B)(0/x), (A\&B)(\dot{s}0/x), \ldots \Rightarrow \forall x A\&B}\ (\&\text{L})}{\forall x(A\&B) \Rightarrow \forall x A\&B}\ (\forall^\omega\text{L})}\ (\&\text{R})$$

They look pretty much the same. Now, an example like this is not sufficient to suggest or conjecture that $\mathcal{C}\text{BCK}_\perp^\forall$ is equivalent to $\mathcal{C}\text{BCK}_\perp^{\forall\omega}$ with regard to sequents with a finite antecedent. After all, there could be sequents with finite antecedent that are derivable by "braiding" numerals for even and odd numbers as illustrated by Petersen (2023) in the context of naive set theory, and as indicated with the phase semantics for a multiplicative infinitary conjunction presented by Pereira and Haeusler (1999). However, it does illustrate the extent to which one is capturing something fundamental with Zardini's rules through ($\forall\text{L}_M$) and thus (MI). It also justifies the description of $\mathcal{C}\text{BCK}_\perp^\forall$ as a preliminary proposal for a finitary variant of $\mathcal{C}\text{BCK}_\perp^{\forall\omega}$, especially since $\mathcal{C}\text{BCK}_\perp^\forall$ has the same issues as $\mathcal{C}\text{BCK}_\perp^{\forall\omega}$ with regard to both ω-inconsistency and inconsistency through vacuous quantification. I end this section by illustrating the latter. The former is the topic for the next section. Further explorations of the relationship between $\mathcal{C}\text{BCK}_\perp^{\forall\omega}$ and $\mathcal{C}\text{BCK}_\perp^\forall$ will be the topic of another paper.

[11] See for example (Ebbinghaus et al., 2021, p.144) for the use of a single-antecedent variant of this rule to define the quantifier "there are uncountably many". Readers familiar with sequent calculus for modal logics will also recognise it as corresponding to the rule defining the modal logic K.

3.3 Inconsistency through vacuous quantification

Before I show that vacuous quantification is an issue for a theory of transparent truth based on $\mathcal{C}\text{BCK}^{\forall}_{\bot}$, I first present the corresponding result regarding a theory of transparent truth based on $\mathcal{C}\text{BCK}^{\forall\omega}_{\bot}$ from Fjellstad and Olsen (2021). See also Petersen (2023).

Let Tr represent a transparent truth-predicate, i.e. defined with the following rules where $\ulcorner A \urcorner = t$:

$$\frac{A, \Gamma \Rightarrow \Delta}{Tr(t), \Gamma \Rightarrow \Delta}\ (Tr\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, Tr(t)}\ (Tr\text{R})$$

Let the nullary function-symbol $\dot\lambda$ be defined with $\dot\lambda = \ulcorner \forall x\, Tr(\dot\lambda) \to \bot \urcorner$. The following derivation is now available with $\mathcal{C}\text{BCK}^{\forall\omega}_{\bot}$ expanded with $(Tr\text{L})$, $(Tr\text{R})$ and that equation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Tr(\dot\lambda) \Rightarrow Tr(\dot\lambda) \qquad Tr(\dot\lambda) \Rightarrow Tr(\dot\lambda) \qquad \ldots}{Tr(\dot\lambda), Tr(\dot\lambda), \ldots \Rightarrow \forall x\, Tr(\dot\lambda)}\ (\forall^\omega\text{R})}{\forall x\, Tr(\dot\lambda) \to \bot, Tr(\dot\lambda), Tr(\dot\lambda), \ldots \Rightarrow \bot}\ (\to\text{L}/\bot)}{Tr(\dot\lambda), Tr(\dot\lambda), Tr(\dot\lambda), \ldots \Rightarrow \bot}\ (Tr\text{L})}{\forall x\, Tr(\dot\lambda) \Rightarrow \bot}\ (\forall^\omega\text{L})}{\Rightarrow \forall x\, Tr(\dot\lambda) \to \bot}\ (\to\text{R})}{\Rightarrow Tr(\dot\lambda)}\ (Tr\text{R}) \qquad \ldots}{\Rightarrow \forall x\, Tr(\dot\lambda)}\ (\forall^\omega\text{R})$$

Vacuous quantification implies inconsistency if Zardini's rules are paired with transparent truth.

One achieves the same result by replacing the infinitary rules with $(\forall\text{L}_M)$ and $(\forall\text{D})$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\, Tr(\dot\lambda)(\dot s x/x) \Rightarrow \forall x\, Tr(\dot\lambda)(0/x)}{\forall x\, Tr(\dot\lambda)(\dot s x/x), \forall x\, Tr(\dot\lambda)(0/x) \to \bot \Rightarrow \bot}\ (\to\text{L}/\bot)}{\forall x\, Tr(\dot\lambda)(\dot s x/x), Tr(\dot\lambda)(0/x) \Rightarrow \bot}\ (Tr\text{L})}{\forall x\, Tr(\dot\lambda) \Rightarrow \bot}\ (\forall\text{L}_M)}{\Rightarrow \forall x\, Tr(\dot\lambda) \to \bot}\ (\to\text{R})}{\Rightarrow Tr(\dot\lambda)}\ (Tr\text{R})}{\Rightarrow \forall x\, Tr(\dot\lambda)}\ (\forall\text{D})$$

The leaf in this proof figure is slightly odd, but the result of replacing a variable that is not present with another expression is the same formula, regardless of what the other expression is. Thus, replacing x in $Tr(\dot\lambda)$ with either $\dot s x$ or 0 results in the very same formula, $Tr(\dot\lambda)$. The reader may thus take note of how the rule $(\forall\text{L}_M)$ actually reduces to the classical rule $(\forall\text{L})$ in the case of vacuous quantification.

4 ω-inconsistency with McGee and Bacon

4.1 A non-classical variant of McGee's result

The purpose of this subsection is to show that a non-classical variant of the ω-inconsistency result by McGee (1985) is obtainable if $\mathcal{C}\text{BCK}_{\perp}^{\text{MI}}$, that is, $\mathcal{C}\text{BCK}_{\perp}+$ (MI), is expanded with some truth-theoretical principles corresponding to those assumed by McGee (1985) and some primitive recursive functions.

To avoid introducing additional principles and formal machinery beyond what is required for the result in this subsection, I do not present the original result by McGee (1985). Instead, a brief informal comparison should suffice. The original ω-inconsistency result by McGee (1985) concerns certain classical theories of truth, and contemporary presentations such as that by Halbach (2010) follow suit by presenting the result with classical logic. The current presentation thus breaks with that tradition by keeping the truth-theoretic principles but restricting the logic to that defined with $\mathcal{C}\text{BCK}_{\perp}^{\text{MI}}$. Now, there is a sense in which my PRAish arithmetical assumptions are stronger even if I do not assume the induction principle of PRA since I simply provide primitive definitions of the required primitive recursive functions as opposed to doing a detour through Q, the arithmetical theory employed by McGee (1985). That being said, I do not actually use more "arithmetic" than what would be definable within Q had the base logic been classical.

For the ω-inconsistency result presented in this subsection, I will use a simplified variant of the primitive recursive function used by McGee (1985). Let $\dot{T}$ represent the function that returns the Gödel-code of the formula $Tr(t)$ upon the input t, $\dot{T}t = \ulcorner Tr(t) \urcorner$. Let furthermore the function-symbol $\dot{\mu}$ be defined recursively with the following equations: $\dot{\mu}0 = \ulcorner \forall x\, Tr(\dot{\mu}x) \to \perp \urcorner$ and $\dot{\mu}\dot{s}n = \dot{T}\dot{\mu}n$.

Regarding the principles of truth, I will more or less follow McGee (1985) and assume the principles of truth corresponding to the modal principles D for Deontic, B for Barcan and N for Necessitation:[12]

(D) $$Tr(\ulcorner A \to \perp \urcorner), Tr(\ulcorner A \urcorner) \Rightarrow C$$

(Bμ) $$\forall x\, Tr(\dot{T}\dot{\mu}x) \Rightarrow Tr(\ulcorner \forall x\, Tr(\dot{\mu}x) \urcorner)$$

(N) $$\frac{\Rightarrow A}{\Rightarrow Tr(t)} \text{ where } t = \ulcorner A \urcorner$$

The Barcan formula for truth becomes slightly complicated when presented in its generality. I thus only present the instance that is required for the proof. The complication with the Barcan-formula is that the universal quantifier in the antecedent quantifies into the truth-predication which in turn requires function-symbols to make the variable available for binding.[13]

[12]Normally, presentations of the result also include a principle for distributing truth over either conjunction or the conditional along the lines of the principle K from modal logic. This is not necessary when the result is presented using sequent calculus as opposed to an axiomatic system.

[13]See Halbach (2010) for a precise and general presentation of the Barcan-formula for a truth-predicate.

Together the principles of truth tell us that *truth is not absurd*, *the ω-rule is truth-preserving* and *theorems are true*. I will call this theory of truth *Deontic Truth with Barcan*, DTB.

With all this in place, I can now present the proof of ω-inconsistency. A theory (as a set of sentences) T is ω-inconsistent if there is a formula φ such that it contains $(\forall x\varphi) \to \bot$ but also $\varphi(n/x)$ for each numeral n. This can for the purposes of this paper be translated into *sequentese* as that the sequents $\forall x\varphi \Rightarrow \bot$ and $\Rightarrow \varphi(n/x)$ for each numeral n for some formula φ are derivable. I will show this for the formula $Tr(\dot{\mu}x)$.

I show first that $\forall x\, Tr(\dot{\mu}x) \Rightarrow \bot$ is derivable. This sequent is derived as follows since $\dot{\mu}0 = \ulcorner \forall x\, Tr(\dot{\mu}x) \to \bot \urcorner$ and $\dot{\mu}\dot{s}n = \dot{T}\dot{\mu}n$:

$$\dfrac{\dfrac{\dfrac{}{Tr(\ulcorner \forall x\, Tr(\dot{\mu}x)\urcorner), Tr(\dot{\mu}0) \Rightarrow \bot}\,(\mathrm{D})}{\forall x\, Tr(\dot{\mu}\dot{s}x), Tr(\dot{\mu}0) \Rightarrow \bot}\,(\mathrm{B}\mu)}{\forall x\, Tr(\dot{\mu}x) \Rightarrow \bot}\,(\mathrm{MI})$$

It is now left to prove that $\Rightarrow Tr(\dot{\mu}n)$ for each numeral n is derivable. To show this, one proceeds as follows since $\dot{\mu}\dot{s}n = \dot{T}\dot{\mu}n$ and $\ulcorner Tr(s)\urcorner = \dot{T}s$:

$$\dfrac{\dfrac{\Rightarrow \forall x\, Tr(\dot{\mu}x) \to \bot}{\Rightarrow Tr(\dot{\mu}0)}\,(\mathrm{N})}{\Rightarrow Tr(\dot{\mu}\dot{s}0)}\,(\mathrm{N})$$
$$\vdots$$

With this, it is established that both $\forall x\, Tr(\dot{\mu}x) \Rightarrow \bot$ and $\Rightarrow Tr(\dot{\mu}n)$ for each numeral n are derivable. It follows that the theory defined with $\mathcal{C}\mathrm{BCK}_{\bot}^{\mathrm{MI}}$ and DTB together with the definitions of the relevant primitive recursive functions is ω-inconsistent.

4.2 A truth-theoretic variant of Bacon's result

The original ω-inconsistency result by Bacon (2013) concerns a theory of transparent truth satisfying certain logical principles involving the conditional that are derivable in $\mathcal{C}\mathrm{BCK}_{\bot}$ together with certain principles for the existential quantifier and the principle that the truth of a conditional distributes over the conditional. This subsection presents a truth-theoretic variation of that result for $\mathcal{C}\mathrm{BCK}_{\bot}$ expanded with DTB from the previous subsection and two quantificational principles that serve the same purpose for the universal quantifier as Bacon's principles for the existential quantifier. Again, I choose not to present the original proof by Bacon (2013) in detail, as that would require the introduction of principles and formal machinery beyond what is otherwise required here.

In addition to a truth-theoretic variant of the ω-inconsistency result by Bacon (2013), this subsection also presents two further observations:

(a) The ω-inconsistency result applies to a fragment of the paraconsistent logic presented by Badia et al. (2022).

(b) The assumptions required to obtain the ω-inconsistency result imply that the principle (MI) holds for the paradoxical formula employed in the proof.

Whereas (a) is merely a corollary of the ω-inconsistency result, (b) can be considered as the main contribution of this subsection since it shows that (MI) is in a particular sense implicit in the assumptions required for the ω-inconsistency result. Before presenting these observations I first introduce the required assumptions.

For the results in this subsection, I will assume the following principles for the universal quantifier:

$$(\forall\dot{s}) \qquad \forall x A \Rightarrow \forall y A(\dot{s}y/x)$$

$$(\forall\&) \qquad \forall x(A\&B) \Rightarrow \forall x A\&B$$

where in $(\forall\&)$ B is free from y in the formula $(A\&B)(y/x)$.

The principle $(\forall\dot{s})$ tells us that whatever follows from that A holds for any successor follows also from that A holds for anything. Assuming that every term equals a numeral, one is basically strengthening the premises to include that A holds for 0. This principle follows from (MI) by "conjunction elimination". The principle $(\forall\&)$ allows us to widen the scope of a quantifier in the antecedent. As demonstrated in subsection 3.2, $(\forall\&)$ is derivable in $\mathcal{C}\textsc{bck}^{\forall}_{\bot}$, that is, our base logic expanded with the rules corresponding to (MI) and its converse together with the rule ($\forall$D).

As in the previous subsection, I present only the instance of the Barcan-formula for the truth-predicate which is relevant for the result. To that purpose, I first define the function-symbol $\dot{\gamma}$ with the equations $\dot{\gamma}0 = \ulcorner \forall x\, Tr(\dot{\gamma}x) \rightarrow \bot \urcorner$ and $\dot{\gamma}\dot{s}n = \dot{\&}\dot{T}\dot{\gamma}n\dot{\gamma}0$. The relevant instance of the Barcan-formula is thus the following:

$$(\mathrm{B}\gamma) \qquad \forall x\, Tr(\dot{\&}\dot{T}\dot{\gamma}x\dot{\gamma}0) \Rightarrow Tr(\ulcorner \forall x(Tr(\dot{\gamma}x)\&(\forall x\, Tr(\dot{\gamma}x) \rightarrow \bot))\urcorner)$$

The principle D, necessitation and cut imply together that the following rule is derivable where $t = \ulcorner A \urcorner$:

$$\frac{A \Rightarrow \bot}{Tr(t) \Rightarrow \bot}$$

This concludes the presentation of the assumptions. I now establish that they suffice for ω-inconsistency.

I establish first that $\forall x\, Tr(\dot{\gamma}x) \Rightarrow \bot$ is derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x\, Tr(\dot{\gamma}x), \forall x\, Tr(\dot{\gamma}x) \rightarrow \bot \Rightarrow \bot}{\forall x(Tr(\dot{\gamma}x)\&(\forall x\, Tr(\dot{\gamma}x) \rightarrow \bot)) \Rightarrow \bot}\,(\forall\&)}{Tr(\ulcorner \forall x(Tr(\dot{\gamma}x)\&(\forall x\, Tr(\dot{\gamma}x) \rightarrow \bot))\urcorner) \Rightarrow \bot}}{\forall x\, Tr(\dot{\gamma}\dot{s}x) \Rightarrow \bot}\,(\mathrm{B}\gamma)}{\forall x\, Tr(\dot{\gamma}x) \Rightarrow \bot}\,(\forall\dot{s})$$

It is left to obtain $\Rightarrow Tr(\dot{\gamma}n)$ for each n. That $\Rightarrow Tr(\dot{\gamma}0)$ is derivable follows immediately from $\forall x\, Tr(\dot{\gamma}x) \Rightarrow \bot$ by ($\rightarrow$R) and necessitation. To secure the rest, it suffices to observe that the following meta-theoretic inductive step is admissible which together with the instance for 0 imply that each instance is derivable:

$$\dfrac{\dfrac{\Rightarrow Tr(\dot\gamma n) \qquad \Rightarrow \forall x\, Tr(\dot\gamma x) \to \bot}{\Rightarrow Tr(\dot\gamma n) \& \forall x\, Tr(\dot\gamma x) \to \bot}}{\Rightarrow Tr(\dot\gamma \dot s n)}$$

This concludes the proof that $\mathcal{C}$BCK$_\bot$expanded with DTB, equations defining the relevant primitive recursive functions, $(\forall\dot s)$ and $(\forall\&)$ defines an ω-inconsistent theory. I now turn to the observations (a) and (b).

Observation (a) is that the ω-inconsistency result presented in this subsection implies that the theory of truth obtained by expanding the paraconsistent logic presented by Badia et al. (2022) with DTB and the equations defining the function-symbols employed in this subsection is ω-inconsistent.

The paraconsistent logic presented by Badia et al. (2022) is defined with a sequent calculus that includes the rules of $\mathcal{C}$BCK$_\bot$and an additive quantifier defined with, among other rules and principles, $(\forall L_A)$, $(\forall R)$ and the following:

$(\forall\&\forall)$ $$\forall x(A\&B) \Rightarrow \forall x A \& \forall x B$$

Badia et al. (2022) describe $(\forall\&\forall)$ it in a footnote as multiplicative, but do not elaborate on that claim. As it turns out, this combination implies $\forall\&$:[14]

$$\dfrac{\dfrac{\forall x A \Rightarrow \forall x A \qquad \forall x B \Rightarrow B}{\forall x A \& \forall x B \Rightarrow \forall x A \& B}}{\forall x(A\&B) \Rightarrow \forall x A \& B}\ (\forall\&\forall)$$

The ω-inconsistency result now follows. Importantly, this result does not concern the paraconsistent part of their logic, as the proof relies on $A \to \bot$ as negation and not their paraconsistent negation. It follows that the theory is trivial if $(\forall R)$ is replaced with an ω-rule.

Observation (b) concerns the relationship between (MI) and $(\forall\&)$ beyond the fact that the latter is derivable in $\mathcal{C}\text{BCK}^{\forall}_{\bot}$. Perhaps surprisingly, one can actually derive (MI) for the paradoxical formula employed in our variation of the ω-inconsistency result by Bacon (2013) presented in this subsection from the assumptions for that result:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\, Tr(\dot\gamma\dot s x), Tr(\dot\gamma 0) \Rightarrow}{\forall x\, Tr(\dot\gamma x), Tr(\dot\gamma 0) \Rightarrow}(\forall\dot s)}{\forall x\, Tr(\dot\gamma x) \& Tr(\dot\gamma 0) \Rightarrow}(\&L)}{\forall x(Tr(\dot\gamma x) \& Tr(\dot\gamma 0)) \Rightarrow}(\forall\&)}{Tr(\ulcorner \forall x(\dot\& \dot T \dot\gamma x \dot\gamma 0)\urcorner) \Rightarrow}(\text{D/N})}{\forall x\, Tr(\dot\& \dot T \dot\gamma n \dot\gamma 0) \Rightarrow}(\text{B})}{\forall x\, Tr(\dot\gamma \dot s x) \Rightarrow}\text{Def } \dot\gamma}{\forall x\, Tr(\dot\gamma x) \Rightarrow}(\forall\dot s)$$

[14]This situation is interestingly analogous to that of light linear logic for naive set theory. The exponential satisfies $!(A\&B) \Rightarrow !A\&!B$ which in turn becomes problematic if one also assumes that $!A \Rightarrow A$. See e.g. Girard (1998) and Terui (2004) for more details.

In the case of the variation on McGee's result, one simply added the principle and proceeded to derive the ω-inconsistency. In the case of the variation on Bacon's result, one added instead sufficient logical and truth-theoretic resources to derive the principle for the paradoxical formula. That the principle is derivable for the paradoxical formula through which ω-inconsistency is generated suggests that (MI) captures an underlying mechanism that is central to the ω-inconsistency results.

5 ω-inconsistency and Hilbert's Hotel

The aim of this section is to take a closer look at this underlying mechanism captured by (MI). In particular, I highlight that the reasoning leading to ω-inconsistency is analogous to the reasoning familiar from Hilbert's Grand Hotel.

Consider again the derivations of (MI) within $\mathcal{C}\text{BCK}_{\bot}^{\forall\omega}$ and $\mathcal{C}\text{BCK}_{\bot}^{\forall}$ above in section 3. The desired sequent is obtained by pooling together *the first* and *the rest* of the infinite sequence from two different branches of the derivation with the help of the multiplicative conjunction which in turn provides the grounds for introducing the universal quantifier. This deductive move encapsulates the inferential role of (MI). Indeed, whereas an additive conjunction in antecedent position weakens in assumptions, a multiplicative conjunction in antecedent position "fuse" together multiple formulas that may have different origins, for example formulas that some multiplicative rule with independent contexts has pooled together so that they can be used for a common purpose. (MI) thus ensures that the logical strength of the "fusion" of that A holds for the first and that A holds for the rest is matched by the logical strength of that A holds for every element of the sequence obtained by appending the first to the beginning of the rest. However, appending an element to the beginning of an infinite sequence results in a new infinite sequence of equal length. That is, the sequence $0, 1, 2, 3, \ldots$ has the same length as the sequence obtained by appending 0 to the beginning of the sequence $1, 2, 3, \ldots$ even if there is already a one-to-one correspondence between the sequences $0, 1, 2, 3, \ldots$ and $1, 2, 3, \ldots$.

As an illustration of how the ω-inconsistency result takes advantage of this feature, it is useful to consider a new variant of the paradoxical reasoning involving the following natural generalisation of (D) to infinitary sequents:

$$\frac{A_0, A_1, \ldots \Rightarrow \bot}{Tr(\ulcorner A_0 \urcorner), Tr(\ulcorner A_1 \urcorner), \ldots \Rightarrow \bot}\ \mathrm{D}^{\omega}$$

With this rule, one may proceed as follows with the infinitary quantifier rules:

$$\dfrac{\dfrac{\dfrac{\dfrac{Tr(\dot{\mu} n) \Rightarrow Tr(\dot{\mu} n) \text{ for every } n}{Tr(\dot{\mu}0), Tr(\dot{\mu}\dot{s}0), \ldots \Rightarrow \forall x\, Tr(\dot{\mu} x)}\ (\forall^{\omega}\mathrm{R})}{Tr(\dot{\mu}0), Tr(\dot{\mu}\dot{s}0), \ldots, \forall x\, Tr(\dot{\mu} x) \to \bot \Rightarrow \bot}\ (\to\mathrm{L}/\bot)}{Tr(\dot{\mu}\dot{s}0), Tr(\dot{\mu}\dot{s}\dot{s}0), \ldots, Tr(\dot{\mu}0) \Rightarrow \bot}\ (\mathrm{D}^{\omega})}{\forall x\, Tr(\dot{\mu} x) \Rightarrow \bot}\ (\forall^{\omega}\mathrm{L})$$

With this rule one simultaneously replaces every instance in the sequence with its successor. This amounts to explicitly pushing each instance up to the next successor to thereby making room for 0. This is just like in Hilbert's Grand Hotel with infinitely many occupied rooms where each guest is asked to move up to the next room when a new guest arrives, thereby making the first room available.[15] The use of (D^ω) corresponds directly to pushing each guest into the next room.

Rather than explicitly contracting two instances as in the liar paradox, the instances are being pushed into something like a black hole. The question "but where did the last instance go?" is not supposed to make sense here since there are infinitely many of them, and there is thus no last instance which is explicitly contracted with its predecessor. In other words, this is not a form or kind of contraction, but it serves the same purpose as far as the paradoxical reasoning is concerned.

Now, what happens if one just keeps applying (D^ω)? In the finitary case, nothing special. However, assume that derivations may be infinitely long in such a way that a sequent in a derivation may have no direct predecessor from which it is obtained, not only as a leaf, but also because it may occur as the limit of an infinitely long branch. With such a notion of a derivation, iterated applications of the rule (D^ω) deliver the empty sequent:

$$\cfrac{\cfrac{\forall x\, Tr(\dot{\mu}x) \to \bot,\, Tr(\dot{\mu}0),\, Tr(\dot{\mu}\dot{s}0), \ldots \Rightarrow}{Tr(\dot{\mu}0),\, Tr(\dot{\mu}\dot{s}0),\, Tr(\dot{\mu}\dot{s}\dot{s}0), \ldots \Rightarrow}\,(\mathrm{D}^\omega)}{Tr(\dot{\mu}\dot{s}0),\, Tr(\dot{\mu}\dot{s}\dot{s}0),\, Tr(\dot{\mu}\dot{s}\dot{s}\dot{s}0), \ldots \Rightarrow}\,(\mathrm{D}^\omega)$$

$$\vdots$$

After $\omega + 1$ applications, the sequent is empty, just like the hotel would be. (D^ω) becomes something like an elimination rule at the limit even if no single step modifies the number of formulas in the sequent because one can just keep pushing the instances into this black hole of Cantorian infinity.

One can construct derivations with (MI) where the relevant behaviour can be observed: With (B), $Tr(\ulcorner \forall x\, Tr(\dot{\mu}x)\urcorner)$ is transformed into $\forall x\, Tr(\dot{\mu}sx)$, thereby making space for $Tr(\dot{\mu}0)$ which together with the latter suffice for $\forall x\, Tr(\dot{\mu}x)$. The analogy is not as clear as in the case of the infinitary quantifier rules and (D^ω), but it is the same thing going on. In that sense, (MI) suffices for a glimpse into Cantor's paradise.

[15]It seems common to refer to Hilbert's 1925 lecture "Über das Unendliche" as the place where this hotel is mentioned. For example, the Wikipedia article on "Hilbert's paradox of the Grand Hotel" claimed so last time I checked (on 24.09.2024). However, I could not find any mention of the hotel in the printed version of that lecture published in *Mathematische Annalen*, volume 95, pages 161-190, 1926. Instead, it is indicated by Moore (2002) that the hotel is used to explain the difference between finite and denumerable sets in his lectures on set theory in 1917.

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