The Best of All Possible Leibnizian Completeness Theorems

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Abstract

Leibniz developed several arithmetical interpretations of the assertoric syllogistic in a series of drafts from April 1679. In this article, I present what I take to be one of his most mature articulations of the arithmetical semantics from that series. I show that the assertoric syllogistic can be characterized exactly not only in the full divisibility lattice, as Leibniz implicitly suggests, but in a certain four-element sublattice thereof. This refinement is also shown to be optimal in the sense that the assertoric syllogistic is not complete with respect to any smaller sublattice using Leibniz's truth conditions.

Keywords: Aristotle, Arithmetic, Divisibility Lattice, Leibniz, Syllogistic.

1 Introduction

In a series of drafts from April 1679, collected in Couturat [4], Leibniz proposed various arithmetical interpretations of the assertoric syllogistic.[1] I will here focus on what I take to be one of his most mature presentations from that series, given in *Regulæ ex quibus de bonitate consequentiarum formisque et modis syllogismorum <categoricum> judicari potest, PER NUMEROS* [4, pp. 77–84] (*Regulæ de bonitate consequentiarum*, for short).[2]

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[1] All translations in this article are my own. Translations from *Regulæ de bonitate consequentiarum* [4, pp. 77–84] to some extent follow Parkinson [14, pp. 25–32]. The critical edition of Aristotle's *Prior Analytics* (*An. Pr.*) is Ross and Minio-Paluello [16].

[2] Many of the key features of the approach in *Regulæ de bonitate consequentiarum*—for example, that terms be represented or interpreted by a pair of coprime integers—are shared

In *Regulæ de bonitate consequentiarum*, Leibniz himself sketches an argument that can reasonably be regarded as establishing (or nearly establishing) the soundness of the assertoric syllogistic with respect to his arithmetical semantics. The question of the adequacy of this semantics has been treated by Słupecki [17, §10, pp. 298–300] and Łukasiewicz [10, §34, pp. 126–129].

Nevertheless, there are good reasons to be dissatisfied with the Słupecki-Łukasiewicz approach. First, the reconstruction of the syllogistic they prove their results with respect to is deficient for reasons which have been articulated by Smiley [18, pp. 136–138] and Corcoran [3, §§1.5–6, pp. 94–98]. Second, while Słupecki and others have been able to obtain completeness theorems of sorts for the syllogistic, as far as I am aware, the issue of what arithmetical completeness theorem is *optimal* has not been addressed. In this article, I will not only obtain the completeness of the syllogistic with respect to Leibniz's semantics, but with respect to an optimal refinement thereof. More specifically, I will identify the smallest arithmetical structure (up to isomorphism) which exactly characterizes the syllogistic.

The plan of the article is as follows. In Section 2, I review the assertoric syllogistic and present it in a convenient Gentzen sequent-style formulation.[3] I present Leibniz's arithmetical interpretation of the syllogistic from *Regulæ de bonitate consequentiarum* in Section 3. I give proofs of soundness, completeness, and other results in Section 4. Some concluding remarks are offered in Section 5.

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with other drafts including *Modus examinandi consequentias per Numeros* [4, pp. 70–77] and the fragment Ariew and Garber [1, pp. 14–17] call "A Fragment on Rules for Drawing Consequences" [4, pp. 89–92]. (On the dating of these, see, e.g., Parkinson [14, pp. xx–xxi] and Ariew and Garber [1, p. 10].)

In fact, some of the ideas in *Regulæ de bonitate consequentiarum* are nearly unchanged even from the first draft in the series. For example, the truth condition for universal affirmative propositions in *Regulæ de bonitate consequentiarum* is the natural extension of a truth condition already articulated in *Elementa Characteristicæ universalis* [4, p. 42]. I should note that suitably corrected versions of Leibniz's earlier arithmetical interpretations have been studied by Sotirov [20].

[3] Thus, my formulation of the syllogistic may not be as 'natural' as, say, that of Smiley [18]. Nevertheless, this reconstruction retains the core idea that syllogisms are basically arguments—not conditionals (pace Łukasiewicz [10])—and provides a convenient framework for proving metatheorems, which is my central interest here.

2 The Syllogistic

I begin with a review of the key features of the traditional assertoric syllogistic.[4] A *categorical proposition* is something of the form AxB where $x \in \{a, e, i, o\}$ and A and B are *terms* (A is the *predicate* term and B is the *subject* term). The categorical propositions may be read as follows:

1. AaB (*universal affirmative*) is read: 'All Bs are As';
2. AeB (*universal negative*) is read: 'No Bs are As';
3. AiB (*particular affirmative*) is read: 'Some B is A';
4. AoB (*particular negative*) is read: 'Some B is not A'.

The *quality* of a categorical proposition refers to whether it is affirmative or negative; the (a) and (i) propositions are affirmative while the (e) and (o) propositions are negative. The *quantity* of a categorical proposition refers to whether it is particular or universal; the (a) and (e) propositions are universal while the (i) and (o) propositions are particular. In accordance with the traditional square of opposition, the *contradictory* of AaB (AeB) is AoB (AiB), and conversely. If p is a categorical proposition, I will often denote its contradictory by $\overline{p}$.

DEFINITION 1 (Deduction Scheme)**.** A *deduction scheme* is something of the form $\varphi, \psi \Rightarrow \theta$, where φ, ψ, and θ range over categorical proposition schemes (i.e., things of the form SxT where x is one of the four logical connectives—$x \in \{a, e, i, o\}$—and S and T are term variables), satisfying the following conditions (where it is supposed θ is of the form PxQ):

1. Exactly three term variables occur among φ, ψ, and θ;
2. Term variable P (the *major* term) must occur in exactly one of φ and ψ (the *major* premise);[5]
3. Term variable Q (the *minor* term) must occur in whichever of φ and ψ term variable P does not occur in (the *minor* premise).

[4]Of course, the locus classicus for the syllogistic is Aristotle's *Prior Analytics* (for a good introduction and translation, consult Smith [19]). The presentation given here will not attempt to follow Aristotle religiously.

[5]The major premise usually comes first, but Aristotle does not always adhere to this convention (for some discussion, consult Rose [15] and Thom [21]). This, in turn, justifies the inclusion of Permutation among the rules of $\mathcal{G}\mathbf{S}$ below.

The remaining term variable is the so-called *middle* term and, from the stipulations, clearly must occur in both φ and ψ.

A *deduction* is a concrete instance of a deduction scheme, that is, something obtained by substituting terms for term variables (n.b., I am not using 'deduction' as a success term here). The countenancing of deduction schemes may seem to the reader to be a superfluous complication. In fact, I think the usual term-requirements imposed on the syllogistic really have to be interpreted schematically if deductions such as $GaS, GeS \Rightarrow SeS$, endorsed by Aristotle himself as a syllogism (*An. Pr.* 2.15, 63b40–64a4), are even to be grammatical.[6] Note that this is perfectly admissible as a deduction (substituting one term for two different term variables in 2-aee), though it would not qualify as a deduction scheme under Definition 1 (interpreting the terms as term variables) even though all of its substitution instances would be deductions.

Deduction schemes may be classified in four *figures* according to the placement of the middle term:[7]

1. First figure: the middle term is the subject in the major premise and the predicate in the minor premise;

2. Second figure: the middle term is the predicate in both premises;

3. Third figure: the middle term is the subject in both premises;

4. Fourth figure: the middle term is the predicate in the major premise and the subject in the minor premise.

The following elementary combinatorial fact is well-known but bears emphasis since it will play a critical role in what follows:

LEMMA 1. *There are exactly* 256 *deduction schemes.*

Proof. In any figure, there are 64 deduction schemes since there are three categorical proposition schemes and four choices for the logical connective for each ($4^3 = 64$). Given that there are four figures, this yields $4 \times 64 = 256$. □

[6] I should note that Corcoran [3, p. 99], in his reconstruction of the syllogistic, *denies* that deductions such as this are well-formed by denying that expressions such as SeS are well-formed; he does this despite acknowledging counterexamples from the second book of the *Analytics*. Here I register my disagreement with Corcoran (and others) on this point (cf. Weiss [28, p. 555, n. 14]).

[7] Aristotle recognized three figures (*An. Pr.* 1.4–6). The existence of a separate fourth figure is the subject of a long and not especially interesting historical debate that appears to hinge on little more than how certain terms are precisely defined.

As a rule, I will be interested in deduction schemes in this article and will not generally labor too hard to distinguish these from deductions hereafter.

I will often call ‘correct’ deductions *syllogisms* (I will mostly try to reserve the word ‘valid’ for semantics). The syllogistic consists of perfect syllogisms (analogous, for deductive purposes anyway, to axioms) and various rules for deriving other syllogisms.[8] The following syllogisms are taken to be perfect (cf. *An. Pr.* 1.4):

$$AaB, BaC \Rightarrow AaC \tag{Barbara}$$

$$AeB, BaC \Rightarrow AeC \tag{Celarent}$$

$$AaB, BiC \Rightarrow AiC \tag{Darii}$$

$$AeB, BiC \Rightarrow AoC \tag{Ferio}$$

The rules for deriving other syllogisms fall into certain natural clusters including conversion rules (which operate on one proposition) and what might be called shift rules (which operate on multiple propositions and involve shifts in position):

$$\frac{AeB, p \Rightarrow q}{BeA, p \Rightarrow q} \tag{e-Conversion}$$

$$\frac{AiB, p \Rightarrow q}{BiA, p \Rightarrow q} \tag{i-Conversion}$$

$$\frac{AiB, p \Rightarrow q}{BaA, p \Rightarrow q} \tag{a-Conversion}$$

$$\frac{p, q \Rightarrow r}{q, p \Rightarrow r} \tag{Permutation}$$

$$\frac{p, q \Rightarrow r}{p, \overline{r} \Rightarrow \overline{q}} \tag{Antilogism}$$

Call the system so-constituted $\mathcal{G}\mathbf{S}$ (“Gentzen Syllogistic”). Formally, then, a *syllogism* is any of the perfect syllogisms or any deduction which can be derived from the perfect syllogisms using the displayed rules. If a deduction $p, q \Rightarrow r$ is a syllogism, I will write $\vdash_{\mathcal{G}\mathbf{S}} p, q \Rightarrow r$.

It is not hard to show that this system is complete in the sense that all of the 24 traditional syllogisms (enumerated in Appendix B) are derivable in it; for an example derivation, see the proof of Theorem 1 below. Moreover, Darii and Ferio are, as Aristotle himself observed (*An. Pr.* 1.7, 29b1ff.), redundant. By $\mathcal{G}\mathbf{RS}$ (“Gentzen reduced Syllogistic”) I mean $\mathcal{G}\mathbf{S}$ without these. Then:

[8] Exactly what is ‘perfect’ about perfect syllogisms for Aristotle is not important for anything here. The reader can find some pertinent discussion in Morison [13].

THEOREM 1 (Reduction). $\vdash_{\mathcal{G}\mathbf{S}} p, q \Rightarrow r$ *if and only if* $\vdash_{\mathcal{G}\mathbf{RS}} p, q \Rightarrow r$.

Proof. It clearly suffices to show that Darii and Ferio are provable in $\mathcal{G}\mathbf{RS}$. Here I just give a derivation of Darii (I suppress permutations):

$$\dfrac{\dfrac{\dfrac{CeA, AaB \Rightarrow CeB}{AeC, AaB \Rightarrow CeB}}{AaB, CiB \Rightarrow AiC}}{AaB, BiC \Rightarrow AiC}$$

This argument proceeds by applying e-Conversion to Celarent, applying Antilogism, and finally applying i-Conversion. □

In view of Theorem 1 and the slight improvement in simplicity afforded thereby I will work with $\mathcal{G}\mathbf{RS}$ hereafter.

3 Leibniz's Arithmetical Interpretation of the Syllogistic

In this section, I discuss Leibniz's arithmetical semantics for the syllogistic from *Regulæ de bonitate consequentiarum.* I begin with a formal presentation of the semantics and then turn to the source material to examine the extent to which the formal apparatus can be found therein. I end the section with some remarks on how to interpret the semantics.

The reader will recall that $\langle \mathbb{N}, | \rangle$ is a *lattice*, where lcm (least common multiple) is join, gcd (greatest common divisor) is meet, and 1 is the least element.[9] A *frame* is any sublattice of $\langle \mathbb{N}, | \rangle$; a *proper frame* is a frame which includes 1, and an *improper frame* is a frame which is not proper. Let $\mathfrak{T}$ be a countable set of terms. If $\mathfrak{F} = \langle S, | \rangle$ with $S \subseteq \mathbb{N}$ is a frame, a *model* (over $\mathfrak{F}$) is a structure $\mathfrak{M} = \langle S, |, \nu \rangle$ where $\nu : \mathfrak{T} \to S \times S$. If $\nu(A) = (i, j)$, then put $\nu^+(A) = i$ and $\nu^-(A) = j$. A model is *admissible* if for all terms A in $\mathfrak{T}$, $\gcd(\nu^+(A), \nu^-(A)) = 1$.[10]

Given an admissible model $\mathfrak{M} = \langle S, |, \nu \rangle$, the relation $\models^{\mathfrak{M}}$ is defined as follows:

1. $\models^{\mathfrak{M}} AaB$ if and only if $\nu^+(A) | \nu^+(B)$ and $\nu^-(A) | \nu^-(B)$;

[9] I will assume some familiarity with lattice theory in this article. The reader not familiar with the basic concepts might consult, for example, Davey and Priestley [5].

[10] Note that if $\mathfrak{F}$ is an improper frame (e.g., if $\mathfrak{F}$ is $\langle \{2, 4\}, | \rangle$), there are *no* admissible models over $\mathfrak{F}$. For the most part, having improper frames around makes no real difference, for which reason I choose not to proscribe them.

2. $\models^{\mathfrak{M}} AeB$ if and only if $\gcd(\nu^+(A), \nu^-(B)) \neq 1$ or $\gcd(\nu^-(A), \nu^+(B)) \neq 1$;

3. $\models^{\mathfrak{M}} AiB$ if and only if $\gcd(\nu^+(A), \nu^-(B)) = 1$ and $\gcd(\nu^-(A), \nu^+(B)) = 1$;

4. $\models^{\mathfrak{M}} AoB$ if and only if $\nu^+(A) \nmid \nu^+(B)$ or $\nu^-(A) \nmid \nu^-(B)$.

DEFINITION 2 (Validity)**.** Extending the definition of $\models^{\mathfrak{M}}$, say that the deduction $p, q \Rightarrow r$ is valid in an admissible model $\mathfrak{M}$ (in symbols, $\models^{\mathfrak{M}} p, q \Rightarrow r$) if whenever $\models^{\mathfrak{M}} p$ and $\models^{\mathfrak{M}} q$, then $\models^{\mathfrak{M}} r$. Where $\mathfrak{F} = \langle S, | \rangle$ is a frame, say that $p, q \Rightarrow r$ is valid in $\mathfrak{F}$ (in symbols, $\models^{\mathfrak{F}} p, q \Rightarrow r$) if $\models^{\mathfrak{M}} p, q \Rightarrow r$ for every admissible model $\mathfrak{M}$ over $\mathfrak{F}$. Then $p, q \Rightarrow r$ is *valid* simpliciter (in symbols, $\models p, q \Rightarrow r$) if $\models^{\mathfrak{F}} p, q \Rightarrow r$ for every frame $\mathfrak{F}$. Where Γ and Δ are deductions, the rule $\frac{\Gamma}{\Delta}$ is *valid* if for every admissible model $\mathfrak{M}$, if $\models^{\mathfrak{M}} \Gamma$, then $\models^{\mathfrak{M}} \Delta$.

I have cleaned up and, in at least one respect, generalized Leibniz's own presentation of his arithmetical interpretation of the syllogistic. Nevertheless, much of this can be found in his writings. Leibniz implicitly works with the full lattice $\langle \mathbb{N}, | \rangle$; I have added the notion of a frame as a sublattice thereof because I will be interested in certain refinements in this article.

Regarding ν, Leibniz writes that any term can be written as two numbers (leaving ν implicit), one modified by a '+' sign and the other by a '-' sign [4, p. 78]. Thus, where I might put $\nu(\text{Pious}) = (10, 3)$, Leibniz would write $+10 - 3$ (n.b., *not* 7).[11] Regarding the restriction on admissible models, Leibniz writes:

> Hoc unum tantùm cavendum est ne duo numeri ejusdem Termini ullum habeant communem divisorem. [4, p. 78]
>
> Only the one thing to be avoided is this: no two numbers of the same term may have any common divisor.

Clearly, as the subsequent example he discusses illustrates, he must mean that they lack a common divisor other than 1; this is obviously equivalent to requiring that their gcd be 1.

Leibniz gives the truth conditions sketched above for each of the categorical propositions, or conditions that are straightforwardly equivalent. For example, a true universal affirmative is one in which:

[11] With Glashoff [7, p. 176], it is hard to disagree that Leibniz's notation is a bit unfortunate.

> numerus characteristicus subjecti [...] per prædicati numerum characteristicum ejusdem notæ [...] exactè (id est ita ut nihil maneat residuum) dividi potest. [4, p. 78]
>
> The characteristic number of the subject can be divided exactly (that is, so that nothing remains) by the characteristic number of the same sign belonging to the predicate.

In *Regulæ de bonitate consequentiarum*, Leibniz does not give a suitable definition of validity and his procedure for showing the validity of certain rules even suggests a misunderstanding. In particular, his argument for the validity of *subalternation*, that is,

$$\frac{AiB, p \Rightarrow q}{AaB, p \Rightarrow q} \qquad \text{(Subalternation)}$$

uses a concrete interpretation (i.e., particular numbers) rather than proceeding generally as required [4, pp. 80–81].[12] In fact, however, the sort of reasoning Leibniz employs in this argument *does* straightforwardly generalize, as I will show below in discussing a-Conversion.[13] Moreover, in other texts, Leibniz endorses approximately the definition of validity that I have given above:

> Si nosse volumus an aliqua figura procedat vi formæ, videmus an contradictorium conclusionis sit compatibile cum præmissis, id est an numeri reperiri possint satisfacientes simul præmissis et contradictoriæ conclusionis; quodsi nulli reperiri possunt, concludet argumentum vi formæ. [4, p. 247]
>
> If we want to know whether some figure proceeds by virtue of its form, we see whether the contradictory of the conclusion is compatible with the premises, that is, whether numbers can be found that simultaneously satisfy the premises and the contradictory of the conclusion; if none such can be found, the argument will conclude by virtue of its form.[14]

[12] Note that Subalternation is plainly admissible in $\mathcal{G}\mathbf{RS}$ by application of i-Conversion and a-Conversion in succession.

[13] This observation is also made by van Rijen [25, p. 201]: "At first sight, Leibniz seems to base these proofs on the properties of the actual pairs of numbers he has assigned to the terms of these sentences. His argumentation ist [sic], however, perfectly general."

[14] This translation to some extent follows Ariew and Garber [1, p. 18].

Some remarks are in order about how Leibniz regarded his semantics. The motivation for producing some kind of arithmetical interpretation appears to be intimately related to a long-held ambition of Leibniz to produce a universal calculus for expressing all thought and reasoning as clearly as arithmetical notions and relations are expressed.[15] The basic orientation of the proposed arithmetical semantics comes out most clearly from considering the truth condition for universal affirmatives (this is also one of the most entrenched features of Leibniz's various arithmetical interpretations of the syllogistic; see Footnote 2).

In an *extensional* semantics, 'all men are rational' would be true if and only if everything in the extension of the subject (man) is in the extension of the predicate (rational), that is, if and only if the subject is contained in the predicate. In Leibniz's *intensional* semantics, the truth condition for universal affirmatives is rather the converse: the predicate is contained in (i.e., divides) the subject. Thus, as the concept man, say, contains the concept rational, it comes out true that all men are rational.[16]

While the intuitions undergirding the *ao*-fragment of the logic are, therefore, pretty clear, it is less clear what intuitions underpin the *ie*-fragment. Historically, it appears that Leibniz came to these other truth conditions—and the dual-integer assignments required for them—in an ad hoc way after discovering inadequacies in his previous arithmetical treatments of particular affirmatives.[17] Whatever might be said for these conditions in retrospect (see, e.g., van Rooij [26, p. 184]), I will leave this matter here.

I have been using the term 'semantics' throughout as logicians of a certain bent are wont to do—that is, as a synonym for *model theory* or, more narrowly, a model theory with some sort of intuition to back it up. To the extent that this model theory tracks at least some intuitions about intensional containment, I take it that Leibniz has offered a semantics for the syllogistic in the narrower sense.[18]

[15]The details of this Leibnizian program and how the arithmetization of the syllogistic fits into it are beyond the scope of this article, but the reader can find some relevant discussion in Gerhardt [6, Ch. XI, pp. 184–189] (translated as 'Preface to a Universal Characteristic' in Ariew and Garber [1, pp. 5–10]), Kneale and Kneale [9, pp. 327–331], Parkinson [14, pp. xvii–xxiii], Marshall, Jr. [11, p. 241], Glashoff [7, pp. 161–162], Uckelman [22, §3], van Rooij [26, p. 181], and Arthur [2, Ch. 2].

[16]For further discussion of extensional vs. intensional semantics in the context of Leibniz's treatment of the syllogistic, consult Glashoff [8, §1] and van Rooij [26].

[17]For some pertinent discussion of this point, consult van Rijen [25, pp. 196–197], Sotirov [20, p. 391], Glashoff [7, p. 174], Uckelman [22, p. 433], and van Rooij [26, pp. 182–183].

[18]I do not claim that he has offered a semantics in any other sense—for example, a semantics of the terms used in syllogistic reasoning. Leibniz at least at some points seems to have thought that it is possible, though very hard, to get *true characteristic numbers*

4 Results

Leibniz gives arguments for the validity of several of the conversion rules in *Regulæ de bonitate consequentiarum*. In some cases, the argument is essentially trivial. For example, in the case of i-Conversion, the result is immediate from the commutativity of gcd. The most sophisticated case he considers, which I have already mentioned, is that of Subalternation; I will largely reproduce his argument for this in tackling a-Conversion in the proof of Theorem 2.

LEMMA 2. *For any admissible model* $\mathfrak{M}$ *and for any categorical proposition* p*, exactly one of* $\models^{\mathfrak{M}} p$ *and* $\models^{\mathfrak{M}} \overline{p}$ *obtains.*

Proof. The truth conditions for a and o-propositions are opposed, and similarly for e and i-propositions, from which the result is immediate. □

THEOREM 2 (Soundness)**.** *If* $\vdash_{\mathcal{G}\mathbf{RS}} p, q \Rightarrow r$*, then* $\models p, q \Rightarrow r$*.*

Proof. It suffices to show that Barbara and Celarent, as well as the rules of $\mathcal{G}\mathbf{RS}$, are valid. I examine just the cases of Celarent, a-Conversion, and Antilogism.

Ad Celarent, suppose there were an admissible model $\mathfrak{M} = \langle S, |, \nu \rangle$ such that $\models^{\mathfrak{M}} AeB$, $\models^{\mathfrak{M}} BaC$, and $\not\models^{\mathfrak{M}} AeC$. Then either $\gcd(\nu^+(A), \nu^-(B)) \neq 1$ or $\gcd(\nu^-(A), \nu^+(B)) \neq 1$ (because $\models^{\mathfrak{M}} AeB$). In the first case, $1 \neq \gcd(\nu^+(A), \nu^-(B)) | \nu^-(B) | \nu^-(C)$ and $\gcd(\nu^+(A), \nu^-(B)) | \nu^+(A)$, from which it follows that $\gcd(\nu^+(A), \nu^-(C)) \neq 1$, contradicting $\not\models^{\mathfrak{M}} AeC$. The second case is symmetric, which yields the result.

Ad a-Conversion, suppose there were an admissible model $\mathfrak{M} = \langle S, |, \nu \rangle$ such that $\models^{\mathfrak{M}} AiB, p \Rightarrow q$ but $\not\models^{\mathfrak{M}} BaA, p \Rightarrow q$. Then, clearly, $\models^{\mathfrak{M}} BaA$ and $\not\models^{\mathfrak{M}} AiB$.[19] From $\models^{\mathfrak{M}} BaA$ and the admissibility requirement, $\nu^+(B) | \nu^+(A)$, $\nu^-(B) | \nu^-(A)$, $\gcd(\nu^+(A), \nu^-(A)) = 1$, and $\gcd(\nu^+(B), \nu^-(B)) = 1$. From $\not\models^{\mathfrak{M}} AiB$, either $\gcd(\nu^+(A), \nu^-(B)) \neq 1$ or $\gcd(\nu^-(A), \nu^+(B)) \neq 1$. In the first case, $\gcd(\nu^+(A), \nu^-(B)) > 1$; so as $\gcd(\nu^+(A), \nu^-(B)) | \nu^-(B) | \nu^-(A)$ and $\gcd(\nu^+(A), \nu^-(B)) | \nu^+(A)$, $\gcd(\nu^+(A), \nu^-(A)) \neq 1$, a contradiction. The second case is symmetric.

Ad Antilogism, suppose there were an admissible model $\mathfrak{M} = \langle S, |, \nu \rangle$ such that $\models^{\mathfrak{M}} p, q \Rightarrow r$ and $\not\models^{\mathfrak{M}} p, \bar{r} \Rightarrow \bar{q}$. Then $\models^{\mathfrak{M}} \bar{r}$ and $\not\models^{\mathfrak{M}} \bar{q}$ imply $\not\models^{\mathfrak{M}} r$ and $\models^{\mathfrak{M}} q$ by Lemma 2. Since $\models^{\mathfrak{M}} p$ and (ex hypothesi) $\models^{\mathfrak{M}} p, q \Rightarrow r$, this is impossible. □

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for all concepts (see, e.g., [6, Ch. XI, p. 189] and [4, p. 85]). But such numbers clearly are not necessary for giving a formal semantics of a formal calculus such as the syllogistic.

[19]From this point the argument is essentially Leibniz's argument for Subalternation, but abstracted from the concrete integers he uses in his treatment [4, pp. 80–81].

I turn now to showing completeness. I will prove completeness with respect to a particular finite frame, namely, $\mathfrak{F}^6 = \langle\{1,2,3,6\}, |\rangle$. That is, I will show that any deduction scheme other than the traditional 24 fails (i.e., has a false instance) in an admissible model over $\mathfrak{F}^6$. Given that the traditional 24 syllogisms are provable in $\mathcal{G}\mathbf{RS}$, this yields the result.

THEOREM 3 (Completeness)**.** *If* $\models p, q \Rightarrow r$*, then* $\vdash_{\mathcal{G}\mathbf{RS}} p, q \Rightarrow r$*.*

Proof. In view of Lemma 1, it is feasible to tackle this by a brute-force procedure. I wrote a Python script that cycled through all admissible models over $\mathfrak{F}^6$ finding countermodels to instances of all 256 deduction schemes other than the traditional 24 syllogisms. A bit intellectually lazy, perhaps, but it also seems to me to be in keeping with the spirit of Leibniz's *Calculemus!* and pioneering efforts in computing. Pseudocode for the script can be found in Appendix A.

For the flavor, anyway, here is a countermodel over $\mathfrak{F}^6$ the program found for $AeB, BeC \Rightarrow AiC$ (1-eei): $\nu(A) = (1,2), \nu(B) = (2,3)$, and $\nu(C) = (6,1)$. Clearly, then, $\models^{\mathfrak{M}} AeB$ because $\gcd(\nu^-(A), \nu^+(B)) = 2$, $\models^{\mathfrak{M}} BeC$ because $\gcd(\nu^-(B), \nu^+(C)) = 3$, and $\not\models^{\mathfrak{M}} AiC$ because $\gcd(\nu^-(A), \nu^+(C)) = 2$, which suffices.

Thus, I have shown that $\models^{\mathfrak{F}^6} p, q \Rightarrow r$ implies $\vdash_{\mathcal{G}\mathbf{RS}} p, q \Rightarrow r$. The theorem as stated now follows from the definitional fact that $\models p, q \Rightarrow r$ implies $\models^{\mathfrak{F}^6} p, q \Rightarrow r$. ☐

Although Theorem 3 is really an improvement on Leibniz's own implicitly suggested result, it bears pointing out that $\mathcal{G}\mathbf{RS}$ is complete with respect to the whole divisibility lattice:

COROLLARY 1. $\mathcal{G}\mathbf{RS}$ *is complete with respect to* $\langle\mathbb{N}, |\rangle$*.*

Proof. Suppose $\not\vdash_{\mathcal{G}\mathbf{RS}} p, q \Rightarrow r$; by Theorem 3, $\not\models^{\mathfrak{F}^6} p, q \Rightarrow r$. If $\mathfrak{M} = \langle\mathfrak{F}^6, \nu\rangle$ is a rejecting countermodel, put $\mathfrak{M}' = \langle\mathbb{N}, |, \nu\rangle$. It is clear that, for any s, $\models^{\mathfrak{M}} s$ if and only if $\models^{\mathfrak{M}'} s$. It follows that $\not\models^{\langle\mathbb{N},|\rangle} p, q \Rightarrow r$, as desired. ☐

I will now show that the completeness of $\mathcal{G}\mathbf{RS}$ with respect to $\mathfrak{F}^6$—or any proper frame isomorphic to $\mathfrak{F}^6$—is the *optimal* result.[20] I show this by demonstrating that a minimum of 4 elements are needed to falsify $AeB, BeC \Rightarrow AiC$ (1-eei); this is not a valid deduction (see the proof of Theorem 3 above). I first show something properly stronger:

[20]Since every deduction is (vacuously) valid over any improper frame, clearly $\mathcal{G}\mathbf{RS}$ cannot be complete with respect to *any* improper frame.

THEOREM 4. *$AeB, BeC \Rightarrow AiC$ (1-eei) is valid in any frame that is a chain.*[21]

Proof. Let $\mathfrak{M} = \langle S, |, \nu \rangle$ be an arbitrary admissible model over a frame that is a chain. Suppose that $\models^{\mathfrak{M}} AeB$ and $\models^{\mathfrak{M}} BeC$. Four cases can be distinguished on the given assumptions:

1. $\gcd(\nu^+(A), \nu^-(B)) \neq 1$ and $\gcd(\nu^+(B), \nu^-(C)) \neq 1$;
2. $\gcd(\nu^+(A), \nu^-(B)) \neq 1$ and $\gcd(\nu^-(B), \nu^+(C)) \neq 1$;
3. $\gcd(\nu^-(A), \nu^+(B)) \neq 1$ and $\gcd(\nu^+(B), \nu^-(C)) \neq 1$;
4. $\gcd(\nu^-(A), \nu^+(B)) \neq 1$ and $\gcd(\nu^-(B), \nu^+(C)) \neq 1$.

Cases 1 and 4 cannot actually occur. Consider the first case. By admissibility, $\gcd(\nu^+(B), \nu^-(B)) = 1$, and so because $\mathfrak{M}$ is a chain, $\nu^+(B) = 1$ or $\nu^-(B) = 1$. Clearly, then, either $\gcd(\nu^+(A), \nu^-(B)) = 1$ or $\gcd(\nu^+(B), \nu^-(C)) = 1$, a contradiction. The fourth case is ruled out in the same way.

Now consider the second case. Clearly, $\nu^+(A) \neq 1$ and $\nu^+(C) \neq 1$. By admissibility and the fact that $\mathfrak{M}$ is a chain, then, $\nu^-(A) = 1$ and $\nu^-(C) = 1$, from which it clearly follows that $\gcd(\nu^+(A), \nu^-(C)) = 1$ and $\gcd(\nu^-(A), \nu^+(C)) = 1$, that is, $\models^{\mathfrak{M}} AiC$. The third case is dispatched similarly. □

In light of Theorem 4, it should be emphasized that I have not shown that $\mathcal{G}$**RS** is complete with respect to just any arbitrary four-element sublattice of $\langle \mathbb{N}, | \rangle$. In particular, $\mathcal{G}$**RS** is not complete with respect to any four-element chain, for example, $\mathfrak{F}^8 = \langle \{1, 2, 4, 8\}, | \rangle$.

COROLLARY 2. $\mathcal{G}$**RS** *is not complete with respect to any sublattice of $\langle \mathbb{N}, | \rangle$ smaller than $\mathfrak{F}^6$.*

Proof. Any lattice (a fortiori, any sublattice of $\langle \mathbb{N}, | \rangle$) with one, two, or three elements is a chain. The result follows immediately from this fact and Theorem 4. □

Corollary 2 shows that the arithmetical completeness theorem I have obtained in Theorem 3 is the best possible result. As far as I know this observation is novel to this article and even other (i.e., non-arithmetical) 'intensional' completeness results for the syllogistic have not delivered the optimal result vis-à-vis cardinality.[22]

[21] A frame $\mathfrak{F} = \langle S, | \rangle$ is a chain if for all $j, k \in S$, $j|k$ or $k|j$.

[22] For example, the completeness theorem of Glashoff [8] (cf. Martin [12]), even in the three-term case, delivers completeness with respect to a larger structure.

5 Concluding Remarks

In this article, I presented one of Leibniz's most mature (relative to April 1679) arithmetical interpretations of the assertoric syllogistic and proved soundness and completeness. I proved that the syllogistic is not only complete with respect to $\langle \mathbb{N}, | \rangle$, as Leibniz implicitly suggested, but also with respect to the four-element sublattice $\mathfrak{F}^6 = \langle \{1, 2, 3, 6\}, | \rangle$. Further, I showed that this is the best of all possible results.

Leibniz's arithmetical semantics is richly suggestive in at least two different ways, both of which already have stimulated further research and could stimulate more yet. While I have followed Leibniz in focusing on the divisibility lattice and sublattices thereof, it is clear that Leibniz's semantics could instead be formulated in a more abstract lattice-theoretic setting. For example, one could focus on abstract semilattices with least or greatest elements. Developments along such lines (and also for richer term logics) have been pursued by, for example, van Rooij [26, §§4, 6].[23]

Another direction for research which Leibniz's pioneering work suggests is to examine other logics characterizable in $\langle \mathbb{N}, | \rangle$. Weiss [27] showed that both positive intuitionistic logic and (semilattice) relevance logic are exactly characterizable in this lattice using truth conditions adapted from Urquhart [23]. It would be of considerable interest to identify the modal logic of this structure with the standard Kripke truth conditions. As far as I am aware, this is an open problem (it clearly must be at least as strong as **S4.2**).

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[23] Interestingly, van Rooij's treatment is partly motivated by work of van Fraassen [24]. The diamond lattice for **FDE** is obviously isomorphic to $\mathfrak{F}^6$.

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A Pseudocode

The "core code" on which Theorem 3 is based is presented below. It has a very straightforward implementation in Python.[24] The program is run by

[24] I have trimmed out certain basic utility functions and made certain parts more colloquial. But what I've given here is very nearly the Python code.

calling 'figures(i)' on a given integer i (e.g., on 6, to yield the main result of this article).

```
def universal_affirmative(predicate,subject):
    if subject[0] % predicate[0] == 0 and subject[1] % predicate[1] == 0:
        return True
    return False

def particular_negative(predicate,subject):
    return not universal_affirmative(predicate,subject)

def particular_affirmative(predicate,subject):
    if gcd(predicate[0],subject[1]) == 1 and gcd(predicate[1],subject[0]) == 1:
        return True
    return False

def universal_negative(predicate,subject):
    return not particular_affirmative(predicate,subject)

def categorical_handler(predicate,subject,constant):
    if constant == 'a':
        return universal_affirmative(predicate,subject)
    elif constant == 'i':
        return particular_affirmative(predicate,subject)
    elif constant == 'o':
        return particular_negative(predicate,subject)
    elif constant == 'e':
        return universal_negative(predicate,subject)
    else:
        return

def first_figure(ints):
    constants = ['a','i','o','e']
    admissibles = coprime_pairs(ints)
    triconstants = constants^3
    triadmissibles = admissibles^3
    for con in triconstants:
        pval = True
        for interpretation in triadmissibles:
            if categorical_handler(interpretation[0],interpretation[1],con[0]) and
            categorical_handler(interpretation[1],interpretation[2],con[1]) and
            not categorical_handler(interpretation[0],interpretation[2],con[2]):
                print("Countermodel to 1st figure " + str(con) + " : " + str(interpretation))
                pval = False
                break
        if pval:
            print("No countermodel to 1st figure " + str(con))
    return

def second_figure(ints):
    constants = ['a','i','o','e']
    admissibles = coprime_pairs(ints)
    triconstants = constants^3
    triadmissibles = admissibles^3
    for con in triconstants:
        pval = True
        for interpretation in triadmissibles:
            if categorical_handler(interpretation[0],interpretation[1],con[0]) and
            categorical_handler(interpretation[0],interpretation[2],con[1]) and
            not categorical_handler(interpretation[1],interpretation[2],con[2]):
                print("Countermodel to 2nd figure " + str(con) + " : " + str(interpretation))
                pval = False
                break
        if pval:
            print("No countermodel to 2nd figure " + str(con))
    return

def third_figure(ints):
    constants = ['a','i','o','e']
    admissibles = coprime_pairs(ints)
    triconstants = constants^3
    triadmissibles = admissibles^3
    for con in triconstants:
        pval = True
        for interpretation in triadmissibles:
            if categorical_handler(interpretation[0],interpretation[1],con[0]) and
            categorical_handler(interpretation[2],interpretation[1],con[1]) and
            not categorical_handler(interpretation[0],interpretation[2],con[2]):
                print("Countermodel to 3rd figure " + str(con) + " : " + str(interpretation))
                pval = False
                break
        if pval:
```

```
            print("No countermodel to 3rd figure " + str(con))
    return

def fourth_figure(ints):
    constants = ['a','i','o','e']
    admissibles = coprime_pairs(ints)
    triconstants = constants^3
    triadmissibles = admissibles^3
    for con in triconstants:
        pval = True
        for interpretation in triadmissibles:
            if categorical_handler(interpretation[0],interpretation[1],con[0]) and
            categorical_handler(interpretation[2],interpretation[0],con[1]) and
            not categorical_handler(interpretation[1],interpretation[2],con[2]):
                print("Countermodel to 4th figure " + str(con) + " : " + str(interpretation))
                pval = False
                break
        if pval:
            print("No countermodel to 4th figure " + str(con))
    return

def figures(i):
    ints = divisors_of(i)
    print("Finding all valid deductions over (" + str(i) + ",|):")
    print()
    first_figure(ints)
    print()
    second_figure(ints)
    print()
    third_figure(ints)
    print()
    fourth_figure(ints)
    return
```

B Valid Deduction Schemes

I present all of the standard valid deduction schemes (syllogisms) here together with their traditional names and shorthands.[25]

[25] This material is fairly standard and is adapted from Thom [21, p. 54] (cf. Kneale and Kneale [9, pp. 67ff.]). This number includes the so-called “subaltern moods.”

Figure	Traditional Name	Shorthand	Scheme
1st	Barbara	1-aaa	$AaB, BaC \Rightarrow AaC$
1st	Celarent	1-eae	$AeB, BaC \Rightarrow AeC$
1st	Darii	1-aii	$AaB, BiC \Rightarrow AiC$
1st	Ferio	1-eio	$AeB, BiC \Rightarrow AoC$
1st	Barbari	1-aai	$AaB, BaC \Rightarrow AiC$
1st	Celaront	1-eao	$AeB, BaC \Rightarrow AoC$
2nd	Cesare	2-eae	$BeA, BaC \Rightarrow AeC$
2nd	Camestres	2-aee	$BaA, BeC \Rightarrow AeC$
2nd	Festino	2-eio	$BeA, BiC \Rightarrow AoC$
2nd	Baroco	2-aoo	$BaA, BoC \Rightarrow AoC$
2nd	Camestrop	2-aeo	$BaA, BeC \Rightarrow AoC$
2nd	Cesaro	2-eao	$BeA, BaC \Rightarrow AoC$
3rd	Darapti	3-aai	$AaB, CaB \Rightarrow AiC$
3rd	Felapton	3-eao	$AeB, CaB \Rightarrow AoC$
3rd	Disamis	3-iai	$AiB, CaB \Rightarrow AiC$
3rd	Datisi	3-aii	$AaB, CiB \Rightarrow AiC$
3rd	Bocardo	3-oao	$AoB, CaB \Rightarrow AoC$
3rd	Ferison	3-eio	$AeB, CiB \Rightarrow AoC$
4th	Bramantip	4-aai	$BaA, CaB \Rightarrow AiC$
4th	Camenes	4-aee	$BaA, CeB \Rightarrow AeC$
4th	Dimaris	4-iai	$BiA, CaB \Rightarrow AiC$
4th	Fesapo	4-eao	$BeA, CaB \Rightarrow AoC$
4th	Fresison	4-eio	$BeA, CiB \Rightarrow AoC$
4th	Camenos	4-aeo	$BaA, CeB \Rightarrow AoC$

Table 1: The Syllogisms