On a Suggested Logic for Paraconsistent Mathematics

John Slaney

Abstract

The logic **subDL** and its quantified extension **subDLQ** were proposed by Badia and Weber (*Dialethism and its Applications*, 2019: 155–176) as a basis for developing a version of mathematics in which paradoxes are harmless. In the present paper, **subDL** as defined in the literature is shown to be too strong to support the theories which motivate it. The crucial point is that contraction is derivable in **subDL**. It follows that the semantic structure used by Badia and Weber to invalidate contraction is not, in fact, a model of **subDL**. Here we identify the axioms responsible for contraction in **subDL** and prove that the logic, weakened by removal of these axioms, is contraction-free and paraconsistent.

1 The logic

The logic **subDL** and its quantificational extension **subDLQ** were proposed by Weber and his collaborators [1, 5] as a vehicle for paraconsistent reconstruction of mathematics, including metamathematics and naïve set theory. In contrast to most logics considered for such purposes, it features two families of connectives, based on two different ontologies of propositions, with axioms linking the two. There is an implication connective, written as a single arrow, and a negation, which behave as in a very weak relevant logic: the implication-negation fragment of **DW**,[1] or **B** with the theorem form of contraposition

$$(A \to B) \to (\neg B \to \neg A)$$

On a different level, there is another implication, written as a double arrow, governed by the principles of **BCK**, which is a very strong substructural logic satisfying all of the pure implication part of linear logic together with weakening. This implication goes along with two other connectives: the **BCK** fusion, or multiplicative conjunction, and the **BCK** additive disjunction. Here I shall follow standard usage in writing the fusion of A and B as $A \circ B$, although Weber writes it as $A \wedge B$ and wants to see it as the one "real" conjunction operator of the logic. In order to make sense of this trio of connectives, we need another ontology on which the space of propositions forms a lattice-ordered[2] **BCK** al-

[1] In the cited works, it is referred to as **DK** rather than **DW**, because it also has a form of the law of the excluded middle. However, the binary connective used in that axiom lacks some properties of **DK** disjunction, so the name is misleading.

[2] The presence of additive disjunction only makes it a join semilattice but since there is also an absurd constant $\perp$, if the semilattice is complete then it is a lattice. In the finite case, it is complete, of course, and in any case it can be completed in familiar ways—e.g. by embedding in the lattice of ideals. Hence it does no harm to think of it as a lattice.

gebra. Negation and the **DW** implication are not operations on propositions at this level. From the **BCK** standpoint, they are intensional connectives in the extreme sense that provably equivalent statements may for instance have non-equivalent negations. Since a **BCK** proposition is some sort of equivalence class of **DW** propositions, the **BCK** connectives are also propositional operators on the **DW** ontology, but what they mean at that level is rather obscure. The fusion connective $\circ$ is an associative and commutative locution satisfying weakening and $\vee$ is its De Morgan dual, but their role as actual fusion and disjunction cannot be seen without reference to the **BCK** propositional structure. As operations on **DW** propositions, they do not in general respect implication, in the sense that $A \to B$ may be true even if neither $(A \circ C) \to (B \circ C)$ nor $(A \vee C) \to (B \vee C)$ is true. At the time of writing, there is no coherent semantic account of **subDL** in the literature.

It is usual to present **subDL** as a Hilbert system (e.g. [5] p. 131) with axioms equivalent to the following (my numbering):

a1	$A \to A$
a2	$(A \to \neg B) \to (B \to \neg A)$
a3	$\neg\neg A \to A$
a4	$(A \circ B) \to A$
a5	$(A \circ B) \to (B \circ A)$
a6	$(A \circ (B \circ C)) \to ((A \circ B) \circ C)$
a7	$((A \to B) \circ (B \to C)) \to (A \to C)$
a8	$(A \vee B) \leftrightarrow \neg(\neg A \circ \neg B)$
a9	$A \vee \neg A$
a10	$(A \circ (B \vee C)) \leftrightarrow ((A \circ B) \vee (A \circ C))$
a11	$(A \vee (B \circ C)) \leftrightarrow ((A \vee B) \circ (A \vee C))$
a12	$(A \to B) \Rightarrow (A \Rightarrow B)$
a13	$\neg(A \Rightarrow B) \Rightarrow \neg(A \to B)$
a14	$(A \Rightarrow (B \Rightarrow C)) \Leftrightarrow ((A \circ B) \Rightarrow C)$
a15	$((A \Rightarrow C) \circ (B \Rightarrow C)) \Rightarrow ((A \vee B) \Rightarrow C)$

The rules of inference are detachment for the double arrow (and hence for the single one as well):

If $A \Rightarrow B$ and A are theorems, then so is B

and replacement

If $A \leftrightarrow B$ is a theorem, then so is $\varphi(A) \to \varphi(B)$

Note that the formula $A \leftrightarrow B$ is defined as $(A \to B) \circ (B \to A)$, and $A \Leftrightarrow B$ similarly as $(A \Rightarrow B) \circ (B \Rightarrow A)$. While these may not be the biconditionals as traditionally understood, they do each have the property of being a theorem iff both component conditionals are theorems.

Some of the above axioms are truly difficult to understand. The De Morgan axiom a8, for instance, is presented as though it recorded the familiar duality between conjunction and disjunction, but of course it does not. Fusion and disjunction make sense as such only within the **BCK** ontology, while negation exists as an operation on propositions only within the **DW** ontology, so it seems that a8 is motivated not by its meaning (whatever that may be) but rather by its utility for proving results that look like classical theorems.

The purpose of the present note, however, is not to provide **subDL** and **subDLQ** with semantics, but to observe that as presented they fail in their primary purpose, and to suggest a correction that has some chance of keeping the project alive.

2 Contraction

The structural rule of contraction—that whatever follows from two assumptions of the same thing also follows from just one—is deeply implicated in a wide range of paradoxical reasoning. Consider Curry's paradox, for example. From somewhere, be it naive set or property comprehension, truth theory or just propositional quantification, we have a formula C which is provably equivalent to $C \rightsquigarrow \bot$, where $\rightsquigarrow$ is an implication connective satisfying contraction. Since $C \rightsquigarrow (C \rightsquigarrow \bot)$ holds, so does $C \rightsquigarrow \bot$; but that is equivalent to C, so C also holds, and so by *modus ponens*, $\bot$. Again, think of the sorites paradox. A plausible assumption is that if n grains can make a heap, so can $n-1$; by appealing to this assumption a mere 999,999 times, if a million grains can make a heap, so can 1 grain. By contraction, then, this grossly incorrect conditional follows from the assumption taken just once. Of course, there is far more to be said about both of these cases, and this is not the place to rehearse the rich literature concerning them. The present point, central to the research program using **subDLQ**, is that the paradoxes have a logical character, and that character is contraction. R.K. Meyer summed it up sharply [3]:

> The problem with these contraction principles is that they contaminate absolutely everything, leading people to confuse sleazy tricks with surprising theorems.
>
> ... Where the sleight-of-hand becomes thickest, the move that pulls the rabbit out of the hat is contraction, whether the bunny be Russell's paradox or just the Tertium Non Datur.

The absence of contraction from **subDL** is crucial to the research program in paraconsistent mathematics. Unfortunately:

Theorem 1. *Contraction is derivable in* **subDL**.

Proof. The following are all theorems of **subDL**:

1.	$A \Rightarrow (A \vee (A \circ A))$	a2, a4, a8, a12
2.	$(A \vee (A \circ A)) \Rightarrow ((A \vee A) \circ (A \vee A))$	a11, a12
3.	$(A \vee A) \Rightarrow A$	a1, a14, a15
4.	$((A \vee A) \circ (A \vee A)) \Rightarrow (A \circ A)$	3, basic **BCK** logic
5.	$A \Rightarrow (A \circ A)$	1, 2, 4, transitivity

□

In the context of **BCK**, contraction in the form of square-increasing (line 5 above) easily implies contraction in the pure implication form

$$(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$$

and in the form of a structural rule.

So **subDL** as it stands is unfit for purpose. Now, where in its axiomatisation should we pin the blame? The theorems at lines 1 and 3 are perfectly normal for additive disjunction, and the reasoning to lines 4 and 5 is scarcely such as to cause any shocks. The one axiom which really stands out as implausible is a11, the distribution of additive disjunction over multiplicative fusion. Hence, we might try dropping axiom 11. The news, however, gets worse before it gets better: axiom a11 is redundant. Consider a10. By contraposition it implies

$$\neg(A \circ (B \vee C)) \leftrightarrow \neg((A \circ B) \vee (A \circ C))$$

Driving negation inside the binary connectives by De Morgan's laws, this amounts to

$$(\neg A \vee \neg(B \vee C)) \leftrightarrow (\neg(A \circ B) \circ \neg(A \circ C))$$

and thus to

$$(\neg A \vee (\neg B \vee \neg C)) \leftrightarrow ((\neg A \vee \neg B) \circ (\neg A \vee \neg C))$$

Substituting $\neg A$, $\neg B$ and $\neg C$ for A, B and C respectively and removing the double negations, this is just a11 again. Hence, if the De Morgan equivalences are to remain, a10 has to go as well.

3 ... but the counter-model!

It has been asserted many times in the literature on **subDL** that contraction is not derivable in that logic. One of the reasons cited, for instance by Weber ([5] pp. 132–133), is a model given by Badia and Weber [1] which purports to show just this. As we know from the previous section, it does not. The technique is to provide a formulation of **subDLQ** as a calculus of sequents (although cut appears to be ineliminable) and to define an algebraic theory intended to correspond directly to this calculus. Then some automated reasoning software (in fact, MACE4 by McCune [2]) is used to find a small algebra satisfying the postulates but clearly not satisfying contraction. The intention is right, but the execution is wrong. The structure given is this:

$\neg$	
0	2
1	3
2	0
3	3

$\cap$	0	1	2	3
0	0	0	0	0
1	0	0	1	3
2	0	1	2	3
3	3	3	3	3

$\cup$	0	1	2	3
0	0	3	2	3
1	3	1	2	3
2	2	2	2	2
3	0	1	2	3

$\Rightarrow$	0	1	2	3
0	2	2	2	2
1	1	2	2	3
2	0	1	2	3
3	3	3	2	3

The implication order is definable as $a \leq b$ iff $a \Rightarrow b \in \{2, 3\}$. That is, 0, 1 and 2 form a chain as one might expect, but 3 is both above and below everything. Thus the order is not transitive.

One minor problem with this structure is that it does not satisfy the postulates of the algebra presented with it ([1] p. 165). Postulate 4, which is supposed to reflect the cut rule, says

$$\forall x, y, z, u, v((x \leq y \ \&\ z \cap (y \cap u) \leq v) \supset z \cap (x \cap u) \leq v)$$

Here & and $\supset$ are the obvious boolean connectives in the metalanguage. But $2 \leq 3$ and $2 \cap 3 \cap 2 \leq 1$, while $2 \cap 2 \cap 2 \not\leq 1$. It is impossible that MACE4 could make this mistake, so it is presumably just a copying error.

The major issue is that the algebraic postulates in the form given are not complete for **subDL** (or **subDL** is not sound for the algebra, if you prefer). The above "Cut" postulate illustrates the problem nicely: it specifies that cut should hold where the context "$z \ldots u$" is non-empty, since the quantifiers range over elements of the algebra so z and u are required to be present. However, in **subDL** cut is required to hold also in the special case where the cut formula is the entire left side of the sequent, and implication is consequently transitive. If the MACE4 input is corrected to reflect this requirement, for instance by adding a null element e such that $e \cap x = x$, then of course there is no model invalidating contraction—there cannot be since contraction is derivable.

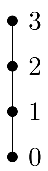
4 The fix

Clearly, if the research program surrounding **subDLQ** is to continue, the logic must be replaced by a better version. At least one of the axioms of **subDL** must be given up. There are several candidates: axiom 11 appears doomed, but we might choose to abandon the idea that **BCK** fusion and **BCK** disjunction are De Morgan duals, or choose to keep that and abandon the distribution of fusion over disjunction, or to admit that $\vee$ is not, after all, additive disjunction, even on the **BCK** ontology. My present suggestion is to drop axiom 11, and to weaken axiom 10 to the double-arrow form:

$$A \circ (B \vee C) \Leftrightarrow (A \circ B) \vee (A \circ C)$$

All else stays as it was. The weakened form of distribution makes perfect sense within the **BCK** ontology, since it is just the standard postulate relating (multiplicative) fusion and (additive) disjunction.

The new logic does have models which show contraction to be underivable. Here, for example, is a simple 4-element model of **DK**:



$\neg$	
0	3
1	2
2	1
3	0

$\to$	0	1	2	3
0	3	3	3	3
1	1	2	2	3
2	0	0	2	3
3	0	0	1	3

The "designated values" are 2 and 3. Since the order is total, the lattice operations corresponding to additive conjunction and disjunction exist in the structure, so those connectives could be added conservatively if desired. They are not desired, however. Instead, we add these two:

$\circ$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	1	2	2
3	0	1	2	3

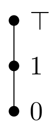
$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	3	3
3	3	3	3	3

At the **DK** level, these are not really fusion and disjunction, but the notation is justified when the **BCK** ontology is superimposed, by conflating values 2 and 3 and adding the double arrow:

$\Rightarrow$	0	1	2	3
0	3	3	3	3
1	1	2	2	3
2	0	1	2	3
3	0	1	2	3

This structure satisfies all axioms of **subDL** except for the discredited a11 and the single-arrow version of a10. Contraction fails, however, because fusion is not square-increasing: $1 \Rightarrow (1 \circ 1) \;=\; 1 \Rightarrow 0 \;=\; 1$. The logic is also paraconsistent: explosion fails in that $(1 \circ \neg 1) \Rightarrow 0 \;=\; (1 \circ 2) \Rightarrow 0 \;=\; 1 \Rightarrow 0 \;=\; 1$.

From the **BCK** perspective, propositions 2 and 3 are the same thing, so let us dub that thing '$\top$'. The model on this view is then 3-valued:



$\circ$	0	1	$\top$
0	0	0	0
1	0	0	1
$\top$	0	1	$\top$

$\vee$	0	1	$\top$
0	0	1	$\top$
1	1	1	$\top$
$\top$	$\top$	$\top$	$\top$

$\Rightarrow$	0	1	$\top$
0	$\top$	$\top$	$\top$
1	1	$\top$	$\top$
$\top$	0	1	$\top$

These tables are characteristic for the 3-valued logic of Łukasiewicz, as might be expected since this is the smallest **BCK** algebra failing contraction. That prompts another question: which other **BCK** algebras are capable of being embedded in a similar way in models of the new logic? All of them? Enough of them to ensure conservative extension by the addition of negation with its subDL-like properties? That is, are there theorems of the new logic in the $\{\Rightarrow, \circ, \vee\}$ vocabulary which are not theorems of **BCK**? These are not necessarily difficult questions, but are hereby left open for future investigation.

5 Conclusion

The attempt to reconstruct mainstream mathematics on a paraconsistent basis is a bold and certainly worthwhile exercise; the present paper should not be construed as antagonistic to it. Naturally, such a research program places conflicting constraints on logic. There is a requirement of sufficient strength to secure not only classical theorems but also classical modes of reasoning about their consequences. This must be balanced against the requirement of sufficient weakness to avoid collapse in the face of paradoxes, or else the paraconsistent position is lost. The valuable contribution of **subDL**-like logical theory is the superposition of two propositional ontologies, allowing an extremely strong positive logic to coexist with a very weak background logic which in particular provides for negation and which can support extreme theories including naïve set theory.

The distinctive features of **subDL** beyond this are less securely motivated. Its striking claim, contrary to the rest of substructural logic, is that the classification of connectives as additive or multiplicative is illusory: there is only one conjunction and only one disjunction, which are multiplicative and additive respectively on the **BCK** level and which are neither (but are each other's De

Morgan duals) on the **DW** level. Weber goes further, adding axioms for the excluded middle, conjunctive syllogism and distributivity (a9, a7, a10 and a11) not because these flow from any antecedently given account of reasoning but apparently because they seem to be useful for the purpose of approximating classical mathematics.

We have seen that the distribution axioms are too strong, as they lead directly to contraction. The law of the excluded middle is also a weak contraction principle in orthodox substructural logic [4] and the conjunctive syllogism (for additive conjunction) trivialises naïve set theory in the presence of fusion[3] so although **subDL** seems to avoid the known derivations of their problematic consequences, the presence of these axioms at least throws into question the suitability of the logic for naïve theories.

This note is not the place to propose yet another logic for paraconsistent mathematics. It has been shown that the published version of **subDL** at least needs revision, and that removing the strong distribution axioms is enough to ensure freedom from contraction. It remains to be investigated whether the weakened logic is sufficiently strong for the purposes of the research program in paraconsistent reasoning. As for a proof of sufficient weakness, without a clear semantic account or good proof-theoretic control that remains less likely.

References

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[6] R. Routley with R. K. Meyer, V. Plumwood, and R. T. Brady. *Relevant logics and their rivals.* Ridgeview, Atascadero, 1982.

[3] For proof, see for example [6] p. 367.