# ON ANTICUT RULES: CLASSICAL, FDE-BASED AND INTUITIONISTIC LOGICS

MARIO PIAZZA AND ANDREA SABATINI

Abstract. In this paper, we investigate certain intricacies and peculiarities of the proof theory of deduction-refutation systems (D-R systems, henceforth), namely systems integrating theorems and antitheorems of a given logic. The logics considered here are classical, first-degree entailment-based (**FDE**-based, henceforth) and intuitionistic logics, formulated within a sequent calculus framework. Our primary focus is on establishing the general conditions under which anticut rules (the contrapositive versions of the familiar cut rule) can be eliminated from D-R sequent calculi, while distinguishing between two main variants of these systems. This proof-theoretic investigation leads to the introduction of a new, Gentzen-style refutation calculus for intuitionistic logic, which is interesting in its own right.

## 1. Introduction

While the idea of a refutation calculus traces back to Łukasiewicz's study of Aristotelian syllogistic in the early 1950s [25], deduction-refutation systems (D-R systems, henceforth) have garnered increasing attention only in recent decades as a burgeoning area within structural proof theory [11, 13][1]. D-R systems are syntactic frameworks designed to derive both valid and refutable formulas. These formulas are filtered by their respective deducibility and refutability relations, denoted by the turnstile symbols ($\vdash$) and ($\dashv$) for sequents and antisequents.

For reasons that will become soon evident, we find it useful to distinguish between *hybrid* and *unmixed* D-R systems. Hybrid systems include (binary) rules that combine both $\vdash$ and $\dashv$, so that an antisequent follows from a sequent and an antisequent as premises, while unmixed systems lack such combined rules, instead featuring rules that involve either sequents or antisequents exclusively.

In this paper, we aim to provide a comprehensive analysis of the so-called anticut rules in D-R sequent calculi for classical, **FDE**-based and intuitionistic logics. This is a topic that has recently been highlighted in the context of D-R sequent calculi for **FDE**-based logics [26]. In traditional sequent calculi, the cut rule is a generalisation of the venerable rule of *modus ponens*:

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Pi \vdash \Sigma}{\Pi, \Gamma \vdash \Delta, \Sigma}\, cut$$

Scuola Normale Superiore di Pisa, Classe di Lettere e Filosofia

*E-mail address*: mario.piazza@sns.it, andrea.sabatini@sns.it.

[1] D-R systems may be considered part of the broader class of bilateral systems, as argued elsewhere [29, 32, 33]. However, we do not intend to pursue this point further here.

Cut rule corresponds to the ubiquitous mathematical tactics of using intermediate lemmas or general theorems within a proof. It composes two sequents while eliminating the cut formula, namely the formula that appears among the conclusions of the first sequent and the premises of the second. In hybrid sequent calculi, the anticut rules have the following form:

$$acut_1\ \frac{\Gamma \vdash \Delta, A \qquad \Pi, \Gamma \dashv \Delta, \Sigma}{A, \Pi \dashv \Sigma} \qquad\qquad \frac{A, \Gamma \vdash \Delta \qquad \Pi, \Gamma \dashv \Delta, \Sigma}{\Pi \dashv \Sigma, A}\ acut_2$$

If $\dashv$ is interpreted as the metatheoretic negation of $\vdash$, then anticut rules can be understood as contrapositive versions of the cut rule for $\vdash$[2]:

$$acut_1\ \frac{\Gamma \vdash \Delta, A \qquad \Pi, \Gamma \nvdash \Delta, \Sigma}{A, \Pi \nvdash \Sigma} \qquad\qquad \frac{A, \Gamma \vdash \Delta \qquad \Pi, \Gamma \nvdash \Delta, \Sigma}{\Pi \nvdash \Sigma, A}\ acut_2$$

The anticut rules operate simultaneously on a sequent and an antisequent, producing a refutational conclusion in which the anticut formula is *introduced*. In the special case where $\Gamma$ and $\Delta$ are empty, the anticut rules effectively function as controlled weakening rules within the framework of refutability:

$$acut_1\ \frac{\vdash A \qquad \Pi \dashv \Sigma}{A, \Pi \dashv \Sigma} \qquad\qquad \frac{A \vdash \qquad \Pi \dashv \Sigma}{\Pi \dashv \Sigma, A}\ acut_2$$

Analogously to the cut rule, anticut rules are impractical for refutation search. Furthermore, it is not immediately clear whether these rules are necessary to guarantee Ł-completeness, that is completeness with respect to both deducibility and refutability[3]. In a recent paper [26], the author examines a family of **FDE**-based propositional logics and raises the question of finding anticut-free hybrid sequent calculi capable of refuting all the antitheorems of these logics[4]. In what follows, we address this unresolved problem by establishing three claims:

(*i*) anticut rules *cannot* be eliminated from hybrid sequent calculi without sacrificing Ł-completeness;

(*ii*) when anticut rules are combined with an appropriate set of axioms, the resulting system is refutation-complete;

(*iii*) in unmixed sequent calculi, anticut rules *can* be constructively eliminated without compromising Ł-completeness.

---

[2]It is worth stressing that our metatheory is grounded in classical logic, ensuring that the metatheoretic negation of $\vdash$ obeys the law of contraposition.

[3]In [12], a *hybrid* D-R system is said to be 'Ł-complete' exactly when it is complete with respect to deducibility and refutability. In this paper, we employ 'Ł-completeness' to refer to completeness with respect to both deducibility and refutability, irrespective of whether *hybrid* or *unmixed* D-R systems are considered.

[4]Note that this issue is specific to sequent-based calculi and does not arise in [12], which presents a hybrid (sequent-style) natural deduction calculus for classical logic. As expected, the system in [12] achieves Ł-completeness without the need for anticut rules.

Unlike [26], this paper also addresses intuitionistic propositional logic. We introduce the first hybrid sequent calculus for intuitionistic propositional logic, combining anticut rules with a suitable set of axioms and an (infinite) set of specific refutational rules. We prove that such rules yield contrapositive, Gentzen-style versions of (restricted) *Visser's rules* – namely, the rules from which any rule which is admissible but not derivable in a Hilbert-style calculus for intuitionistic logic can be derived [38]. Moreover, we show that the anticut-elimination strategy designed for **FDE**-based logics can be adapted to a (terminating) unmixed sequent calculus for intuitionistic logic.

The paper is organized as follows. In Section 2, we set the stage by introducing basic concepts and an array of auxiliary notions serving our machinery. In Section 3, we show that the anticut rules cannot be eliminated from hybrid sequent calculi for classical propositional logic, and that the system resulting from the complementarity axiom and the anticut rules is refutation-complete for classical propositional logic. Moreover, we introduce a strategy for eliminating anticut rules from an unmixed sequent calculus for classical propositional logic. In Section 4, we build on the basic concepts of hybrid and unmixed sequent calculi for **FDE**, **K3**, and **LP** to establish that results analogous to those presented in Section 3 can similarly be achieved for these logics. In Section 5, we present a structural analysis of our hybrid calculus for intuitionistic propositional logic, together with a detailed exposition of the anticut-elimination argument for the unmixed sequent calculus for the same logic. Finally, in Section 6, we outline potential directions for future research. Appendices A, B and C provide all the proof systems discussed throughout the paper.

## 2. Preliminary notions and basic results

We use capital Greek letters $\Gamma, \Delta, \Pi, \Sigma, \ldots$ to denote finite multisets of formulas; in the present and the following section, we employ $\Theta, \Lambda \ldots$ to denote multisets of *atomic* formulas. The *logical complexity* $\mathcal{C}(A)$ of a formula $A$ is 0 if $A$ is atomic, $\mathcal{C}(B)+1$ if $A$ is of the form $\neg B$ and $\mathcal{C}(B)+\mathcal{C}(C)+1$ if $A$ is of the form $B \otimes C$, with $\otimes \in \{\wedge, \vee, \rightarrow\}$.

We shall be dealing with Gentzen-style sequents $\Gamma \vdash \Delta$ as well as *antisequents* $\Gamma \dashv \Delta$, where $\Gamma \dashv \Delta$ is valid if, and only if, $\Gamma \vdash \Delta$ is *in*valid [11, 13]. In the case of classical logic, an antisequent is valid if and only if there is some Boolean valuation verifying all the formulas in $\Gamma$ and falsifying all those in $\Delta$. The measure $\mathcal{C}$ can be easily extended to any multiset $\Gamma = A_1, \ldots, A_n$ by writing $\mathcal{C}(\Gamma) = \mathcal{C}(A_1) + \ldots + \mathcal{C}(A_n)$, as well as to any sequent $\Gamma \vdash \Delta$ and antisequent $\Gamma \dashv \Delta$ by writing $\mathcal{C}(\Gamma \vdash \Delta) = \mathcal{C}(\Gamma \dashv \Delta) = \mathcal{C}(\Gamma) + \mathcal{C}(\Delta)$.

A deduction-refutation sequent calculus features rules dealing with sequents and antisequents. A derivation $\pi$ in the calculus may end either in a sequent $\Gamma \vdash \Delta$ or in an antisequent $\Gamma \dashv \Delta$: in the first case, we say that $\pi$ is a *proof* for $\Gamma \vdash \Delta$; in the second, $\pi$ qualifies as a *refutation* for $\Gamma \vdash \Delta$. As usual, the *height* of a derivation $\pi$ – denoted by, $h(\pi)$ – is defined as the number of nodes in its longest branch.

Kleene's $\mathsf{G4}$ calculus for classical propositional logic[5] is imported from [23] (see Appendix A.1). For our purposes, it suffices to recall the following facts about $\mathsf{G4}$:

[5]Kleene's $\mathsf{G4}$ is the same as the $\mathsf{G3}$ calculus for classical propositional logic in [43].

**Theorem 1.** G4 *enjoys the following properties.*

(*i*) *Each logical rule of* G4 *is height-preserving invertible.*

(*ii*) *The weakening, contraction and cut rules are admissible in* G4.

*Proof.* For proofs see [4, 43, 29]. □

The hybrid calculus $\mathsf{G4}_{H1}$ is obtained from G4 by incorporating the complementarity axiom $ax_{cl} \dashv$ along with the contrapositive versions of the logical and cut rules (see Appendix A.2). The method of adding contrapositive versions of deductive rules to design deduction-refutation calculi is illustrated in [12].

Conversely, the hybrid calculus $\mathsf{G4}_{H2}$ is constructed from G4 by including the complementarity axiom $ax_{cl} \dashv$, refutational versions of unary logical rules, hybrid versions of binary logical rules, and contrapositive versions of the cut rule (see Appendix A.3). To our knowledge, the $\mathsf{G4}_{H2}$ calculus is not explicitly labeled in the literature. Since the logical rules of G4, when read bottom-up, correspond to tableaux rules for provability [42], any refutational branch of an anticut-free derivation in $\mathsf{G4}_{H2}$ can be viewed as an open branch of the corresponding tableaux.

Finally, the unmixed system $\mathsf{G4}_{\dashv\vdash}$ is imported from [29]. The method for constructing refutational calculi that underlies the design of $\mathsf{G4}_{\dashv\vdash}$ offers an alternative approach to the one presented by [12]. Beginning with an invertible and terminating proof calculus, we obtain a refutational calculus by first converting unary rules into their refutational counterparts. Then, for each binary rule, we introduce a unary rule featuring one of the original premises as refutational premise.

We write $\Gamma \models^{*} \Delta$ to generalize over the union of sequents and antisequent. The rules of $\mathsf{G4}_{\dashv\vdash}$ can be understood as a two-step procedure for decomposing any (anti)sequent $\Gamma \models^{*} \Delta$ into a set of atomic (anti)sequents, as follows:

(1) (bottom-up) decompose the (anti)sequent $\Gamma \models^{*} \Delta$ using the logical rules of G4, with $\models^{*}$ in place of $\vdash$, until each leaf of the resulting tree ends with an atomic (anti)sequent;

(2) (top-down) turn each occurrence of $\models^{*}$ into an occurrence of $\vdash$ or $\dashv$, following the rules of $\mathsf{G4}_{\dashv\vdash}$.

Let us recall a crucial feature of the $\mathsf{G4}_{\dashv\vdash}$ proof system:

**Theorem 2.** *Maximal* $\mathsf{G4}_{\dashv\vdash}$*-decomposition yields a unique set of atomic (anti)sequents.*

*Proof.* For a proof see [4, 29]. □

*Example* 1. This is a $\mathsf{G4}_{\dashv\vdash}$-based decomposition-tree of $p \to q, p \vee s \models^{*} r$:

$$\cfrac{p \to q, p \vee s \models^{*} r}{\vee\!\models^{*}\ \cfrac{p \vee s \models^{*} p, r}{ax_{cl} \vdash\ \cfrac{p \models^{*} p, r}{p \vdash p, r} \qquad ax_{cl} \dashv\ \cfrac{s \models^{*} p, r}{s \vdash p, r}} \qquad \cfrac{p \vee s, q \models^{*} r}{\cfrac{p, q \models^{*} r}{p, q \dashv r}\ ax_{cl} \dashv \qquad \cfrac{s, q \models^{*} r}{s, q \dashv r}\ ax_{cl} \dashv}\ \vee\!\models^{*}}\ \to\!\models^{*}$$

## 3. Anticut and classical logic

In this section, we illustrate the deductive power of anticut by proving a cluster of results: to this aim, we employ the hybrid $\mathsf{G4}_{H1\cap 2}$ calculus, which gathers all the rules shared by $\mathsf{G4}_{H1}$ and $\mathsf{G4}_{H2}$ (see Appendix A.4).

We begin by showing that the contrapositive versions of standard weakening rules can be seen as anticut applications in disguise.

**Lemma 3.** *The rules of strengthening*

$$str\ \frac{A, \Gamma \dashv \Delta}{\Gamma \dashv \Delta} \qquad\qquad \frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta}\ str$$

*are admissible in the* $\mathsf{G4}_{H1\cap 2}$ *calculus.*

*Proof.* If $\Gamma \cup \Delta = \varnothing$, the conclusion is in both cases immediate. On the other hand, if $\Gamma = \Gamma' \cup [B]$ or $\Delta = \Delta' \cup [C]$, we proceed as follows:

$$acut_1\ \frac{\begin{array}{c}\vdots\\ A, B \vdash B\end{array} \qquad A, B, \Gamma' \dashv \Delta}{B, \Gamma' \dashv \Delta} \qquad\qquad \frac{\begin{array}{c}\vdots\\ A, C \vdash C\end{array} \qquad A, \Gamma \dashv \Delta', C}{\Gamma \dashv \Delta', C}\ acut_2$$

$$acut_1\ \frac{\begin{array}{c}\vdots\\ B \vdash B, A\end{array} \qquad B, \Gamma' \dashv \Delta, A}{B, \Gamma' \dashv \Delta} \qquad\qquad \frac{\begin{array}{c}\vdots\\ C \vdash C, A\end{array} \qquad \Gamma \dashv \Delta', C, A}{\Gamma \dashv \Delta', C}\ acut_2$$

□

Moreover, the contrapositive versions of standard contraction rules are admissible in $\mathsf{G4}_{H1\cap 2}$.

**Lemma 4.** *The rules of duplication*

$$dup\ \frac{A, \Gamma \dashv \Delta}{A, A, \Gamma \dashv \Delta} \qquad\qquad \frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta, A, A}\ dup$$

*are admissible in* $\mathsf{G4}_{H1\cap 2}$.

*Proof.* We focus on Left Duplication, the other case being analogous. Reasoning by induction on the height of the $\mathsf{G4}_{H1\cap 2}$-derivation $\pi$ of $A, \Gamma \dashv \Delta$, we prove the existence of a $\mathsf{G4}_{H1\cap 2}$-derivation $\rho$ of $A, A, \Gamma, \Gamma \dashv \Delta, \Delta$ (cf. the *copy* rule in Lemma 9). Hence, we apply Lemma 3 to $\rho$ to get a $\mathsf{G4}_{H1\cap 2}$-derivation of $A, A, \Gamma \dashv \Delta$.

If $h(\pi) = 1$, the conclusion is trivial. If $h(\pi) > 1$ and the last rule applied is $acut_1$, we reason by cases over $A$. If $A$ is principal in the $acut_1$-application, then $\rho$ has the following form:

$$\frac{\dfrac{\begin{array}{c}\vdots\\ \Pi \vdash \Sigma, A\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma, \Pi \dashv \Sigma, \Delta\end{array}}{A, \Gamma \dashv \Delta}\ acut_1}{A, A, \Gamma, \Gamma \dashv \Delta, \Delta}\ copy$$

We consider the following derivation:

$$\dfrac{\begin{matrix}\vdots\\ \Pi \vdash \Sigma, A\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \Pi \vdash \Sigma, A\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \Gamma, \Pi \dashv \Sigma, \Delta\end{matrix}}{\Gamma, \Gamma, \Pi, \Pi \dashv \Sigma, \Sigma, \Delta, \Delta}\,\mathit{copy}}{A, \Gamma, \Gamma, \Pi \dashv \Sigma, \Delta, \Delta}\,\mathit{acut}_1}{A, A, \Gamma, \Gamma \dashv \Delta, \Delta}\,\mathit{acut}_1$$

The *copy* application can be removed by inductive hypothesis. On the other hand, if $A$ is not principal in the $\mathit{acut}_1$-application, then $\rho$ has the following form:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Pi \vdash \Sigma, B\end{matrix} \qquad \begin{matrix}\vdots\\ A, \Gamma', \Pi \dashv \Sigma, \Delta\end{matrix}}{A, B, \Gamma' \dashv \Delta}\,\mathit{acut}_1}{A, A, B, B, \Gamma', \Gamma' \dashv \Delta, \Delta}\,\mathit{copy}$$

with $\Gamma = \Gamma' \cup [A]$. We take the following derivation:

$$\dfrac{\begin{matrix}\vdots\\ \Pi \vdash \Sigma, B\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \Pi \vdash \Sigma, B\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ A, \Gamma', \Pi \dashv \Sigma, \Delta\end{matrix}}{A, A, \Gamma', \Gamma', \Pi, \Pi \dashv \Sigma, \Sigma, \Delta, \Delta}\,\mathit{copy}}{A, A, B, \Gamma', \Gamma', \Pi \dashv \Sigma, \Delta, \Delta}\,\mathit{acut}_1}{A, A, B, B, \Gamma', \Gamma' \dashv \Delta, \Delta}\,\mathit{acut}_1$$

Finally, if the last rule applied is $\mathit{acut}_2$, $\rho$ has the following form:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ B, \Pi \dashv \Sigma\end{matrix} \qquad \begin{matrix}\vdots\\ A, \Gamma, \Pi \dashv \Sigma, \Delta'\end{matrix}}{A, \Gamma \dashv \Delta', B}\,\mathit{acut}_2}{A, A, \Gamma, \Gamma \dashv \Delta', \Delta', B, B}\,\mathit{copy}$$

with $\Delta = \Delta' \cup [B]$. We consider the following derivation to get the conclusion:

$$\dfrac{\begin{matrix}\vdots\\ B, \Pi \dashv \Sigma\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ B, \Pi \dashv \Sigma\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ A, \Gamma, \Pi \dashv \Sigma, \Delta'\end{matrix}}{A, A, \Gamma, \Gamma, \Pi, \Pi \dashv \Sigma, \Sigma, \Delta', \Delta'}\,\mathit{copy}}{A, A, \Gamma, \Gamma, \Pi \dashv \Sigma, \Delta', \Delta', B}\,\mathit{acut}_2}{A, A, \Gamma, \Gamma \dashv \Delta', \Delta', B, B}\,\mathit{acut}_2$$

□

Next, we show that any application of the contrapositive of a $\mathsf{G4}$ logical rule can be turned into a number of *acut* and *dup* applications:

**Theorem 5.** *Each logical rule of* $\mathsf{G4}_{H1}$ *is admissible in the* $\mathsf{G4}_{H1\cap 2}$ *calculus.*

*Proof.* We consider the refutational rules for $\wedge$ and $\rightarrow$, leaving the other cases to the reader.

$a\wedge$ Consider the following derivations, with $i = 1, 2$[6]:

$$\dfrac{\begin{matrix}\vdots\\ A \wedge B \vdash B\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ A \wedge B \vdash A\end{matrix} \qquad \dfrac{A \wedge B, \Gamma \dashv \Delta}{A \wedge B, A \wedge B, \Gamma \dashv \Delta}\,\mathit{dup}}{A \wedge B, A, \Gamma \dashv \Delta}\,\mathit{acut}_1}{A, B, \Gamma \dashv \Delta}\,\mathit{acut}_1$$

$$\dfrac{\vdash\wedge\,\dfrac{\mathit{wk}\,\dfrac{\Gamma \vdash \Delta, A_i}{A_{3-i}, \Gamma \vdash \Delta, A_i} \qquad \begin{matrix}\vdots\\ A_{3-i}, \Gamma \vdash \Delta, A_{3-i}\end{matrix}}{A_{3-i}, \Gamma \vdash \Delta, A_1 \wedge A_2} \qquad \dfrac{\Gamma \dashv \Delta, A_1 \wedge A_2}{\Gamma, \Gamma \dashv \Delta, \Delta, A_1 \wedge A_2}\,\mathit{dup}}{\Gamma \dashv \Delta, A_{3-i}}\,\mathit{acut}_2$$

$a \to$ Consider the following derivations:

$$\dfrac{\to\vdash\,\dfrac{\mathit{wk}\,\dfrac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A, B} \qquad \begin{matrix}\vdots\\ B, \Gamma \vdash \Delta, B\end{matrix}}{A \to B, \Gamma \vdash \Delta, B} \qquad \dfrac{A \to B, \Gamma \dashv \Delta}{A \to B, \Gamma, \Gamma \dashv \Delta, \Delta}\,\mathit{dup}}{B, \Gamma \dashv \Delta}\,\mathit{acut}_1$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A, \Gamma \vdash \Delta, A\end{matrix} \qquad \dfrac{B, \Gamma \vdash \Delta}{B, A, \Gamma \vdash \Delta}\,\mathit{wk}}{A \to B, A, \Gamma \vdash \Delta}\,\to\vdash \qquad \dfrac{A \to B, \Gamma \dashv \Delta}{A \to B, \Gamma, \Gamma \dashv \Delta, \Delta}\,\mathit{dup}}{\Gamma \dashv \Delta, A}\,\mathit{acut}_2$$

$$\dfrac{\begin{matrix}\vdots\\ B \vdash A \to B\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \vdash A \to B, A\end{matrix} \qquad \dfrac{\Gamma \dashv \Delta, A \to B}{\Gamma \dashv \Delta, A \to B, A \to B}\,\mathit{dup}}{A, \Gamma \dashv \Delta, A \to B}\,\mathit{acut}_1}{A, \Gamma \dashv \Delta, B}\,\mathit{acut}_2$$

□

On the other hand, any application of a $\mathsf{G4}_{H2}$ refutational logical rule can be turned into a number of *acut* and *dup* applications:

**Theorem 6.** *Each logical rule of* $\mathsf{G4}_{H2}$ *is admissible in the* $\mathsf{G4}_{H1\cap 2}$ *calculus.*

*Proof.* We consider the refutational rules for $\vee$ and $\to$, leaving the other cases to the reader.

$a'\vee$ Take the following derivations:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A_{3-i} \vdash A_1 \vee A_2\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ A_i, \Gamma \vdash \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ A_{3-i}, \Gamma \dashv \Delta\end{matrix}}{A_{3-i}, \Gamma, \Gamma \dashv \Delta, \Delta}\,\mathit{dup}}{A_{3-i}, \Gamma \dashv \Delta, A_i}\,\mathit{acut}_2}{A_1 \vee A_2, \Gamma \dashv \Delta, A_i}\,\mathit{acut}_1}{A_1 \vee A_2, \Gamma \dashv \Delta}\,\mathit{str}$$

[6]We use dashed lines to denote admissible rules, and doubled (dashed) lines to refer to multiple applications of (admissible) rules.

$$\dfrac{\overset{\vdots}{A \lor B \vdash A, B} \qquad \overset{\vdots}{\Gamma \dashv \Delta, A, B}}{\Gamma \dashv \Delta, A \lor B}\ acut_2$$

$a' \to$ Consider the following derivations:

$$\dfrac{\overset{\vdots}{B \vdash A \to B} \qquad \dfrac{\overset{\vdots}{\Gamma \vdash \Delta, A} \qquad \dfrac{\overset{\vdots}{B, \Gamma \dashv \Delta}}{B, \Gamma, \Gamma \dashv \Delta, \Delta}\ dup}{A, B, \Gamma \dashv \Delta}\ acut_1}{\dfrac{A \to B, A, \Gamma \dashv \Delta}{A \to B, \Gamma \dashv \Delta}\ str}\ acut_1$$

$$\dfrac{\overset{\vdots}{\vdash A \to B, A} \qquad \dfrac{\overset{\vdots}{B, \Gamma \vdash \Delta} \qquad \dfrac{\overset{\vdots}{\Gamma \dashv \Delta, A}}{\Gamma, \Gamma \dashv \Delta, \Delta, A}\ dup}{\Gamma \dashv \Delta, A, B}\ acut_2}{\dfrac{A \to B, \Gamma \dashv \Delta, B}{A \to B, \Gamma \dashv \Delta}\ str}\ acut_1$$

$$\dfrac{\overset{\vdots}{A \to B, A \vdash B} \qquad \overset{\vdots}{A, \Gamma \dashv \Delta, B}}{\Gamma \dashv \Delta, A \to B}\ acut_2$$

□

We conclude this section by showing that any application of a refutational logical rule of $\mathsf{G4}_{\dashv\vdash}$ can be replaced by applications of *acut*:

**Theorem 7.** *Each refutational logical rule of* $\mathsf{G4}_{\dashv\vdash}$ *is admissible in the* $\mathsf{G4}_{H1\cap 2}$ *calculus.*

*Proof.* We consider only the $\dashv \land, \lor \dashv$ and $\to \dashv$ rules.

$\dashv \land$ Consider the following derivations:

$$acut_2\ \dfrac{\overset{\vdots}{A \land B \vdash A} \qquad \overset{\vdots}{\Gamma \dashv \Delta, A}}{\Gamma \dashv \Delta, A \land B} \qquad acut_2\ \dfrac{\overset{\vdots}{A \land B \vdash B} \qquad \overset{\vdots}{\Gamma \dashv \Delta, B}}{\Gamma \dashv \Delta, A \land B}$$

$\lor \dashv$ Take the following derivations:

$$acut_1\ \dfrac{\overset{\vdots}{A \vdash A \lor B} \qquad \overset{\vdots}{A, \Gamma \dashv \Delta}}{A \lor B, \Gamma \dashv \Delta} \qquad \dfrac{\overset{\vdots}{B \vdash A \lor B} \qquad \overset{\vdots}{B, \Gamma \dashv \Delta}}{A \lor B, \Gamma \dashv \Delta}\ acut_1$$

$\to \dashv$ Consider the following derivations:

$$acut_1\ \dfrac{\overset{\vdots}{\vdash B \to C, B} \qquad \overset{\vdots}{\Gamma \dashv \Delta, B}}{B \to C, \Gamma \dashv \Delta} \qquad acut_1\ \dfrac{\overset{\vdots}{C \vdash B \to C} \qquad \overset{\vdots}{C, \Gamma \dashv \Delta}}{B \to C, \Gamma \dashv \Delta}$$

□

*Remark* 1. In the proofs of Theorems 6 and 7 we establish the admissibility of the refutational logical rules of $\mathsf{G4}_{\dashv\vdash}$ in $\mathsf{G4}_{H1\cap 2}$ leveraging the following provable sequents:

$$A, B \vdash A \wedge B \qquad A \wedge B \vdash A \qquad A \wedge B \vdash B \tag{3.1}$$

$$A \vdash A \vee B \qquad B \vdash A \vee B \qquad A \vee B \vdash A, B \tag{3.2}$$

$$\vdash A \to B, A \qquad B \vdash A \to B \qquad A \to B, A \vdash B \tag{3.3}$$

$$\vdash A, \neg A \qquad \neg A, A \vdash \tag{3.4}$$

This approach to logical rules is analogous to that of Gentzen's **LDK** calculus: there, the sequents (3.1), (3.2) and (3.4) played the role of axioms (under the label of 'logical groundsequents') and standard structural rules were sufficient to recover each logical rule for deducibility [44, 27].

3.1. **Anticut elimination in classical logic.** In this subsection, we focus on the topic of anticut elimination from $\mathsf{D\text{-}R}$ sequent calculi for classical logic. In order to illustrate a constructive approach to *acut* elimination in $\mathsf{G4}_{\dashv\vdash}$, we first prove two preliminary results.

**Theorem 8.** *For any sequent $\Gamma \vdash \Delta$, either $\mathsf{G4}_{\dashv\vdash}$ proves $\Gamma \vdash \Delta$ or refutes $\Gamma \vdash \Delta$ – but not both.*

*Proof.* Since the complexity of each of the premises of any logical rule of $\mathsf{G4}_{\dashv\vdash}$ is strictly smaller than that of the conclusion, the decomposition procedure for a given sequent always terminates on a set of atomic (anti)sequents. By Theorem 2, this set is unique – and this suffices to the conclusion. □

**Lemma 9.** *The copy rule*

$$\dfrac{\Gamma \dashv \Delta \qquad \Gamma \dashv \Delta}{\Gamma, \Gamma \dashv \Delta, \Delta}\,\mathit{copy}$$

*is admissible in $\mathsf{G4}_{\dashv\vdash}$.*

*Proof.* Let us assume (without loss of generality) that for any *copy* application the two premisses have the same derivation $\pi$: we reason by induction on $2h(\pi)$ to get the conclusion. If $h(\pi) = 1$, the conclusion is immediate; otherwise, we reason by cases over the last rule applied in $\pi$.

∨ If the last rule applied in $\pi$ is $\vee\dashv$, we have e.g.

$$\mathit{copy}\,\dfrac{\vee\dashv\dfrac{\begin{matrix}\vdots\\ A, \Gamma \dashv \Delta\end{matrix}}{A \vee B, \Gamma \dashv \Delta} \qquad \vee\dashv\dfrac{\begin{matrix}\vdots\\ A, \Gamma \dashv \Delta\end{matrix}}{A \vee B, \Gamma \dashv \Delta}}{A \vee B, A \vee B, \Gamma, \Gamma \dashv \Delta, \Delta} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ A, \Gamma \dashv \Delta\end{matrix} \qquad \begin{matrix}\vdots\\ A, \Gamma \dashv \Delta\end{matrix}}{A, A, \Gamma, \Gamma \dashv \Delta, \Delta}\,\mathit{copy}}{A \vee B, A, \Gamma, \Gamma \dashv \Delta, \Delta}\,\vee\dashv}{A \vee B, A \vee B, \Gamma, \Gamma \dashv \Delta, \Delta}\,\vee\dashv$$

On the other hand, if the last rule applied in $\pi$ is $\dashv\vee$, we have

$$\mathit{copy}\,\dfrac{\dashv\vee\dfrac{\begin{matrix}\vdots\\ \Gamma \dashv \Delta, A, B\end{matrix}}{\Gamma \dashv \Delta, A \vee B} \qquad \dashv\vee\dfrac{\begin{matrix}\vdots\\ \Gamma \dashv \Delta, A, B\end{matrix}}{\Gamma \dashv \Delta, A \vee B}}{\Gamma, \Gamma \dashv \Delta, \Delta, A \vee B, A \vee B} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma \dashv \Delta, A, B\end{matrix} \qquad \begin{matrix}\vdots\\ \Gamma \dashv \Delta, A, B\end{matrix}}{\Gamma, \Gamma \dashv \Delta, \Delta, A, B, A, B}\,\mathit{copy}}{\Gamma, \Gamma \dashv \Delta, \Delta, A, B, A \vee B}\,\dashv\vee}{\Gamma, \Gamma \dashv \Delta, \Delta, A \vee B, A \vee B}\,\dashv\vee$$

¬ If the last rule applied in $\pi$ is $\neg\dashv$, we have

$$
{\scriptstyle copy}\dfrac{{\scriptstyle \neg\dashv}\dfrac{\overset{\vdots}{\Gamma\dashv\Delta,A}}{\neg A,\Gamma\dashv\Delta}\qquad {\scriptstyle \neg\dashv}\dfrac{\overset{\vdots}{\Gamma\dashv\Delta,A}}{\neg A,\Gamma\dashv\Delta}}{\neg A,\neg A,\Gamma,\Gamma\dashv\Delta,\Delta}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\overset{\vdots}{\Gamma\dashv\Delta,A}\qquad\overset{\vdots}{\Gamma\dashv\Delta,A}}{\Gamma,\Gamma\dashv\Delta,\Delta,A,A}{\scriptstyle copy}}{\neg A,\Gamma,\Gamma\dashv\Delta,\Delta,A}{\scriptstyle \neg\dashv}}{\neg A,\neg A,\Gamma,\Gamma\dashv\Delta,\Delta}{\scriptstyle \neg\dashv}
$$

On the other hand, if the last rule applied in $\pi$ is $\dashv\neg$, we have

$$
{\scriptstyle copy}\dfrac{{\scriptstyle \dashv\neg}\dfrac{\overset{\vdots}{A,\Gamma\dashv\Delta}}{\Gamma\dashv\Delta,\neg A}\qquad {\scriptstyle \dashv\neg}\dfrac{\overset{\vdots}{A,\Gamma\dashv\Delta}}{\Gamma\dashv\Delta,\neg A}}{\Gamma,\Gamma\dashv\Delta,\Delta,\neg A,\neg A}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\overset{\vdots}{\Gamma\dashv\Delta,A}\qquad\overset{\vdots}{\Gamma\dashv\Delta,A}}{\Gamma,\Gamma\dashv\Delta,\Delta,A,A}{\scriptstyle copy}}{\neg A,\Gamma,\Gamma\dashv\Delta,\Delta,A}{\scriptstyle \dashv\neg}}{\neg A,\neg A,\Gamma,\Gamma\dashv\Delta,\Delta}{\scriptstyle \dashv\neg}
$$

□

We are ready to prove the main result of this section:

**Theorem 10.** *There exists an algorithm which turns any* $\mathsf{G4}_{\dashv\vdash}+acut_i$*-derivation of* $\Pi\dashv\Sigma$ *into a* $\mathsf{G4}_{\dashv\vdash}$*-derivation of* $\Pi\dashv\Sigma$*, with* $i=1,2$.

*Proof.* We focus on the topmost $acut_i$ application, proceeding by primary induction on the logical complexity of the $acut_i$ formula and by secondary induction on the height of the derivation of the right premise.

First, there are cases of reduction of the size of the $acut_i$ formula where the inductive hypothesis applies to smaller $acut_{3-i}$ formulas:

$$
{\scriptstyle acut_1}\dfrac{\overset{\vdots}{\Gamma\vdash\Delta,\neg B}\qquad\overset{\vdots}{\Pi,\Gamma\dashv\Delta,\Sigma}}{\neg B,\Pi\dashv\Sigma}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{{\scriptstyle inv}\dfrac{\overset{\vdots}{\Gamma\vdash\Delta,\neg B}}{B,\Gamma\vdash\Delta}\qquad\overset{\vdots}{\Pi,\Gamma\dashv\Delta,\Sigma}}{\Pi\dashv\Sigma,B}{\scriptstyle acut_2}}{\neg B,\Pi\dashv\Sigma}{\scriptstyle \neg\dashv}
$$

$$
{\scriptstyle acut_2}\dfrac{\overset{\vdots}{\neg B,\Gamma\vdash\Delta}\qquad\overset{\vdots}{\Pi,\Gamma\dashv\Delta,\Sigma}}{\Pi\dashv\Sigma,\neg B}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{{\scriptstyle inv}\dfrac{\overset{\vdots}{\neg B,\Gamma\vdash\Delta}}{\Gamma\vdash\Delta,B}\qquad\overset{\vdots}{\Pi,\Gamma\dashv\Delta,\Sigma}}{B,\Pi\dashv\Sigma}{\scriptstyle acut_1}}{\Pi\dashv\Sigma,\neg B}{\scriptstyle \dashv\neg}
$$

On the other hand, there are cases of reduction where a single $acut_i$ application is replaced by multiple $acut_i$ and $acut_{3-i}$ applications on smaller formulas. Take e.g. the following derivations:

$$
\dfrac{\overset{\vdots}{\Gamma\vdash\Delta,B\wedge C}\qquad\overset{\vdots}{\Pi,\Gamma\dashv\Delta,\Sigma}}{B\wedge C,\Pi\dashv\Sigma}{\scriptstyle acut_1}
\qquad\qquad
\dfrac{\overset{\vdots}{B\vee C,\Gamma\vdash\Delta}\qquad\overset{\vdots}{\Pi,\Gamma\dashv\Delta,\Sigma}}{\Pi\dashv\Sigma,B\vee C}{\scriptstyle acut_2}
$$

$$\dfrac{\begin{matrix}\vdots\\ B \to C, \Gamma \vdash \Delta\end{matrix} \qquad \begin{matrix}\vdots\\ \Pi, \Gamma \dashv \Delta, \Sigma\end{matrix}}{\Pi \dashv \Sigma, B \to C}\ acut_2$$

We transform these derivations as follows:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma \vdash \Delta, B \wedge C\end{matrix}}{\Gamma \vdash \Delta, C}\ inv \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma \vdash \Delta, B \wedge C\end{matrix}}{\Gamma \vdash \Delta, B}\ inv}{\Pi, \Gamma \vdash \Delta, \Sigma, B}\ wk \qquad \dfrac{\begin{matrix}\vdots\\ \Pi, \Gamma \dashv \Delta, \Sigma\end{matrix} \qquad \begin{matrix}\vdots\\ \Pi, \Gamma \dashv \Delta, \Sigma\end{matrix}}{\Pi, \Pi, \Gamma, \Gamma \dashv \Delta, \Delta, \Sigma, \Sigma}\ copy}{B, \Pi, \Gamma \dashv \Delta, \Sigma}\ acut_1}{B, C, \Pi \dashv \Sigma}\ acut_1}{B \wedge C, \Pi \dashv \Sigma}\ \wedge\dashv$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ B \vee C, \Gamma \vdash \Delta\end{matrix}}{C, \Gamma \vdash \Delta}\ inv \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ B \vee C, \Gamma \vdash \Delta\end{matrix}}{B, \Gamma \vdash \Delta}\ inv}{B, \Pi, \Gamma \vdash \Delta, \Sigma}\ wk \qquad \dfrac{\begin{matrix}\vdots\\ \Pi, \Gamma \dashv \Delta, \Sigma\end{matrix} \qquad \begin{matrix}\vdots\\ \Pi, \Gamma \dashv \Delta, \Sigma\end{matrix}}{\Pi, \Pi, \Gamma, \Gamma \dashv \Delta, \Delta, \Sigma, \Sigma}\ copy}{\Pi, \Gamma \dashv \Delta, \Sigma, B}\ acut_2}{\Pi \dashv \Sigma, B, C}\ acut_2}{\Pi \dashv \Sigma, B \vee C}\ \dashv\vee$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ B \to C, \Gamma \vdash \Delta\end{matrix}}{\Gamma \vdash \Delta, B}\ inv \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ B \to C, \Gamma \vdash \Delta\end{matrix}}{C, \Gamma \vdash \Delta}\ inv}{C, \Pi, \Gamma \vdash \Delta, \Sigma}\ wk \qquad \dfrac{\begin{matrix}\vdots\\ \Pi, \Gamma \dashv \Delta, \Sigma\end{matrix} \qquad \begin{matrix}\vdots\\ \Pi, \Gamma \dashv \Delta, \Sigma\end{matrix}}{\Pi, \Pi, \Gamma, \Gamma \dashv \Delta, \Delta, \Sigma, \Sigma}\ copy}{\Pi, \Gamma \dashv \Delta, \Sigma, C}\ acut_1}{B, \Pi \dashv \Sigma, C}\ acut_1}{\Pi \dashv \Sigma, B \to C}\ \dashv\to$$

A different strategy must be employed to deal with the following case:

$$\dfrac{\begin{matrix}\vdots\\ \Gamma \vdash \Delta, B \vee C\end{matrix} \qquad \begin{matrix}\vdots\\ \Pi, \Gamma \dashv \Delta, \Sigma\end{matrix}}{B \vee C, \Pi \dashv \Sigma}\ acut_1$$

Take a $\mathsf{G4}_{\dashv\vdash}$-proof $\pi$ of $\Pi, \Gamma \vdash \Delta, \Sigma, B, C$: if we unthread (say) $B$ from $\pi$, we obtain either a $\mathsf{G4}_{\dashv\vdash}$-proof $\pi'$ of $\Pi, \Gamma \vdash \Delta, \Sigma, C$ or (at least) one $acut_i$-free derivation $\pi''$ of $\Pi, \Gamma \dashv \Delta, \Sigma, C$[7]. We reason by cases to reach the conclusion:

[7]Intuitively, a formula $D$ occurring in the conclusion of a derivation $\rho$ is *unthread* from $\rho$ whenever $D$ and its ancestors are deleted from $\rho$, up to the initial (anti)sequents of $\rho$. For a formal treatment of the notion of unthreading, see [36].

$$\dfrac{\dfrac{\overset{\vdots}{C, \Gamma \vdash \Delta, C} \qquad \dfrac{\overset{\vdots_{\pi'}}{\Pi, \Gamma \vdash \Delta, \Sigma, C} \qquad \dfrac{\overset{\vdots}{\Pi, \Gamma \dashv \Delta, \Sigma} \qquad \overset{\vdots}{\Pi, \Gamma \dashv \Delta, \Sigma}}{\Pi, \Pi, \Gamma, \Gamma \dashv \Delta, \Delta, \Sigma, \Sigma}\,copy}{C, \Pi, \Gamma \dashv \Delta, \Sigma}\,acut_1}{C, \Pi \dashv \Sigma}\,acut_1}{B \vee C, \Pi \dashv \Sigma}\,\vee\dashv$$

$$\dfrac{\dfrac{inv\,\dfrac{\overset{\vdots}{\Gamma \vdash \Delta, B \vee C}}{\Gamma \vdash \Delta, B, C} \qquad \overset{\vdots_{\pi''}}{\Pi, \Gamma \dashv \Delta, \Sigma, C}}{B, \Pi \dashv \Sigma}\,acut_1}{B \vee C, \Pi \dashv \Sigma}\,\vee\dashv$$

The same argument applies to the following cases:

$$acut_2\,\dfrac{\overset{\vdots}{B \wedge C, \Gamma \vdash \Delta} \qquad \overset{\vdots}{\Pi, \Gamma \dashv \Delta, \Sigma}}{\Pi \dashv \Sigma, B \wedge C} \qquad\qquad \dfrac{\overset{\vdots}{\Gamma \vdash \Delta, B \to C} \qquad \overset{\vdots}{\Pi, \Gamma \dashv \Delta, \Sigma}}{B \to C, \Pi \dashv \Sigma}\,acut_1$$

We can thus reduce ourselves to the case where the topmost $acut_i$ formula is atomic. Notice that if $\pi$ is the derivation of the right premiss $\Pi, \Gamma \dashv \Delta, \Sigma$ of the topmost $acut_i$ application, then $\pi$ is $acut_i$-free. Now, we proceed by induction on $h(\pi)$ to show that the topmost $acut_i$ application can be eliminated.

If $h(\pi) = 1$, the antisequent $\Pi, \Gamma \vdash \Delta, \Sigma$ has the form $\Theta \dashv \Lambda$.

(*i*) If $i = 1$, then the left premise has the form $\Theta' \vdash \Lambda', A$, with $\Theta' \subseteq \Theta$ and $\Lambda' \subseteq \Lambda$. This implies that $A \in \Theta'$, and thus that $A \notin \Lambda$: as a result, we infer that $\mathsf{G4}_{\dashv\vdash}$ derives $A, \Theta'' \dashv \Lambda''$, with $\Theta'' = \Theta \setminus \Theta'$ and $\Lambda'' = \Lambda \setminus \Lambda'$.

(*ii*) If $i = 2$, then the left premise has the form $A, \Theta' \vdash \Lambda'$, with $\Theta' \subseteq \Theta$ and $\Lambda' \subseteq \Lambda$. This implies that $A \in \Lambda'$, and thus that $A \notin \Theta$: as a result, we infer that $\mathsf{G4}_{\dashv\vdash}$ derives $\Theta'' \dashv \Lambda'', A$, with $\Theta'' = \Theta \setminus \Theta'$ and $\Lambda'' = \Lambda \setminus \Lambda'$.

On the other hand, if $h(\pi) > 1$ and some formula in $\Gamma \cup \Delta$ is principal in the last rule applied in $\pi$, we exploit the secondary inductive hypothesis as follows:

$$acut_1\,\dfrac{\overset{\vdots}{\Gamma \vdash \Delta, A} \qquad \dashv\,\dfrac{\overset{\vdots}{\Pi, \Gamma' \dashv \Delta', \Sigma}}{\Pi, \Gamma \dashv \Delta, \Sigma}}{A, \Pi \dashv \Sigma} \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\vdots}{\Gamma \vdash \Delta, A}}{\Gamma' \vdash \Delta', A}\,inv \qquad \overset{\vdots}{\Pi, \Gamma' \dashv \Delta', \Sigma}}{A, \Pi \dashv \Sigma}\,acut_1$$

$$acut_1\,\dfrac{\overset{\vdots}{A, \Gamma \vdash \Delta} \qquad \dashv\,\dfrac{\overset{\vdots}{\Pi, \Gamma' \dashv \Delta', \Sigma}}{\Pi, \Gamma \dashv \Delta, \Sigma}}{\Pi \dashv \Sigma, A} \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\vdots}{A, \Gamma \vdash \Delta}}{A, \Gamma' \vdash \Delta'}\,inv \qquad \overset{\vdots}{\Pi, \Gamma' \dashv \Delta', \Sigma}}{\Pi \dashv \Sigma, A}\,acut_1$$

Finally, if $h(\pi) > 1$ and some formula in $\Pi \cup \Sigma$ is principal in the last rule applied in $\pi$, we leverage the secondary inductive hypothesis as follows:

$$\mathit{acut}_1\ \dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \Delta, A\end{array} \qquad \dashv\ \dfrac{\begin{array}{c}\vdots\\ \Pi', \Gamma \dashv \Delta, \Sigma'\end{array}}{\Pi, \Gamma \dashv \Delta, \Sigma}}{A, \Pi \dashv \Sigma} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \Delta, A\end{array} \qquad \begin{array}{c}\vdots\\ \Pi', \Gamma \dashv \Delta, \Sigma'\end{array}}{A, \Pi' \dashv \Sigma'}\ \mathit{acut}_1}{A, \Pi \dashv \Sigma}\ \dashv$$

$$\mathit{acut}_2\ \dfrac{\begin{array}{c}\vdots\\ A, \Gamma \vdash \Delta\end{array} \qquad \dashv\ \dfrac{\begin{array}{c}\vdots\\ \Pi', \Gamma \dashv \Delta, \Sigma'\end{array}}{\Pi, \Gamma \dashv \Delta, \Sigma}}{\Pi \dashv \Sigma, A} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ A, \Gamma \vdash \Delta\end{array} \qquad \begin{array}{c}\vdots\\ \Pi', \Gamma \dashv \Delta, \Sigma'\end{array}}{\Pi' \dashv \Sigma', A}\ \mathit{acut}_2}{\Pi \dashv \Sigma, A}\ \dashv$$

□

The strategy for *acut*-elimination does not work for either $\mathsf{G4}_{H1}$ or $\mathsf{G4}_{H2}$: it is instructive to highlight the features hindering *acut* elimination in both cases.

For $\mathsf{G4}_{H1}$, it is clear that unthreading formulae from $\mathsf{G4}$-proofs is insufficient to produce $\mathsf{G4}_{H1}$ refutations. Take e.g. the following $\mathsf{G4}$-proof $\pi$:

$$\dfrac{\vdash\neg\ \dfrac{ax_{cl}\vdash\ \dfrac{}{p \vdash p, q}}{\vdash p, \neg p, q} \qquad \dfrac{\dfrac{}{p, q \vdash q}\ ax_{cl}\vdash}{q \vdash \neg p, q}\ \vdash\neg}{p \to q \vdash \neg p, q}\ \to\vdash$$

If we unthread $q$ from the conclusion of $\pi$, we get the following derivation $\pi'$:

$$\dfrac{\vdash\neg\ \dfrac{ax_{cl}\vdash\ \dfrac{}{p \vdash p}}{\vdash p, \neg p} \qquad \dfrac{\dfrac{}{p, q \dashv}\ ax_{cl}\dashv}{q \dashv \neg p}\ a'R\neg}{p \to q \dashv \neg p}\ a'L\to_1$$

Since $a'R\neg$ and $a'L \to_1$ are admissible but not derivable in $\mathsf{G4}_{H1}$, the derivation $\pi'$ does not count as a $\mathsf{G4}_{H1}$-refutation. Instead, a $\mathsf{G4}_{H1}$-refutation of the same end-antisequent must include (at least) one *acut*-application – as witnessed by the following derivation:

$$\dfrac{\vdash\to\ \dfrac{ax_{cl}\vdash\ \dfrac{}{q, p \vdash q}}{q \vdash p \to q} \qquad \dfrac{\neg\vdash\ \dfrac{ax_{cl}\vdash\ \dfrac{}{p \vdash p}}{p, \neg p \vdash} \qquad \dfrac{}{q, p \dashv}\ ax_{cl}\dashv}{q \dashv \neg p}\ \mathit{acut}_2}{p \to q \dashv \neg p}\ \mathit{acut}_1$$

Additionally, the secondary inductive hypothesis cannot be applied – as witnessed e.g. by the following configurations, where the $\mathit{acut}_i$ formula is atomic:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \Delta, A\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \Pi, \Gamma \vdash \Delta, \Sigma, C\end{array} \qquad \begin{array}{c}\vdots\\ \Pi, \Gamma \dashv \Delta, \Sigma, B \wedge C\end{array}}{\Pi, \Gamma \dashv \Delta, \Sigma, B}}{A, \Pi \dashv \Sigma, B}$$

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \Delta, B, A\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \Pi, \Gamma \vdash \Delta, \Sigma, C\end{array} \qquad \begin{array}{c}\vdots\\ \Pi, \Gamma \dashv \Delta, \Sigma, B \wedge C\end{array}}{\Pi, \Gamma \dashv \Delta, \Sigma, B}}{A, \Pi \dashv \Sigma}$$

On the other hand, the fact that *acut* rules cannot be eliminated from $\mathsf{G4}_{H1}$ without compromising completeness is not unexpected. As a matter of fact, the refutational logical rules of $\mathsf{G4}_{H1}$ do not allow for the introduction of logically complex formulas without assuming the introduction of even more complex formulas: since initial antisequents involve only atomic formulas, it is essential to have rules (actually, the *acut* rules) which introduce logically complex formulas following a distinct pattern.

As for $\mathsf{G4}_{H2}$, although unthreading remains well-defined, the elimination process cannot be fully executed. For instance, consider the following scenario:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \Delta, B \vee C\end{array} \qquad \begin{array}{c}\vdots\\ \Pi, \Gamma \dashv \Delta, \Sigma\end{array}}{B \vee C, \Pi \dashv \Sigma}$$

If $\mathsf{G4}_{H2}$ proves $\Gamma \vdash \Delta, C$, we have

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \Delta, C\end{array} \qquad \begin{array}{c}\vdots\\ \Pi, \Gamma \dashv \Delta, \Sigma\end{array}}{C, \Pi \dashv \Sigma}$$

If $\mathsf{G4}_{H2}$ refutes $B, \Pi \dashv \Sigma$, one cannot infer $B \vee C, \Pi \dashv \Sigma$ from $C, \Pi \dashv \Sigma$. Again, this is unsurprising. The binary refutational rules of $\mathsf{G4}_{H2}$ are sufficiently strong to ensure soundness, but they are too weak to achieve completeness, because they do not account for scenarios where both premises are invalid.

## 4. Anticut and **FDE**-based logics

In this section, we shall use capital Greek letters $\Theta, \Lambda \ldots$ to denote multisets of *literals* – i.e., atoms and negated atoms. Moreover, we take the *logical complexity* $\mathcal{C}(A)$ of a formula $A$ to be 0 if $A$ is a literal, $\mathcal{C}(B) + 1$ if $A$ is of the form $\neg\neg B$, $\mathcal{C}(B) + \mathcal{C}(C) + 1$ if $A$ is of the form $B \otimes C$ and $\mathcal{C}(\neg B) + \mathcal{C}(\neg C) + 1$ if $A$ is of the form $\neg(B \otimes C)$, with $\otimes \in \{\wedge, \vee\}$.

In this section, we shall deal with the following logics: Belnap's first-degree entailment logic [1, 2], Kleene's strong three-valued logic [22] and Priest's logic of paradox [35] – in short, **FDE**, **K3** and **LP**. The $\mathsf{Gfde}, \mathsf{Gk3}$ and $\mathsf{Glp}$ systems are sequent calculi for **FDE**, **K3** and **LP** (see Appendices B.1, B.6 and B.7 for detailed expositions)[8]. On the other hand, the $\mathsf{Gfde}_{H1}, \mathsf{Gfde}_{H2}, \mathsf{Gfde}_{H1\cap 2}$ and $\mathsf{Gfde}_{\dashv\vdash}$ calculi – as well as their counterparts for **K3** and **LP** – are analogous to the $\mathsf{G4}_{H1}$, $\mathsf{G4}_{H2}$, $\mathsf{G4}_{H1\cap 2}$ and $\mathsf{G4}_{\dashv\vdash}$ calculi, respectively (see Appendices B.2, B.3, B.4, B.5, B.6 and B.7).

Let $\mathsf{G}$ be any system among $\mathsf{Gfde}, \mathsf{Gk3}, \mathsf{Glp}$: we recall some basic facts about $\mathsf{G}_{\dashv\vdash}$.

[8]These systems are *multi-succedent* sequent calculi, modeled after the multi-succedent sequent calculus for **FDE** introduced in [37]. In contrast, [26] employs single-succedent calculi, based on the single-succedent sequent calculus for **FDE** first presented in [9]

**Theorem 11.** $\mathsf{G}_{\dashv\vdash}$ *enjoys the following properties.*

(*i*) *Each logical rule of* $\mathsf{G}$ *is height-preserving invertible.*

(*ii*) *The weakening, contraction and cut rules are admissible in* $\mathsf{G}_{\dashv\vdash}$.

(*iii*) *Maximal* $\mathsf{G}_{\dashv\vdash}$*-decomposition yields a unique set of atomic (anti)sequents.*

*Proof.* For proofs we refer to [31]. □

As a preliminary result, we establish that applications of the contrapositives of standard weakening and contraction rules are admissible in the $\mathsf{G}_{H1\cap 2}$ calculi.

**Lemma 12.** *The strengthening rules are admissible in the* $\mathsf{G}_{H1\cap 2}$ *calculus.*

*Proof.* We argue as in the proof of Lemma 3: it suffices to prove that $\mathsf{G}$ proves $\Gamma, A \vdash A, \Delta$ for any formula $A$ – proceeding by induction on the logical complexity of $A$. □

**Lemma 13.** *The duplication rules are admissible in the* $\mathsf{G}_{H1\cap 2}$ *calculus.*

*Proof.* We argue as in the proof of Lemma 4. □

Now, we show that any application of the contrapositive of a logical rule in $\mathsf{G}$ can be interpreted as a series of *acut* and *dup* applications.

**Theorem 14.** *Each logical rule of* $\mathsf{G}_{H1}$ *is admissible in the* $\mathsf{G}_{H1\cap 2}$ *calculus.*

*Proof.* It suffices to consider the refutational rules which are not featured by $\mathsf{G4}_{H1}$: we focus on $a\neg\wedge$, leaving the others to the reader.

$a\neg\wedge$ Consider the following derivations:

$$\dfrac{\neg\wedge\vdash\,\dfrac{wk\,\dfrac{\neg A, \Gamma \vdash \Delta}{\neg A, \Gamma \vdash \Delta, \neg B} \qquad \begin{matrix}\vdots\\ \neg B, \Gamma \vdash \Delta, \neg B\end{matrix}}{\neg(A\wedge B), \Gamma \vdash \Delta, \neg B} \qquad \dfrac{\neg(A\wedge B), \Gamma \dashv \Delta}{\neg(A\wedge B), \Gamma, \Gamma \dashv \Delta, \Delta}\,dup}{\neg B, \Gamma \dashv \Delta}\,acut_1$$

$$\dfrac{\neg\wedge\vdash\,\dfrac{wk\,\dfrac{\neg B, \Gamma \vdash \Delta}{\neg B, \Gamma \vdash \Delta, \neg A} \qquad \begin{matrix}\vdots\\ \neg A, \Gamma \vdash \Delta, \neg A\end{matrix}}{\neg(A\wedge B), \Gamma \vdash \Delta, \neg A} \qquad \dfrac{\neg(A\wedge B), \Gamma \dashv \Delta}{\neg(A\wedge B), \Gamma, \Gamma \dashv \Delta, \Delta}\,dup}{\neg A, \Gamma \dashv \Delta}\,acut_1$$

$$\dfrac{\begin{matrix}\vdots\\ \neg B \vdash \neg(A\wedge B)\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \neg A \vdash \neg(A\wedge B)\end{matrix} \qquad \dfrac{\Gamma \dashv \Delta, \neg(A\wedge B)}{\Gamma \dashv \Delta, \neg(A\wedge B), \neg(A\wedge B)}\,dup}{\Gamma \dashv \Delta, \neg A, \neg(A\wedge B)}\,acut_2}{\Gamma \dashv \Delta, \neg A, \neg B}\,acut_2$$

□

We can also reduce applications of logical rules in $\mathsf{G}_{H2}$ to applications of *acut* and *dup* rules.

**Theorem 15.** *Each logical rule of* $\mathsf{G}_{H2}$ *is admissible in the* $\mathsf{G}_{H1\cap 2}$ *calculus.*

*Proof.* It suffices to consider the refutational rules which do not belong to $\mathsf{G4}_{H2}$: we focus on $a'R\neg\vee$, leaving the other cases to the reader.

$a'R\neg\vee$ Take the following derivations, with $i = 1, 2$:

$$\dfrac{\dfrac{\vdots}{\neg(A_1 \vee A_2) \vdash \neg A_{3-i}} \quad \dfrac{\Gamma \vdash \Delta, \neg A_i \quad \dfrac{\Gamma \dashv \Delta, \neg A_{3-i}}{\Gamma, \Gamma \dashv \Delta, \Delta, \neg A_{3-i}}\,dup}{\neg A_i, \Gamma \dashv \Delta, \neg A_{3-i}}\,acut_1}{\dfrac{\neg A_i, \Gamma \dashv \Delta, \neg(A_1 \vee A_2)}{\Gamma \dashv \Delta, \neg(A_1 \vee A_2)}\,str}\,acut_2$$

□

Finally, we prove the following.

**Theorem 16.** *Each refutational rule in* $\mathsf{G}_{\dashv\vdash}$ *is admissible in the* $\mathsf{G}_{H1\cap 2}$ *calculus.*

*Proof.* We consider only the $\neg\wedge\dashv$ and $\dashv\neg\vee$ rules.

$\neg\wedge\dashv$ Consider the following derivations:

$$acut_1\,\dfrac{\dfrac{\vdots}{\neg A \vdash \neg(A \wedge B)} \quad \dfrac{\vdots}{\neg A, \Gamma \dashv \Delta}}{\neg(A \wedge B), \Gamma \dashv \Delta} \qquad \dfrac{\dfrac{\vdots}{\neg B \vdash \neg(A \wedge B)} \quad \dfrac{\vdots}{\neg B, \Gamma \dashv \Delta}}{\neg(A \wedge B), \Gamma \dashv \Delta}\,acut_1$$

$\dashv\neg\vee$ Take the following derivations:

$$acut_2\,\dfrac{\dfrac{\vdots}{\neg(A \vee B) \vdash \neg A} \quad \dfrac{\vdots}{\Gamma \dashv \Delta, \neg A}}{\Gamma \dashv \Delta, \neg(A \vee B)} \qquad \dfrac{\dfrac{\vdots}{\neg(A \vee B) \vdash \neg B} \quad \dfrac{\vdots}{\Gamma \dashv \Delta, \neg B}}{\Gamma \dashv \Delta, \neg(A \vee B)}\,acut_2$$

□

4.1. **Anticut elimination in FDE-based logics.** We extend now the anticut-elimination strategy presented in Section 3 to logics based on **FDE**. First, we state a basic fact about all these systems:

**Theorem 17.** *For any sequent* $\Gamma \vdash \Delta$*, either* $\mathsf{G}_{\dashv\vdash}$ *proves* $\Gamma \vdash \Delta$ *or refutes* $\Gamma \vdash \Delta$ *– but not both.*

*Proof.* The argument follows the same reasoning as in the proof of Theorem 8. □

We prove the desired result exploiting the following.

**Lemma 18.** *The copy rule is admissible in* $\mathsf{G}_{\dashv\vdash}$.

*Proof.* We argue as in the proof of Lemma 9. □

**Theorem 19.** *There exists an algorithm which turns any* $\mathsf{G}_{\dashv\vdash} + acut_i$*-derivation of* $\Pi \dashv \Sigma$ *into* $\mathsf{G}_{\dashv\vdash}$*-derivation of* $\Pi \dashv \Sigma$*, with* $i = 1, 2$.

*Proof.* We argue as in the proof of Theorem 10: here, we consider only one case arising when the $acut_i$ formula is a literal – leaving the others to the reader.
Let $A$ be the $acut_i$ formula and $\pi$ the derivation of the right premise. If $\mathcal{C}(A) = 0$, $h(\pi) = 1$, $i = 1$ and $\mathsf{G}$ is $\mathsf{Gfde}$, consider the derivation

$$\dfrac{ax_{fde}\vdash \dfrac{}{\Theta\vdash\Lambda, A} \qquad \dfrac{}{\Phi,\Theta\dashv\Lambda,\Psi}\, ax_{fde}\dashv}{A,\Phi\dashv\Psi}\, acut_1$$

We have that $\Theta\cap\Lambda = \varnothing$ and thus that $A \in \Theta$: since $\Theta\cap\Psi = \varnothing$, we infer that $A \notin \Psi$. Hence, we can derive $A, \Phi \dashv \Psi$ by a single application of $ax_{fde} \dashv$. Notice that if $\mathsf{G}$ is $\mathsf{Gk3}$ ($\mathsf{Glp}$) and $A \notin \Theta$, then $[p, \neg p] \subseteq \Theta$ ($[p, \neg p] \subseteq \Lambda$, respectively) – contrary to the hypothesis that $\Phi, \Theta \vdash \Lambda, \Psi$ is refutable. □

For the same reasons we detailed in the case of $\mathsf{G4}_{H1}$ and $\mathsf{G4}_{H2}$, the elimination strategy just presented does not apply to either $\mathsf{G}_{H1}$ or $\mathsf{G}_{H2}$ calculi.

4.2. **A glimpse on anticut and Ł-adequacy.** Let $\mathsf{H}$ denote any system among Kleene's $\mathsf{G4}$, $\mathsf{Gfde}$, $\mathsf{Gk3}$, and $\mathsf{Glp}$. Theorems 8 and 17 state that, for any pair of multisets $\Gamma$ and $\Delta$, there exists either an $\mathsf{H}{\dashv\vdash}$-derivation of $\Gamma\vdash\Delta$ or an $\mathsf{H}{\dashv\vdash}$-derivation of $\Gamma\dashv\Delta$ (but not both). Furthermore, $\mathsf{H}{\dashv\vdash}$ derives $\Gamma\vdash\Delta$ precisely when $\bigvee\Delta$ is a logical consequence of $\bigwedge\Gamma$. Consequently, Theorems 8 and 17 guarantee that $\mathsf{H}{\dashv\vdash}$ derives $\Gamma\dashv\Delta$ exactly when $\bigvee\Delta$ is not a logical consequence of $\bigwedge\Gamma$. This establishes that the (strong) Ł-adequacy of unmixed $\mathsf{H}_{\dashv\vdash}$ systems can be proven without relying on anticut rules. Moreover, it is important to observe that the anticut-elimination strategy employed in the proofs of Theorems 10 and 19 presupposes Ł-adequacy.

With respect to hybrid systems, Theorems 5, 6, 10, as well as 14, 15, and 19, establish the equivalence of these systems with the corresponding unmixed systems for the same logic. This equivalence indirectly confirms the (strong) Ł-adequacy of each hybrid system. A direct argument for establishing the Ł-adequacy of hybrid calculi for both classical and **FDE**-based logics can be uniformly constructed by leveraging *acut* rules.

Specifically, concerning refutational completeness, one can provide an argument which differs from the one detailed in [26, pp. 610–612]. Whenever $\mathsf{H}$ is one of classical, **FDE**, **K3**, or **LP** logic, if $\Gamma \not\models A$ in $\mathsf{H}$, we can proceed by induction on $\mathcal{C}(\Gamma\cup[A])$ to show that $\mathsf{H}_{H1\cap H2}$ derives $\Gamma\dashv A$ (details omitted).

## 5. Anticut and intuitionistic logic

In this section, we examine anticut rules in $\mathsf{D}$-$\mathsf{R}$ systems for intuitionistic propositional logic[9]. We begin by considering the unmixed sequent calculus $\mathsf{G4ip}_{\dashv\vdash}$ (see Appendices C.1 and C.2), and start our exposition by revisiting some fundamental results.

**Theorem 20.** $\mathsf{G4ip}_{\dashv\vdash}$ *enjoys the following properties.*

- ($i$) *Each logical rule of* $\mathsf{G4ip}$ *is height-preserving invertible, except for* $\to\to\vdash$ *(which fails to be invertible with respect to the left premise) and* $\vdash\to$ *.*
- ($ii$) *The weakening, contraction and cut rules are admissible in* $\mathsf{G4ip}$.

[9]We emphasize that this work adheres to Łukasiewicz's approach to refutation within an intuitionistic framework. As highlighted by a referee, this approach diverges from that of [24], where the author introduces informal conditions on the concept of refutation that are analogous to the BHK interpretation of constructive proof.

(*iii*) *Maximal* $\mathsf{G4ip}_{\dashv\vdash}$*-decomposition yields (at least) one set of irreducible (anti)sequents – namely, (anti)sequents of the form* $A_1 \to B_1, \ldots, A_l \to B_l, q_1, \ldots, q_m \mid\!\!\overset{*}{-} r_1, \ldots, r_n$ *with* $A_1, \ldots, A_l$ *being atoms or* $\bot$ *and* $[A_1, \ldots, A_l] \cap [q_1, \ldots, q_m] = \varnothing$.

*Proof.* For proofs see [7, 34]. □

To the best of our knowledge, no hybrid sequent calculus for intuitionistic logic has been formulated so far[10]. One may be tempted to design Gentzen-style hybrid systems for intuitionistic logic modeled after the hybrid sequent calculi for classical and **FDE**-based logics. Let us consider the following candidates:

(*i*) $\mathsf{G4ip}_{H1}$, obtained from $\mathsf{G4ip}$ by adding $ax_{int} \dashv$, the *acut* rules and the contrapositive versions of the logical rules of $\mathsf{G4ip}$;

(*ii*) $\mathsf{G4ip}_{H2}$, defined as the extension of $\mathsf{G4ip}$ with $ax_{int} \dashv$, the *acut* rules and the hybrid rules obtained from the logical rules of $\mathsf{G}$ by turning one of the premises and the conclusion into an antisequent.

The following results show that these candidates are not qualified for the job.

**Proposition 21.** *The* $\mathsf{G4ip}_{H1}$ *is not Ł-complete.*

*Proof.* We prove a stronger statement – namely, that if $\mathsf{G4ip}_{H1}$ derives $\Gamma \dashv \Delta$, then $\mathsf{G4}_{H1}$ derives $\Gamma \dashv \Delta$: since $\mathsf{G4}_{H1}$ does not derive (say) $\dashv p \vee (p \to \bot)$, this implies that $\mathsf{G4ip}_{H1}$ is not complete with respect to refutability.
We proceed by induction on the height of the $\mathsf{G4ip}_{H1}$-derivation $\pi$ of $\Gamma \dashv \Delta$. Suppose by contradiction that $\Gamma \dashv \Delta$ is an initial antisequent $A_1 \to B_1, \ldots, A_l \to B_l, q_1, \ldots, q_m \dashv r_1, \ldots, r_n$ of $\mathsf{G4ip}_{H1}$, and that $\mathsf{G4}_{H1}$ derives $A_1 \to B_1, \ldots, A_l \to B_l, q_1, \ldots, q_m \vdash r_1, \ldots, r_n$. By Theorem 2, point (*i*) we infer that $\mathsf{G4}_{H1}$ proves $q_1, \ldots, q_m \vdash r_1, \ldots, r_n, A_1, \ldots, A_l$: since $[q_1, \ldots, q_m] \cap [r_1, \ldots, r_n] = [q_1, \ldots, q_m] \cap [A_1, \ldots, A_l] = \varnothing$, we get the contradiction. If $h(\pi) > 1$, we reason by cases over the last rule applied, exploiting the inductive hypothesis as well as Lemma 3, Theorem 5 and the following derivations to reach the conclusion.

$$\dfrac{\begin{array}{c}\vdots\\ p, p \to B \vdash B\end{array} \qquad \dfrac{p, p \to B, \Gamma \dashv \Delta}{p, p, p \to B, \Gamma \dashv \Delta}\,dup}{p, B, \Gamma \dashv \Delta}\,acut_1$$

$$\dfrac{\begin{array}{c}\vdots\\ (C \wedge D) \to B \vdash C \to (D \to B)\end{array} \qquad (C \wedge D) \to B, \Gamma \dashv \Delta}{C \to (D \to B), \Gamma \dashv \Delta}\,acut_1$$

[10] This is not true of Hilbert-style $\mathsf{D}$-$\mathsf{R}$ systems for intuitionistic logic, as witnessed by [39] and [40].

$$\dfrac{\overset{\vdots}{(C \lor D) \to B \vdash D \to B} \qquad \dfrac{\overset{\vdots}{(C \lor D) \to B \vdash C \to B} \qquad \dfrac{(C \lor D) \to B, \Gamma \dashv \Delta}{(C \lor D) \to B, (C \lor D) \to B, \Gamma \dashv \Delta}\,dup}{C \to B, (C \lor D) \to B, \Gamma \dashv \Delta}\,acut_1}{C \to B, D \to B, \Gamma \dashv \Delta}\,acut_1$$

$$\dfrac{\overset{\vdots}{C \to D, (C \to D) \to B \vdash B} \qquad \dfrac{\vdash \to \dfrac{cut\,\dfrac{\overset{\vdots}{(C \to D) \to B \vdash D \to B} \qquad C, D \to B, \Gamma \vdash D}{C, (C \to D) \to B, \Gamma \vdash D}}{(C \to D) \to B, \Gamma \vdash C \to D} \qquad dup\,\dfrac{(C \to D) \to B, \Gamma \dashv \Delta}{(C \to D) \to B, (C \to D) \to B, \Gamma, \Gamma \dashv \Delta}}{C \to D, (C \to D) \to B, \Gamma \dashv \Delta}\,acut_1}{B, \Gamma \dashv \Delta}\,acut_1$$

$$\dfrac{\overset{\vdots}{(C \to D) \to B \vdash D \to B} \qquad \dfrac{\dfrac{\overset{\vdots}{(C \to D) \to B, C \to D \vdash B} \qquad \dfrac{B, \Gamma \vdash \Delta \qquad \dfrac{(C \to D) \to B, \Gamma \dashv \Delta}{(C \to D) \to B, (C \to D) \to B, \Gamma, \Gamma \dashv \Delta}\,dup}{(C \to D) \to B, (C \to D) \to B, \Gamma \dashv B}\,acut_1}{(C \to D) \to B, \Gamma \dashv C \to D}\,acut_2}{(C \to D) \to B, \Gamma, C \dashv D}\,aR \to}{D \to B, \Gamma, C \dashv D}\,acut_1$$

□

**Proposition 22.** *The* $\mathsf{G4ip}_{H2}$ *is not Ł-sound.*

*Proof.* Theorem 20, point $(ii)$ ensures that the following rule is not sound:

$$\dfrac{C, D \to B, \Gamma \dashv D \qquad B, \Gamma \vdash \Delta}{(C \to D) \to B, \Gamma \dashv \Delta}\,a'L \to\to$$

□

To obtain Ł-adequate, hybrid sequent calculi for intuitionistic logic we introduce new refutational rules – namely, the $adp_m$ rules (see Appendix C.3). These are Gentzen-style reformulations of the refutational principles presented in [39] and [40]. On this basis, we define the minimal $\mathsf{G4ip}_H$ calculus as the extension of $\mathsf{G4ip}$ with $ax_{int} \dashv$, the *acut* rules and the $adp_m$ rules (see Appendix C.3). Since this calculus makes its first appearance in this paper, we devote the first part of this section to the investigation of its structural properties.

Henceforth, we shall consider $\bot$ as a 0-ary connective instead of an atomic formula. We use capital Greek letters $\Theta, \Lambda, \ldots$ to denote multisets of atomic formulas and $\bot$. Moreover, we take the logical complexity $\mathcal{C}(A)$ of a formula $A$ to be defined as follows [7]: 0 if $A$ is $\bot$, 1 if $A$ is atomic, $\mathcal{C}(B) + \mathcal{C}(C) + 1$ if $A$ is $B \to C$, $\mathcal{C}(B) + \mathcal{C}(C) + 2$ if $A$ is $B \land C$ and $\mathcal{C}(B) + \mathcal{C}(C) + 3$ if $A$ is $B \lor C$. The other proof-theoretic notions are defined in the same way as in the previous sections of this paper.

**Lemma 23.** *The strengthening and duplication rules are admissible in the minimal* $\mathsf{G4ip}_H$ *calculus.*

*Proof.* We argue as in the proofs of Lemmas 3 and 4. □

**Theorem 24.** *Each refutational rule of* $\mathsf{G4ip}_{\dashv\vdash}$ *is admissible in the minimal* $\mathsf{G4ip}_H$ *calculus.*

*Proof.* We focus on the most interesting case, leaving the others to the reader.

$\to\dashv\to$ Take the following derivations $\pi_i$, $\pi'_j$ and $\pi''_i$, for any $1 \leq i \leq m$ and $1 \leq j \leq m$:

$$\dfrac{\begin{array}{c}\vdots\\ C_i \to D_i, C_i \vdash D_i\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ D_i \to B_i, C_i \vdash (C_i \to D_i) \to B_i\end{array} \qquad \dfrac{D_i \to B_i, C_i, \Gamma_i, \Theta \dashv D_i}{D_i \to B_i, C_i, C_i, \Gamma_i, \Theta \dashv D_i}\,dup}{(C_i \to D_i) \to B_i, C_i, \Gamma_i, \Theta \dashv D_i}\,acut}{(C_i \to D_i) \to B_i, \Gamma_i, \Theta \dashv C_i \to D_i}\,acut$$

$$\dfrac{\begin{array}{c}\vdots\\ \vdash (E_n \to F_n) \to (E_n \to F_n)\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \vdash (E_1 \to F_1) \to (E_1 \to F_1)\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ E_j \to F_j, E_j \vdash F_j\end{array} \qquad \Gamma, \Theta, E_j \dashv F_j}{\Gamma, \Theta \dashv E_j \to F_j}\,acut}{(E_1 \to F_1) \to (E_1 \to F_1), \Gamma, \Theta \dashv E_j \to F_j}\,acut\\ \vdots\end{array}}{(E_1 \to F_1) \to (E_1 \to F_1), \ldots, (E_n \to F_n) \to (E_n \to F_n), \Gamma, \Theta \dashv E_j \to F_j}\,acut$$

$$\dfrac{\begin{array}{c}\vdots\\ \vdash (E_n \to F_n) \to (E_n \to F_n)\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \vdash (E_1 \to F_1) \to (E_1 \to F_1)\end{array} \qquad \begin{array}{c}\vdots\pi_i\\ (C_i \to D_i) \to B_i, \Gamma_i, \Theta \dashv C_i \to D_i\end{array}}{(E_1 \to F_1) \to (E_1 \to F_1), (C_i \to D_i) \to B_i, \Gamma_i, \Theta \dashv C_i \to D_i}\,acut\\ \vdots\end{array}}{(E_1 \to F_1) \to (E_1 \to F_1), \ldots, (E_n \to F_n) \to (E_n \to F_n), (C_i \to D_i) \to B_i, \Gamma_i, \Theta \dashv C_i \to D_i}\,acut$$

Let $\Pi = (E_1 \to F_1) \to (E_1 \to F_1), \ldots, (E_n \to F_n) \to (E_n \to F_n)$. We obtain the conclusion by considering the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\{\pi''_i\}_{1\leq i\leq m}\\ \{\Pi, \Gamma, \Theta \dashv C_i \to D_i\}_{1\leq i\leq m}\end{array} \qquad \begin{array}{c}\vdots\{\pi'_j\}_{1\leq j\leq n}\\ \{\Pi, \Gamma, \Theta \dashv E_j \to F_j\}_{1\leq j\leq n}\end{array}}{\Pi, \Gamma, \Theta \dashv C_1 \to D_1, \ldots, C_m \to D_m, E_1 \to F_1, \ldots, E_n \to F_n, \Lambda}\,adp}{\Pi, \Gamma, \Theta \dashv E_1 \to F_1, \ldots, E_n \to F_n, \Lambda}\,str}{\Gamma, \Theta \dashv E_1 \to F_1, \ldots, E_n \to F_n, \Lambda}\,str$$

$\square$

*Remark* 2. In the proof of Theorem 24 we establish the admissibility of the refutational logical rules of $\mathsf{G4ip}_{\dashv\vdash}$ in the minimal $\mathsf{G4ip}_H$ calculus leveraging the provable sequents (3.1) and (3.2) of Section 3 as well as the following ones:

$$p, p \to B \vdash p \land B \tag{5.1}$$

$$(C \land D) \to B \vdash C \to (D \to B) \tag{5.2}$$

$$(C \lor D) \to B \vdash (C \to B) \land (C \to D) \tag{5.3}$$

$$B \vdash (C \to D) \to B \tag{5.4}$$

It should be noticed that the provability of sequents (5.1) – (5.3) can be reduced to the provability of the following instances of the sequent $A, A \to B \vdash B$:

$$p, p \to B \vdash B \qquad (C \wedge D) \to B, C \wedge D \vdash B \qquad (C \vee D) \to B, C \vee D \vdash B$$

This approach to logical rules is analogous to that of Gentzen's **LIG** calculus: there, the sequents (3.1), (3.2) and $A, A \to B \vdash B$ played the role of axioms (under the label of 'logical groundsequents'), and standard structural rules together with the single-succedent version of $\vdash \to$ were sufficient to recover each logical rule for deducibility [44, 3].

**Lemma 25.** *Let $A_1, \ldots, A_m$ be formulas such that $A_i$ is not an implication, for some $1 \leq i \leq m$, $\Gamma^{\to} = C_1 \to D_1, \ldots, C_n \to D_n$ with $C_1, \ldots, C_n$ being atoms or $\bot$ and $\Theta \cap [C_1, \ldots, C_n] = \Theta \cap \Lambda = \varnothing$. The rules*

$$\dfrac{A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_1 \qquad \ldots \qquad A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_m}{A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_1, \ldots, A_m, \Lambda}\, adp'_m$$

*are admissible in the $\mathsf{G4ip}_H$ calculus.*

*Proof.* For any $A_i$ which is not implicational we consider the following derivations $\pi_j$ and $\pi'_i$, with $1 \leq j \leq m$:

$$\dfrac{\vdash\to\ \dfrac{cut\ \dfrac{\to\to\vdash\ \dfrac{\begin{array}{c}\vdots\\ p, p \to A_i \vdash p\end{array} \qquad \begin{array}{c}\vdots\\ A_i \vdash A_i\end{array}}{(p \to p) \to A_i \vdash A_i} \qquad \begin{array}{c}\vdots\\ A_i \to B_i, A_i \vdash B_i\end{array}}{A_i \to B_i, (p \to p) \to A_i \vdash B_i}}{A_i \to B_i \vdash ((p \to p) \to A_i) \to B_i} \qquad A_1 \to B_1, \ldots, A_i \to B_i, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_j}{A_1 \to B_1, \ldots, ((p \to p) \to A_i) \to B_i, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_j}\, acut_1$$

$$\dfrac{\to\to\vdash\ \dfrac{\begin{array}{c}\vdots\\ p, p \to A_i \vdash p\end{array} \qquad \begin{array}{c}\vdots\\ A_i \vdash A_i\end{array}}{(p \to p) \to A_i \vdash A_i} \qquad \begin{array}{c}\vdots\,\pi_{j=i}\\ A_1 \to B_1, \ldots, ((p \to p) \to A_i) \to B_i, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_i\end{array}}{A_1 \to B_1, \ldots, ((p \to p) \to A_i) \to B_i, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv (p \to p) \to A_i}\, acut_2$$

A single application of $adp_m$ yields $A_1 \to B_1, \ldots, ((p \to p) \to A_i) \to B_i, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_1, \ldots, (p \to p) \to A_i, \ldots, A_m, \Lambda$. If $\Pi = A_1 \to B_1, \ldots, A_{i-1} \to B_{i-1}, A_{i+1} \to B_{i+1}, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta$ and $\Sigma = A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_m, \Lambda$, we take the following derivation $\rho$ to reach the conclusion:

$$\dfrac{\begin{array}{c}\vdots\\ A_i \vdash (p \to p) \to A_i\end{array} \qquad acut_1\ \dfrac{\vdash\to_i\ \dfrac{\to\to\vdash\ \dfrac{\begin{array}{c}\vdots\\ p \to p, A_i \to B_i, A_i \vdash A_i\end{array} \qquad \begin{array}{c}\vdots\\ B_i, A_i \vdash B_i\end{array}}{((p \to p) \to A_i) \to B_i, A_i \vdash B_i}}{((p \to p) \to A_i) \to B_i \vdash A_i \to B_i} \qquad \begin{array}{c}\vdots\\ ((p \to p) \to A_i) \to B_i, \Pi \dashv (p \to p) \to A_i, \Sigma\end{array}}{A_i \to B_i, \Pi \dashv (p \to p) \to A_i, \Sigma}}{A_i \to B_i, \Pi \dashv A_i, \Sigma}\, acut_2$$

□

**Theorem 26.** *Let $\Gamma^{\to} = D_1 \to E_1, \ldots, D_n \to E_n$, $D_1, \ldots, D_n$ being atoms or $\bot$, $\Theta \cap [D_1, \ldots, D_n] = \varnothing$ and $m \geq 1$. Any rule of the form*

$$\dfrac{\{A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_i\}_{1 \leq i \leq m} \qquad \{A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv C_h\}_{1 \leq h \leq k}}{A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv C_1, \ldots, C_k}\, vis_m$$

*is admissible in the minimal* $\mathsf{G4ip}_H$ *calculus, with* $A_i \neq C_h$ *for any* $1 \leq i \leq m$ *and* $1 \leq h \leq k$*.*

*Proof.* Suppose that each $A_i$ is implicational and each $C_h$ atomic: for any $1 \leq h \leq k$, if the minimal $\mathsf{G4ip}_H$ calculus refutes $A_1 \to B_1, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv C_h$, then $C_h \notin \Theta$. Hence, it suffices to consider the following derivation:

$$\dfrac{\dfrac{\{A_1 \to B_1, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_i\}_{1 \leq i \leq m}}{A_1 \to B_1, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_1, \dots, A_m, C_1, \dots, C_k}\,adp_m}{A_1 \to B_1, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv C_1, \dots, C_k}\,str$$

If there exists an $A_i$ which is not implicational, we replace $A_i \to B_i$ with $((p \to p) \to A_i) \to A_i$ and $A_i$ with $(p \to p) \to A_i$ as in Lemma 25: hence, we obtain a derivation $\pi_i$ of $A_1 \to B_1, \dots, ((p \to p) \to A_i) \to B_i, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv (p \to p) \to A_i$ as well as, for any $h$, a derivation $\pi'_h$ of $A_1 \to B_1, \dots, ((p \to p) \to A_i) \to B_i, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv C_h$. Moreover, if there exists a $C_h$ which is not atomic, we take the following derivations $\rho_h$ and $\rho'_i$:

$$\dfrac{\to\to\vdash\,\dfrac{p, p \to C_h \vdash p \qquad C_h \vdash C_h}{(p \to p) \to C_h \vdash C_h} \qquad \dfrac{\vdash\to\,\dfrac{\to\to\vdash\,\dfrac{p, p \to C_h \vdash p \qquad C_h \vdash C_h}{(p \to p) \to C_h \vdash C_h}}{\vdash ((p \to p) \to C_h) \to C_h} \qquad \begin{matrix}\vdots\,\pi'_h \\ A_1 \to B_1, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv C_h\end{matrix}}{((p \to p) \to C_h) \to C_h, A_1 \to B_1, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv C_h}\,acut_1}{((p \to p) \to C_h) \to C_h, A_1 \to B_1, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv (p \to p) \to C_h}\,acut_2$$

$$\dfrac{\vdash\to\,\dfrac{\to\to\vdash\,\dfrac{p, p \to C_h \vdash p \qquad C_h \vdash C_h}{(p \to p) \to C_h \vdash C_h}}{\vdash ((p \to p) \to C_h) \to C_h} \qquad \begin{matrix}\vdots\,\pi_i \\ A_1 \to B_1, \dots, ((p \to p) \to A_i) \to B_i, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv (p \to p) \to A_i\end{matrix}}{((p \to p) \to C_h) \to C_h, A_1 \to B_1, \dots, ((p \to p) \to A_i) \to B_i, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv (p \to p) \to A_i}\,acut_1$$

Let $\Pi = ((p \to p) \to C_h) \to C_h, A_1 \to B_1, \dots, ((p \to p) \to A_i) \to B_i, \dots, A_m \to B_m$, $\Pi' = \Pi \setminus [((p \to p) \to C_h) \to C_h]$, $\Pi'' = \Pi' \setminus [((p \to p) \to A_i) \to B_i]$ and $\Sigma = C_1, \dots, C_{h-1}, C_{h+1}, \dots, C_k$. We plug $\rho_h$ and $\rho'_i$ into the following derivations $\rho$ and $\rho'$:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \{\Pi, \Gamma^{\to}, \Theta \dashv A_j\}_{1 \leq j \neq i \leq m}\end{matrix} \qquad \begin{matrix}\vdots\,\rho_h \\ \Pi, \Gamma^{\to}, \Theta \dashv (p \to p) \to C_h\end{matrix} \qquad \begin{matrix}\vdots\,\rho'_i \\ \Pi, \Gamma^{\to}, \Theta \dashv (p \to p) \to A_i\end{matrix}}{\Pi, \Gamma^{\to}, \Theta \dashv A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_m, C_1, \dots, C_{h-1}, C_{h+1}, \dots, C_k, (p \to p) \to A_i, (p \to p) \to C_h}\,adp_{m+1}}{\Pi', \Gamma^{\to}, \Theta \dashv C_1, \dots, C_{h-1}, C_{h+1}, \dots, C_k, (p \to p) \to C_h}\,str$$

$$\dfrac{C_h \vdash (p \to p) \to C_h \qquad \dfrac{((p \to p) \to A_i) \to B_i \vdash A_i \to B_i \qquad \begin{matrix}\vdots\,\rho \\ ((p \to p) \to A_i) \to B_i, \Pi'', \Gamma^{\to}, \Theta \dashv \Sigma, (p \to p) \to C_h\end{matrix}}{A_i \to B_i, \Pi'', \Gamma^{\to}, \Theta \dashv \Sigma, (p \to p) \to C_h}\,acut_1}{A_i \to B_i, \Pi'', \Gamma^{\to}, \Theta \dashv \Sigma, C_h}\,acut_2$$

Notice that $A_i \to B_i, \Pi'', \Gamma^{\to}, \Theta \dashv \Sigma, C_h$ is identical to $A_1 \to B_1, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv C_1, \dots, C_k$, as desired. The same argument applies whenever there are multiple non-implicational $A_i$'s and non-atomic $C_h$'s. $\square$

*Remark* 3. The $vis_m$ rules are Gentzen-style formulations of the contrapositive versions of (restricted) Visser's rules [18, 19][11]. If we drop the requirement that $A_i \neq C_h$, then $vis_m$ and $adp'_m$ rules are equivalent in presence of *acut* rules (cf. [21, p. 256] and [14, p. 339]).

We say that $B$ is an *Harrop formula* whenever $B$ is either an atom, or $\bot$, or a conjunction of Harrop formulas, or an implication whose consequent is Harrop [15]. It is well-known that the disjunction property of intuitionistic logic is preserved under Harrop formulas: if $\Gamma$ contains only Harrop formulas, then $A \vee B$ is a logical consequence of $\Gamma$ only if either $A$ or $B$ is a logical consequence of $\Gamma$. The following is a refutation-theoretic formulation of this result[12]:

**Theorem 27.** *Let $\Gamma$ be a multiset of Harrop formulas. The rule*

$$\frac{\Gamma \dashv \Delta_1 \qquad \Gamma \dashv \Delta_2}{\Gamma \dashv \Delta_1, \Delta_2}\; ahdp$$

*is admissible in the minimal* $\mathsf{G4ip}_H$ *calculus.*

*Proof.* If $\Gamma = \varnothing$, $\Delta_1 = A_{11}, \dots, A_{1m}$ and $\Delta_2 = A_{21}, \dots, A_{2m}$ where $m \geq 1$, we take the following derivation $\pi_i$, with $i = 1, 2$:

$$\dfrac{\dfrac{\vdash A_{i1} \to A_{i1} \qquad \dashv A_{i1}, \dots, A_{im}}{A_{i1} \to A_{i1} \dashv A_{i1}, \dots, A_{im}}\; acut_1}{\vdots}$$

$$\frac{\vdash A_{(3-i)m} \to A_{(3-i)m} \qquad A_{i1} \to A_{i1}, \dots, A_{(3-i)(m-1)} \to A_{(3-i)(m-1)} \dashv A_{i1}, \dots, A_{im}}{A_{i1} \to A_{i1}, \dots, A_{(3-i)(m-1)} \to A_{(3-i)(m-1)}, A_{(3-i)m} \to A_{(3-i)m} \dashv A_{i1}, \dots, A_{im}}\; acut_1$$

Hence, for any $1 \leq j \leq m$ we consider the following derivation $\pi'_j$:

$$\dfrac{\begin{array}{c}\vdots\,\pi_i \\ A_{i1} \to A_{i1}, \dots, A_{(3-i)m} \to A_{(3-i)m} \dashv A_{i1}, \dots, A_{im}\end{array}}{A_{i1} \to A_{i1}, \dots, A_{(3-i)m} \to A_{(3-i)m} \dashv A_{ij}}\; str$$

Let $\Pi = A_{i1} \to A_{i1}, \dots, A_{(3-i)m} \to A_{(3-i)m}$: we plug the $\pi'_j$'s into the following derivation $\rho$ to reach the conclusion.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_{11} \\ \Pi \dashv A_{11}\end{array} \quad \cdots \quad \begin{array}{c}\vdots\,\pi_{2m} \\ \Pi \dashv A_{2m}\end{array}}{\Pi \dashv \Delta_1, \Delta_2}\; adp'_{2m}}{\dashv \Delta_1, \Delta_2}\; str$$

[11]It suffices to notice that the minimal $\mathsf{G4ip}_H$ calculus derives $A_1 \to B_1, \dots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv D_j$ for any $1 \leq j \leq n$.

[12]Let us recall that the disjunction property of intuitionistic logic extends to the broader class of *projective formulas*, which includes non-Harrop formulas such as $p \to (q \vee r)$ [10]. Consequently, we assert that the generalized version of *ahdp*, where $\Gamma$ is a multiset of projective formulas, is admissible in the minimal $\mathsf{G4ip}_H$ calculus. However, we leave this claim without a proof, as a complete inductive characterization of the class of projective formulas (if any) remains an open problem [5].

If $\Gamma \neq \varnothing$, then $\Gamma = A_1 \to B_1, \ldots, A_m \to B_m, \Gamma'$, with $m \geq 0$, $A_1, \ldots, A_m$ being atoms or $\bot$ and $\Gamma'$ lacking implications with atoms or $\bot$ as antecedent: to reach the conclusion, we reason by cases over $deg(\Gamma')$, defined as the maximal logical complexity of a formula in $\Gamma'$, and proceed by induction on the logical complexity of $\Gamma$.

If $deg(\Gamma') = 1$, then $\Gamma' = q_1, \ldots, q_n$. If $A_i \neq q_j$ for any $1 \leq i \leq m$ and $1 \leq j \leq n$, then for any $A \in \Delta_1, \Delta_2$ we apply $acut_2$ to $\vdash A \to A$ and $\Gamma \dashv \Delta_1$ as well as $\vdash A \to A$ and $\Gamma \dashv \Delta_2$: hence, we exploit $adp'_m$ to get the conclusion. On the other hand, if $A_i = q_j$ for some $1 \leq i \leq m$ and $1 \leq j \leq n$, we apply $at \to \dashv_{inv}$ and exploit the inductive hypothesis.

If $deg(\Gamma') > 1$, we reason by cases over $\Gamma'$. If $\Gamma' = B \wedge C, \Gamma''$, we leverage $acut_1$ to replace $B \wedge C$ with (say) $B$, and apply the inductive hypothesis. If $\Gamma' = B \to C, \Gamma''$ and $B = D \wedge E$, we exploit $acut_1$ to replace $(D \wedge E) \to C$ with $D \to (E \to C)$; if $B = D \vee E$ or $D \to E$, we leverage $acut_1$ to replace $B \to C$ with $D \to C$. In all cases, we apply the inductive hypothesis to reach the conclusion.

□

*Remark* 4. Theorems 26 and 27 establish that, in presence of *acut* rules, the admissibility of $adp'_m$ rules entails the admissibility of contrapositive versions of the disjunction property and Visser rules $vis_m$, for any $m$. Since the only intermediate logic where the disjunction property and all the Visser rules hold is intuitionistic logic [17], we infer that for any intermediate logic $\mathbf{L}$ distinct from the latter there must exist (at least) one $n \geq 2$ such that some $adp'_n$ rules are not admissible in any hybrid calculus for $\mathbf{L}$. A(n algebraic) proof of the fact that there exists (at least) one $n \geq 2$ such that *all* $adp'_n$ rules are not admissible in any hybrid calculus for $\mathbf{L}$ is detailed in [40, pp. 79-80].

5.1. **Anticut elimination in intuitionistic logic.** In this subsection, we illustrate a constructive approach to *acut* admissibility in $\mathsf{G4ip}_{\dashv\vdash}$. First, we recall a crucial result:

**Theorem 28.** *For any sequent $\Gamma \vdash \Delta$, either $\mathsf{G4ip}_{\dashv\vdash}$ proves $\Gamma \vdash \Delta$ or refutes $\Gamma \vdash \Delta$ – but not both.*

*Proof.* For a proof see [34, pp. 227-228]. □

In what follows, we establish a number of intermediate lemmas, which we will exploit in the proof of the main result – namely, Theorem 36.

**Lemma 29.** *The rules*

$$str\ \frac{A, \Gamma \dashv \Delta}{\Gamma \dashv \Delta} \qquad \frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta}\ str$$

$$\wedge \dashv_{inv}\ \frac{B \wedge C, \Gamma \dashv \Delta}{B, C, \Gamma \dashv \Delta} \qquad \frac{\Gamma \dashv \Delta, B \vee C}{\Gamma \dashv \Delta, B, C}\ \dashv \vee_{inv}$$

$$at \to \dashv_{inv}\ \frac{p, p \to B, \Gamma \dashv \Delta}{p, B, \Gamma \dashv \Delta} \qquad \frac{(B \wedge C) \to D, \Gamma \dashv \Delta}{B \to (C \to D), \Gamma \dashv \Delta}\ \wedge \to \dashv_{inv}$$

$$\vee \to \dashv_{inv}\ \frac{(B \vee C) \to D, \Gamma \dashv \Delta}{B \to D, C \to D, \Gamma \dashv \Delta} \qquad \frac{\Gamma \dashv A \to B}{\Gamma, A \dashv B}\ \dashv \to_{inv}$$

*are admissible in* $\mathsf{G4ip}_{\dashv\vdash}$.

*Proof.* For each rule distinct from $\dashv \rightarrow_{inv}$, we establish the conclusion proceeding by induction on the height of the derivation $\pi$ of the premise: in the cases of *inv* rules, we exploit the fact that the last rule applied in $\pi$ cannot be $\rightarrow \dashv \rightarrow$. As far as the rule $\dashv \rightarrow_{inv}$ is concerned, let $\pi$ be a $\mathsf{G4ip}_{\dashv\vdash}$-derivation of $\Gamma \dashv A \rightarrow B$: we define the *right rank* of $A \rightarrow B$ in $\pi$ as the number of consecutive antisequents in $\pi$ where $A \rightarrow B$ occurs on the right-hand side, from the conclusion upwards. We reach the conclusion proceeding by induction on the right rank of $A \rightarrow B$ in $\pi$ (we omit the details). $\square$

**Lemma 30.** *Let $\vec{A} = A_1 \rightarrow \cdots \rightarrow A_m$, with $m \geq 1$. If the rule*

$$\frac{B, \Gamma \dashv \Delta}{C, \Gamma \dashv \Delta}\, r$$

*is admissible in $\mathsf{G4ip}_{\dashv\vdash}$, then the rule*

$$\frac{\vec{A} \rightarrow B, \Gamma \dashv \Delta}{\vec{A} \rightarrow C, \Gamma \dashv \Delta}\, nr$$

*is admissible in $\mathsf{G4ip}_{\dashv\vdash}$.*

*Proof.* We argue by primary induction on $m$, secondary induction on the height of the derivation $\pi$ of the premise $\vec{A} \rightarrow B, \Gamma \dashv \Delta$ and ternary induction on the logical complexity of $A_m$. If $m = 1$, $h(\pi) = 1$ and $A_m$ is $\bot$ or an atom, the conclusion is immediate. If $m = 1$ and $h(\pi) > 1$, we reason by cases over the last rule applied in $\pi$. If $A_1 \rightarrow B$ is not principal in it, we apply the secondary inductive hypothesis. If $A_1$ is principal in it, we distinguish multiple cases according to $A_1$'s principal connective (if any). If $A_1$ is an atom $p$ belonging to $\Gamma$, we apply $r$ and $at \rightarrow \dashv$ to get the conclusion; if $A_1 = D \wedge E$, we exploit the secondary inductive hypothesis. Whenever $A_1 = D \vee E$, or $A_1 = D \rightarrow E$ with $\rightarrow \dashv \rightarrow$ being the last rule applied, we exploit the secondary as well as the ternary inductive hypothesis. If $m > 1$, we leverage the primary inductive hypothesis and replicate the argument we made for $m = 1$ to reach the conclusion. $\square$

**Lemma 31.** *Let $A$ be a non-implicational formula. The rule*

$$\frac{A \rightarrow B, A \rightarrow C, \Gamma \dashv \Delta}{A \rightarrow (B \wedge C), \Gamma \dashv \Delta} \rightarrow \wedge \dashv$$

*is admissible in the $\mathsf{G4ip}_{\dashv\vdash}$ calculus.*

*Proof.* We reason by primary induction on the logical complexity of $A$ and secondary induction on the height of the derivation $\pi$ of the premise. If $A$ is $\bot$ or an atom and $h(\pi) = 1$, the conclusion is immediate. If $h(\pi) > 1$ and neither $p \rightarrow B$ nor $p \rightarrow C$ is principal in the last rule applied, we exploit the secondary inductive hypothesis. If (say) $p \rightarrow B$ is principal and $p \rightarrow C$ not, we consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{p, B, p \rightarrow C, \Gamma \dashv \Delta}{p, B, C, \Gamma \dashv \Delta}\, at \rightarrow \dashv_{inv}}{p, B \wedge C, \Gamma \dashv \Delta}\, \wedge \dashv}{p, p \rightarrow (B \wedge C), \Gamma \dashv \Delta}\, at \rightarrow \dashv}{p \rightarrow (B \wedge C), \Gamma \dashv \Delta}\, str$$

If $A = D \wedge E$, we consider the following derivation:

$$\dfrac{\dfrac{\dfrac{(D \wedge E) \to B, (D \wedge E) \to C, \Gamma \dashv \Delta}{D \to (E \to B), (D \wedge E) \to C, \Gamma \dashv \Delta}\ \wedge \to \dashv_{inv}}{D \to (E \to B), D \to (E \to C), \Gamma \dashv \Delta}\ \wedge \to \dashv_{inv}}{D \to ((E \to B) \wedge (E \to C)), \Gamma \dashv \Delta}\ \to \wedge \dashv$$

By primary inductive hypothesis and $\wedge \dashv_{inv}$, the rule leading from $(E \to B) \wedge (E \to C), \Gamma \dashv \Delta$ to $E \to (B \wedge C), \Gamma \dashv \Delta$ is admissible in $\mathsf{G4ip}_{\dashv\vdash}$: hence, Lemma 30 ensures the admissibility of the rule leading from $D \to ((E \to B) \wedge (E \to C)), \Gamma \dashv \Delta$ to $D \to (E \to (B \wedge C)), \Gamma \dashv \Delta$: an application of $\wedge \to \dashv$ suffices to the conclusion.
If $A = D \vee E$, we take the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(D \vee E) \to B, (D \vee E) \to C, \Gamma \dashv \Delta}{D \to B, E \to B, (D \vee E) \to C, \Gamma \dashv \Delta}\ \vee \to \dashv_{inv}}{D \to B, E \to B, D \to C, E \to C, \Gamma \dashv \Delta}\ \vee \to \dashv_{inv}}{D \to (B \wedge C), E \to B, E \to C, \Gamma \dashv \Delta}\ \to \wedge \dashv}{D \to (B \wedge C), E \to (B \wedge C), \Gamma \dashv \Delta}\ \to \wedge \dashv}{(D \vee E) \to (B \wedge C), \Gamma \dashv \Delta}\ \vee \to \dashv$$

□

**Lemma 32.** *Let* $\Gamma^{\to} = A_1 \to B_1, \ldots, A_k \to B_k$*, with* $A_1, \ldots, A_k$ *atoms or* $\bot$*, and* $\Theta \cap [A_1, \ldots, A_k] = \varnothing$*. The restricted copy rule*

$$\dfrac{\Pi, \Gamma^{\to}, \Theta \dashv \Lambda, \Delta \qquad \Pi, \Gamma^{\to}, \Theta \dashv \Lambda, \Delta}{\Pi, \Gamma, \Gamma \dashv \Lambda, \Lambda, \Delta}\ rcopy$$

*is admissible in* $\mathsf{G4ip}_{\dashv\vdash}$*.*

*Proof.* We assume (without loss of generality) that for any *rcopy* application the two premises have the same derivation $\pi$: we reason by induction on $2h(\pi)$ to get the conclusion. If $h(\pi) = 1$, the conclusion is immediate; otherwise, we reason by cases over the last rule applied in $\pi$. The only interesting case arises when the last rule applied in $\pi$ is

$$\dfrac{\overset{\vdots\{\pi_i\}_{1 \le i \le m}}{\{D_i, E_i \to C_i, \Gamma_i, \Gamma^{\to}, \Theta \dashv E_i\}_{1 \le i \le m}} \qquad \overset{\vdots\{\pi'_j\}_{1 \le j \le n}}{\{\Gamma, \Gamma^{\to}, \Theta, F_j \dashv G_j\}_{1 \le j \le n}}}{\Gamma, \Gamma^{\to}, \Theta \dashv F_1 \to G_1, \ldots, F_n \to G_n, \Lambda}\ \to \dashv \to$$

where $\Gamma_i = \Gamma \setminus [(D_i \to E_i) \to C_i]$. We apply the inductive hypothesis to remove the *rcopy* applications from the following derivations $\rho_i$ and $\rho'_j$, for any $1 \le i \le m$ and $1 \le j \le m$:

$$\dfrac{\overset{\vdots\pi_i}{D_i, E_i \to C_i, \Gamma_i, \Gamma^{\to}, \Theta \dashv E_i} \qquad \overset{\vdots\pi_i}{D_i, E_i \to C_i, \Gamma_i, \Gamma^{\to}, \Theta \dashv E_i}}{D_i, E_i \to C_i, \Gamma_i, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta \dashv E_i}\ rcopy$$

$$\dfrac{\dfrac{\vdots \pi'_j}{\Gamma, \Gamma^{\to}, \Theta, F_j \dashv G_j} \quad \dfrac{\vdots \pi'_j}{\Gamma, \Gamma^{\to}, \Theta, F_j \dashv G_j}}{\Gamma, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta, F_j \dashv G_j} \, rcopy$$

Hence, we plug $\rho_i$ and $\rho'_i$ into the following derivation:

$$\dfrac{\dfrac{\vdots \{\rho_i\}_{1 \le i \le m}}{\{D_i, E_i \to C_i, \Gamma_i, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta \dashv E_i\}_{1 \le i \le m}} \quad \dfrac{\vdots \{\pi'_j\}_{1 \le j \le n}}{\{\Gamma, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta, F_j \dashv G_j\}_{1 \le j \le n}}}{\Gamma, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta \dashv F_1 \to G_1, \ldots, F_n \to G_n, \Lambda, \Lambda} \to \dashv \to$$

□

**Lemma 33.** *The copy rule*

$$\dfrac{\Gamma \dashv \Delta \qquad \Gamma \dashv \Delta}{\Gamma, \Gamma \dashv \Delta, \Delta} \, copy$$

*is admissible in* $\mathsf{G4ip}_{\dashv\vdash}$.

*Proof.* We argue as in the proof of Lemma 9: we focus on the case where the last rule applied in the derivation $\pi$ of each premise is $\to \dashv \to$. We have that $\pi$ has the following form:

$$\dfrac{\dfrac{\vdots \{\pi_i\}_{1 \le i \le m}}{\{D_i, E_i \to C_i, \Gamma_i, \Gamma^{\to}, \Theta \dashv E_i\}_{1 \le i \le m}} \quad \dfrac{\vdots \{\pi'_j\}_{1 \le j \le n}}{\{\Gamma, \Gamma^{\to}, \Theta, F_j \dashv G_j\}_{1 \le j \le n}}}{\Gamma, \Gamma^{\to}, \Theta \dashv F_1 \to G_1, \ldots, F_n \to G_n, \Lambda} \to \dashv \to$$

where $\Gamma_i$, $\Gamma^{\to}$ and $\Theta$ are defined as in the proof of Lemma 32. First, we apply *rcopy* to the $\pi_i$ and $\pi'_j$'s to obtain the derivations $\rho_i$ and $\rho'_j$ of $D_i, E_i \to C_i, \Gamma_i, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta \dashv E_i$ and $\Gamma, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta, F_j \dashv G_j$, respectively. For each $1 \le i \le m$ and $1 \le j \le n$ we consider a copy $\rho_{m+i}$ and $\rho'_{n+j}$ of $\rho_i$ and $\rho'_j$ such that $D_i = D_{m+i}$, $E_i = E_{m+i}$, $C_i = C_{m+i}$, $\Gamma_i = \Gamma_{m+1}$, $F_j = F_{n+j}$ and $G_j = G_{n+j}$. Hence, we plug the $\rho_1, \ldots, \rho_{2m}$ and $\rho'_1, \ldots, \rho'_{2n}$'s into the following derivation:

$$\dfrac{\dfrac{\vdots \{\rho_{i'}\}_{1 \le i' \le 2m}}{\{D_{i'}, E_{i'} \to C_{i'}, \Gamma_{i'}, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta \dashv E_i\}_{1 \le i' \le 2m}} \quad \dfrac{\vdots \{\rho'_{j'}\}_{1 \le j' \le 2n}}{\{\Gamma, \Gamma^{\to}, \Theta, F_{j'} \dashv G_{j'}\}_{1 \le j' \le 2n}}}{\Gamma, \Gamma, \Gamma^{\to}, \Gamma^{\to}, \Theta, \Theta \dashv F_1 \to G_1, \ldots, F_n \to G_n, F_1 \to G_1, \ldots, F_n \to G_n, \Lambda, \Lambda} \to \dashv \to$$

□

**Lemma 34.** *The rules*

$$dup \, \dfrac{A, \Gamma \dashv \Delta}{A, A, \Gamma \dashv \Delta} \qquad \dfrac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta, A, A} \, dup$$

*are admissible in the* $\mathsf{G4ip}_{\dashv\vdash}$ *calculus.*

*Proof.* It suffices to consider the following derivations:

$$copy \, \dfrac{A, \Gamma \dashv \Delta}{\dfrac{A, A, \Gamma, \Gamma \dashv \Delta, \Delta}{A, A, \Gamma \dashv \Delta} \, str} \qquad \dfrac{\Gamma \dashv \Delta, A}{\dfrac{\Gamma, \Gamma \dashv \Delta, \Delta, A, A}{\Gamma \dashv \Delta, A, A} \, str} \, copy$$

□

**Lemma 35.** *The rule*

$$\frac{B, \Gamma \dashv \Delta}{A \to B, \Gamma \dashv \Delta} \to \dashv$$

*is admissible in the* $\mathsf{G4ip}_{\dashv\vdash}$ *calculus.*

*Proof.* We assume (without loss of generality) that $A$ is non-implicational. We reason by primary induction on the height of the derivation $\pi$ of the premise, secondary induction on the logical complexity of $A$ and ternary induction on $B$. If $h(\pi) = 1$ and $A$ is $\perp$ or an atom $p$, the conclusion is straightforward. If $A = C \wedge D$, we take the following derivation:

$$\cfrac{\cfrac{\cfrac{B, \Gamma \dashv \Delta}{D \to B, \Gamma \dashv \Delta} \to \dashv}{C \to (D \to B), \Gamma \dashv \Delta} \to \dashv}{(C \wedge D) \to B, \Gamma \dashv \Delta} \wedge \to \dashv$$

On the other hand, if $A = C \vee D$, we consider the following derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{B, \Gamma \dashv \Delta}{B, B, \Gamma \dashv \Delta}\ dup}{C \to B, B, \Gamma \dashv \Delta} \to \dashv}{C \to B, D \to B, \Gamma \dashv \Delta} \to \dashv}{(C \vee D) \to B, \Gamma \dashv \Delta} \vee \to \dashv$$

If $h(\pi) > 1$ and $A$ is an atom $p$ such that $p \notin \Gamma$, we reason by cases over the last rule applied. If $B$ is not principal in the last rule applied, it suffices to apply the first inductive hypothesis. If $B$ is principal in the last rule applied, and the last rule applied is neither $\wedge \dashv$ nor $\to \dashv \to$, we apply the primary inductive hypothesis and Lemma 30 to reach the conclusion. Here, we focus on the remaining cases.

If $B = C \wedge D$, we consider the following derivation:

$$\cfrac{\cfrac{\cfrac{C, D, \Gamma \dashv \Delta}{A \to C, D, \Gamma \dashv \Delta} \to \dashv}{A \to C, A \to D, \Gamma \dashv \Delta} \to \dashv}{A \to (C \wedge D), \Gamma \dashv \Delta} \to \wedge \dashv$$

On the other hand, if $B = (C \to D) \to E$ and the last rule applied is $\to \dashv \to$, we apply the first inductive hypothesis. Finally, if $h(\pi) > 1$ and $A$ is neither atomic nor $\perp$, we proceed in the same way as in the case where $h(\pi) = 1$ and $A$ is neither atomic nor $\perp$. □

Now, we are ready to prove the existence of an *acut* elimination strategy for $\mathsf{G4ip}_{\dashv\vdash}$.

**Theorem 36.** *There exists an algorithm which turns any* $\mathsf{G4ip}_{\dashv\vdash} + acut_i$*-derivation of* $\Pi \dashv \Sigma$ *into a* $\mathsf{G4ip}_{\dashv\vdash}$*-derivation of* $\Pi \dashv \Sigma$*, with* $i = 1, 2$*.*

*Proof.* We focus on the topmost $acut_i$ application, proceeding by primary induction on the logical complexity of the $acut_i$ formula and by secondary induction on the height of the derivation $\pi$ of the right premise. If the $acut_i$ formula is an atom, a conjunction or a disjunction, the argument runs as in the proof of Theorem 10. The only different case arises whenever $A$ is an atom and the last rule applied in $\pi$ is $\to \dashv \to$: we leverage the secondary inductive hypothesis as well as the fact that $\Gamma \vdash \Delta, A$ can be derived with a single $ax_{int} \vdash$ application to reach the conclusion.
Here, we discuss in detail only the case where the $acut_i$ formula has the form $A \to B$: we assume without loss of generality that $A$ is either an atom or an implication.

($i$) Consider the following derivation:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \Delta, A \to B\end{array} \qquad \begin{array}{c}\vdots\,\pi\\ \Pi, \Gamma \dashv \Delta, \Sigma\end{array}}{A \to B, \Pi \dashv \Sigma}\ acut_1$$

We reason by cases over the last rule applied in $\pi$ to reach the conclusion. If $\Pi, \Gamma \dashv \Delta, \Sigma$ is an initial antisequent, then $\mathsf{G4ip}_{\dashv\vdash}$ proves $\Gamma, A \vdash B$, and thus $\Pi, \Gamma, A \vdash B, \Delta, \Sigma$. On the other hand, Theorem 28 ensures that $\mathsf{G4ip}_{\dashv\vdash}$ derives either $\Pi, \Gamma \vdash B, \Delta, \Sigma$ or $\Pi, \Gamma \dashv B, \Delta, \Sigma$. In the first case, we consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Pi, \Gamma \vdash B, \Delta, \Sigma\end{array} \qquad \dfrac{\dfrac{}{\Pi, \Gamma \dashv \Delta, \Sigma}\ ax_{int} \dashv}{\Pi, \Pi, \Gamma, \Gamma \dashv \Delta, \Delta, \Sigma, \Sigma}\ copy}{B, \Gamma, \Pi \dashv \Delta, \Sigma}\ acut_1}{B, \Pi \dashv \Sigma}\ str}{A \to B, \Pi \dashv \Sigma}\ \to \dashv$$

In the second case we consider the following derivation:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma, A \vdash B, \Delta\end{array} \qquad \begin{array}{c}\vdots\\ \Pi, \Gamma \dashv B, \Delta, \Sigma\end{array}}{\Pi \dashv \Sigma, A}\ acut_2$$

If $A$ is atomic, it suffices to notice that $A \notin \Pi$: this implies that $A \to B, \Pi \dashv \Sigma$ can be derived with a single application of $ax_{int} \dashv$. If $A = C \to D$, we consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, C \to D \vdash B\end{array}}{\Gamma, D \vdash B}\ inv}{\Gamma \vdash D \to B}\ \vdash \to \qquad \dfrac{\begin{array}{c}\vdots\\ \Gamma, C \to D \vdash B, \Delta\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \Pi, \Gamma \dashv B, \Delta, \Sigma\end{array}}{\Pi, \Gamma, \Gamma \dashv B, \Delta, \Sigma}\ dup}{\Pi, \Gamma \dashv \Sigma, C \to D}\ acut_2}{\Pi, D \to B \dashv \Sigma, C \to D}\ acut_1}{\dfrac{\Pi, D \to B \dashv C \to D}{\Pi, D \to B, C \dashv D}\ \dashv \to_{inv}}\ str}{(C \to D) \to B, \Pi \dashv \Sigma}\ \to \dashv \to$$

If $h(\pi) > 1$ and the last rule applied in $\pi$ is not $\to\dashv\to$, it suffices to exploit the secondary inductive hypothesis to reach the conclusion. If the last rule applied in $\pi$ is $\to\dashv\to$ we argue as in the case where $h(\pi) = 1$.

$(ii)$ Consider the following derivation:

$$\dfrac{\begin{matrix}\vdots\\ A\to B,\Gamma\vdash\Delta\end{matrix}\qquad \begin{matrix}\vdots_{\pi}\\ \Pi,\Gamma\dashv\Delta,\Sigma\end{matrix}}{\Pi\dashv\Sigma,A\to B}\,acut_2$$

If $\Pi,\Gamma\dashv\Delta,\Sigma$ is an initial antisequent, we reason by cases over $A$. If $A$ is atomic, we infer that the last rule applied in the derivation of $A\to B,\Gamma\vdash\Delta$ is $at\to\vdash$ : if $\Gamma=\Gamma',A$, then $\mathsf{G4ip}_{\dashv\vdash}$ proves $B,A,\Gamma'\vdash\Delta$. Take the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ B,A,\Gamma'\vdash\Delta\end{matrix}\qquad \dfrac{\dfrac{}{\Pi,A,\Gamma'\dashv\Delta,\Sigma}\,ax_{int}\dashv}{\Pi,A,A,\Gamma'\dashv\Delta,\Sigma}\,dup}{A,\Pi\dashv\Sigma,B}\,acut_2}{A,\Pi\dashv B}\,str}{\Pi\dashv\Sigma,A\to B}\,\to\dashv\to$$

If $A=C\to D$, the last rule applied in the derivation of $A\to B,\Gamma\vdash\Delta$ is $\to\to\vdash$. As a result, we have that $\mathsf{G4ip}_{\dashv\vdash}$ proves $C,D\to B,\Gamma\vdash D$ and $B,\Gamma\vdash\Delta$: the provability of $C,D\to B,\Gamma\vdash D$ entails the provability of $B,\Pi,\Gamma\vdash C\to D,\Delta,\Sigma$. Theorem 28 ensures that $\mathsf{G4ip}_{\dashv\vdash}$ derives either $\Pi,\Gamma\vdash C\to D,\Delta,\Sigma$ or $\Pi,\Gamma\dashv C\to D,\Delta,\Sigma$. In the first case, we leverage the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ B,\Gamma\vdash\Delta\end{matrix}\qquad \dfrac{\begin{matrix}\vdots\\ \Pi,\Gamma\vdash\Delta,\Sigma,C\to D\end{matrix}\qquad \dfrac{\dfrac{}{\Pi,\Gamma\dashv\Delta,\Sigma}\,ax_{int}\dashv}{\Pi,\Pi,\Gamma,\Gamma\dashv\Delta,\Delta,\Sigma,\Sigma}\,copy}{C\to D,\Pi,\Gamma\dashv\Delta,\Sigma}\,acut_1}{C\to D,\Pi\dashv\Sigma,B}\,acut_2}{C\to D,\Pi\dashv B}\,str}{\Pi\dashv\Sigma,(C\to D)\to B}\,\to\dashv\to$$

If $h(\pi) > 1$ and the last rule applied in $\pi$ is not $\to\dashv\to$, it suffices to exploit the secondary inductive hypothesis to get the conclusion. If the last rule applied in $\pi$ is $\to\dashv\to$ we argue as in the case where $h(\pi) = 1$.

□

*Remark* 5. Consider the following rule:

$$\dfrac{\Gamma\dashv\Delta}{\bot\to B,\Gamma\dashv\Delta}\,\bot\to\dashv$$

In [28], the authors state that $\bot\to\dashv$ is redundant in $\mathsf{G4ip}_{\dashv\vdash}$, due to the fact that the context $\Gamma^{\to}$ in the rules $ax_{int}\dashv$ and $\to\dashv\to$ is allowed to contain implications with $\bot$ as antecedent. Theorem 36 provides a constructive argument for the same claim, since any $\bot\to\dashv$ application can be seen as an *acut* application:

$$\dfrac{\begin{array}{c}\vdots\\ \vdash \bot \to B\end{array} \qquad \Gamma \dashv \Delta}{\bot \to B, \Gamma \dashv \Delta}\ acut_1$$

Notice that if we disallow implications with antecedent $\bot$ from $\Gamma^{\to}$ in the rules $ax_{int} \dashv$ and $\to \dashv \to$ of $\mathsf{G4ip}_{\dashv\vdash}$, there can be no *acut*-free $\mathsf{G4ip}_{\dashv\vdash}$-derivation of (say) $(p \to \bot) \to (q \vee r) \dashv (p \to \bot) \to q, (p \to \bot) \to r$, and thus Theorem 36 ceases to hold.

We conclude this section with the following result, establishing that $\mathsf{G4ip}_{\dashv\vdash}$ and the minimal $\mathsf{G4ip}_H$ calculus derive the same (anti)sequents:

**Theorem 37.** *The rules $adp_m$ are admissible in $\mathsf{G4ip}_{\dashv\vdash}$.*

*Proof.* Let $\Gamma = (C_1 \to D_1) \to B_1, \ldots, (C_m \to D_m) \to B_m$ and $\Gamma_i = \Gamma \setminus [(C_i \to D_i) \to B_i]$, for any $1 \leq i \leq m$. Take the following derivations $\pi_i$ and $\pi_i'$:

$$\dfrac{(C_i \to D_i) \to B_i, \Gamma_i, \Gamma^{\to}, \Theta \dashv C_i \to D_i}{(C_i \to D_i) \to B_i, \Gamma_i, \Gamma^{\to}, \Theta, C_i \dashv D_i}\ \dashv \to_{inv}$$

$$\dfrac{(C_i \to D_i) \to B_i \vdash D_i \to B_i \qquad \begin{array}{c}\vdots_{\pi_i}\\ (C_i \to D_i) \to B_i, \Gamma_i, \Gamma^{\to}, \Theta, C_i \dashv D_i\end{array}}{D_i \to B_i, \Gamma_i, \Gamma^{\to}, \Theta, C_i \dashv D_i}\ acut$$

We plug the derivations $\pi_i$ and $\pi_i'$ into the following derivation:

$$\dfrac{\begin{array}{c}\vdots_{\{\pi_i\}_{1\leq i\leq m}}\\ \{D_i \to B_i, \Gamma_i, \Gamma^{\to}, \Theta, C_i \dashv D_i\}_{1\leq i\leq m}\end{array} \qquad \begin{array}{c}\vdots_{\{\pi_i'\}_{1\leq i\leq m}}\\ \{(C_i \to D_i) \to B_i, \Gamma_i, \Gamma^{\to}, \Theta, C_i \dashv D_i\}_{1\leq i\leq m}\end{array}}{(C_1 \to D_1) \to B_1, \ldots, (C_m \to D_m) \to B_m, \Gamma^{\to}, \Theta \dashv C_1 \to D_1, \ldots, C_m \to D_m}\ \to \dashv \to$$

□

## 6. Concluding remarks

In this paper, we conducted a proof-theoretic investigation of D-R calculi for classical and non-classical logics, such as **FDE** and intuitionistic logic. Specifically, we have proved that anticut rules cannot be eliminated from hybrid sequent calculi without compromising Ł-completeness. Moreover, when anticut rules are combined with an appropriate set of rules, the resulting system attains refutational completeness. Finally, we have explored the role of anticut rules in unmixed sequent calculi, showing that anticut rules can be constructively eliminated by leveraging Ł-completeness.

From a methodological standpoint, combining anticut rules with an appropriate set of additional refutational principles proves to be a flexible and robust approach, adaptable to a wide range of logics. For instance, this method is equally applicable to various modal logics [11, 20] and intermediate logics [41, 14]. The minimal hybrid calculi developed for these logics can serve as effective frameworks for constructing new unmixed calculi tailored to the same logics, with **S4** being a notable example [16, 8].

From a broader perspective, it would be worthwhile to investigate whether there exist logics whose antitheorems can be recursively axiomatized only through hybrid, rather than unmixed, D-R systems. An especially intriguing case study in this context is Medvedev's logic of finite frames, given the long-standing open problem concerning its decidability [6, 41].

Finally, a further promising line of research appears to be the connection between anticut rules and the formulation of abductive problems viewed as deductive arguments in reverse [30].

## Appendix A. Proof systems for classical propositional logic

In this appendix we present the proof systems for classical logic employed in the paper. We use the capital Greek letters $\Gamma, \Delta, \Pi, \Sigma$ to refer to multisets of formulas, and $\Theta, \Lambda, \ldots$ to denote multisets of *atomic* formulas.

### A.1. Kleene's G4.

$$ax_{cl}\vdash \frac{}{\Gamma, p \vdash p, \Delta}$$

$$\wedge\vdash \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \vdash\wedge \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee\vdash \frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \qquad \vdash\vee \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\neg\vdash \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \qquad \vdash\neg \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$$

$$\to\vdash \frac{\Gamma \vdash \Delta, B \qquad A, \Gamma \vdash \Delta}{A \to B, \Gamma \vdash \Delta} \qquad \vdash\to \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}$$

### A.2. The $\mathsf{G4}_{H1}$ calculus.

The $\mathsf{G4}_{H1}$ calculus is obtained from G4 by adding the following rules:

$$ax_{cl}\dashv \frac{}{\Theta \dashv \Lambda}$$

$$aL\wedge \frac{A \wedge B, \Gamma \dashv \Delta}{A, B, \Gamma \dashv \Delta} \qquad aR\wedge_1 \frac{\Gamma \vdash \Delta, A \qquad \Gamma \dashv \Delta, A \wedge B}{\Gamma \dashv \Delta, B} \qquad aR\wedge_2 \frac{\Gamma \vdash \Delta, B \qquad \Gamma \dashv \Delta, A \wedge B}{\Gamma \dashv \Delta, A}$$

$$aL\vee_1 \frac{A, \Gamma \vdash \Delta \qquad A \vee B, \Gamma \dashv \Delta}{B, \Gamma \dashv \Delta} \qquad aL\vee_2 \frac{B, \Gamma \vdash \Delta \qquad A \vee B, \Gamma \dashv \Delta}{A, \Gamma \dashv \Delta} \qquad aR\vee \frac{\Gamma \dashv \Delta, A \vee B}{\Gamma \dashv \Delta, A, B}$$

$$aL\to_1 \frac{\Gamma \vdash \Delta, A \qquad A \to B, \Gamma \dashv \Delta}{B, \Gamma \dashv \Delta} \qquad aL\to_2 \frac{B, \Gamma \vdash \Delta \qquad A \to B, \Gamma \dashv \Delta}{\Gamma \dashv \Delta, A} \qquad aR\to \frac{\Gamma \dashv \Delta, A \to B}{A, \Gamma \dashv \Delta, B}$$

$$aL\neg \frac{\neg A, \Gamma \dashv \Delta}{\Gamma \dashv \Delta, A} \qquad aR\neg \frac{\Gamma \dashv \Delta, \neg A}{A, \Gamma \dashv \Delta}$$

$$acut_1 \frac{\Gamma \vdash \Delta, A \qquad \Pi, \Gamma \dashv \Delta, \Sigma}{A, \Pi \dashv \Sigma} \qquad acut_2 \frac{A, \Gamma \vdash \Delta \qquad \Pi, \Gamma \dashv \Delta, \Sigma}{\Pi \dashv \Sigma, A}$$

In the rule $ax_{cl} \dashv$ , $\Theta \cap \Lambda = \varnothing$.

A.3. **The $\mathsf{G4}_{H2}$ calculus.** The $\mathsf{G4}_{H2}$ calculus is obtained from $\mathsf{G4}$ by adding $ax_{cl} \dashv$ as well as the following rules:

$$a'\wedge\dashv \frac{A, B, \Gamma \dashv \Delta}{A \wedge B, \Gamma \dashv \Delta} \qquad \dashv a'\wedge \frac{\Gamma \vdash \Delta, A \qquad \Gamma \dashv \Delta, B}{\Gamma \dashv \Delta, A \wedge B} \qquad \dashv a'\wedge \frac{\Gamma \vdash \Delta, B \qquad \Gamma \dashv \Delta, A}{\Gamma \dashv \Delta, A \wedge B}$$

$$a'\vee\dashv \frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \dashv \Delta}{A \vee B, \Gamma \dashv \Delta} \qquad a'\vee\dashv \frac{B, \Gamma \vdash \Delta \qquad A, \Gamma \dashv \Delta}{A \vee B, \Gamma \dashv \Delta} \qquad \dashv a'\vee \frac{\Gamma \dashv \Delta, A, B}{\Gamma \dashv \Delta, A \vee B}$$

$$a'\neg\dashv \frac{\Gamma \dashv \Delta, A}{\neg A, \Gamma \dashv \Delta} \qquad \dashv a'\neg \frac{A, \Gamma \dashv \Delta}{\Gamma \dashv \Delta, \neg A}$$

$$a' \rightarrow \dashv \frac{\Gamma \vdash \Delta, A \qquad B, \Gamma \dashv \Delta}{A \rightarrow B, \Gamma \dashv \Delta} \qquad a' \rightarrow \dashv \frac{B, \Gamma \vdash \Delta \qquad \Gamma \dashv \Delta, A}{A \rightarrow B, \Gamma \dashv \Delta} \qquad \dashv a' \rightarrow \frac{A, \Gamma \dashv \Delta, B}{\Gamma \dashv \Delta, A \rightarrow B}$$

$$acut_1 \frac{\Gamma \vdash \Delta, A \qquad \Pi, \Gamma \dashv \Delta, \Sigma}{A, \Pi \dashv \Sigma} \qquad acut_2 \frac{A, \Gamma \vdash \Delta \qquad \Pi, \Gamma \dashv \Delta, \Sigma}{\Pi \dashv \Sigma, A}$$

A.4. **The $\mathsf{G4}_{H1\cap 2}$ calculus.** The $\mathsf{G4}_{H1\cap 2}$ calculus is obtained from $\mathsf{G4}$ by adding $ax_{cl} \dashv$ and the *acut* rules.

A.5. **The $\mathsf{G4}_{\dashv\vdash}$ calculus.** The $\mathsf{G4}_{\dashv\vdash}$ is obtained from $\mathsf{G4}$ by adding $ax_{cl} \dashv$ as well as the following rules:

$$\wedge\dashv \frac{A, B, \Gamma \dashv \Delta}{A \wedge B, \Gamma \dashv \Delta} \qquad \dashv\wedge \frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta, A \wedge B} \qquad \dashv\wedge \frac{\Gamma \dashv \Delta, B}{\Gamma \dashv \Delta, A \wedge B}$$

$$\vee\dashv \frac{A, \Gamma \dashv \Delta}{A \vee B, \Gamma \dashv \Delta} \qquad \vee\dashv \frac{B, \Gamma \dashv \Delta}{A \vee B, \Gamma \dashv \Delta} \qquad \dashv\vee \frac{\Gamma \dashv \Delta, A, B}{\Gamma \dashv \Delta, A \vee B}$$

$$\neg\dashv \frac{\Gamma \dashv \Delta, A}{\neg A, \Gamma \dashv \Delta} \qquad \dashv\neg \frac{A, \Gamma \dashv \Delta}{\Gamma \dashv \Delta, \neg A}$$

$$\rightarrow\dashv \frac{\Gamma \dashv \Delta, A}{A \rightarrow B, \Gamma \dashv \Delta} \qquad \rightarrow\dashv \frac{B, \Gamma \dashv \Delta}{A \rightarrow B, \Gamma \dashv \Delta} \qquad \dashv\rightarrow \frac{A, \Gamma \dashv \Delta, B}{\Gamma \dashv \Delta, A \rightarrow B}$$

## Appendix B. Proof systems for **FDE**-based logics

In this appendix we gather the proof systems for **FDE**-based logics employed in the paper, using the capital Greek letters $\Gamma, \Delta, \Pi, \Sigma$ to refer to multisets of formulas, and $\Theta, \Lambda, \ldots$ to denote multisets of *literals*.

B.1. **The $\mathsf{Gfde}$ calculus.**

$$ax_{fde}\vdash \frac{}{\Gamma, p \vdash p, \Delta} \qquad ax_{fde}\vdash \frac{}{\Gamma, \neg p \vdash \neg p, \Delta}$$

$$\wedge\vdash \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \vdash\wedge \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee\vdash \frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \qquad \vdash\vee \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\neg\wedge\vdash \frac{\neg A, \Gamma \vdash \Delta \qquad \neg B, \Gamma \vdash \Delta}{\neg(A \wedge B), \Gamma \vdash \Delta} \qquad \vdash\neg\wedge \frac{\Gamma \vdash \Delta, \neg A, \neg B}{\Gamma \vdash \Delta, \neg(A \wedge B)}$$

$$\neg\vee\vdash \frac{\neg A, \neg B, \Gamma \vdash \Delta}{\neg(A \vee B), \Gamma \vdash \Delta} \qquad \vdash\neg\vee \frac{\Gamma \vdash \Delta, \neg A \qquad \Gamma \vdash \Delta, \neg B}{\Gamma \vdash \Delta, \neg(A \vee B)}$$

$$\neg\neg\vdash \frac{A, \Gamma \vdash \Delta}{\neg\neg A, \Gamma \vdash \Delta} \qquad \vdash\neg\neg \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, \neg\neg A}$$

B.2. **The $\mathsf{Gfde}_{H1}$ calculus.** The $\mathsf{Gfde}_{H1}$ calculus is obtained from the $\mathsf{Gfde}$ calculus by adding the $a\wedge, a\vee$ and *acut* rules as well as the following ones:

$$ax_{fde}\dashv \frac{}{\Theta \dashv \Lambda} \qquad ax_{fde}\dashv \frac{}{\Theta, p, \neg p \dashv \Lambda} \qquad ax_{fde}\dashv \frac{}{\Theta \dashv p, \neg p, \Lambda}$$

$$aL\neg\wedge_1 \frac{\neg A, \Gamma \vdash \Delta \qquad \neg(A \wedge B), \Gamma \dashv \Delta}{\neg B, \Gamma \dashv \Delta} \qquad aL\neg\wedge_2 \frac{\neg B, \Gamma \vdash \Delta \qquad \neg(A \wedge B), \Gamma \dashv \Delta}{\neg A, \Gamma \dashv \Delta} \qquad aR\neg\wedge \frac{\Gamma \dashv \Delta, \neg(A \wedge B)}{\Gamma \dashv \Delta, \neg A, \neg B}$$

$$aR\neg\vee_1 \frac{\Gamma \vdash \Delta, \neg A \qquad \Gamma \dashv \Delta, \neg(A \vee B)}{\Gamma \dashv \Delta, \neg B} \qquad aR\neg\vee_2 \frac{\Gamma \vdash \Delta, \neg B \qquad \Gamma \dashv \Delta, \neg(A \vee B)}{\Gamma \dashv \Delta, \neg A} \qquad aL\neg\vee \frac{\neg(A \vee B), \Gamma \dashv \Delta}{\neg A, \neg B, \Gamma \dashv \Delta}$$

$$aL\neg\neg \frac{\neg\neg A, \Gamma \dashv \Delta}{A, \Gamma \dashv \Delta} \qquad aR\neg\neg \frac{\Gamma \dashv \Delta, \neg\neg A}{\Gamma \dashv \Delta, A}$$

In the rules $ax_{fde}\dashv$, $\Theta \cap \Lambda = \varnothing$.

B.3. **The $\mathsf{Gfde}_{H2}$ calculus.** The $\mathsf{Gfde}_{H2}$ is obtained from $\mathsf{Gfde}$ by adding the $ax_{fde}\dashv$, $a'\wedge$, $a'\vee$ and *acut* rules, as well as the following ones:

$$a'\neg\wedge\dashv \frac{\neg A, \Gamma \vdash \Delta \qquad \neg B, \Gamma \dashv \Delta}{\neg(A \wedge B), \Gamma \dashv \Delta} \qquad a'\neg\wedge\dashv \frac{\neg B, \Gamma \vdash \Delta \qquad \neg A, \Gamma \dashv \Delta}{\neg(A \wedge B), \Gamma \vdash \Delta} \qquad \dashv a'\neg\wedge \frac{\Gamma \dashv \Delta, \neg A, \neg B}{\Gamma \dashv \Delta, \neg(A \wedge B)}$$

$$a'\neg\vee\dashv \frac{\neg A, \neg B, \Gamma \dashv \Delta}{\neg(A \vee B), \Gamma \dashv \Delta} \qquad \dashv a'\neg\vee \frac{\Gamma \vdash \Delta, \neg A \qquad \Gamma \dashv \Delta, \neg B}{\Gamma \dashv \Delta, \neg(A \vee B)} \qquad \dashv a'\neg\vee \frac{\Gamma \vdash \Delta, \neg B \qquad \Gamma \dashv \Delta, \neg A}{\Gamma \dashv \Delta, \neg(A \vee B)}$$

$$a'\neg\neg\dashv \frac{A, \Gamma \dashv \Delta}{\neg\neg A, \Gamma \dashv \Delta} \qquad \dashv a'\neg\neg \frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta, \neg\neg A}$$

B.4. **The $\mathsf{Gfde}_{H1\cap 2}$ calculus.** The $\mathsf{Gfde}_{H1\cap 2}$ calculus is obtained from $\mathsf{Gfde}$ by adding the $ax_{fde}\dashv$, the *acut* rules and the *dup* rules.

B.5. **The $\mathsf{Gfde}_{\dashv\vdash}$ calculus.** The $\mathsf{Gfde}_{\dashv\vdash}$ is obtained from $\mathsf{Gfde}$ by adding the $ax_{fde} \dashv$, $\wedge \dashv$, $\dashv \wedge$, $\vee \dashv$ and $\dashv \vee$ rules, as well as the following rules:

$$\neg\wedge\dashv\frac{\neg A, \Gamma \dashv \Delta}{\neg(A \wedge B), \Gamma \vdash \Delta} \qquad \neg\wedge\dashv\frac{\neg B, \Gamma \dashv \Delta}{\neg(A \wedge B), \Gamma \vdash \Delta} \qquad \vdash\neg\wedge\frac{\Gamma \dashv \Delta, \neg A, \neg B}{\Gamma \dashv \Delta, \neg(A \wedge B)}$$

$$\neg\vee\dashv\frac{\neg A, \neg B, \Gamma \dashv \Delta}{\neg(A \vee B), \Gamma \dashv \Delta} \qquad \dashv\neg\vee\frac{\Gamma \vdash \Delta, \neg A}{\Gamma \vdash \Delta, \neg(A \vee B)} \qquad \dashv\neg\vee\frac{\Gamma \vdash \Delta, \neg B}{\Gamma \vdash \Delta, \neg(A \vee B)}$$

$$\neg\neg\dashv\frac{A, \Gamma \dashv \Delta}{\neg\neg A, \Gamma \dashv \Delta} \qquad \dashv\neg\neg\frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta, \neg\neg A}$$

B.6. **The $\mathsf{Gk3}, \mathsf{Gk3}_{H1}, \mathsf{Gk3}_{H2}, \mathsf{Gk3}_{H1\cap 2}$ and $\mathsf{Gk3}_{\dashv\vdash}$ calculi.** The $\mathsf{Gk3}$ ($\mathsf{Gk3}_{H1}$, $\mathsf{Gk3}_{H2}$, $\mathsf{Gk3}_{H1\cap 2}$ and $\mathsf{Gk3}_{\dashv\vdash}$) calculus is just like the $\mathsf{Gfde}$ ($\mathsf{Gfde}_{H1}$, $\mathsf{Gfde}_{H2}$, $\mathsf{Gfde}_{H1\cap 2}$ and $\mathsf{Gfde}_{\dashv\vdash}$, respectively), except that we have the following axioms:

$$ax_{k3}\vdash\frac{}{\Gamma, p \vdash p, \Delta} \qquad ax_{k3}\vdash\frac{}{\Gamma, \neg p \vdash \neg p, \Delta} \qquad ax_{k3}\vdash\frac{}{\Gamma, p, \neg p \vdash \Delta}$$

$$ax_{k3}\dashv\frac{}{\Theta \dashv \Lambda} \qquad ax_{k3}\dashv\frac{}{\Theta \dashv p, \neg p, \Lambda}$$

In the rules $ax_{k3} \dashv$ , $\Theta \cap \Lambda = \varnothing$ and no atom $p$ is such that $[p, \neg p] \subseteq \Theta$.

B.7. **The $\mathsf{Glp}, \mathsf{Glp}_{H1}, \mathsf{Glp}_{H2}, \mathsf{Glp}_{H1\cap 2}$, and $\mathsf{Glp}_{\dashv\vdash}$ calculi.** The $\mathsf{Glp}$ ($\mathsf{Glp}_{H1}$, $\mathsf{Glp}_{H2}$, $\mathsf{Glp}_{H1\cap 2}$ and $\mathsf{Glp}_{\dashv\vdash}$) calculus is just like the $\mathsf{Gfde}$ ($\mathsf{Gfde}_{H1}$, $\mathsf{Gfde}_{H2}$, $\mathsf{Gfde}_{H1\cap 2}$ and $\mathsf{Gfde}_{\dashv\vdash}$, respectively), except that we have the following axioms:

$$ax_{lp}\vdash\frac{}{\Gamma, p \vdash p, \Delta} \qquad ax_{lp}\vdash\frac{}{\Gamma, \neg p \vdash \neg p, \Delta} \qquad ax_{lp}\vdash\frac{}{\Gamma \vdash p, \neg p, \Delta}$$

$$ax_{lp}\dashv\frac{}{\Theta \dashv \Lambda} \qquad ax_{lp}\dashv\frac{}{\Theta, p, \neg p \dashv \Lambda}$$

In the rules $ax_{lp} \dashv$ , $\Theta \cap \Lambda = \varnothing$ and no atom $p$ is such that $[p, \neg p] \subseteq \Lambda$.

## Appendix C. Proof systems for intuitionistic propositional logic

In this appendix we present the proof systems for intuitionistic logic employed in the paper. We use the capital Greek letters $\Gamma, \Delta, \Pi, \Sigma$ to refer to multisets of formulas, and $\Theta, \Lambda, \dots$ to denote multisets of *atomic* formulas and $\bot$.

C.1. **Vorob'ev's multisuccedent $\mathsf{G4ip}$.**

$$ax_{int} \vdash \frac{}{\Gamma, p \vdash p, \Delta} \qquad ax_{\bot} \vdash \frac{}{\bot, \Gamma \vdash \Delta}$$

$$\wedge \vdash \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \vdash \wedge \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee \vdash \frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \qquad \vdash \vee \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$at \to \vdash \frac{A, B, \Gamma \vdash \Delta}{A, A \to B, \Gamma \vdash \Delta} \qquad \wedge \to \vdash \frac{C \to (D \to B), \Gamma \vdash \Delta}{(C \wedge D) \to B, \Gamma \vdash \Delta}$$

$$\vee \to \vdash \frac{C \to B, D \to B, \Gamma \vdash \Delta}{(C \vee D) \to B, \Gamma \vdash \Delta} \qquad \to\to \vdash \frac{D \to B, C, \Gamma \vdash D \qquad B, \Gamma \vdash \Delta}{(C \to D) \to B, \Gamma \vdash \Delta}$$

$$\vdash \to_i \frac{\Gamma, A \vdash B}{\Gamma \vdash \Delta, A \to B}$$

In the rule $at \to \vdash$ , $A$ is an atom or $\bot$.

C.2. **The $\mathsf{G4ip}_{\dashv\vdash}$ calculus.** The $\mathsf{G4ip}_{\dashv\vdash}$ is obtained from $\mathsf{G4ip}$ by adding the following rules:

$$ax_{int} \dashv \frac{}{\Gamma^{\to}, \Theta \dashv \Lambda}$$

$$\wedge \dashv \frac{A, B, \Gamma \dashv \Delta}{A \wedge B, \Gamma \dashv \Delta} \qquad \dashv \wedge \frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta, A \wedge B} \qquad \dashv \wedge \frac{\Gamma \dashv \Delta, B}{\Gamma \dashv \Delta, A \wedge B}$$

$$\vee \dashv \frac{A, \Gamma \dashv \Delta}{A \vee B, \Gamma \dashv \Delta} \qquad \vee \dashv \frac{B, \Gamma \dashv \Delta}{A \vee B, \Gamma \dashv \Delta} \qquad \dashv \vee \frac{\Gamma \dashv \Delta, A, B}{\Gamma \dashv \Delta, A \vee B}$$

$$at \to \dashv \frac{p, B, \Gamma \dashv \Delta}{p, p \to B, \Gamma \dashv \Delta} \qquad \wedge \to \dashv \frac{C \to (D \to B), \Gamma \dashv \Delta}{(C \wedge D) \to B, \Gamma \dashv \Delta}$$

$$\vee \to \dashv \frac{C \to B, D \to B, \Gamma \dashv \Delta}{(C \vee D) \to B, \Gamma \dashv \Delta} \qquad \to\to \dashv \frac{B, \Gamma \dashv \Delta}{(C \to D) \to B, \Gamma \dashv \Delta}$$

$$\to \dashv \to \frac{D_1 \to B_1, C_1, \Gamma_1, \Theta \dashv D_1 \quad \dots \quad D_m \to B_m, C_m, \Gamma_m, \Theta \dashv D_m \qquad \Gamma, \Theta, E_1 \dashv F_1 \quad \dots \quad \Gamma, \Theta, E_n \dashv F_n}{\Gamma, \Theta \dashv E_1 \to F_1, \dots, E_n \to F_n, \Lambda}$$

In the rules $ax_{int} \dashv$ and $\to \dashv \to$ the following conditions hold:

(*i*) $\Gamma^{\to} = A_1 \to B_1, \dots, A_l \to B_l$, with $A_1, \dots, A_l$ being atoms or $\bot$;

(*ii*) $\bot \notin \Theta$;

(*iii*) $\Theta \cap [A_1, \dots, A_l] = \Theta \cap \Lambda = \varnothing$;

(*iv*) $\Gamma = (C_1 \to D_1) \to B_1, \dots, (C_m \to D_m) \to B_m, \Gamma^{\to}$;

(*v*) $\Gamma_i = \Gamma \setminus [(C_i \to D_i) \to B_i]$, for any $1 \leq i \leq m$.

C.3. **The minimal $\mathsf{G4ip}_H$ calculus.** The minimal $\mathsf{G4ip}_H$ calculus is obtained by extending $\mathsf{G4ip}$ with $ax_{int}$ ⊣, the *acut* rules and the following rule schema:

$$adp_m \frac{A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_1 \qquad \ldots \qquad A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_m}{A_1 \to B_1, \ldots, A_m \to B_m, \Gamma^{\to}, \Theta \dashv A_1, \ldots, A_m, \Lambda}$$

In the rule schema *adp* the following conditions hold:

(*i*) $\Gamma^{\to} = E_1 \to F_1, \ldots, E_n \to F_n$, with $E_1, \ldots, E_n$ being atoms or $\bot$;

(*ii*) $\bot \notin \Theta$;

(*iii*) $\Theta \cap [E_1, \ldots, E_n] = \Theta \cap \Lambda = \varnothing$;

(*iv*) $m \geq 2$;

(*v*) $A_i$ has the form $C_i \to D_i$, for any $1 \leq i \leq m$.

## References


[1] Alan R. Anderson and Nuel Belnap. Tautological entailments. *Philosophical Studies*, 13(1-2):9–24, 1962. https://www.jstor.org/stable/4318404.

[2] Alan R. Anderson and Nuel Belnap. First degree entailments. *Mathematische Annalen*, 149:302–319, 1963. https://doi.org/10.1007/BF01471125.

[3] Michael Arndt. Eight inference rules for implication. *Studia Logica*, 107:781–808, 2019. https://doi.org/10.1007/s11225-018-9821-9.

[4] Arnon Avron. Gentzen-type systems, resolution and tableaux. *Journal of Automated Reasoning*, 10(2):265–281, 1993. https://doi.org/10.1007/BF00881838.

[5] Nick Bezhanishvili and Dick de Jongh. Extendible formulas in two variables in intuitionistic logic. *Studia Logica*, 100:61–89, 2012. https://doi.org/10.1007/s11225-012-9389-8.

[6] Alexander Chagrov and Michael Zakharyaschev. *Modal logic*. Oxford University Press, 1997.

[7] Roy Dyckhoff and Sara Negri. Admissibility of structural rules for contraction-free systems of intuitionistic logic. *The Journal of Symbolic Logic*, 65:1499–1518, 2000. https://doi.org/10.2307/2695061.

[8] Camillo Fiorentini. Terminating sequent calculi for proving and refuting formulas in S4. *Journal of Logic and Computation*, 25(1):179–205, 2012. https://doi.org/10.1093/logcom/exs053.

[9] Josep M. Font and Ventura Verdú. Algebraic logic for some non-protoalgebraizable logics. In Hajnal Andréka, J. Donald Monk, and István Németi, editors, *Algebraic logic*, pages 183–188. North-Holland Publishing Company, 1991.

[10] Silvio Ghilardi. Unification in intuitionistic logic. *The Journal of Symbolic Logic*, 64:859–880, 1999. https://doi.org/10.2307/2586506.

[11] Valentin Goranko. Refutation systems in modal logic. *Studia Logica*, 53(2):299–324, 1994. https://doi.org/10.1007/BF01054714.

[12] Valentin Goranko. Hybrid deduction-refutation systems. *Axioms*, 8(4):1–19, 2019. https://doi.org/10.3390/axioms8040118.

[13] Valentin Goranko, Gabriele Pulcini, and Tomasz Skura. Refutation systems: an overview and some applications to philosophical logics. In Fenrong Liu and Hiroakira Ono, editors, *Logic in Asia: Studia Logica Library*, pages 173–197. Springer, 2020. https://doi.org/10.1007/978-981-15-2221-5_9.

[14] Jeroen P. Goudsmit. The admissible rules of $BD_2$ and *GSc*. *Notre Dame Journal of Formal Logic*, 59:325–353, 2018. https://doi.org/10.1215/00294527-3838972.

[15] Ronald Harrop. On disjunctions and existential statements in intuitionistic systems of logic. *Mathematische Annalen*, 132:347–361, 1956. https://doi.org/10.1007/BF01360048.

[16] Jörg Hudelmaier. A contraction-free sequent calculus for S4. In Heinrich Wansing, editor, *Proof theory of modal logic*, pages 3–16. Springer, 1996. https://doi.org/10.1007/978-94-017-2798-3_1.

[17] Rosalie Iemhoff. A(nother) characterization of intuitionistic propositional logic. *Annals of Pure and Applied Logic*, 113:161–173, 2001. https://doi.org/10.1016/S0168-0072(01)00056-2.

[18] Rosalie Iemhoff. On the admissible rules of intuitionistic propositional logic. *The Journal of Symbolic Logic*, 66:281–294, 2001. https://doi.org/10.2307/2694922.

[19] Rosalie Iemhoff. Intermediate logics and Visser's rules. *Notre Dame Journal of Formal Logic*, 46:65–81, 2005. https://doi.org/10.1305/ndjfl/1107220674.

[20] Emil Jeřábek. Admissible rules of modal logics. *Notre Dame Journal of Formal Logic*, 15:411–431, 2005. https://doi.org/10.1093/logcom/exi02.

[21] Emil Jeřábek. Independent bases of admissible rules. *Logic Journal of the IGPL*, 16:249–267, 2008. https://doi.org/10.1093/jigpal/jzn004.

[22] Stephen C. Kleene. *Introduction to metamathematics*. North-Holland Publishing Co, 1952.

[23] Stephen C. Kleene. *Mathematical logic*. John Wiley and Sons, 1967.

[24] E. G. K. López-Escobar. Refutability and elementary number theory. *Indagationes Mathematicae*, 75:362–374, 1972. https://doi.org/10.1016/1385-7258(72)90053-4.

[25] Jan Łukasiewicz. *Aristotle's syllogistic from the standpoint of modern formal logic*. Clarendon Press, Oxford, 1951.

[26] Eoin Moore. Hybrid deduction-refutation systems for FDE-based logics. *The Australasian Journal of Logic*, 18(6):599–615, 2021. https://doi.org/10.26686/ajl.v18i6.7080.

[27] Enrico Moriconi. Early structural reasoning: Gentzen 1932. *The Review of Symbolic Logic*, 8(4):662–679, 2015. https://doi.org/10.1017/S1755020315000222.

[28] Sara Negri and Jan von Plato. From Kripke models to algebraic counter-valuations. In Harrie de Swart, editor, *Automated Reasoning with Analytic Tableaux and Related Methods. TABLEAUX 1998*, pages 247–261. Springer, 1998. https://doi.org/10.1093/10.1007/3-540-69778-0_26.

[29] Mario Piazza and Gabriele Pulcini. Fractional semantics for classical logic. *The Review of Symbolic Logic*, 13(4):810–828, 2020. https://doi.org/10.1017/S1755020319000431.

[30] Mario Piazza, Gabriele Pulcini, and Andrea Sabatini. Abduction as deductive saturation: a proof-theoretic inquiry. *Journal of Philosophical Logic*, 52(6):1575–1602, 2023.

[31] Mario Piazza, Gabriele Pulcini, and Andrea Sabatini. Content-based entailment. *draft*, 2025.

[32] Mario Piazza, Gabriele Pulcini, and Matteo Tesi. Fractional-valued modal logic. *The Review of Symbolic Logic*, pages 1–20, 2021. https://doi.org/10.1017/S1755020321000411.

[33] Mario Piazza, Gabriele Pulcini, and Matteo Tesi. Linear logic in a refutational setting. *Journal of Logic and Computation*, 2023. https://doi.org/10.1093/logcom/exad048.

[34] Luis Pinto and Roy Dyckhoff. Loop-free construction of counter-models in intuitionistic propositional logic. In Minaketan Behara, Rudolf Fritsch, and Rubens G. Lintz, editors, *Symposia Gaussiana. Conference A: Mathematics and Theoretical Physics*, pages 225–232. De Gruyter, 1995.

[35] Graham Priest. The logic of paradox. *Journal of Philosophical Logic*, 8(1):219–241, 1979. https://doi.org/10.1007/BF00258428.

[36] Gabriele Pulcini. Cut elimination by unthreading. *Archive for Mathematical Logic*, 63:211–223, 2023. https://doi.org/10.1007/s00153-023-00892-4.

[37] Alexej P. Pynko. Characterizing Belnap's logic via De Morgan's laws. *Mathematical Logic Quarterly*, 41(4):442–454, 1995. https://doi.org/10.1002/malq.19950410403.

[38] Vladimir Rybakov. *Admissibility of logical inference rules*. Elsevier, 1997.

[39] Dana Scott. Completeness proofs for the intuitionistic sentential calculus. In *Summaries of talks presented at the Summer Institute for Symbolic Logic, Cornell University, 1957*, pages 231–241. Communications Research Division, Institute for Defense Analyses, Princeton, 1960.

[40] Tomasz Skura. A complete syntactical characterization of the intuitionistic logic. *Bulletin of the Section of Logic*, 23:75–80, 1989. https://www.researchgate.net/publication/239552062_A_COMPLETE_SYNTACTICAL_CHARACTERIZATION_OF_THE_INTUITIONISTIC_LOGIC.

[41] Tomasz Skura. Refutation calculi for certain intermediate logics. *Notre Dame Journal of Formal Logic*, 33:522–560, 1992. https://doi.org/10.1305/ndjfl/1093634486.

[42] Raymond M. Smullyan. *First-order logic*. Courier Corporation, 1995.

[43] Anne S. Troelstra and Helmut Schwichtenberg. *Basic proof theory*. Cambridge University Press, 2000.

[44] Jan von Plato. Gentzen's proof systems: byproducts in a work of a genius. *The Bulletin of Symbolic Logic*, 18(3):313–367, 2012. https://doi.org/10.2178/BSL/1344861886.