The calculus of the compatibility of propositions

I. E. Orlov
(Translated from the Russian by Werner Stelzner)

1. Tranlator's Preliminary remarks

The following translation was carried out at the suggestion of David Makinson, with whom the translator also discussed successive drafts. I would like to thank David for the discussions, and an anonymous reviewer for some helpful suggestions. On those occasions when I did not follow their suggestions, it was to secure the closest possible correspondence between the original Russian version and the translation. My aim has been to render Orlov's paper, published in Russian in 1928, as precisely as possible in meaning, terminology, style, and even layout. In particular, I wished to avoid suggesting to the reader interpretations that are not supported by the Russian text in the context of the 1920s. Some remarks on interpretation can be found in the translator's footnotes and, of course, in the accompanying *Guide*.

The translation of the Russian title *Исчисление совместности предложений* [*Ischislenie sovmestnosti predlozhenij*] as *Calculus of the compatibility of propositions* already raises questions. They concern both the rendering of «совместность» [sovmestnost'] as "compatibility" and «предложение» [predloženie] as "proposition". In settling on these terms, the translator was guided by both the Russian text and the French summary that accompanied the original (reproduced here at the end of the translation).

The title of this summary, *Sur la théorie de la compatibilité des propositions*, presumably approved by Orlov himself, suggests using the analogous wording in English as well.

The aim of reproducing Orlov's original text as precisely as possible also implies that the translator has refrained from correcting obvious errors (whether arising in the printing process or from Orlov's inattention). When such errors seem significant, they are noted in the translator's footnotes and/or the *Guide*.

Some technical remarks on formatting:

- The indentations of lines of text are taken from the Russian original. So too are some typographic conventions, notably the form of inverted commas in Orlov's footnotes and wide spaces between letters to convey emphasis.
- Orlov's bibliographical references are left as in the original text but are given in greater detail in the bibliography at the end of the *Guide*.

- The symbolic representation used by Orlov has largely been followed. Only those for negation and compatibility have been changed, as described in the translator's footnotes.
- The translations of Orlov's footnotes, which are numbered page by page in Orlov's work, are marked consecutively across pages with superscripted 1), 2), ...
  Supplementary footnotes of the translator are marked by T1, T2, ... in superscript.
- The bold numbers inside square brackets indicate new pages in Orlov's original text.

2. Translation

И. Е. Орлов, Исчисление совместности предложений, *Математический сборник*, 1928, том 35, номер 3-4, 263–286. [I. E. Orlov, Ischislenie sovmestnosti predlozhenij, Matematicheskij sbornik, 1928, tom 35, nomer 3-4, 263–286.]

[263]

The calculus of the compatibility of propositions.

I. E. Orlov (Moscow)

§ 1. By excluding certain "axioms" from the so-called "calculus of propositions", which is the most general part of mathematical logic, one can obtain a symbolic system that is essentially different in its basic conception from the classical theory. Classical mathematical logic is based on the well-known notion of material implication, which can connect in one formula two propositions that have no internal sense connection; however, the system that we have in mind can treat in symbolic form the connections of sense between propositions. Together with that, this system does not deal primarily with the question of the truth or falsity of the given propositions, but with the question of their compatibility or incompatibility with each other. Therefore, in contrast to the classical "calculus of propositions", we call the system developed here "the calculus of compatibility of propositions". One can also assert that the elaboration of the indicated theory is a live issue, given the need to accommodate symbolic logic with the new methods introduced by intuitionism.

An important peculiarity of intuitionism consists in the fact that in the works of intuitionists, defined notions do not depend directly on the defining propositions a, b, c ..., but on functions of the latter, of the following kind: "a is reliably known", "a is provable", "a is reducible to absurdity", "the absurdity of a is absurd", etc.[4)]

The use of such functions by intuitionists is a fact, and this fact cannot be ignored in the elaboration of mathematical logic. However, the introduction of functions of this kind into classical mathematical logic leads to contradiction with the law "tertium non datur"; in the following, it will be shown that such an introduction is, indeed, impossible. However, the "calculus of compatibility of propositions" admits the introduction of the mentioned functions and symbolic operations over these functions so that the provisions of intuitionism are obtained without negating the law of "tertium non datur", i. e., the provisions are brought into full agreement with that law.

Before proceeding to the formal presentation of our theory, let us make a few preliminary explanatory remarks.

In cases where in classical mathematical logic the inference is made from two or more propositions, the premises forming a "logical product" are stated

[4)] See e.g. L. E. J. Brouwer, Zur Begründung der intuitionistischen Mathematik, *Math. Ann.*, Bd. 93, 1925.

together as true; but such requirement of conjoint **[264]** assertion of the premises introduces into logic an element alien to the essence of the matter. For the possibility of a deductive inference, the requirement of truth of the premises, generally speaking, is not necessary; it is sufficient that the weaker requirement of compatibility of the premises be satisfied. From false propositions true consequences can be inferred, but from premises that are incompatible with each other inferences are not possible at all. From this it follows that the requirement of compatibility of propositions is all that we need, and the requirement of their joint truth is exaggerated.

The notion of compatibility of propositions can be defined as follows: proposition a is "compatible" with proposition b if the negation of b does not follow from a. Thus, the definition of "compatibility" is based on the notions of "consequence" and "negation". We take these last two notions as the initial ones, and formally they remain undefined; nevertheless, we will dwell on the notion of inference (or "consequence"), in order to clarify how we understand them.

The expression $a \to b$ [T1] (from a follows b) in the system developed here should not be confused with "material implication". On the basis of this expression one can say: "either a is false, or b is true", but that is not sufficient, and the sense of the given symbolic expression is not exhausted by it. In order to be able to write $a \to b$, there must also be a certain connection by sense between propositions a and b.[5)]

The indicated "connection by sense" covers all those cases where the preceding component presupposes the following one, or is impossible without it. The notion of inference can be clarified more sharply only in the course of the development of the theory.

Thus, the starting point of the theory does not consist of data about truth and falsity of separate propositions, but data about the presence or absence of sense connections of the indicated type between propositions; other relations of an analogous type are derived as consequences of the data that serve as premises for the relations of inference and compatibility.

The names "inference", "logical product", "logical sum" are given to certain symbolic formulas, not to one or another interpretation of those formulas. Therefore, we retain these names, while giving a different sense to the formulas. In particular, the logical product of two propositions no longer means their joint assertion, but means the assertion of their compatibility. For this reason, the sense of the symbolic expressions accepted as axioms also changes, and a number of the axioms of the classical "calculus of propositions" have to be recognized as false. First of all, the so-called "principle of simplification": "from a and b, taken together, follows a" has to be eliminated. In fact, we interpret the expression "a and b" as an assertion of the compatibility of propositions a and b with each other. But the truth of those propositions does not follow from their

[T1] Orlov usually (not always) encloses compound formulas in lines of text in double quotation marks. In the translation, such quotation marks are omitted.

[5)] For example, the proposition "Socrates is human, therefore Socrates is mortal" is true; but the proposition "Socrates is human, therefore two times two is four" is false. However, in classical mathematical logic both expressions are regarded as true.

compatibility, since false propositions can also be compatible.

The principle of simplification can also be expressed in an equivalent form: "from a follows either b or a".[6] This second form has been adopted by *Russell* as an axiom under the name of "principle of addition".[7] According to the modified interpretation of the notion of logical sum, this proposition **[265]** should also be considered false.[8] Equally, the proposition $a \to (b \to a)$, accepted as an axiom in the *Hilbert* system,[9] must be excluded.

On the technical side, our task consists in developing a logical calculus, without applying the "principle of simplification" or any other proposition equivalent to it. Since the proofs of most of the theorems of the calculus of propositions were based on the application of that principle or its consequences, its exclusion leads to considerable changes in the presentation of the symbolic theory. However, such an exclusion reveals a remarkable fact unknown to *Peano* and *Russell*, namely, that the most important theorems founding deduction, which have always been proved with the help of the principle of simplification (or propositions equivalent to it), in reality do not depend on that principle. Once the principle of simplification is excluded, it is impossible to derive propositions of a paradoxical character, such as: "a true proposition follows from every proposition", "every proposition follows from a false proposition", "all true propositions are equivalent", and so on.

In the formal theory we have to distinguish between axioms and propositions derived from them, on the one hand, and those propositions a, b, c... that may be substituted in the axioms and to which the axioms are applied. The former are to be regarded as truthful and assertible. With regard to the latter, in the general part of our work we disregard the question of the truthfulness and in general of the provability of the given premises a, b, c..., which can be substituted in axioms, and regard them as conditionally adopted hypotheses. In this connection, the assertion of separate propositions is not possible. In the second part of the paper, we introduce special functions of propositions, and it is only there that the assertion of isolated propositions becomes possible.

We will follow the symbolic notations employed by *Hilbert*, as the simplest and most convenient, denoting, for the sake of simplicity, propositions in small Latin letters and keeping the traditional point for the logical product. Into this symbolism, however, we will put a changed content.

The sign "=" will indicate everywhere only agreement upon the writing of the symbols, and can be read: "otherwise written". We will write the defined expression on the right, the defining expression on the left. We will keep separate numbering for axioms, definitions, and derived propositions.

§ 2. We regard a "proposition" as an elementary fact, not being a subject of

[6] See Peano, *Notations de Logique Mathematique*. Turin. 1894, p. 9.

[7] Whitehead and Russell, *Principia Mathematica*. Cambridge. 1910, v. 1, p. 100.

[8] The mentioned sentences are false as general formulas; in particular cases (which we will encounter) they may be true.

[9] „Ueber das Unendliche", *Math. Ann.*, Bd. 95, 1925. Of the twelve logical axioms given by Ackermann in „Begründung des tertium non datur", *Math. Ann.*, Bd. 93. p. 4, only numbers 2, 3, 4, 10, and 12 can be accepted as true sentences, from our point of view.

analysis, and admit it, without definition, as a primitive concept. We also take "inference" and "negation" as primitive concepts, denoting them in the usual way.

The relation of compatibility of two propositions (logical product) can be defined formally as follows:

$$\neg(a \to \neg b) = a{\cdot}b \quad \text{[T2]} \qquad \text{Def. (1).}$$

The negation of the compatibility relation is the incompatibility relation:

$$\neg(a{\cdot}b) = a|b \quad \text{[T3]} \qquad \text{Def. (2).}$$

[266]

Now we provide a list of six formal axioms, which are sufficient to develop the general part of the "calculus," i. e., for the foundation of the general theory of deduction.

$$a \to \neg\neg a \qquad \text{Ax. (1)}$$

$$\neg\neg a \to a \qquad \text{Ax. (2)}$$

$$a \to a{\cdot}a \quad \text{[T4]} \qquad \text{Ax. (3)}$$

$$(a \to b) \to (\neg b \to \neg a) \qquad \text{Ax. (4)}$$

$$\{a \to (b \to c)\} \to \{b \to (a \to c)\} \qquad \text{Ax. (5)}$$

$$(b \to c) \to \{(a \to b) \to (a \to c)\} \qquad \text{Ax. (6).}$$

Ax. (4) is known as the "principle of contraposition". On the basis of Ax. (4) we can propose the following interpretation which best corresponds to the essence of the concept of inference: consequence b is a necessary condition for the assumption a, since from the falsity of b follows the falsity of a.[10)]

If, however, the inference is made not from one but from two or more premises, then the derived proposition is no longer a condition of the truth of the

[T2] Orlov writes Def. (1) in the following way: $\overline{(a \to \bar{b})} = a.b$. He expresses negation throughout by overlining the negated formula.

For typographic convenience we depart from this notation and express the negation of a formula by preceding hook. So instead of $\overline{H}$ and of $(\overline{H})$ (with both notations Orlov expresses the negation of H) we write $\neg(H)$. We use the bracket convention that hook $\neg$ binds stronger than all other connective symbols. Accordingly, no parentheses are placed around negated propositional variables. It also avoids parentheses between multiple negations: Instead of $\neg(\neg(H))$, $\neg\neg(H)$ is written (as usual).

We also replace the dot . used by Orlov as the symbol for compatibility with the multiplication sign $\cdot$. So instead of Orlov's $H.G$ we write $H{\cdot}G$.

[T3] Thus, according to definitions (1) and (2), the expression $a|b$ is written in basic symbols as $\neg\neg(a \to \neg b)$ and $a|b$ is definitional abbreviation for $\neg\neg(a \to \neg b)$, respectively.

[T4] At certain points Orlov uses the following unspoken bracketing conventions:

- Inference $\to$ and equivalence $\rightleftarrows$ separate more strongly than all other compounds.
- Incompatibility $|$ connects more strongly than compatibility $\cdot$.
- Negation connects stronger than the other compounds.
- Outside parentheses are generally omitted.

Accordingly, Ax. (3) without parenthesis conventions is to be written as follows: $(a \to (a{\cdot}a))$.

[10)] Or otherwise: we interpret "$a \to b$" as "a presupposes b". The interpretation of the notion of inference cannot be otherwise, since we abstract from the question of the trustworthiness of the sentences a, b substituted in the axioms - and accept them conditionally.

premises, but only of the compatibility of the premises. Let us consider the expression $a \cdot b \to c$, which is a general scheme of syllogistic inference, i. e. inference from two premises. Propositions a and b can be true or false, but that is, generally speaking, irrelevant, because in a correctly constructed syllogism a true conclusion may follow both from true and false premises. But one condition is necessary for a syllogism to exist. There must be a "connection by sense" between the compatibility of the premises, on the one hand, and the truth of the conclusion, on the other. "Syllogism" takes place if the following conditions are met: if proposition c is true, the hypothesis of the truth of a does not depend on b. If c is false, then the hypothesis that a is true is admissible only under the condition that b is false. The same is true for the converse relation from b to a. So, we say that c follows from the compatibility, not from the truth of a and b. If c is false, then a and b are incompatible, i. e. in this case from a follows $\neg b$ and from b follows $\neg a$. This is expressed by the formula obtained from Ax. (4) and Def. (2):

$$(a \cdot b \to c) \to (\neg c \to (a|b)).^{\mathrm{T}5}$$

Thus, the conclusions that can be derived from any premises, generally speaking, can only be regarded as necessary conditions for the compatibility of the premises, and nothing more. However, the specific case in which the premises are axioms or propositions that have been proven must be evaluated in a completely different way. This is what we will now do.

In order to be able to use the given axioms to construct a system, it is necessary to adopt one axiom of a non-formal character:

"Axioms, as well as propositions derived from axioms,
may be omitted as part of symbolic formulas
if they serve as premises for some conclusion." Ax. (7).

[267]

Thus, if we have the expression $a \to b$, where a is an axiom (or a proposition previously proved), then in this case we can assert proposition b separately.

Based on Ax. (2) and (6) we have

$$(\neg\neg a \to a) \to \{(a \to \neg\neg a) \to (a \to a)\},$$

so that the expression $a \to a$, or the "law of identity", on the basis of Ax. (7) can be written separately.

Based on Ax. (2) we write

$$\neg\neg(a \to \neg b) \to (a \to \neg b)$$

and on the basis of Defs. (1) and (2)

$$a|b = \neg\neg(a \to \neg b),^{\mathrm{T}6}$$

[T5] This formula, written with iterated round brackets, does not follow Hilbert's convention of round-within-curly brackets, under which it would be written as $(a \cdot b \to c) \to \{\neg c \to (a|b)\}$.

[T6] This expression should not occur in Orlov's system as a formula, because it contains the character $=$, which is not a basic character of the system and was not introduced by definition.

The sign $=$ is used by Orlov to formulate definitions. In Orlov's definitions (deviating from their usual formulation) the definiens stands to the left of $=$, while the definiendum stands to the right. In this formula certainly no definition is to be supplied for the $\neg\neg(a \to \neg b)$ standing from which we get:[T7]

$$\neg(a{\cdot}b) \to (a \to \neg b). \tag{1a}$$

Analogously we get the expression:

$$(a \to \neg b) \to \neg(a{\cdot}b). \tag{1b}$$

Let us now prove the expression:

$$\neg(a{\cdot}\neg b) \to (a \to b). \tag{1c}$$

Proof: Based on Ax. (6):

$$(\neg\neg b \to b) \to \{(a \to \neg\neg b) \to (a \to b)\},$$

where the expression $(\neg\neg b \to b)$ can be omitted (Ax. 7). Substituting in (1a) $\neg b$ in place of b we have:

$$\neg(a{\cdot}\neg b) \to (a \to \neg\neg b).$$

Based on Ax. (6):

$$[(a \to \neg\neg b) \to (a \to b)] \to \{[\neg(a{\cdot}\neg b) \to (a \to \neg\neg b)] \to [\neg(a{\cdot}\neg b) \to (a \to b)]\}$$

And, omitting the proved expressions, we obtain (1c).

In the same way we can prove:

$$(a \to b) \to \neg(a{\cdot}\neg b) \tag{1d}$$

Applying Ax. (4) to (1c) and (1d), we get:

$$\neg(a \to b) \to (a{\cdot}\neg b) \tag{1e}$$

$$(a{\cdot}\neg b) \to \neg(a \to b) \tag{1f}$$

If we have two propositions a and b between which there is no internal connection at all, then all four possible inference relations will fail: $a \to b$, $a \to \neg b$, $\neg a \to b$ and $\neg a \to \neg b$. Based on Def. (1) **[268]** and expression (1e) it is easy to show that all four expressions of compatibility $a{\cdot}b$, $a{\cdot}\neg b$, $\neg a{\cdot}b$, and $\neg a{\cdot}\neg b$ will hold simultaneously in this case .[11)]

Ax. (3) requires that any true proposition be compatible with itself, i. e. does not contradict itself. On the basis of Ax. (3) and (4) and expression. (1a) can also be written:

$$a|a \to \neg a; \tag{2a}$$

on the right. And vice versa the left-standing $a|b$ cannot be defined here for the second time.

What is valid according to definitons (1) and (2) (and what Orlov could have written as sound formulas of his system) is: $a|b \to \neg\neg(a \to \neg b)$, $\neg\neg(a \to \neg b) \to a|b$ (and with the later introduced $\rightleftarrows$): $a|b \rightleftarrows \neg\neg(a \to \neg b)$.

However, (on the basis of definitions (1) and (2) the expression $a|b = \neg\neg(a \to \neg b)$ listed by Orlov can also be understood as a formulation of a metatheorem which says: The expressions $a|b$ and $\neg\neg(a \to \neg b)$ are replaceable by each other in arbitrary formulas without change of provability.

[T7] The following formula (1a) is obtained by substituting $(a \to \neg b)$ for a in Ax. (2), giving $\neg\neg(a \to \neg b) \to (a \to \neg b)$, and (according to Def. (1) by replacing $\neg(a \to \neg b)$ by $(a{\cdot}b)$. The expression $a|b = \neg\neg(a \to \neg b)$ (resp. the metatheorem we mentioned in footnote T6) does not play any role.

[11)] For example, this is what happens when sentences are taken from different fields of knowledge. From this it does not follow, of course, that compatible propositions cannot be connected by sense.

It is also easy to derive the expressions:

$$(a|a)|a, \tag{2b}$$

$$(a \to \neg a) \to \neg a \tag{2c}$$

The expression $a|a$, from which $a \to \neg a$ follows, means that the proposition is incompatible with itself, or contradictory. Such a proposition can be called absurd.

(2c) expresses a situation, often occurring in mathematics, when from the assumption a follows $\neg a$, which serves as a proof of the falsity of a.[12)]

This is the "reductio ad absurdum" principle.

By substituting $\neg a$ in place of a in (2c), it is easy to get the expression:

$$(\neg a \to a) \to a \tag{2d}$$

The expression "$a{\cdot}a \to a$" is, generally speaking, false, since it does not follow from the fact that a proposition is not contradictory that it is true.[13)] In the same way, the expression "$\neg a \to a|a$" is false.

Let us prove a few simple formulas that we will need in the following.

$$(a \to b) \to (\neg\neg a \to b) \tag{3a}$$

Proof: On the basis of Ax. (6)

$$(a \to b) \to \{(\neg\neg a \to a) \to (\neg\neg a \to b)\},$$

which on the basis of Ax. (5) can be written as:

$$(\neg\neg a \to a) \to \{(a \to b) \to (\neg\neg a \to b)\},$$

from where one gets (3a).

$$(a|b) \to (\neg\neg a|b) \tag{3b}$$

Proof:

On the basis of (3a):

$$(a \to \neg b) \to (\neg\neg a \to \neg b)$$

In addition, we have:

$$(a|b) \to (a \to \neg b) \text{ and } (\neg\neg a \to \neg b) \to (\neg\neg a|b);$$

from here, by double application of Ax. (6) we obtain (3b).

[269]

Analogously, we can prove the following expressions:

$$(\neg\neg a \to b) \to (a \to b) \tag{3c}$$

$$(a \to \neg\neg b) \to (a \to b) \tag{3d}$$

$$\neg\neg a|b \to a|b \tag{3e}$$

Substituting $\neg b$ for b in (3c) and applying Ax. (4), we obtain:

12) The case when from a sentence its negation follows is a particular case of absurdity. Any sentence which is brought into contradiction with an axiom can be called absurd.

13) In some particular cases, an expression of such a type may be true.

$$\neg(a \to \neg b) \to \neg(\neg\neg a \to \neg b),$$

which, on the basis of Def. (1), we can write in the form:

$$a{\cdot}b \to \neg\neg a{\cdot}b; \qquad (3f)$$

analogously one obtains:

$$\neg\neg a{\cdot}b \to a{\cdot}b. \qquad (3g)$$

On the basis of ($3c$) we write:

$$(\neg\neg a \to \neg\neg b) \to (a \to \neg\neg b)$$

By joining with ($3d$) and applying Ax. (6). we obtain:

$$(\neg\neg a \to \neg\neg b) \to (a \to b). \qquad (3h)$$

Also easily derived are:

$$(\neg a \to b) \to (\neg b \to a), \qquad (3i)$$

$$(a \to \neg b) \to (b \to \neg a), \qquad (3k)$$

From Ax. (4) the commutative law follows:

$$a{\cdot}b \to \ b{\cdot}a. \qquad (4a)$$

Proof: On the basis of ($3k$) and Ax. (4):

$$\{(b \to \neg a) \to (a \to \neg b)\} \to \{\neg(a \to \neg b) \to \neg(b \to \neg a)\}.$$

Based on Def. (1) and omitting the expression on the left-hand side on the basis of Ax. (7), we obtain ($4a$).

Based on ($1c$) and ($1d$), Ax. (4) can be represented in the form $a|\neg b \to \neg b|a$, from which, replacing b by $\neg b$,[T8] we obtain:

$$a|b \to b|a. \qquad (4b)$$

Let us proceed to the consequences of Ax. (5). Substitute $\neg c$ for c in Ax. (5):

$$\{a \to (b \to \neg c)\} \to \{b \to (a \to \neg c)\}.$$

Applying Ax. (4) and Def. (1) and Def. (2) to the above expression, we obtain:

$$a|b{\cdot}c \to b|a{\cdot}c.^{T9} \qquad (5a)$$

From (4b) follows (5b):

$$a|b{\cdot}c \to b{\cdot}c|a. \qquad (5b)$$

[270]

Let us prove the expression:

$$a|b{\cdot}c \to a|c{\cdot}b. \qquad (5c)$$

Proof: According to ($4b$) we have $b|c \to c|b$, from which, on the basis of Ax. (6), follows $(a \ \to b|c) \to (a \to c|b)$. On the basis of definitions (1) and (2), this last expression can be written in the form ($5c$).

[T8] Orlov writes: "replacing $\neg b$ by b", which is obviously an error: Replacing $\neg b$ by b is not a legitimate act of substitution, while replacing b by $\neg b$ is, and gives ($4b$) by deleting double negations.

[T9] According to the bracketing conventions, read as: $(a|b){\cdot}c \to \ (b|a){\cdot}c$.

$$a|b{\cdot}c \to a{\cdot}b|c. \qquad (5d)$$

Proof: Applying the transformations (5c), (5a) and (5b) successively[T10] to the expression $a|b.c$, we obtain (5d).

In the same way, one can easily derive the expressions:

$$a{\cdot}b|c \to a|b{\cdot}c, \qquad (5e)$$

$$a{\cdot}b|c \to a{\cdot}c|b, \qquad (5f)$$

etc.

Applying Ax. (4) to expressions (5a)–(5f), we obtain expressions of the associative law:

$$(a{\cdot}b){\cdot}c \to a{\cdot}(b{\cdot}c). \qquad (5g)$$

$$(a{\cdot}b){\cdot}c \to c{\cdot}(b{\cdot}a). \qquad (5h)$$

etc.

Hence, given Ax. (4), the axiom (5) is just another expression of the associative law. Let us dwell on the meaning of the associative law in our "calculus of the compatibility of propositions".

The expression $(a{\cdot}b){\cdot}c$ represents the "logical product" of two propositions — proposition $(a{\cdot}b)$ and proposition c. Or, in our interpretation, it is the hypothesis of compatibility of proposition c with the hypothesis of compatibility of propositions a and b.

Let's conventionally write expressions (5g), (5h) etc. without brackets:

$$(a{\cdot}b){\cdot}c = a{\cdot}b{\cdot}c \qquad \text{Def. (3)}$$

The expression $a.b.c$ can be interpreted as "the compatibility of the three propositions"; essentially it means that any one of these three propositions is compatible with the hypothesis of the compatibility of the other two.

We point out the following inferences, which at first sight may seem paradoxical: the proposition $a{\cdot}b{\cdot}c \to a{\cdot}b$ is, generally speaking, false; the proposition $a{\cdot}\neg a{\cdot}b$ can be true. Let us demonstrate that this is indeed the case.

Suppose we have the proposition $a{\cdot}b \to d$, and consequently also $\neg d \to (a|b)$. Let us now choose a proposition c, which has no connection by sense with the propositions a, b, d. In this case the expressions $c{\cdot}d$, $c{\cdot}(a{\cdot}b)$, $(c{\cdot}a){\cdot}b$ etc., and hence the proposition $a{\cdot}b{\cdot}c$ will be correct. But at the same time the expressions $c{\cdot}\neg d$ and $c{\cdot}(a|b)$ are correct. All the mentioned propositions remain correct, regardless of whether proposition d is true or false. But the expression $a{\cdot}b$ becomes false if d is false. Thus it is possible that the expression $a{\cdot}b{\cdot}c$ remains true while $a{\cdot}b$ is false. It follows that the expression $a{\cdot}b{\cdot}c \to a{\cdot}b$ is not true.

The truth of the expression $a{\cdot}\neg a{\cdot}b$ follows from the rejection of the "principle of simplification". Indeed: on the basis of (1e):

$$\neg(a{\cdot}b \to a) \to a{\cdot}b{\cdot}\neg a,$$

[271]

[T10] Orlov is also using Ax. (6).

from the consequent of which, on the basis of the associative law,[T11] $a{\cdot}\neg a{\cdot}b$ follows. We note that the expression $\neg(a{\cdot}\neg a \rightarrow b)$ is true in all cases where a and b are not connected by sense.

§ 3. Let us prove the "principle of importation" and the "principle of exportation".

$$\{a \rightarrow (b \rightarrow c)\} \rightarrow (a{\cdot}b \rightarrow c) \qquad (6a)$$

Proof: On the basis of (1d):

$$(b \rightarrow c) \rightarrow \neg(b{\cdot}\neg c)$$

from which, on the basis of Ax. (6) we have:

$$\{a \rightarrow (b \rightarrow c)\} \rightarrow \{a \rightarrow \neg(b{\cdot}\neg c)\}$$

Further we write (1d) in the form

$$(a \rightarrow b) \rightarrow a|\neg b$$

and substitute the expression $\neg(b{\cdot}\neg c)$ in place of b. In the result we get:

$$\{a \rightarrow \neg(b{\cdot}\neg c)\} \rightarrow (a|\neg\neg(b{\cdot}\neg c)).$$

$$a|\neg\neg(b{\cdot}\neg c) \rightarrow a|b{\cdot}\neg c \qquad \text{by (1}b\text{) and (3}e\text{)},$$

$$a|b{\cdot}\neg c \rightarrow a{\cdot}b|\neg c \qquad \text{by (5}d\text{)}.$$

$$a{\cdot}b|\neg c \rightarrow (a{\cdot}b \rightarrow c) \qquad \text{by (1}c\text{)}.$$

Applying successively Ax. (6) to the series of the above expressions, we obtain (6a).

$$(a{\cdot}b \rightarrow c) \rightarrow \{a \rightarrow (b \rightarrow c)\} \qquad (6b)$$

Proof:

$$(a{\cdot}b \rightarrow c) \rightarrow a{\cdot}b|\neg c \qquad \text{by (1}d\text{)},$$

$$a{\cdot}b|\neg c \rightarrow a|b{\cdot}\neg c \qquad \text{by (5}e\text{)},$$

$$a|b{\cdot}\neg c \rightarrow (a \rightarrow b|\neg c) \qquad \text{by (1}c\text{)}.$$

On the basis of (1c) we write:

$$b|\neg c \rightarrow (b \rightarrow c),$$

from where, on the basis of Ax. (6), we obtain:

$$(a \rightarrow b|\neg c) \rightarrow (a \rightarrow (b \rightarrow c)).$$

Applying sequentially Ax. (6), we obtain (6b).
[272]

$$(a{\cdot}b \rightarrow c) \rightarrow (a{\cdot}\neg c \rightarrow \neg b.^{T12} \qquad (6c)$$

Proof:

$$a{\cdot}b|\neg c \rightarrow a{\cdot}\neg c|b \qquad \text{by (5}f\text{)},$$

[T11] Orlov here confuses "associative law" with "commutative law".
[T12] Right bracket before the punctuation mark omitted in the original text.

$$a{\cdot}\neg c|b \rightarrow (a{\cdot}\neg c \ \rightarrow \neg b) \qquad \text{by } (1c),$$

and then applyingAx. (6), we obtain (6*c*).

$$(a{\cdot}b \rightarrow c) \rightarrow (b \rightarrow (a \rightarrow c)). \qquad (6d)$$

Proof: The above expression follows from:

$$(a{\cdot}b \rightarrow c) \rightarrow a{\cdot}b|\neg c \qquad \text{by } (1d),$$

$$a{\cdot}b|\neg c \rightarrow b|a{\cdot}\neg c \qquad \text{by } (5a),$$

$$b|a{\cdot}\neg c \rightarrow (b \rightarrow (a \rightarrow c)) \qquad \text{by } (1a).$$

From the "principle of importation" we get the "principle of syllogism"

$$(a \rightarrow b){\cdot}(b \rightarrow \ c) \rightarrow (a \rightarrow c). \qquad (7a)$$

Proof: Replacing in expression (6a):
a by $(b \rightarrow c)$,
b by $(a \rightarrow b)$,
c by $(a \rightarrow c)$:

$$\{(b \rightarrow c) \rightarrow [(a \rightarrow b) \rightarrow (a \rightarrow c)]\} \rightarrow \{(b \rightarrow c){\cdot}(a \rightarrow b) \rightarrow (a \rightarrow c)\}.$$

The left-hand side can be omitted as Ax. (6), and the right part may be written in the form of (7*a*).

As already pointed out, in the syllogism the conclusion follows not from the truth of the premises, but from their compatibility. Therefore, the syllogism remains correct even when both its premises and its conclusion are false. In the given case, the falsehood of the propositions $a \rightarrow b$, $b \rightarrow c$, $a \rightarrow c$ denotes the absence of an internal connection between the terms a, b and c. Despite this, the right and left parts of expression (7*a*) remain internally connected (at least by the generality of the terms), and consequently formula (7*a*) remains always correct whatever a, b, and c are. If the conclusion $a \rightarrow c$ is false, i. e. if the terms a and c have no internal connection, then from the expression

$$\neg(a \rightarrow c) \rightarrow (a \rightarrow b)|(b \ \rightarrow c),$$

which is obtained by applying Ax. (4) to (7),[T13] it follows that in such a case the premises are incompatible, independently of whether they[T14] are true or false. Thus, the interpretation of the notion of compatibility is made more precise. We see that only those propositions can be compatible from which no false conclusion can be drawn on the basis of the axioms we have adopted.

[273]

Axiom (6) is only another expression of the formula of syllogism, and therefore also represents no more than the known relation between compatibility and incompatibility of the terms that are contained in it. Ax. (6) is also always true, whatever a, b and c are, and the case when the expressions $(b \rightarrow c)$ and

[T13] "(7)" is a misprint and it should be "(7*a*)", because the given expression is obtained by contraposition (i. e. by Ax. (4)) from (7*a*).

[T14] Orlov refers here with the word "they" (in the Russian original "они") not to the premises, but to the expressions a and c, with which independently of their truth value a false conclusion is formed, if between a and c no sense connection exists.

$(a \to b)$, being parts of the formula, are true and asserted, is only a special case of the application of Ax. (6).

Since expressions of the form $(a{\cdot}b \to c)$, $(a \to (b \to c))$, and $(b \to (a \to c))$ can be derived from one another, the application of Ax. (7) is extended. When we infer from two or more premises, we can omit those premises that have already been proved, e.g., we can write $a \to c$ instead of $a{\cdot}b \to c$ when b is a proven proposition.

Applying the principle of importation to expressions:

$$(a \to b) \to (a \to b) \text{ and } (a \to b) \to (\neg b \to \neg a),$$

we obtain, respectively:

$$(a \to b){\cdot}a \to b, \tag{7b}$$

$$(a \to b){\cdot}\neg b \to \neg a. \tag{7c}$$

Expression (7b) has a formal similarity to the "scheme of logical inference" by *Hilbert*; but they cannot have the same meaning. Indeed, since the propositions a and $a \to b$ are not asserted in (7b); the truth of b is only a condition of the compatibility of those propositions.

From (7b) we obtain the proposition:

$$\{a \to (b{\cdot}\neg b)\} \to \neg a, \tag{7d}$$

adopted by *Hilbert* as an axiom.[14)]

Proof: Substitute in (7b)[T15] the expression "$b{\cdot}\neg b$" in place of b:

$$\{a \to (b{\cdot}\neg b)\}{\cdot}(b|\neg b) \to \neg a;$$

from here, we get (7d), since the expression $b|\neg b$ can be omitted on the basis of Ax. (7).

$$(a \to (a \to b)) \to (a \to b) \tag{8}$$

Proof:

$$(a \to (a \to b)) \to (a{\cdot}a \to b)$$

on the basis of the principle of importation, while

$$(a \to a{\cdot}a){\cdot}(a{\cdot}a \to b) \to (a \to b)$$

based on the principle of syllogism; omitting $(a \to a{\cdot}a)$ from the latter by Ax. (3) and applying the principle of syllogism, we get (8a).[T16]

From the principle of syllogism we also obtain the expression:

$$(a \to b){\cdot}a{\cdot}c \to b{\cdot}c. \tag{9a}$$

[274]

Proof: By the principle of syllogism we obtain

$$(a \to b){\cdot}(b \to \neg c) \to (a \to \neg c)$$

$$(a \to b){\cdot}\neg(a \to \neg c) \to \neg(b \to \neg c) \qquad \text{using (6c)}$$

---

14) „Ueber das Unendliche", *Math. Ann.*, Bd. 95, p. 178.

T15 In fact, Orlov carries out the substitution in (7c).

T16 Orlov should have referred to (8) instead of (8a).

$(a \to b)\cdot(a\cdot c) \to (b\cdot c)$ is obtained from the last expression by Def. (1), and by Def. (3) we can rewrite that as (9a).

By applying the principle of exportation to (9*a*) we obtain:

$$(a \to b) \to (a\cdot c \to b\cdot c). \tag{9b}$$

On the basis of (9*b*) the associative law can be generalized:

$$(a\cdot b\cdot c)\cdot d \to a\cdot(b\cdot c\cdot d) \tag{9c}$$

Proof: Substitute in (9*b*)

in place of a — $(a\cdot b)\cdot c$,
in place of b — $a\cdot(b\cdot c)$,
in place of c — d.

Applying (5*g*) on the left-hand side, we write:

$$((a\cdot b)\cdot c)\cdot d \to (a\cdot(b\cdot c))\cdot d.$$

But, on the basis of Def. (3) the expression $((a\cdot b)\cdot c)\cdot d$ can be written as $(a\cdot b\cdot c)\cdot d$.
Moreover, based on (5*g*) again:

$$(a\cdot(b\cdot c))\cdot d \to a\cdot((b\cdot c)\cdot d).$$

We write the right-hand part in the form $a\cdot(b\cdot c\cdot d)$ and, applying the principle of syllogism, we obtain (9*c*).

It follows from (9*c*) that if the associative law is valid for n propositions, then it is also valid for $n+1$ propositions. On this basis we will write without parentheses the expression for the compatibility of any number of propositions.

(9*b*) can be generalized in the following way:

$$((k\cdot l\cdot m\cdot \ldots) \to n) \to \{(k\cdot l\cdot m\cdot \ldots)\cdot(p\cdot q\cdot r\cdot ...) \to (p\cdot q\cdot r\cdot ...)\cdot n\}. \tag{9d}$$

This is proved by substituting the corresponding expressions in (9*b*).

Let us also prove the expression:

$$(a \to b) \to \{(\neg a \to b) \to b\}, \tag{10}$$

called by *Hilbert* the *Princip des* "tertium non datur".[15]

Proof: Substitute in (9*b*) the expression $a \to b$ in place of a, the expression $\neg b \to \neg a$ in place of b and the expression $\neg a \to b$ in place of c. Then, from the expression:

$$(a \to b) \to (\neg b \to \neg a)$$

[275]
we obtain:

$$(a \to b)\cdot(\neg a \to b) \to (\neg b \to \neg a)\cdot(\neg a \to b);$$

further

$$(\neg b \to \neg a)\cdot(\neg a \to b) \to (\neg b \to b) \text{ (principle of syllogism)},$$

$$(\neg b \to b) \to b \qquad \text{on the basis of (2}d\text{).}$$

By double application of the principle of syllogism we get the expression

[15] *Math. Ann.*, Bd. 88, p. 153.

$$(a \to b)\cdot(\neg a \to b) \to b,$$

which can be written in the form (10) on the basis of the principle of extraction.

§ 4. We define the notions of "equivalence" and "logical sum"

$$(a \to b)\cdot(b \to a) = (a \rightleftarrows b) \qquad \text{Def. (4)},$$

$$\neg a|\neg b = a \vee b, \qquad \text{Def. (5)}.$$

We have pointed out that the expression "$a\cdot b \to a$" generally speaking is not correct; but it may also be correct in certain special cases. Namely, such a case occurs for proposition equivalence, as the following theorem shows:

$$(a \rightleftarrows b) \to (a \to b) \qquad (11a),$$

Proof: Let us substitute in (9b)

in the place of a – $(a \to b)$,
in the place of b – $(a \to b)\cdot(a \to b)$,
in the place of c – $(b \to a)\cdot a$.

Then from the expression

$$(a \to b) \to (a \to b)\cdot(a \to b),$$

which is true on the basis of Ax. (3), an application of Ax. (7) permits us to separate:

$$(a \rightleftarrows b)\cdot a \to (a \to b)\cdot(a \rightleftarrows b)\cdot a.$$

In (9a) we substitute in place of c the expression $(a \rightleftarrows b)$, to obtain:

$$(a \to b)\cdot(a \rightleftarrows b)\cdot a \to (a \rightleftarrows b)\cdot b.$$

Then substitute in (9a)

in place of a — b,
in place of b — a,
in place of c — $a \to b$,

and we obtain:

$$(a \rightleftarrows b)\cdot b \to (a \to b)\cdot a;$$

then according to (7b):

$$(a \to b)\cdot a \to b.$$

[276]

Applying the principle of syllogism three times, we get:

$$(a \rightleftarrows b)\cdot a \to b \qquad (11b),$$

Finally, applying the principle of exportation to the last expression, we obtain (11a).

Since the equivalence relation is symmetric, obviously also the following expression will be correct:

$$(a \rightleftarrows b) \rightleftarrows (b \to a).$$ [T17]

In view of (11a) and (11b), it is advantageous to connect the forward and inverse derivations into one equivalence formula. On the basis of Def. (4), some expressions we are already familiar with can take the following form:

$$a \rightleftarrows \neg\neg a,$$

$$(a \to b) \rightleftarrows (\neg b \to \neg a),$$

$$(a{\cdot}b){\cdot}c \rightleftarrows a{\cdot}(b{\cdot}c) \rightleftarrows (a{\cdot}c){\cdot}b,$$

$$a{\cdot}b \rightleftarrows b{\cdot}a,$$

$$\{a \to (b \to c)\} \rightleftarrows (a{\cdot}b \to c),$$

etc.

The notion of logical sum must be interpreted differently than in the classical theory. The expression $a \vee b$ tells us that at least one of these two propositions is true, but not every two propositions, of which one is true, can form a logical sum, for here too, a “connection by sense” similar to the relation of inference is necessary. The following formulas express the relation between the logical sum, on the one hand, and notions with which we are already familiar, on the other.

$$(\neg a \to b) \rightleftarrows (a \vee b), \qquad (12a)$$

$$(a \to b) \rightleftarrows (\neg a \vee b), \qquad (12b)$$

$$a|b \rightleftarrows (\neg a \vee \neg b), \qquad (12c)$$

$$a{\cdot}b \rightleftarrows \neg(\neg a \vee \neg b), \qquad (12d)$$

$$\neg a{\cdot}\neg b \rightleftarrows \neg(a \vee b). \qquad (12e)$$

The given expressions are easily derived on the basis of Def. (5) and expressions (1c), (1d), (3b), etc.

Let us derive some formulas which include the notion of logical sum.The formula:

$$a \vee b \rightleftarrows b \vee a \qquad (13a)$$

follows from (3i) and Def. (5).

$$(a \vee b)(a \to c) \to c \vee b \qquad (13b)$$ [T18]

Proof:

Using the syllogism principle:

$$(\neg b \to a){\cdot}(a \to c) \to (\neg b \to c)$$

This expression is converted to (13b), using Def. (5) and (13a).

[T17] In this expression, Orlov has confused the middle equivalence with implication and the following implication with equivalence. The expression intended by Orlov is perhaps $(a \rightleftarrows b) \to (b \rightleftarrows a)$, or $(a \rightleftarrows b) \rightleftarrows (b \rightleftarrows a)$.

[T18] In this formula the sign for compatibility is missing between $(a \vee b)$ and $(a \to c)$.

[277]

$$(b \vee c) \rightarrow \{(a \vee b) \rightarrow (a \vee c)\} \qquad (13c)$$

Proof:

$$(b \rightarrow c) \rightarrow (\neg c \rightarrow \neg b) \qquad \text{by Ax. (4)}$$

$$(\neg c \rightarrow \neg b) \rightarrow \{(\neg a \cdot \neg c) \rightarrow (\neg a \cdot \neg b)\} \qquad \text{by (9}b\text{)}$$

Using Ax. (4) and Def. (5):

$$\{(\neg a \cdot \neg c) \rightarrow (\neg a \cdot \neg b)\} \rightarrow \{(a \vee b) \rightarrow (a \rightarrow c)\}.$$

Applying the principle of the syllogism, we obtain (13c).

$$a \vee (b \vee c) \rightleftarrows (a \vee b) \vee c \qquad (14)$$

Proof: According to (12a):

$$a \vee (b \vee c) \rightleftarrows \{\neg a \rightarrow (b \vee c)\},$$

and also

$$\{\neg a \rightarrow (b \vee c)\} \rightarrow \{\neg a \rightarrow (\neg b \rightarrow c)\},$$

According to the importation principle:[T19]

$$\{\neg a \rightarrow (\neg b \rightarrow c) \rightarrow \{\neg a \cdot \neg b \rightarrow c\};$$

in addition:

$$(\neg a \cdot \neg b \rightarrow c) \rightarrow (\neg c \rightarrow a \vee b) \quad \text{using Ax. (4) and Def. (5),}$$

$$(\neg c \rightarrow a \vee b) \rightarrow (c \vee (a \vee b)) \qquad \text{by (12}a\text{),}$$

$$c \vee (a \vee b) \rightarrow (a \vee b) \vee c \qquad \text{by (13}a\text{).}$$

Applying the principle of the syllogism, we get:

$$a \vee (b \vee c) \rightarrow (a \vee b) \vee c.$$

Analogously, it is proved:

$$(a \vee b) \vee c \rightarrow a \vee (b \vee c).$$

As for the distributive law $a \cdot (b \vee c) \rightleftarrows a \cdot b \vee a \cdot c$, in our system it should be recognized as false.

§ 5. Let us prove the well-known "*praeclarum theorem*" of *Leibniz*:

$$(a \rightarrow b) \cdot (c \rightarrow d) \rightarrow (a \cdot c \rightarrow b \cdot d) \qquad (15a)$$

Proof: We substitute $c \cdot (c \rightarrow d)$ for c in (9a):

$$(a \rightarrow b) \cdot a \cdot c \cdot (c \rightarrow d) \rightarrow b \cdot c \cdot (c \rightarrow d).$$

Substitute again in (9a)

c for a,
d for b,
b for c,

[T19] There is a confusion of brackets in the following expression. It should be read: $\{\neg a \rightarrow (\neg b \rightarrow c)\} \rightarrow (\neg a \cdot \neg b \rightarrow c)$.

and we get:

$$(c \to d)\cdot c\cdot b \to b\cdot d.$$ [T20]

[278]

Applying the principle of syllogism and the commutative law, we obtain the expression:

$$(a \to b)\cdot(c \to d)\cdot a\cdot c \to b\cdot d, \tag{15b}$$

whence, on the basis of the principle of extraction, we obtain (15a).

Leibniz' theorem is needed to justify deduction. It has a very general character; all possible forms of deductive inference can be obtained as special cases of (15a) by simple transformations. Expression (15a) can be interpreted in the following manner: "if the premises (a and c) are compatible, then the conclusions (b and d) are compatible";[T21] or: "compatibility of the conclusions is a condition of compatibility of the premises".[16)]

In works on mathematical logic, *Leibniz'* theorem is usually deduced by means of the principle of simplification,[17)] but it does not depend on that principle at all. Since the symbolic theory does not depend on one interpretation or the other, and since we have not yet introduced any symbolic expression that cannot hold in the systems of *Russell* or *Peano*, the manner of derivation specified in the text can also be applied in the classical calculus of propositions.

(15a) can be expressed in the form:

$$(a \to b)\cdot(c \to d) \to \{(a \vee c) \to (b \vee d)\} \tag{15c}$$

Proof: On the basis of Ax. (4) and (9b) we have the following expressions:

$$(a \to b)\cdot(c \to d) \to (\neg b \to \neg a)\cdot(c \to d),$$

$$(\neg b \to \neg a)\cdot(c \to d) \to (\neg b \to a)\cdot(\neg d \to \neg c),$$

Further, from (15a):

$$(\neg b \to \neg a)\cdot(\neg d \to \neg c) \to (\neg b\cdot\neg d \to \neg a\cdot\neg c).$$

On the basis of Ax. (4) and Def. (5):

$$(\neg b\cdot\neg d \to \neg a\cdot\neg c) \to \{(a \vee c) \to (b \vee d)\}.$$

By applying the principle of syllogism several times, we obtain (15c).

From (15a), as a special case, the so-called "composition principle" follows:

$$(a \to b)\cdot(a \to c) \to (a \to b\cdot c) \tag{16a}$$

Proof:

[T20] If the mentioned substitution in (9a) is carried out, one obtains (deviating from Orlov) the formula $(c \to d)\cdot c\cdot b \to d\cdot b$. Taking into account commutativity for $d\cdot b$, one obtains the formula mentioned by Orlov.

[T21] It seems that Orlov wants to say that from the compatibility of the premises a and c of two inferences $(a \to b)$ and $(c \to d)$ follows the compatibility of the conclusions b and d of these two inferences.

[16)] One should keep in mind that $a\cdot c$, i. e. "the compatibility of propositions a and c" is the third independent premise in expression (15a), and that the expression $a|c$ is compatible with the expression $(a \to b)\cdot(c \to d)\cdot b\cdot d$.

[17)] See e. g. Whitehead and Russell, *Princ. Math.*, p. 115–116.

$$(a \to b) \cdot (a \to c) \to (a \cdot a \to b \cdot c) \qquad \text{by (15}a\text{)},$$

Using the principle of syllogism:

$$(a \to a \cdot a) \cdot (a \cdot a \to b \cdot c) \to (a \to b \cdot c)$$

[279]

Omitting the expression $a \to a \cdot a$, which is Ax. (3) and applying the syllogism principle, we obtain (16*a*).

$$a \to \{(a \to b) \to (a \cdot b)\} \tag{16b}$$

Proof: Based on (16*a*), we write:

$$(a \to a) \cdot (a \to b) \to (a \to a \cdot b);$$

we omit the expression $(a \to a)$ and get (16*b*) on the basis of (Ax. 5).

$$\{(a \to d) \cdot (b \to e) \cdot (c \to f)\} \to \{a \cdot b \cdot c \to d \cdot e \cdot f\} \tag{17a}$$

Proof: In (15*a*) we substitute:

in place of a the expression $a \cdot b \cdot (a \to d) \cdot (b \to e)$,
in place of b the expression $d \cdot e$,
in place of c the expression c,
in place of d the expression f,

Then the left-hand side of the obtained expression will have the form:

$$\{a \cdot b \cdot (a \to d) \cdot (b \to e) \to d \cdot e\} \cdot (c \to f)$$

and the right-hand side:

$$a \cdot b \cdot c \cdot (a \to d) \cdot (b \to e) \to d \cdot e \cdot f.$$

Since in the left part the expression in curly brackets can be omitted, by (15*b*), the whole expression can be written in the form:

$$(c \to f) \to \{a \cdot b \cdot c \cdot (a \to d) \cdot (b \to e) \to (d \cdot e \cdot f)\},$$

from where, applying the principles of insertion and extraction successively, we obtain (17*a*).

From (17*a*) follows that if *Leibniz*' theorem is correct for n premises, then it is also correct for $n+1$. On the basis of (17*a*) we also get the following generalized formula:

$$\left.\begin{array}{l} \{(a \cdot b \cdot c \cdot \ldots \to f) \cdot (k \cdot l \cdot m \cdot \ldots \to p) \cdot (p \cdot q \cdot r \cdot \ldots \to t) \ldots\} \to \\ \quad \to \{(a \cdot b \cdot c \cdot \ldots \cdot k \cdot l \cdot m \cdot \ldots \cdot p \cdot q \cdot r \cdot \ldots) \to (f \cdot p \cdot t \cdot \ldots)\} \end{array}\right\} \tag{17b}$$

By combining the expressions (9*d*) and (17*b*) we obtain the following law: "if a system of n compatible premises $a \cdot b \cdot c \cdot \ldots \cdot k \cdot l \cdot m \cdot \ldots$ is given, then the conclusion arising from any combination of premises is compatible with all other premises and with the conclusions arising from any other combination of premises".

§ 6. With the derivation of *Leibniz*' theorem, the problem of the general justification of deduction without applying the principle of simplification can be considered to be solved. Let us now turn to the consideration of special functions.

Up to now, we have abstracted from the question of what the propositions a, b, c,... represent, which we substitute in our formulas, and we have abstracted from the question of their reliability and provability. However, true propositions **[280]** in mathematics matter insofar as they can be proved. For symbolic logic, therefore, it is necessary to consider not only propositions a, b, c, ... but also propositions of their provability or unprovability. If we consider a as a proposition which is true or false, irrespective of the question of provability, then the propositions "a is provable" or "a is nonprovable" will be functions of proposition a. Let us now denote "a is provable" by $\Phi(a)$ and "the falsity of a is provable" by $X(a)$. The expressions $\neg\Phi(a)$ and $\neg X(a)$ will denote, respectively: "a is unprovable" and "the falsity of a is unprovable".

The proposition a is provable if there are no fundamental difficulties for solving the question, even if the solution would be practically impossible due to the duration and tediousness of the calculations. For example, a question about some property of a large finite number should be considered solvable if all difficulty lies in the cumbersomeness of calculations. We point out that $\Phi(a)$ also covers cases where a has already been proved or is obvious.[18)] If, on the other hand, the difficulties in solving the question are of a fundamental character, insurmountable in the current state of science, then the proposition is unprovable, and we have the expression $\neg\Phi(a)$.

The law of "tertium non datur" can be applied to expressions $\Phi(a)$ and $\neg\Phi(a)$ as well as to $X(a)$ and $\neg X(a)$, but to expressions $\Phi(a)$ and $X(a)$ the said law obviously cannot be directly applied. It also follows from the given explanations that $\Phi(a)$ and $X(a)$ form a complete disjunction in the finite domain, but do not form a disjunction in the transfinite domain and may be false together.

We assert that such functions, though without special symbolic notations, are in fact introduced into mathematics by intuitionists. The expression "any property for any system is either true or impossible" is evaluated differently in purely formal mathematics and in the works of intuitionists precisely because it is not understood in the same sense. If in the first case it is accepted that every property is definite "in itself," in the second case the question of the provability of the property arises. In the first case the question about the existence of a property is treated in the sense of the expression "a", whereas in the second case it is treated in the sense of "$\Phi(a)$". Equally, the expression "absurdity of a proposition" corresponds to the function $X(a)$. *Brouwer* considers the case where a property entails its negation only as a special case of "absurdity," understanding the latter in a broader sense.[19)] He uses the expression "absurd" in all those cases where it can be proved that a contradicts some axiom or a proved

[18)] Thus, we can write $\Phi(a)$ whenever the sentence a is acceptable. But $\Phi(a)$ is a more encompassing notion than acceptability, because $\Phi(a)$ can also refer to truths that are available only in principle, but in fact remain unknown. The negation of $\Phi(a)$, i. e. $\neg\Phi(a)$, covers both cases where a is true, but unprovable, as well as those cases where a is false.

[19)] L. E. J. Brouwer, „Intuitionistische Zerlegung mathematischer Grundbegriffe“, *Jahresbericht d. deutsch. Math.-Ver.*, 33 Bd.

proposition. But our expression $X(a)$ has the same sense. From this *Brouwer*'s statement, that the correctness of the property and its absurdity form a complete disjunction only in a certain finite system, becomes understandable. Although the properties of such a system may remain **[281]** partially unknown, it requires only a sufficient amount of time to fully elucidate them.[20)]

Further, one can form functions from functions, such as $X(\Phi(a))$ or $X(X(a))$. Let us point out how such expressions should be interpreted.

While the expressions $\Phi(a)$ and $\Phi(\Phi(a))$ ("a is provable" and "It is provable, that a is provable") have the same meaning,[21)] the expressions $\neg\Phi(a)$ and $\Phi(\neg\Phi(a))$ differ significantly with respect to sense.

$\neg\Phi(a)$ expresses only the fact that some mathematical problem is undecidable, whereas $\Phi(\neg\Phi(a))$ means that this fact can be introduced into mathematics as provable. The unprovability of a proposition needs a proof like any other non-obvious truth. The proof of the undecidability of a mathematical problem is, in turn, a mathematical problem, the solution of which can be of great importance.

The expression $X(X(a))$ can be interpreted in the sense of the notion used by *Brouwer* "the absurdity of the absurdity of a". It means not only that the falsity of a is unprovable, but that this unprovability can be proved.[22)]

From the formal side, the matter reduces to introducing into the logical calculus the notion $\Phi(a)$ which we regard as formally undefinable, as well as a group of three "axioms of provability":

$$\Phi(a) \to a \qquad \text{Ax. (8)}$$

"a is provable, consequently a is true"[23)]

$$\Phi(a) \to \Phi(\Phi(a)) \qquad \text{Ax. (9)}$$

"a is provable, consequently, it is provable that a is provable"[24)]

$$\Phi(a \to b) \to \{\Phi(a) \to \Phi(b)\} \qquad \text{Ax. (10)}$$

"If it is provable that from a follows b, and if the proposition a is also provable, then b is also provable."

The expression $X(a)$ can be defined the following manner:

$$\Phi(\neg a) = X(a) \qquad \text{Def. (6)}$$

On the basis of Def. (6) the expression $\Phi(\neg\Phi(a))$ will be written as $X(\Phi(a))$, the expression $\Phi(\neg X(a))$ as $XX(a)$, etc.

[282]

20) „Ueber die Bedeutung des Satzes vom ausgeschl. Dritten" *Journ. f. die reine u. angew. Math.*, Bd. 154.

21) In fact, the proof of the truth of the predicate and the proof of the provability of the predicate are one and the same.

22) That is, the assumption that "the falsity of a can be proved" leads to a contradiction with some obvious or provable sentence.

23) Or differently: "the provability of a supposes the truth of this proposition". The converse proposition $a \to \Phi(a)$ has no place in our system. We thus accept the existence of truths that are unprovable under the current state of science. Such an assumption obviously has nothing to do with "ignorabimus".

24) From "a is obvious" follows "it is obvious that a is obvious".

Since we consider all axioms and the propositions derived from them to be assertible and consequently trustworthy, every expression admitted as an axiom, or derived in the preceding paragraphs, can be written as a function $\Phi(a)$.

For simplicity, let us stipulate that expressions $\Phi(a)$ and $X(a)$ should henceforth be written without brackets: Φa and Xa. Thus, instead of the expression $\Phi(X(a))$ we will write ΦXa, etc. An expression following Φ or X to the end of a formula or to the sign of some operation, we will regard as the argument of given function.

Combining Def. (6) and Ax. (8), we obtain:

$$Xa \to \neg a, \tag{18a}$$

wherefrom also:

$$a \to \neg Xa. \tag{18b}$$

On the basis of Ax. (8) and (18b):

$$\Phi a \to \neg Xa, \tag{18c}$$

whence, on the basis of Ax. (4):

$$Xa \to \neg \Phi a. \tag{18d}$$

From Ax. (8) and (4) we obtain:

$$\neg a \to \neg \Phi a. \tag{18e}$$

$$\Phi a \rightleftarrows \Phi\neg\neg a \tag{18f}$$

Proof: The proposition $a \rightleftarrows \neg\neg a$ can be written in the form $\Phi(a \rightleftarrows \neg\neg a)$; then on the basis of Ax. (10):

$$\Phi(a \rightleftarrows \neg\neg a) \to \{\Phi a \rightleftarrows \Phi\neg\neg a\},$$

from which the left part can be omitted on the basis of the Ax. (7).

Since, according to Def. (6), $\Phi\neg\neg a = X\neg a$, so from (18f) follows also:

$$\Phi a \rightleftarrows X\neg a. \tag{18g}$$

Substituting in Ax. (8) Φa in place of a and combining it with Ax. (9), we obtain:

$$\Phi a \rightleftarrows \Phi\Phi a. \tag{19a}$$

In place of a we substitute $\neg a$ and, taking into account Def. (6), we obtain:

$$Xa \rightleftarrows \Phi Xa. \tag{19b}$$

We substitute Xa for a in (18b):

$$Xa \to \neg XXa. \tag{19c}$$

We derive the proposition:

$$\Phi(a \to b) \to \{Xb \to Xa\} \tag{20}$$

Proof:

We substitute in Ax. (10) $a \to b$ in place of a and $\neg b \to \neg a$ in place of b.

$\Phi\{(a \to b) \to (\neg b \to \neg a)\} \to \{\Phi(a \to b) \to \Phi(\neg b \to \neg a)\}$,

[283]

from which the left part on the basis of the Ax. (4) and (7) can be omitted.

On the basis of Ax. (10) and Def. (6):

$$\Phi(\neg b \to \neg a) \to \{Xb \to Xa\}.$$

Applying the principle of the syllogism, we obtain proposition (20).

From (20) follows a whole series of formulas that we need:

$$Xa \to X\Phi a \qquad (21a)$$

follows from Ax. (8) on the basis of (20);

$$XX\Phi a \to XXa \qquad (21b)$$

follows from (21a) on the basis of (20);

$$\Phi Xa \to X\Phi a. \qquad (21c)$$

follows from (19b) and (21a).

The converse inferences from the right-hand to the left-hand sides are impossible because the right-hand parts of expressions (21a), (21b), (21c) are weaker assertions than the expressions on the left side.

$$\Phi a \to XXa \qquad (21d)$$

"From the provability of a follows the absurdity of the absurdity of a".

Proof: On the basis of (21a) we write:

$$X\neg a \to X\Phi\neg a.$$

Using (18g) and Def. (6), the last expression can be written as (21d).

$$\Phi a \to XX\Phi a. \qquad (21e)$$

Proof: In (21d) we substitute Φa for a; taking into account Ax. (9) and the syllogism principle we obtain (21e).

(21d) and (21e) can be interpreted as follows: "the proof of proposition a is at the same time a proof that the falsity of a is unprovable and that a proof of its falsity is impossible". We note that the expression "$a \to XXa$" cannot be derived from the axioms that we have adopted.[25)]

$$X\Phi\Phi a \rightleftarrows X\Phi a \qquad (21f)$$

Proof: In (20) we substitute Φa in place of a and $\Phi\Phi a$ in place of b. Then (21f) follows from (19a).

Suppose we have an infinite series S and we ask ourselves whether in this series there exists a member with property E. Such a member either exists or it does not exist; the law of "tertium non datur" is true; but as long as the truth remains hidden, it cannot serve to define concepts. When we turn to the question of provability, the law of "tertium non datur" cannot be directly applied, and we already have not two but three possibilities.

[284]

1) If we have found a member with property E, then we have Φa.

25) This sentence is false, because even if a is true, and we have no corresponding proof, we cannot always prove that the falsity of a cannot be proved. Let, for example, a be the existence of Martians; a or $\neg a$ is true, but we do not have either XXa or $XX\neg a$.

2) If, from the law defining the properties of series S it follows that a member with property E is impossible, then we have Xa.

3) It is possible that both are false, i. e., that we have $\neg\Phi a$ and $\neg Xa$. In this case it is possible to find a third completely definite solution, namely XXa, since obviously it in this case it is impossible to prove by a finite number of operations the non-existence of a member with property E.

Expression (18d) can be written as "$\Phi a|Xa$" and (19c) as "$Xa|XXa$". Taking into account also (21d), we obtain a three-membered scheme coinciding with that of *Brouwer*.[26])

$$\Phi a \to XXa \longleftrightarrow Xa \qquad (22)$$

The above scheme thus in no way contradicts with the law of "tertium non datur".

§ 7. In *Brouwer*'s works, the usage of the predicates "absurdity" and "absurdity of absurdity" is of great importance for the formation of concepts. *Brouwer* often resorts to the following method of defining concepts: "let us call a real number g rational if two integers p and q can be defined (bestimmt werden können) such that $g{=}p{:}q$; and let us call g irrational when the assumption that g is rational leads to absurdity."[27])

Here it is obvious that a rational number is defined by means of the provability of the existence of two integers, in other words, by means of a function of the form Φa. If the assumption that the existence of the numbers p and q is provable leads to absurdity, then the number g is irrational. Hence, irrationality is defined using $X\Phi a$. If we apply the absurdity predicate to the latter expression, we obtain the expression $XX\Phi a$, from which XXa follows on the basis of (21b). In this case the number g will be neither rational, in the previously defined sense, nor irrational; it will be rational in another sense. *Brouwer* uses the predicate of "absurdity of absurdity" to define another type of rationality of a number, different from the former one. The question is, if we lengthen the series of predicates of absurdity, won't we get new concepts all the time. *Brouwer* points out that if we have a series of predicates "absurdity absurdity . . . absurdity," then striking out two neighboring absurdity predicates is possible not only in classical but also in intuitionist theory; in the latter case with the restriction that the last absurdity predicate in the series cannot be deleted.[28]) In our system, this conclusion can also be obtained; however, the condition is further complicated by the fact that the last predicate of absurdity in the series does not belong directly to the proposition a, but to Φa. In this case, the exclusion of the last predicate of absurdity from the rule of deleting two adjacent predicates does not contradict the law of "tertium non datur".

Brouwer proves the theorem "the absurdity of absurdity of absurdity is equivalent to absurdity"[29]).

---

[26]) By the sign "$\longleftrightarrow$" Brouwer expresses a contradiction between the sentences.

[27]) L. E. J. Brouwer, „Intuitionistische Zerlegung mathematischer Grundbegriffe", *Jahr. Ber. d. deutsch. Math. Ver.*, Bd. 33.

[28]) Ibid.

[29]) Ibid.

[285]

The indicated theorem in the system of *Brouwer* plays the role, as if it were a law of "quartum non datur", limiting the process of formation of concepts .[30)]

We can also derive a analogous theorem, provided that the last predicate of absurdity refers to Φa

$$XXX\Phi a \rightleftarrows X\Phi a. \tag{23}$$

Proof:

A) From (21d) on the basis of (20) we have:

$$XXXa \rightarrow X\Phi a.$$

Replacing a in the last expression with Φa, we obtain:

$$XXX\Phi a \rightarrow X\Phi\Phi a,$$

whence by (21f) and the principle of syllogism, we obtain the first half of the expression to be proved:

$$XXX\Phi a \rightarrow X\Phi a.$$

B) Replace a in (21d) by Xa:

$$\Phi Xa \rightarrow XXXa.$$

In connection with (19b), and applying the syllogism principle, we obtain:

$$Xa \rightarrow XXXa.$$

We replace a with Φa:

$$X\Phi a \rightarrow XXX\Phi a$$

The theorem is proved.

The above theorem cannot be proved in the form:

$$XXXa \rightleftarrows Xa,$$

since the premise component

$$XXXa \rightarrow Xa$$

is false.[31)]

Thus, the "calculus of the compatibility of propositions" allows us to perform operations not only directly on propositions a, b, c ..., but also on their functions Φa, Xa, etc. The introduction of such functions opens a special way to the foundation of the law of "Tertium non datur," which consists in reconciling the tenets of intuitionism with that law.

[30)] "Quartum non datur" should not be understood in the sense of a complete alternative of three members, but in the sense of excluding superfluous notions from the scheme of the kind (22). The alternative is impossible since all three members of expression (22) can be false at the same time.

[31)] Taking (18a) into consideration, $\neg a$ must follow from $XXXa$. However, let us assume the following: both proposition a and proposition $\neg a$ are unprovable, but in order to be convinced of this it would be necessary to run through the whole infinite series of provable propositions, which is impossible. In this case, the expression $XXXa$ will be true in the case where a is false, but also in the case where a is true. We would have such a case, for example, if by a we mean the existence of Martians.

[286]

The introduction of the above functions in classical mathematical logic is impossible, since the interpretation of the notion of "follow" as a material inference deprives of sense all the expressions proved for the functions we have introduced. In addition, in classical theory, propositions are accepted and derived which, from our point of view, cannot be evaluated otherwise than as false. For example, from the proposition proved in classical theory, "all true propositions are equivalent," the following consequence emerges:

$$a \rightleftarrows \Phi a \rightleftarrows X\Phi a \rightleftarrows XX\Phi a.$$ [T22]

Such a consequence renders the introduction of the above kind of functions devoid of any meaning; in this case, in order to construct schemes of transfinite conclusions there would be no other way but to deny the law "tertium non datur".

Sur la théorie de la compatibilité des propositions.

Par **M. J. Orloff** (Moscou).

(*Résumé.*)

1) Dans la théorie symbolistique de compatibilité des propositions la notion du produit logique doit être interprêtée dans le sens d'une hypothèse de compatibilité des propositions — et non dans celui de coïncidence de leur affirmation. „Implication" doit être interprêtée dans le sens de la liaison intérieure des propositions. Les interprétations des autres notions changent de ce fait.

2) Le „principe de simplification" et toutes autres propositions similaires doivent être considerés comme faux. Cela mène à la découverte d'un fait remarquable que les théorèmes fondamentaux de la déduction, déduits jusqu'ici au moyen du principe susindiqué, n'en dépendent nullement.

3) La théorie de compatibilité des propositions admet la déduction des principes de „l'intuitionismus" sans négation du principe „tertium non datur".

(Rec. Math. XXXV: 3–4; 1928)

[T22] There is a misprint or error in this formula. Obviously, a negation sign is missing in the third term. This is discussed in more detail in section 3.2 of the in the introductory guide "Orlov Ninety-Six Years On".