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The Metaphysics of Routley Star*

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Abstract

This paper investigates two forms of the Routley star operation, one in Routley & Routley 1972 and the other in Leitgeb 2019. We use object theory (OT) to define both forms and show that in OT's hyperintensional logic, (a) the two forms aren't equivalent, but (b) become equivalent under certain conditions. We verify our definitions by showing that the principles governing both forms become derivable and need not be stipulated. Since no mathematics is assumed in OT, the existence of the Routley star image s^* of a situation s is therefore guaranteed not by set theory but by a theory of abstract objects. The work in the paper integrates Routley star into a more general theory of (partial) situations that has previously been used to develop the theory of possible worlds and impossible worlds.

1 Introduction

The Routley 'star' operation was introduced in Routley & Routley 1972. Their study of the semantics of entailment assumed the existence of situations ('set-ups') that are neither consistent nor maximal (*ibid.*, 335–339).[1] When the Routleys set up the star operator on situations, they used 'H' to range over set-ups (i.e, "a class of propositions or wff") and used 'A' to range over propositions or wffs (*ibid.*, 337). Then they considered the following condition (*ibid.*, 338) on the star (*) operation, which they labeled as (iv):

(iv) $\sim A$ is in H iff A is not in H^*

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[1] Some logicians use the term 'non-normal worlds' to describe situations that are neither maximal (complete) nor consistent. The Routleys, however, used the term 'world' for consistent and maximal situations (1972, 339). In what follows, we reserve the term 'world' for maximal situations, some of which are possible worlds and some of which are impossible worlds.

They subsequently stipulated that a set-up is $\sim$-*normal* if it satisfies (iv) for every A and $H = H^{**}$ (*ibid*., 338).

That was then. Although the Routley star has been studied and applied in a number of subsequent works, it was recently used in Leitgeb 2019 (321ff) to build a semantics for a system of hyperintensional logic ('HYPE'). Leitgeb first builds a propositional language $\mathcal{L}$ that includes propositional letters, with some standard logical connectives, but with a non-standard conditional. Leitgeb then constructs HYPE-models for $\mathcal{L}$ in terms of structures whose elements include a non-empty set of states S and a valuation function V from S to the power set of the set of literals of the language $\mathcal{L}$, so that each state s in S is associated with a set of literals $V(s)$. I'll describe HYPE models in fuller detail below, but for the purposes of this introduction, it is important to note that the various elements of HYPE models are simultaneously constrained by the requirements of a Routley star operation having the following properties, among others (Leitgeb 2019, 322):

- $V(s^*) = \{\overline{v} \,|\, v \notin V(s)\}$
- $s^{**} = s$

Leitgeb then discusses the properties of the star operation and uses HYPE models to define various truth conditions for hyperintensional operators.

These two bookend cases, Routley & Routley 1972 and Leitgeb 2019, demonstrate how the Routley star operation has been deployed to help us understand various non-classical, and more fine-grained, semantic phenomena. But despite their similarities, a study of the two definitions in a hyperintensional background logic (a) shows that they aren't equivalent and (b) reveals the assumption under which they become so.

Moreover, a metaphysician looking at the body of literature inclusive between these papers would find that relatively little attention has been paid to the question: What kind of metaphysics is represented by a semantics making use of Routley star, and how are we to understand the Routley star operation given that metaphysics? Questions about the meaning of the Routley star operation were raised early on, in Copeland 1979 and van Benthem 1979. Restall 1999 (54) raised this question when he wrote:

> The operator * was introduced to relevant logic by Routley and Routley [23]. If $x \neq x^*$, then certainly we can get both $A \wedge {\sim}A \rightarrow B$ and $A \rightarrow B \vee {\sim}B$ to fail, but there is a price. The price is the obligation to explain the meaning of the operator *.

But even though we may now be more comfortable with Routley star and recognize how interesting and efficacious it is (given the work that has been done), there is still an open question about what, exactly, is the proper metaphysical framework for defining and studying the Routley star operation?

In our two case studies, and for most studies in between, one typically finds the Routley star introduced into semantic models constructed with the help of set theory, domains of primitive entities (set-ups, situations, states, possible or impossible worlds), and functions defined on those domains, etc. Most authors don't spend time considering the metaphysics of the entities used in their semantic models, and quite rightly, given their goals. For their purposes, it is sufficient to adopt another attitude expressed in Restall 1999 (57):

> It would be interesting to chart the connections between states as we have sketched them and other entities like ... objects, states of affairs, propositions, and many other things besides. However, this is neither the time nor the place for that kind of metaphysics. Suffice it to say that a coherent comprehensive view of states ought to tell us how these things fit together. For now, we will use *states* as the points in our frames for relevant logics.

For example, Leitgeb writes (2019, 323, footnote 9):

> I want to leave open in this paper whether states are interpreted (i) in a metaphysically robust manner, or (ii) in a looser informational manner. In the first case, states would be "chunks of reality" that are "located in the world", while in the second case they might be some kind of abstract entities corresponding to "pieces of thought".

Mares (2004, 4.4–4.11) does attempt to develop an intuitive understanding of the assumptions concerning properties, states of affairs, situations, propositions, etc., that are used in the semantic models. But (a) the focus of Mares 2004 is to interpret the ternary relation R used in Routley-Meyer semantics for relevant logic (Routley & Meyer 1972, 1973), and (b) Mares assumes that some background theory of situations such as Barwise and Perry 1983 is available, for he takes a number of principles about situations as given.

By contrast, in what follows, we plan to develop the metaphysics of Routley star without any mathematics, set theory, primitive domains of situations, states, or worlds (possible or impossible), or functions on domains. We won't identify propositions as sets of possible worlds, as functions from possible worlds to truth values, as sets of situations, or as classes of wffs. Nor will we assume any axioms governing primitive set-ups, situations, possible worlds, or impossible worlds. Instead, we shall *define* the Routley star operator metaphysically in *object theory* (= OT), where situations are defined and their first principles derived. And we employ a theory of propositions (= 0-ary relations) that is part of a larger, hyperintensional theory of n-ary relations – one on which necessarily equivalent relations and propositions aren't identified. Basic OT allows us to define a unique Routley star image s^* for each situation s, as in Routley & Routley 1972. Moreover, with a minimal, additional assumption (or axiom, if you prefer), s^* can be defined as in Leitgeb 2019. Our goals, then are to show that, in such a setting, (a) the metaphysical entities needed to formulate and understand the Routley star image can be defined and proved to exist, (b) the principles governing Routley star, as formulated in both Routley & Routley 1972 and Leitgeb 2019, can be *derived* rather than stipulated, and (c) a reconciliation between the two definitions of Routley star can be simply and precisely articulated.

The present effort may be distinguished from other recent discussions of Routley star by its methodology and the focus of the investigation. Relatively recent papers such as Restall 2000, Berto 2015, and Berto & Restall 2019 (and going back to Došen 1986 and Dunn 1993) are about the semantic analysis of various forms of negation and, as such, the action takes place in the semantics. In each of these papers, a frame semantics involving a primitive relation of *compatibility* (or *incompatibility*) on *points* or primitive *worlds* is introduced and used to interpret an uninterpreted language with a negation symbol.[2] By contrast, it is *not* a goal of the present paper to study or define non-classical negation semantically. The reader will find no semantics in what follows. Rather, situations, possible worlds, impossible worlds, and the Routley star are all defined. The definitions are cast in a logic and metaphysics that is systematized proof-theoretically. Moreover, notions very much like the notions of incompatibility and compatibility utilized in the papers by Restall and Berto will also be *defined*, and the key principles that govern them will be derived rather than stipulated.[3]

Another distinguishing feature of the present work is its stated goal of reconciling Routley star as developed Routley & Routley 1972 and Leitgeb 2019. OT's hyperintensional logic makes it clear that the two definitions of Routley star are not equivalent. We'll examine a principle under which the two definitions become equivalent.[4] So by investigating the metaphysics of Routley star in the manner below, the present effort may help us better understand the domain of application for Routley star and thereby better prepare us for understanding the uses to which it has been put in the semantics of non-classical negation, both in the two papers that serve as the focus of our study and in other papers on the semantics of negation. In particular, relevant logicians may find it of interest that if one adopts a hyperintensional logic and metaphysics, in which propositions are not identified with their double negations,

[2] In Restall 2000, semantic frames and a primitive relation of compatibility on points are introduced on the first page. In Berto 2015, frames are introduced (766ff), and negation is analyzed as a modality (767) that is interpreted by a distinguished accessibility relation on worlds, R_N, understood as a compatibility relation (768ff). In Berto & Restall 2019, the semantic analysis occurs in Section 3, where frames and the primitive compatibility relation on worlds are introduced (1127).

[3] See Section 4.3 for the definition of *incompatibility* and see footnote 19 for (a) a way to define the *compatibility* relation that Restall and Berto take as primitive, and (b) a derivation of the principle they stipulate to characterize that relation (Restall 2000, 853, Definition 1.1; Berto 2015, 768, 'Backward'; and Berto & Restall 2019, 1129, 'Backwards').

A second principle, the Heredity Principle (Restall 2000, Definition 1.2; Berto 2015, 767; and Berto & Restall 2019, 1128), was previously derived in OT as the *Persistence* principle (Zalta 1993, 413, Theorem 8); this settled a choice point in Barwise 1989 (265) in favor of Alternative 6.1.

Finally, see footnote 16 below for a discussion of how the reflexivity, anti-symmetry, and transitivity principles governing the relation $\sqsubseteq$ on the points of compatibility frames, stipulated in Restall 2000 (853, Definition 1.1), was previously derived in OT (Zalta 1993, 413, Theorem 7) in terms of the condition $s \trianglelefteq s'$ on object-theoretic situations.

[4] Papers published subsequent to Leitgeb 2019 have had other goals. See Odintsov & Wansing 2020 for a comparison of the hyperintensional propositional logic in HYPE with a number of other logics, and Punčochář & Sedlár 2022 for a discussion of the Routley star operation in information-based semantics rather than truth-conditional semantics.

then it matters how one defines the Routley star situation.[5]

1.1 The Background Theory

The background theory needed to achieve these goals has been motivated and published elsewhere (see below) and we shall draw on those published results. In what follows, the reader should be familiar with the fact that OT is expressed in a syntactically second-order, quantified modal language (without identity) that includes two kinds of atomic formulas: standard *exemplification* formulas of the form $F^n x_1 \dots x_n$ and *encoding* formulas of the form xF. This language is extended with complex individual terms, namely (rigid) definite descriptions of the form $\imath x\varphi$, and with complex n-ary relation terms of the form $[\lambda x_1 \dots x_n\, \varphi]$ ($n \geq 0$). A primitive unary predicate $E!$ (*being concrete*) is used to distinguish *ordinary* objects ($O!x \equiv_{df} \Diamond E!x$) and *abstract* objects ($A!x \equiv_{df} \neg\Diamond E!x$). Identity for objects is defined: $x = y$ holds if and only if either x and y are both ordinary objects that necessarily exemplify the same properties or both abstract objects that necessarily encode the same properties.

The underlying logic of OT includes: (a) classical propositional logic, (b) classical predicate logic for the constants and variables but negative free logic for the complex terms (i.e., descriptions and λ-expressions may fail to denote), and (c) full S5 modal logic, including the Barcan (1946) and converse Barcan formulas for both the first and second-order variables (i.e., there are fixed domains of objects and relations).

OT's underlying logic also includes a hyperintensional theory of n-ary relations that encompasses properties (i.e., unary relations) and propositions (i.e., 0-ary relations). A λ-expression of the form $[\lambda x_1 \dots x_n\, \varphi]$ is guaranteed to denote a relation if none of the variables bound by the λ occur in 'encoding position', i.e., as an argument term in an encoding formula in φ. The well-known principle of λ-Conversion (aka β-Conversion) holds for such λ-expressions, as does α-Conversion.[6] λ-Converion implies a comprehension principle for n-ary relations ($n \geq 0$), namely:

> $\exists F^n \Box \forall x_1 \dots \forall x_n (F^n x_1 \dots x_n \equiv \varphi)$, provided F^n isn't free in φ and none of the x_i occur in encoding position in φ.

These existence conditions for relations are supplemented with a definition of hyperintensional identity conditions for n-ary relations that do *not* imply that necessarily equivalent properties, relations, or propositions are identical.[7]

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[5] I'm indebted to an anonymous referee of this journal for suggesting that I make this point explicit.

[6] The principle of α-Conversion simply states that an identity holds between two alphabetically variant λ-expressions (assuming at least one of them denotes). Note also that for *elementary* λ-expressions of the form $[\lambda x_1 \dots x_n F x_1 \dots x_n]$, η-Conversion holds. It asserts $[\lambda x_1 \dots x_n F x_1 \dots x_n] = F$, where F is an n-ary relation variable.

[7] The key definition is for the identity of properties (Zalta 1993, 407):

$$F = G \equiv_{df} \Box \forall x (xF \equiv xG)$$

In OT, $\Box \forall x (Fx \equiv Gx)$ doesn't imply $F = G$. If we substitute 'p' and 'q' for the propositional variables

Finally, there are three axioms in OT that govern the logic of encoding. One asserts that ordinary objects necessarily fail to encode properties ($O!x \rightarrow \Box\neg\exists FxF$). A second asserts that the modal logic of encoding is rigid ($xF \rightarrow \Box xF$). And the main axiom is a comprehension schema for abstract objects, which asserts that for any condition φ with no free xs, there is an abstract object that encodes all and only the properties such that φ:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \varphi)), \text{ provided } x \text{ isn't free in } \varphi \tag{1}$$

Further details of the system will be brought to bear as the occasion arises.

In Zalta 1993 and 1997, OT was deployed to develop the theory of situations, possible worlds, and impossible worlds. The theory begins with the definitions:

- a *situation* is any abstract object that encodes only properties of the form *being such that p* (i.e., properties of the form $[\lambda x\, p]$, where x is vacuously bound by the λ, and p is a variable ranging over propositions):

$$Situation(x) \equiv_{df} A!x \,\&\, \forall F(xF \rightarrow \exists p(F = [\lambda x\, p])) \tag{2}$$

- *p is true in situation s* ('$s \models p$'), or *s makes p true*, is defined as s encodes the propositional property *being such that p*:

$$s \models p \equiv_{df} s[\lambda x\, p] \tag{3}$$

In OT, '$\models$' always takes the *smallest* scope; so $s \models p \rightarrow p$ is to be parsed $(s \models p) \rightarrow p$; otherwise, we write $s \models (p \rightarrow p)$. Also, we sometimes read $s \models p$ as s encodes p, thereby extending the notion of encoding.

In Zalta 1993 (410–414), it was shown that the basic principles of situation theory are derivable from the definition of *situation* given above. Indeed, 15 of the 19 principles outlined in Barwise 1989 were derived. Possible world theory was then shown to be an extension of situation theory and was developed via the following definitions:

- a *possible world* is any situation s that might be such that all and only true propositions are true in s:

$$PossibleWorld(s) \equiv_{df} \Diamond\forall p(s \models p \equiv p) \tag{4}$$

  Given our convention, the subformula $s \models p \equiv p$ is to be parsed as $(s \models p) \equiv p$.

The basic principles of possible world theory are derivable from the definition of *possible world* given above (Zalta 1993, 414–419). These include formal versions of the following principles:

'F^0' and 'G^0', then propositions p and q are defined to be identical just in case the propositional properties $[\lambda x\, p]$ and $[\lambda x\, q]$ are identical (Zalta 1993, 409). And n-ary relations F and G ($n \geq 2$) are identical just in case each way of 'plugging' $n-1$ objects into both F and G yields identical properties. Note that in the context of the present paper, in which a negative free logic applies to complex relation terms that may fail to denote, we can only instantiate the above definition of property identity to denoting property terms.

- every possible world is maximal, consistent, and modally closed;
- there is a unique actual world;
- possibly p iff there is a possible world in which p is true; and
- necessarily p iff p is true in every possible world.

This was extended further in Zalta 1997 to include impossible world theory:

- an *impossible world* is any maximal situation (i.e., for every proposition p, either s makes p true or s makes $\neg p$ true) for which it is not possible that every proposition true in s is true:

$$Maximal(s) \equiv_{df} \forall p(s \models p \vee s \models \neg p) \quad (5)$$

$$ImpossibleWorld(s) \equiv_{df} Maximal(s) \,\&\, \neg\Diamond\forall p(s \models p \rightarrow p) \quad (6)$$

The basic principles of impossible world theory can be derived from the definition of *impossible world* given above (Zalta 1997, 646–649). These include formal versions of:

- there are impossible worlds;
- if it is not possible that p, then there exists a non-trivial impossible world in which p is true;[8]
- there exist impossible worlds where the principle *ex contradictione quodlibet* (ECQ) fails; and
- there exist impossible worlds where disjunctive syllogism fails.

The above principles were all shown to be theorems. Familiarity with the foregoing results will be presupposed in what follows, since we now plan to extend and build upon them.

1.2 The Recent Developments We'll Need

Among the recent developments of OT we'll need for the analysis of Routley star are the following definition and theorem schema:

$$\overline{p} =_{df} \neg p \quad (7)$$

$$\vdash \exists s \forall p(s \models p \equiv \varphi), \text{ provided } s \text{ isn't free in } \varphi \quad (8)$$

Definition (7) lets us denote the negation of a proposition more simply as $\overline{p}$. As a theorem schema, (8) is in fact a comprehension schema for situations and is derivable from axiom (1). A derivation of (8) is given in the Appendix. It is also provable that situations s and s' are identical just in case they make the same propositions true (Zalta 1993, 412):

[8]Cf. Nolan (1997, 542), who similarly suggests that impossible worlds are governed by the comprehension principle: for every proposition that cannot be true, there is an impossible world where that proposition is true.

$$\vdash s = s' \equiv \forall p(s \models p \equiv s' \models p) \tag{9}$$

Consequently, it follows immediately from (8) that there is a unique situation that makes true all and only the propositions satisfying φ:

$$\vdash \exists! s \forall p(s \models p \equiv \varphi), \text{ provided } s \text{ isn't free in } \varphi \tag{10}$$

It is a consequence of (10) that every definite description having the form $\imath s \forall p(s \models p \equiv \varphi)$ is always well-defined (i.e., provably has a denotation), provided s isn't free in φ. These are, therefore, *canonical* descriptions for situations.

1.3 Some Other Non-classical Situations

Earlier we described how OT implies the existence of impossible worlds in which certain classical laws of logic fails to hold. But it is important to remember, as we work through the results below, that we don't have to consider impossible worlds to find situations in which the laws of classical logic fail. Classical laws may fail in situations that aren't impossible worlds. Consider the law *ex contradictione quodlibet* (ECQ) and let q_1 be any proposition. Then the following is an instance of (10), which asserts the existence of a unique situation that makes exactly one proposition true, namely, the conjunction $q_1 \& \neg q_1$:

$$\exists! s \forall p(s \models p \equiv p = (q_1 \& \neg q_1))$$

Call this situation s_1, so that we know $\forall p(s_1 \models p \equiv p = (q_1 \& \neg q_1))$. Clearly, the conjunction $q_1 \& \neg q_1$ is true in s_1, i.e., $s_1 \models (q_1 \& \neg q_1)$. Now consider any proposition that is distinct from the conjunction $q_1 \& \neg q_1$, say, r_1. It then follows that $\neg s \models r_1$. So we've established that for any propositions q and r such that $(q \& \neg q) \neq r$, there is a situation in which ECQ fails:

$$\vdash \forall q \forall r\big((q \& \neg q) \neq r \rightarrow \exists s(s \models (q \& \neg q) \& \neg s \models r)\big)$$

Indeed, we can use the above to define a condition that isolates precisely those situations that fail ECQ:

$$s \textit{ is an ECQ-falsifier} \equiv_{df} \exists q \exists r(s \models (q \& \neg q) \& \neg s \models r)$$

Similarly, we can define a group of situations in which disjunctive syllogism (DS) fails. Let q_1 and r_1 be any propositions such that the propositions $q_1 \vee r_1$, $\neg q_1$, and r_1 are all pairwise distinct. Then consider following instance of (10), which asserts the existence of a unique situation that encodes exactly two propositions, namely, $q_1 \vee r_1$ and $\neg q_1$:

$$\exists! s \forall p\big(s \models p \equiv p = (q_1 \vee r_1) \vee p = \neg q_1\big)$$

Call this s_2, so that we know $\forall p\big(s_2 \models p \equiv p = (q_1 \vee r_1) \vee p = \neg q_1\big)$. Then it is easy to establish all of the following: $s_2 \models (q_1 \vee r_1)$, $s_2 \models \neg q_1$, and $\neg s_2 \models r_1$. So DS fails with respect to s_2. And, in general, we have established that for any pairwise distinct propositions $q \vee r$, $\neg q$, and r, there is a situation in which DS fails:

$$\vdash \forall q \forall r((q \vee r) \neq \neg q \;\&\; \neg q \neq r \;\&\; (q \vee r) \neq r) \rightarrow \exists s(s \models (q \vee r) \;\&\; s \models \neg q \;\&\; \neg s \models r))$$

Again, we can use the above to define a group of situations in which DS fails:

$$s \textit{ is a DS-falsifier} \equiv_{df} \exists q \exists r(s \models (q \vee r) \;\&\; s \models \neg q \;\&\; \neg s \models r)$$

These examples are of interest because they show that OT already has the capacity to develop counterexamples to classical logical laws without Routley star, once those classical laws are interpreted within the domain of situations. We don't need to formulate a separate language and define truth for the formulas of that language with respect to the domain of situations. The condition $s \models \varphi$ (i.e., φ is *true in* s) is defined for all situations s and formulas φ. That's because *every* formula φ denotes a proposition,[9] and so each φ (with no free xs) can be instantiated for p in definition (3) to obtain an instance of the form $s \models \varphi \equiv_{df} s[\lambda x\, \varphi]$.[10] So our notion of *true in* situation s applies to arbitrary formulas and we can directly evaluate the truth of formulas relative to any distinguished (i.e., definable) group of situations.

Moreover, one can define, for example, the *conjunction-normal* situations as those situations that make $p \& q$ true whenever they both make p true and make q true. Formally:

$$s \textit{ is conjunction-normal} \equiv_{df} \forall p \forall q\big(s \models (p \;\&\; q) \equiv (s \models p \;\&\; s \models q)\big)$$

And s is *double-negation normal* just in case s makes $\overline{\overline{p}}$ true if and only if it makes p true. And so on. One may therefore precisely define, for some particular application, the group of situations to be studied.

1.4 Canonical Descriptions and Modality

At the end of Section 1.2, we identified *canonical* descriptions of the form $\imath s \forall p(s \models p \equiv \varphi)$. Though canonical descriptions are always logically proper, one must take care when deploying them in a modal context, given that, in OT, the formal definite description $\imath x \varphi$ rigidly denotes the unique object, if there is one, that satifies φ at the distinguished actual world. It is worth digressing a moment to understand the issues that arise and why the present paper will be able ignore them. We conclude the digression and this section by formulating a theorem schema involving descriptions that will play an important role in the paper.

[9] To see this, recall that in OT, propositions are 0-ary relations. So let Π be an arbitrary 0-ary relation term. In the latest developments of object theory (Zalta m.s.), we define: $\Pi\!\downarrow$ (read: Π exists) just in case $[\lambda \nu\, \Pi]\!\downarrow$, where ν is some variable not free in Π. But the definiens, which asserts that the propositional property $[\lambda \nu\, \Pi]$ exists, is axiomatic, since it meets the condition that the bound variable ν doesn't occur as an argument in an encoding formula anywhere in Π. So, it is provable that $\Pi\!\downarrow$. But, in OT, formulas are 0-ary relation terms and since Π was arbitrary, it is a theorem of OT that $\varphi\!\downarrow$, for any formula φ. So every formula denotes a proposition.

[10] The restriction that x not be free in φ is no real restriction. If φ has a free variable x, then choose a variable that is not free in φ. Without loss of generality, suppose it is y. Then as an instance of an alphabetic variant of definition (3), we have $s \models \varphi \equiv_{df} s[\lambda y\, \varphi]$. So the definition holds for any formula φ.

Note that in a modal logic with rigid definite descriptions, one can produce logical theorems that are not necessary. For example, the conditional $y = \iota x Gx \rightarrow Gy$ will be false at a world, say w_1, when y (is assigned an object that) fails to be G at w_1 but is the unique G at the actual world w_0 (in such a case, the the antecedent is true at w_1 but the consequent false at w_1). More generally, where φ^y_x is the result of substituting y for all the free occurrences of x in φ, the claim $y = \iota x\varphi \rightarrow \varphi^y_x$ is not a necessary truth, though it is logically true (i.e., true at the distinguished actual world of every model, for every assignment to y) given the semantics of rigid definite descriptions.

In a fuller presentation of OT, we could axiomatize rigid definite descriptions by introducing an actuality operator $\mathcal{A}$ and asserting, as an axiom:

$$y = \iota x\varphi \equiv \forall x(\mathcal{A}\varphi \equiv x = y) \tag{11}$$

This is a form of the Hintikka principle (1959); it is a necessary truth and it immediately implies the following as a necessary truth, in which $\mathcal{A}\varphi^y_x = (\mathcal{A}\varphi)^y_x = \mathcal{A}(\varphi^y_x)$:

$$\vdash y = \iota x\varphi \rightarrow \mathcal{A}\varphi^y_x, \text{ provided } y \text{ is substitutable for } x \text{ in } \varphi \tag{12}$$

If we then adjust the original example, it should be easy to see that $y = \iota xGx \rightarrow \mathcal{A}Gy$ is a necessary truth. But though (11), (12), and their instances are necessary truths, the axiomatization of the actuality operator includes an axiom, namely $\mathcal{A}\varphi \rightarrow \varphi$, that is a logical truth which isn't necessary (Zalta 1988).[11] So the Rule of Necessitation has to be slightly adjusted; one may not apply the rule to necessitate a theorem whose proof depends on the axiom $\mathcal{A}\varphi \rightarrow \varphi$.

In what follows, though, we won't need to worry about illicit applications of the Rule of Necessitation since all of the definite descriptions we'll deploy involve a special class of formulas for which we can derive the conditional $y = \iota x\varphi \rightarrow \varphi^y_x$ without appealing to the contingent axiom for actuality. The formulas in question are *modally collapsed*, i.e., any formula φ for which it is provable that $\Box(\varphi \rightarrow \Box\varphi)$. When a formula having this form is provable, one can prove $\mathcal{A}\varphi \equiv \varphi$ without appealing to the contingent axiom $\mathcal{A}\varphi \rightarrow \varphi$.[12] If φ is modally collapsed, then $y = \iota x\varphi \rightarrow \varphi^y_x$ is a necessary truth:

$$\vdash y = \iota x\varphi \rightarrow \varphi^y_x, \tag{13}$$
provided φ is modally collapsed and y is substitutable for x in φ

[11] To see why the formula schema $\mathcal{A}\varphi \rightarrow \varphi$ can't be necessitated, note that the conditional is true at the actual world: if φ is true at the actual world, then the conditional is true at the actual world (by truth of the consequent), and if φ is false at the actual world, then the conditional is true at the actual world (by failure of the antecedent). However, the conditional is false at any world w_1 whenever φ is true at the actual world but false at w_1.

[12] Assume $\Box(\varphi \rightarrow \Box\varphi)$. Then by the K$\Diamond$ principle, i.e., $\Box(\psi \rightarrow \chi) \rightarrow (\Diamond\psi \rightarrow \Diamond\chi)$, it follows that $\Diamond\varphi \rightarrow \Diamond\Box\varphi$. But in S5, $\Diamond\Box\varphi \rightarrow \Box\varphi$. So by hypothetical syllogism, we've established:

(θ) $\Diamond\varphi \rightarrow \Box\varphi$

Now to see that $\mathcal{A}\varphi \equiv \varphi$, we prove both directions. ($\rightarrow$) Assume $\mathcal{A}\varphi$. Then $\Diamond\varphi$. So by (θ), $\Box\varphi$. Hence φ, by the T schema. ($\leftarrow$) Assume φ. Then $\Diamond\varphi$. But again by (θ), it follows that $\Box\varphi$. Hence $\mathcal{A}\varphi$.

(See the Appendix for the proof.) In this paper, we shall appeal only to definite descriptions $\imath x\varphi$ in which φ is modally collapsed, and so we won't need to worry about mistakenly applying the Rule of Necessitation to theorems derived from a logical truth that is not necessary.

In particular, as a special case of (13) for modally collapsed φ, it is a theorem that if a situation s is identical to the situation that makes true all and only the propositions satisfying φ, then s makes true all and only the propositions satisfying φ, i.e.,

$$\vdash s = \imath s'\forall p(s' \models p \equiv \varphi) \rightarrow \forall p(s \models p \equiv \varphi), \quad (14)$$
provided s' isn't free in φ and φ is modally collapsed

The keys to the proof in the Appendix are the facts that $s' \models p$ is, by definition (3), an instance of the formula xF and that the modal logic of encoding is $xF \rightarrow \Box xF$. So by the Rule of Necessitation, $\Box(xF \rightarrow \Box xF)$ and, as an instance, $\Box(s' \models p \rightarrow \Box s' \models p)$. This fact, and the fact that φ is modally collapsed, lets us validly infer that the formula $\forall p(s' \models p \equiv \varphi)$ is modally collapsed. So the description $\imath s'\forall p(s' \models p \equiv \varphi)$ will be governed by (13).

(14) plays a crucial role in what follows. All of descriptions of the form $\imath s'\forall p(s' \models p \equiv \varphi)$ used in the present work will be constructed in terms of formulas φ that are modally collapsed; it is provable that their truth necessarily implies their own necessity. This should forestall any concerns about the fact that we shall be working within a modal context in which definite descriptions are interpreted rigidly.

2 Routley Star 1972: Definitions and Theorems

To capture the definition in Routley & Routley 1972, we say that *the Routley star image of* situation s, written s^*, is the situation s' that makes true all and only those propositions whose negations aren't true in s:

$$s^* =_{df} \imath s'\forall p(s' \models p \equiv \neg s \models \overline{p}) \quad (15)$$

Clearly, the definiens has a denotation: it is a canonical description for which s' doesn't occur free in $\neg s \models \overline{p}$. So s^* is well-defined. Since it can be shown that $\neg s \models \overline{p}$ is a modally collapsed formula, it follows from (15) by (14) that p is true in s^* iff $\overline{p}$ fails to be true in s:

$$\vdash \forall p(s^* \models p \equiv \neg s \models \overline{p}) \quad (16)$$

This holds for any situation s. (The first part of the proof in the Appendix establishes that $\neg s \models \overline{p}$ is a modally collapsed formula.)

We now establish a number of facts that show (15) and theorem (16) properly capture the definition of s^* in Routley & Routley 1972. Since formulas of the form $\varphi \equiv \neg\psi$ are necessarily equivalent to formulas of the form $\neg\varphi \equiv \psi$, (16) implies that, for any proposition p, $\overline{p}$ is true in s if and only if p fails to be true in s^*:

$$\vdash \forall p(s \models \overline{p} \equiv \neg s^* \models p) \tag{17}$$

Again, this holds for any situation s. (17) is an analogue of the Routleys' principle (iv), as formulated in the opening paragraph of Section 1 above.

To set up the next confirmation that (15) is correct, let us say that *s has a glut with respect to p*, written $GlutOn(s,p)$, if and only if both p and $\overline{p}$ are true in s; and that *s has a gap with respect to p*, written $GapOn(s,p)$, if and only if neither p nor $\overline{p}$ is true in s:

$$GlutOn(s,p) \equiv_{df} s \models p \;\&\; s \models \overline{p} \tag{18}$$

$$GapOn(s,p) \equiv_{df} \neg s \models p \;\&\; \neg s \models \overline{p} \tag{19}$$

Then it follows that the condition $s = s^{**}$ implies that if s has a glut with respect to p, then s^* has a gap with respect to p:

$$\vdash s = s^{**} \rightarrow (GlutOn(s,p) \rightarrow GapOn(s^*,p)) \tag{20}$$

And $s = s^{**}$ also implies that if s has a gap with respect to p, then s^* has a glut with respect to p:

$$\vdash s = s^{**} \rightarrow (GapOn(s,p) \rightarrow GlutOn(s^*,p)) \tag{21}$$

Moreover, it can be shown, without the assumption that $s = s^{**}$, that if s neither has a glut nor a gap w.r.t. p, then s^* makes p true if and only if s makes p true:

$$\vdash (\neg GlutOn(s,p) \;\&\; \neg GapOn(s,p)) \rightarrow (s^* \models p \equiv s \models p) \tag{22}$$

It then follows that if, for every proposition p, s neither has a glut nor a gap w.r.t. p, then $s^* = s$ (since they make the same propositions true); and for every proposition p, s neither has a glut nor a gap w.r.t. p if and only if for every proposition p, s makes p true if and only if s fails to make $\overline{p}$ true:

$$\vdash \forall p(\neg GlutOn(s,p) \;\&\; \neg GapOn(s,p)) \rightarrow s^* = s \tag{23}$$

$$\vdash \forall p(\neg GlutOn(s,p) \;\&\; \neg GapOn(s,p)) \equiv \forall p(s \models p \equiv \neg s \models \overline{p}) \tag{24}$$

Intuitively, (24) tells us that if s is free of gluts and gaps, then it is coherent with respect to negation.

We conclude this section by additionally deriving three interesting facts, the first two of which require us to define *the null situation* ($\boldsymbol{s_\emptyset}$), in which no propositions are true, and *the trivial situation* ($\boldsymbol{s_V}$), in which every proposition is true:

$$\boldsymbol{s_\emptyset} =_{df} \imath s' \forall p(s' \models p \equiv p \neq p) \tag{25}$$

$$\boldsymbol{s_V} =_{df} \imath s' \forall p(s' \models p \equiv p = p) \tag{26}$$

The facts are that: if $s^{**} = s$ holds universally, then the Routley star image of the null situation is the trivial situation; if $s^{**} = s$ holds universally, then the Routley star image of the trivial situation is the null situation; and s^{**} is identical to s if and only if, for every proposition p, p is true in s iff $\overline{\overline{p}}$ is true in s:

$$\vdash \forall s(s^{**} = s) \rightarrow \boldsymbol{s_\emptyset}^* = \boldsymbol{s_V} \tag{27}$$

$$\vdash \forall s(s^{**} = s) \rightarrow \boldsymbol{s_V}^* = \boldsymbol{s_\emptyset} \tag{28}$$

$$\vdash s^{**} = s \equiv \forall p(s \models p \equiv s \models \overline{\overline{p}}) \tag{29}$$

(29) becomes interesting when we consider the passage in Routley & Routley 1972 (338) in which they discuss their principle (iv), which we reproduced above in the opening paragraph of Section 1:

> Requirement (iv) on its own does not suffice for the normality of the negation, since it does not assume such characteristic negation features as double negation features. For these features it is, however, unnecessary to adopt the over-restrictive condition $H = H^*$, which would take us back to (ii); it suffices to require that $H = H^{**}$.

The Routleys don't say here exactly which double negation features they are referring to. But (29) tells us that the condition $s^{**} = s$ is equivalent to a specific double negation feature. As we've seen, the Routleys go on to suggest that a 'set-up', i.e., a situation s, is classical ('normal') with respect to double negation when $s^{**} = s$. Even if the fact expressed by (29) has been made explicit somewhere else in the literature, it has now been derived from general principles that don't assume any mathematics, and the derivation occurs in a hyperintensional logical and metaphysical system in which propositions have been axiomatized, and situations and their Routley star images have been defined.

3 An Alternative Definition

In Section 5 below, we investigate an alternative definition of the Routley star image, given in Leitgeb 2019. Instead of defining s^* as the situation that makes true all and only the propositions whose negations aren't true in s, the alternative defines s^* as the situation that makes true all and only the negations of propositions that aren't true in s:

$$s^* =_{df} \imath s' \forall p(s' \models p \equiv \exists q(\neg s \models q \ \& \ p = \overline{q})) \tag{ϑ}$$

Since the condition $\exists q(\neg s \models q \ \& \ p = \overline{q})$ is modally collapsed,[13] (ϑ) immediately implies, by (14):

[13] We can show this by first noting that both conjuncts of this quantified conjunction are modally collapsed. Since $s \models q$ is modally collapsed (see the discussion immediately following (14)), so is $\neg s \models q$. Moreover, as noted in footnote 7, $p = q$ holds just in case the property identity $[\lambda x\, p] = [\lambda x\, q]$ holds, where the identity of properties $F = G$ is defined as $\Box\forall x(xF \equiv xG)$. Given the S4 axiom then, it is easy to show $F = G \rightarrow \Box F = G$. So by the Rule of Necessitation $\Box(F = G \rightarrow \Box F = G)$. Instantiating F and G to $[\lambda x\, p]$ and $[\lambda x\, q]$ and applying the definition of identity for propositions, we have the instance $\Box(p = q \rightarrow \Box p = q)$, which holds for any propositions p and q. Hence $\Box(p = \overline{q} \rightarrow \Box p = \overline{q})$.

So it remains to show that the quantified conjunction is modally collapsed. But if φ and ψ are modally collapsed, it follows that $\varphi \,\&\, \psi$ is modally collapsed, i.e., if $\Box(\varphi \rightarrow \Box\varphi)$ and $\Box(\psi \rightarrow \Box\psi)$, then $\Box((\varphi \,\&\, \psi) \rightarrow \Box(\varphi \,\&\, \psi))$. From these facts it doesn't take much more work to show $\Box\big(\exists q(\neg s \models q \,\&\, p = \overline{q}) \rightarrow \Box\exists q(\neg s \models q \,\&\, p = \overline{q})\big)$.

$$\forall p(s^* \models p \equiv \exists q(\neg s \models q \;\&\; p = \overline{q})) \qquad (\xi)$$

(ϑ) and its consequence (ξ) are of interest because the key condition, $\exists q(\neg s \models q \;\&\; p = \overline{q})$, is *not* equivalent to the condition $\neg s \models \overline{p}$ used in (15); in other words, (ϑ) and (15) don't always define the same s^* for any given situation s.[14]

To see why, consider a simple situation, say s_1, in which a single proposition, say p_1, is true. Let's ignore all other propositions and consider what propositions are true in s_1^* according to (16) vs. what propositions are true s_1^* according to consequence (ξ). According to (16), the following propositions are true in s_1^*:

- p_1 (since $\neg s_1 \models \overline{p_1}$),
- $\overline{p_1}$ (since $\neg s_1 \models \overline{\overline{p_1}}$),
- $\overline{\overline{p_1}}$ (since $\neg s_1 \models \overline{\overline{\overline{p_1}}}$),
- and so on.

But according to (ξ), neither p_1 nor $\overline{p_1}$ are true in s_1^* (neither p_1 nor $\overline{p_1}$ is the negation of a proposition that s_1 *fails* to encode). Instead, the following propositions are true in s_1 according to (ξ):

- $\overline{\overline{p_1}}$ (since $\neg s_1 \models \overline{p_1}$ and $\overline{\overline{p_1}}$ is the negation of $\overline{p_1}$),
- $\overline{\overline{\overline{p_1}}}$ (since $\neg s_1 \models \overline{\overline{p_1}}$ and $\overline{\overline{\overline{p_1}}}$ is the negation of $\overline{\overline{p_1}}$),
- and so on.

So the two alternative ways of defining s_1^*, namely (15) and (ϑ), yield different situations even in this very simple case. They aren't equivalent. Clearly, then, one can't simply replace the definiens of (15) with the definiens:

$$\imath s' \forall p(s' \models p \equiv \exists q(\neg s \models q \;\&\; p = \overline{q}))$$

This won't preserve the results we've established thus far.

Though there may be multiple ways one could bring (15) and (ϑ) into alignment and force them into defining the same Routley star situation for a given s, the simplest way is to limit some of the hyperintensionality in propositions. In particular, we can show that $\exists q(\neg s \models q \,\&\, p = \overline{q})$ and $\neg s \models \overline{p}$ become equivalent whenever propositions are identical to their double negations, i.e., whenever:

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[14] In what follows, it is important to distinguish the following two conditions:

(a) $\exists q(\neg s \models q \;\&\; p = \overline{q})$

(b) $\exists q(\neg s \models q \;\&\; q = \overline{p})$

Condition (b) is equivalent to $\neg s \models \overline{p}$, by the following argument:

> ($\rightarrow$) Assume $\exists q(\neg s \models q \;\&\; q = \overline{p})$ and suppose r is such a propositions, so that we know both $\neg s \models r$ and $r = \overline{p}$. Then $\neg s \models \overline{p}$. ($\leftarrow$) Assume $\neg s \models \overline{p}$. Then $\neg s \models \overline{p} \;\&\; \overline{p} = \overline{p}$, by the reflexivity of identity and &I. Hence, $\exists q(\neg s \models q \;\&\; q = \overline{p})$.

But we're now going to focus on condition (a), to see why it isn't equivalent to $\neg s \models \overline{p}$.

$$\forall p(\overline{\overline{p}} = p) \tag{ζ}$$

Consider how (ζ) plays a role in the proof of both directions of the biconditional asserting the following equivalence:

$$\exists q(\neg s \models q \,\&\, p = \overline{q}) \equiv \neg s \models \overline{p} \tag{ω}$$

Proof: ($\rightarrow$) Assume $\exists q(\neg s \models q \,\&\, p = \overline{q})$ and let r be such a proposition, so that we know both $\neg s \models r$ and $p = \overline{r}$. The latter implies that $\overline{p} = \overline{\overline{r}}$, for if propositions are identical, so are their negations. But by (ζ), $\overline{\overline{r}} = r$. Hence, $\overline{p} = r$ and so $\neg s \models \overline{p}$. ($\leftarrow$) Assume $\neg s \models \overline{p}$. Then by ($\zeta$), $\neg s \models \overline{p} \,\&\, p = \overline{\overline{p}}$. By existentially generalizing on $\overline{p}$ we have: $\exists q(\neg s \models q \,\&\, p = \overline{q})$. ⋈

Note that OT does *not* imply (ζ) since the identity conditions of relations and propositions are hyperintensional; one may consistently claim that propositions and their double negations are distinct despite being necessarily equivalent. That's because in OT, propositions p and q are identical just in case the corresponding propositional properties $[\lambda x\, p]$ and $[\lambda x\, q]$ are identical, where property identity is, in turn, defined in terms of being necessarily encoded by the same objects, not in terms of being necessarily exemplified by the same objects (see footnote 7). Since necessarily equivalent properties and propositions are not identified, one may regard properties and propositions as more fine-grained. But one can easily and consistently add (ζ) as an axiom to OT or, in the alternative, derive consequences from the *assumption* that (ζ) and thereby derive conditional theorems in which (ζ) is the antecedent, via the Deduction Theorem.

But we don't even have to go as far as adding (ζ) an axiom. In Section 4.1, we'll (a) define a group of propositions that are identical with their double negations, (b) assert only that there are at least some such propositions (without asserting that every proposition is identical to its double negation), and then (c) focus our attention on situations that are built out of such propositions. Then, in Section 5, we'll use (ϑ) to define Routley star relative to that group of situations.

4 HYPE

Leitgeb (2019, 321ff) builds a semantics for a system of hyperintensional propositional logic ('HYPE'). He first builds a propositional language $\mathcal{L}$ by starting with atomic propositional letters $p_1, p_2, \ldots$, and logical symbols $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\top$ (where $\rightarrow$ does not express the material conditional). He writes $\overline{p_i}$ for $\neg p_i$, and uses $\overline{\overline{p_i}}$ as an abbreviation for p_i. The proposition letters and their negations constitute the *literals*. Leitgeb then constructs HYPE-models for $\mathcal{L}$ in terms of structures $\langle S, V, \circ, \bot \rangle$, where the elements of the models are simultaneously constrained by the requirements of a Routley star operation *. He describes the elements of the models as follows (Leitgeb 2019, 321–22):

- S is a non-empty set of states.

- V is a function (the valuation function) from S to the power set of the set of literals of the language $\mathcal{L}$, so that each state s in S is associated with a set of literals $V(s)$.

- $\circ$ is a partial fusion function on states that is idempotent and, when defined, commutative and partially associative.

- $\perp$ is a relation of incompatibility that relates states s and s' when some proposition p is true at one and its negation $\overline{p}$ is true at the other. [Note: The *relation* $\perp$ among HYPE states is not to be confused with the symbol $\perp$ that Leitgeb uses as a metalinguistic abbreviation of the proposition $\neg\top$ (2019, 321). In what follows, the context should make it clear whether $\perp$ denotes the relation or the proposition.]

The Routley star operation that constrains these models will be discussed and defined later, in Section 5.

Consequently, in the remainder of this paper, we use OT to reconstruct the above elements of HYPE models and we'll see that the reconstruction comports with both of the suggestions for understanding HYPE states quoted above in Leitgeb 2019 (323, footnote 9). In Section 4.1 we develop basic definitions and show how to interpret the V function in HYPE; in Section 4.2 we show how to interpret the HYPE fusion operation $\circ$; and in Section 4.3, we show how to interpret the HYPE incompatibility relation $\perp$. Finally, in Section 5, we define the HYPE version of Routley star and prove that it has the expected features.

4.1 HYPE Propositions and HYPE States

First, we work our way towards a definition of a *Hype*-state by defining *Hype*-propositions. We say that (30) a *Hype*-proposition is any proposition p that is identical to its double negation:

$$Hype(p) \equiv_{df} \overline{\overline{p}} = p \tag{30}$$

Clearly, then it follows that if p is a *Hype*-proposition, then so is its negation $\overline{p}$:

$$\vdash Hype(p) \rightarrow Hype(\overline{p}) \tag{31}$$

Though OT guarantees the existence of propositions (by 0-ary relation comprehension) and provides identity conditions for them (footnote 13), it doesn't guarantee the existence of *Hype*-propositions. The identity conditions for propositions in OT leave one free to assert the existence of *Hype*-propositions and the existence of propositions that are more fine-grained, e.g., by asserting $\exists p(\overline{\overline{p}} \neq p)$. Though $\Box(\overline{\overline{p}} \equiv p)$ is a theorem, it doesn't follow that $\overline{\overline{p}} = p$.

However, it is a trivial matter to extend OT by asserting the existence of at least some *Hype*-propositions. This hypothesis shouldn't be controversial to logicians who have worked in systems where propositions are represented as functions from possible worlds to truth values; such logicians have implicitly accepted that *all* propositions are identical with their double negations in those systems, since such propositions have the same truth value at every possible world. Nor does such a hypothesis significantly increase the resources OT needs for its analysis; it only ensures the existence of a subclass of propositions from among those that already exist. More importantly, by asserting there are *Hype*-propositions, we are asserting a metaphysical counterpart of Leitgeb's stipulation that the symbol '$\overline{\overline{p}}$' is to be an abbreviation of 'p' (i.e., when he defines the HYPE language; 2019, 321). Every proposition letter in HYPE's language is thereby identified with its double negation, but here we require only that some propositions are so identifiable.

Moreover, we need not even assert this hypothesis as an axiom; it is sufficient to take it on board as an assumption. Of course, by asserting the hypothesis as an axiom, the results below would all become theorems. But to accomplish the goals of this paper, we need only show *what follows* in OT from the assumption that there are *Hype*-propositions. This demonstrates that if we extend OT with a principle (i.e., $\exists p Hype(p)$) used to frame the target logic (HYPE), the result is a metaphysical system for defining HYPE's Routley star and deriving the principles that govern it. By collapsing at least some propositions and their negations, and deriving the basic principles of HYPE relative to those propositions, we establish that one can analyze the metaphysics of HYPE in terms of a domain of propositions that are not themselves hyperintensional entities; the hyperintensionality will arise via other means.

So, in what follows, we shall work under the assumption that there are *Hype*-propositions:

$$\exists p Hype(p) \tag{32}$$

Then we may define x is a *HypeState* just in case x is a situation such that every proposition true in x is a *Hype*-proposition:

$$HypeState(x) \equiv_{df} Situation(x) \,\&\, \forall p(x \models p \rightarrow Hype(p)) \tag{33}$$

So we're identifying *HypeStates* not as primitive entities but as situations. Thus when Leitgeb speaks of the members of $V(s)$ as the facts or states of affairs obtaining at s (2019, 322), we may interpret this in terms of our defined notion, *p is true in s*, as follows:

- $p \in V(s) \equiv_{df} s \models p$

Now it is easy to prove the existence of *HypeStates*; (32) guarantees there are *Hype*-propositions and (8) guarantees that for any condition on *Hype*-propositions, there are situations that make true all and only such propositions. Clearly, any such situation is a *HypeState*.

Indeed, we now invoke (8) to derive comprehension conditions for *HypeStates* with the help of some new variables. Note that the conditions $Hype(p)$ and $HypeState(x)$, defined respectively in (30) and (33), are modally collapsed conditions. So may use introduce rigid restricted variables to range over them. For clarity, we use special new variables in a distinguished, sans-serif font:

- $\mathsf{p}, \mathsf{q}, \ldots$ are restricted variables ranging over *Hype*-propositions.

- $\mathsf{s}, \mathsf{s}', \ldots$ are restricted variables ranging over *HypeStates*.

Using these variables we may formulate Simplified Comprehension for *HypeStates* as follows:

$$\vdash \exists \mathsf{s} \forall \mathsf{p}(\mathsf{s} \models \mathsf{p} \equiv \varphi), \text{ provided } \mathsf{s} \text{ isn't free in } \varphi \quad (34)$$

Clearly, in the usual way, it is provable that there is a unique such *HypeState* for each such instance:

$$\vdash \exists! \mathsf{s} \forall \mathsf{p}(\mathsf{s} \models \mathsf{p} \equiv \varphi), \text{ provided } \mathsf{s} \text{ isn't free in } \varphi \quad (35)$$

And this yields that *canonical* descriptions for *HypeStates* of the form $\imath \mathsf{s} \forall \mathsf{p}(\mathsf{s} \models \mathsf{p} \equiv \varphi)$ are always well-defined, provided s isn't free in φ.

It should also be clear that there are *HypeStates* that are counterexamples to classical laws, just like the ones described in Section 1.3. Those show us how to construct *HypeStates* that falsify Explosion (ECQ) and Disjunctive Syllogism, i.e., two of the laws mentioned in the last bullet point of Observation 14 in Leitgeb 2019 (333). And there are *HypeStates* that falsify some of the others mentioned there as well, such as Excluded Middle, Law of Non-Contradiction, and (General) Contraposition.[15] In the next section, we'll specifically discuss the last example of a HYPE-invalid inference mentioned in Observation 14 (involving the proposition $\bot$), namely, $\varphi \rightarrow \bot \models \neg\varphi$.

4.2 The HYPE Fusion Operation

If we put aside, for the moment, the fact that the fusion function $\circ$ in HYPE is a *partial* binary operation on *HypeStates*, then we can represent the HYPE $\circ$ operation as the following (total) *summation* operation $\oplus$ on situations generally:

$$s \oplus s' =_{df} \imath s'' \forall p\big(s'' \models p \equiv (s \models p \vee s' \models p)\big) \quad (36)$$

In other words, $s \oplus s'$ is the situation that makes a proposition p true just in case either s makes p true or s' makes p true. Since $s \models p \vee s' \models p$ is modally collapsed, it follows that a proposition p is true in $s \oplus s'$ just in case either p is true in s or p is true in s':

$$\vdash \forall p\big(s \oplus s' \models p \equiv (s \models p \vee s' \models p)\big) \quad (37)$$

To see that $\oplus$ captures additional features about the partial nature of situations generally, let us say that *s is a part of s'* just in case every proposition true in s is true in s':[16]

[15] For example, let p_1 and q_1 be arbitrary *Hype*-propositions and consider any *HypeState*, say s_1, that makes just q_1 true and no other propositions true. Then $\neg \mathsf{s}_1 \models (\mathsf{p}_1 \vee \neg \mathsf{p}_1)$. So in OT there is a *HypeState* in which Excluded Middle doesn't hold. In general, it is not provable that $\mathsf{s} \models (\varphi \vee \neg\varphi)$ for arbitrary s and φ. And any *HypeState* s that isn't maximal will be such that there are *Hype*-propositions q such that neither $\mathsf{s} \models \mathsf{q}$ nor $\mathsf{s} \models \neg \mathsf{q}$. We leave it to the reader to construct *HypeStates* in which the Law of Non-Contradiction and (General) Contraposition are false.

[16] The definition that follows was derived as a theorem in Zalta 1993 (412), as a consequence of the more general definition $x \trianglelefteq y \equiv_{df} \forall F(xF \rightarrow yF)$ and the fact that situations encode only

$$s \trianglelefteq s' \equiv_{df} \forall p(s \models p \rightarrow s' \models p) \quad (38)$$

It follows relatively straightforwardly that s is a part of s' if and only if the sum of s and s' just is s':

$$\vdash s \trianglelefteq s' \equiv s \oplus s' = s' \quad (39)$$

A further consequence of these definitions and theorems is that $\oplus$ is idempotent, commutative, and associative with respect to situations *generally*. Since *HypeStates* are situations, it follows that:

$\vdash \oplus$ is idempotent, commutative, and associative on *HypeStates*. (40)

Formally:

$$\vdash \mathsf{s} \oplus \mathsf{s}$$
$$\vdash \mathsf{s} \oplus \mathsf{s}' = \mathsf{s}' \oplus \mathsf{s}$$
$$\vdash \mathsf{s} \oplus (\mathsf{s}' \oplus \mathsf{s}'') = (\mathsf{s} \oplus \mathsf{s}') \oplus \mathsf{s}''$$

So by ignoring the partiality of $\circ$, we may interpret $s \circ s'$ in Leitgeb 2019 as $\mathsf{s} \oplus \mathsf{s}'$.

But the $\circ$ operation is in fact partial while $\mathsf{s} \oplus \mathsf{s}'$ is defined for *any HypeStates* s and s'. So one might be concerned that our results don't properly capture the metaphysics of the fusion operation in HYPE. I don't think that concern is justified, however, for we could in fact model the partiality of $\circ$ by introducing a partial ternary relation R^3 (*not* the ternary relation R of Routley-Meyer 1972, 1973) that may or may not relate a pair of *HypeStates* s and s' to a unique third *HypeState*.[17] But we shall leave further details for some other occasion and continue with our total fusion operation $\oplus$.

Moreover, it should be remembered that Leitgeb explicit labels the partiality of $\circ$ a 'design choice' and concludes with the hope that the "success of the system as a whole is going to justify the design choice" (2019, 323, footnote 9). I similarly suggest that the success of our metaphysical analysis of the two approaches to Routley star and analysis of HYPE has to be judged by the success of what the system can represent as a whole. So let me further suggest that the

propositional properties. But for the present investigation, we may simply take the following as a definition. It follows from our definition that $\trianglelefteq$ is reflexive, anti-symmetric, and transitive on situations generally (Zalta 1993, 413, Theorem 7), and on HypeStates in particular (exercise). Compare these theorems about $\trianglelefteq$ with Barwise 1989b (185) and 1989a (259), where they are taken as axioms of situation theory. Similarly, Restall 2000 (853), stipulates that there is an analogous relation $\sqsubseteq$ that is reflexive, anti-symmetric and transitive on the points of *compatibility frames*.

[17] Intuitively, R would be a partial relation that is idempotent and commutative when $\imath \mathsf{s}_0 R \mathsf{s}\mathsf{s}'\mathsf{s}_0$ exists. Then we could re-define $\oplus$ for *HypeStates* so that it meets the following condition:

$$\mathsf{s} \oplus \mathsf{s}' =_{df} \imath \mathsf{s}'' \forall \mathsf{p}(\mathsf{s}'' \models \mathsf{p} \equiv \mathsf{s} \models \mathsf{p} \vee \mathsf{s}' \models \mathsf{p} \vee \imath \mathsf{s}_0(R\mathsf{s}\mathsf{s}'\mathsf{s}_0) \models \mathsf{p})$$

The intuition here is that R ensures that $\mathsf{s} \oplus \mathsf{s}'$ makes true *Hype*-propositions other than the ones true in s and s'. Moreover, we must also require:

$$\imath \mathsf{s}_0 R \mathsf{s}_1((\mathsf{s}_1 \oplus \mathsf{s}_2) \oplus \mathsf{s}_3)\mathsf{s}_0 \trianglelefteq (\mathsf{s}_1 \oplus \mathsf{s}_2) \oplus \mathsf{s}_3$$

The extra constraint on R guarantees partial associativity. Thus, constraints on R validate idempotence, commutativity when defined, and partial associativity when defined.

loss of partiality in the analysis of $\circ$ by $\oplus$ isn't serious because the system of OT as a whole reconstructs the partiality to which $\circ$ is put to use in HYPE by other means. For example, in Example 3 (2019, 335), Leitgeb makes use of the partiality of $\circ$ to build a countermodel in HYPE to the inference $\varphi \to \bot \models \neg\varphi$. In that countermodel, there are two HYPE states s_a and s_b, and though $\circ$ is idempotent, both $s_a \circ s_b$ and $s_b \circ s_a$ are undefined. Leitgeb then shows this model yields a counterexample to $A \to \bot \models \neg A$.[18]

By contrast, in OT, $\oplus$ needn't be partial to construct a *HypeState* s and a *Hype*-proposition p such that $\mathsf{s} \models (\mathsf{p} \to \bot)$ and $\mathsf{s} \not\models \neg\mathsf{p}$. To see this, just take on board the assumption made in HYPE's language that $\top$ (or $\neg\top$ or $\bot$) is a distinguished proposition term. Then let p_2 be an arbitrary *Hype*-proposition. Note that OT doesn't require the identity $\mathsf{p}_2 \to \bot = \neg\mathsf{p}_2$. So consider the *HypeState* s_2 that makes just $\mathsf{p}_2 \to \bot$ true:

$$\mathsf{s}_2 =_{df} \imath \mathsf{s} \forall \mathsf{p}\big(\mathsf{s} \models \mathsf{p} \equiv \mathsf{p} = (\mathsf{p}_2 \to \bot)\big)$$

One cannot validly infer in OT that $\mathsf{s}_2 \models \neg\mathsf{p}_2$. And if given the hyperintensional claim that $\mathsf{p}_2 \to \bot \neq \neg\mathsf{p}_2$, it does follow that $\neg\mathsf{s}_2 \models \neg\mathsf{p}_2$. So, we don't need partiality in $\oplus$ to build a countermodel of the inference $\varphi \to \bot \models \neg\varphi$; instead, we just exploit the hyperintensionality already built into OT's theory of propositions. But if one is, nevertheless, still convinced that the metaphysics of HYPE can't be reconstructed with the partiality of $\circ$, then see footnote 17 for a means of doing so.

4.3 The HYPE Explicit Incompatibility Relation

Next we define, in object-theoretic terms, the HYPE *explicit incompatibility* condition $\bot$ (not to be confused with the proposition $\bot$ just discussed) that holds between *HypeStates*, by first defining it on situations generally. We say s is *explicitly incompatible* with s' ($s\,!\,s'$) just in case there is a proposition p such that s makes p true and s' makes the negation of p true:

$$s\,!\,s' \equiv_{df} \exists p(s \models p \;\&\; s' \models \overline{p}) \tag{41}$$

Since explicit incompatibility is now defined for all situations, it is defined on *HypeStates*, i.e., we may henceforth write $\mathsf{s}\,!\,\mathsf{s}'$ when *HypeStates* are explicitly incompatible.

Now the first principle governing $\bot$ in HYPE is (Leitgeb 2019, 322):

- If there is a v with $v \in V(s)$ and $\overline{v} \in V(s')$, then $s \perp s'$.

Given our interpretation of $\bot$ in terms of !, this becomes represented and derivable as the following theorem governing *HypeStates* and *Hype*-propositions:

$$\vdash (\mathsf{s} \models \mathsf{p} \;\&\; \mathsf{s}' \models \overline{\mathsf{p}}) \to \mathsf{s}\,!\,\mathsf{s}' \tag{42}$$

[18] For those familiar with HYPE, the argument is this (mostly quoting from 2019, 335): [It holds that] $s_a \models p_2 \to \bot$, as there is no p_2 state with which s_a could be fused. However, $s_a \not\models \neg p_2$, since $\overline{p_2}$ has not been assigned to s_a, or, equivalently (by Lemma 8), because s_a does not stand in the $\perp_3$ relation to the p_2-satisfying state s_b.

And the second principle governing $\perp$ in HYPE is (Leitgeb 2019, 322):

- If $s \perp s'$ and both $s \circ s''$ and $s' \circ s'''$ are defined, then $s \circ s'' \perp s' \circ s'''$.

Given our interpretation of $\circ$ as $\oplus$ and the fact that $s \oplus s'$ is always defined for any situations s and s', this becomes represented and derivable as the following theorem regarding *HypeStates*:

$$\vdash \mathrm{s}\,!\,\mathrm{s}' \rightarrow (\mathrm{s} \oplus \mathrm{s}'')\,!\,(\mathrm{s}' \oplus \mathrm{s}''') \tag{43}$$

The proofs of both (42) and (43) are in the Appendix.[19]

5 Routley Star in HYPE

We continue to use our restricted variables 'p' and 's' to range over *Hype*-propositions and *HypeStates*, respectively. Our next goal, then, is to reconstruct and derive the principles that govern the HYPE Routley star operator. That is, we must reconstruct and derive the following conditions laid down in Leitgeb 2019 (322), in which we've replaced Leitgeb's variable '*s*' by our restricted variable 's':

For every s in S,

(A) there is a unique $\mathrm{s}^* \in S$ (the star image of s) such that:

[19]In addition to analyzing the HYPE incompatibility condition $\perp$ in terms of the object-theoretic definition of ! in (41), we may also analyze the compatibility relation C used in Restall 2000 (853), Berto 2015 (767), and Berto & Restall 2019 (1127) as the negation of !, i.e., via the following definition, cast in terms of situations generally:

$$sCs' \equiv_{df} \neg\exists p(s \models p \,\&\, s' \models \overline{p})$$

That is, s is compatible with s' just in case there is no proposition p that s makes p true and s' makes $\overline{p}$ true. (Depending on the purpose at hand, one might prefer to revise this definition to ensure that compatibility holds only when no proposition p is such that p is true in the modal closure of s while $\overline{p}$ is true in the modal closure of s'. But we don't need this more sophisticated understanding of compatibility in what follows.) Then, the semantic principle governing C stipulated in Restall 2000 (853, Definition 1.1), namely:

for any x, y, x', and y', if xCy, $x' \sqsubseteq x$, and $y' \sqsubseteq y$, then $x'Cy'$,

becomes derivable in OT, with $\trianglelefteq$ instead of $\sqsubseteq$. To see how, let us temporarily use x, y as restricted variables ranging over situations. Then we have:

$$\vdash (xCy \,\&\, x' \trianglelefteq x \,\&\, y' \trianglelefteq y) \rightarrow x'Cy'$$

Proof: Assume xCy, $x' \trianglelefteq x$, and $y' \trianglelefteq y$. These assumptions imply, respectively:

(A) $\neg\exists p(x \models p \,\&\, y \models \overline{p})$

(B) $\forall p(x' \models p \rightarrow x \models p)$

(C) $\forall p(y' \models p \rightarrow y \models p)$

To show $x'Cy'$, suppose not, for reductio. Then $\exists p(x' \models p \,\&\, y' \models \overline{p})$. Suppose p_1 is such a proposition, so that we know both $x' \models p_1$ and $y' \models \overline{p_1}$. Then by (B) and (C), respectively, these entail that $x \models p_1$ and $y \models \overline{p_1}$. But this contradicts (A).

As noted in footnote 3, this same principle, labeled 'Backwards' (compatibility), is stipulated in Berto 2015 (768) and Berto & Restall 2019 (1129).

(B) $V(s^*) = \{\overline{v} | v \notin V(s)\}$,

(C) $s^{**} = s$,

(D) s and s^* are not incompatible, i.e., $\neg(s \perp s^*)$, and

(E) s^* is the largest state compatible with s, i.e., if s is not incompatible with s', then the fusion of s' and s^* is defined and the fusion of $s' \circ s^* = s^*$.

Note that s^* is defined in HYPE as $V(s^*) = \{\overline{v} | v \notin V(s)\}$, instead of as $V(s^*) = \{v | \overline{v} \notin V(s)\}$. However, as we saw in Section 3, these two definitions become equivalent if propositions and their double negations are generally identified. And as we saw in Section 4, Leitgeb does identify p and $\overline{\overline{p}}$ in his propositional language $\mathcal{L}$. Since we've defined *Hype*-propositions as ones that exhibit this behavior, let us examine how the HYPE Routley star and the principles governing it can be defined or derived given our analysis of *Hype*-propositions and *HypeStates*.

For any *HypeState* s, we may define the HYPE Routley star image of s, written s^*, as the *HypeState* s' that makes a *Hype*-proposition p true just in case p is the negation of a proposition not true in s:[20]

$$s^* =_{df} \imath s' \forall p(s' \models p \equiv \exists q(\neg s \models q \,\&\, p = \overline{q})) \tag{44}$$

We take (44) to be a reconstruction of principle (B) above. Now although the HYPE principle (A) requires that there be a unique s^* satisfying (B) – (E), it should be clear that s^* is already uniquely defined; for any s, exactly one s^* has been identified by a canonical description.

So we may immediately conclude that s^* exists, for any s. Before we show that the unique star image s^* of a *HypeState* s also satisfies constraints (C) – (E), it proves useful to first confirm a few facts that follow from (44). By now familiar reasoning, we may infer from (44) that the *Hype*-propositions true in s^* are precisely the negations of the *Hype*-propositions that fail to be true in s:

$$\vdash \forall p(s^* \models p \equiv \exists q(\neg s \models q \,\&\, p = \overline{q})) \tag{45}$$

Moreover, we may verify that the principle proved in Section 3 holds for *HypeStates*, namely that p is the negation of some proposition that s fails to make true if and only if s fails to make $\overline{p}$ true:

$$\vdash \exists q(\neg s \models q \,\&\, p = \overline{q}) \equiv \neg s \models \overline{p} \tag{46}$$

Clearly, then, (45) and (46) imply both that s^* makes p true if and only if s fails to make $\overline{p}$ true and, by a simple logical consequence of this fact, $\overline{p}$ is true in a *HypeState* s if and only if it is not the case that p is true in s^*:

$$\vdash \forall p(s^* \models p \equiv \neg s \models \overline{p}) \tag{47}$$

[20]The following should be considered a *redefinition* of the Routley star image. That's because *HypeStates* are situations and, in Section 2, (15) defines the Routley star on situations. So to avoid conflicting definitions, just consider the following as a redefinition of this operator.

$$\vdash \forall p(s \models \overline{p} \equiv \neg s^* \models p) \tag{48}$$

(48) is a direct analogue of the Routley & Routley condition (iv) described in the Introduction above, and so corresponds directly to (17).

Note next that we can make use of the definitions of gaps and gluts in (18) and (19), respectively; these notions were defined generally for any situations and propositions and so apply to *HypeStates* and *Hype*-propositions. We may then further confirm that (44) is correct by establishing that if s has a glut w.r.t. p, then s^* has a gap w.r.t. p; if s has a gap w.r.t. p, then s^* has a glut w.r.t. p; and if s has neither a glut nor a gap w.r.t. p, then s^* agrees with s^* on p:

$$\vdash \mathit{GlutOn}(s,p) \to \mathit{GapOn}(s^*,p) \tag{49}$$

$$\vdash \mathit{GapOn}(s,p) \to \mathit{GlutOn}(s^*,p) \tag{50}$$

$$\vdash (\neg \mathit{GlutOn}(s,p) \,\&\, \neg \mathit{GapOn}(s,p)) \to (s^* \models p \equiv s \models p) \tag{51}$$

Now that we have confirmed that (44) is a definition of s^* that yields the latter's desired characteristics, we turn to the derivation of principle (C) governing HYPE s^*, namely, that s^{**} is identical to s:

$$\vdash s^{**} = s \tag{52}$$

Cf. Leitgeb 2019 (322). So, whereas (29) establishes that the stipulation $s^{**} = s$ in Routley & Routley 1972 is equivalent to the double-negation condition $\forall p(s \models p \equiv s \models \overline{\overline{p}})$, (52) establishes that the analogous stipulation $s^{**} = s$ in Leitgeb 2019 can be derived from the double-negation fact about *Hype*-propostions that $\forall p(p = \overline{\overline{p}})$. These results give us a deeper understanding of the connection between the two ways of defining the Routley star image of a situation.

Principles (D) and (E) of HYPE s^* may be derived as follows. (D) can be captured as the theorem that s is not explicitly incompatible with s^*:

$$\vdash \neg s!s^* \tag{53}$$

And since $s' \oplus s^*$ is always defined in our reconstruction, we can reconstruct and derive (E) as the simpler claim if s is not incompatible with s', then the sum/fusion of s' and s^* just is s^*:

$$\vdash \neg s!s' \to (s' \oplus s^* = s^*) \tag{54}$$

(54) guarantees that s^* is the largest state compatible with s.

Finally, if we recall definition (38) of $s \trianglelefteq s'$ and fact (39) that $s \trianglelefteq s' \equiv \forall p(s \models p \to s' \models p)$, we may prove that the HYPE Routley star operation reverses $\trianglelefteq$:

$$\vdash s \trianglelefteq s' \to s'^* \trianglelefteq s^* \tag{55}$$

Cf. Observation 3, Leitgeb 2019 (325). This completes the derivation of the principles stipulated in HYPE for the Routley star operation, modulo the partiality of the HYPE fusion operation.

6 Conclusion

We've now answered the question: What kind of metaphysics is represented by a semantics making use of Routley star? Without assuming any mathematical entities or theory of sets and functions, we've used OT to define two forms of the Routley star operation and derive the principles that govern these forms. And the better we understand the theorems that are implied by the two ways of defining it, the better we understand how the star operation might be used. The existence of the Routley star image s^* of a situation s is guaranteed not by set theory but by a theory of abstract objects. And out reconstruction shows that situations have both a metaphysical character and an informational character, at least as these are described in the quote above from Leitgeb 2019 (footnote 9). One can view situations in OT as "chunks of reality" that are "located in the world" if one takes the Aristotelian view that they are (abstracted, logical) forms that are immanent in what there is. Alternatively, one can view situations informationally, as abstract entities that systematize information contents about which we might communicate. But these metaphilosophical considerations about how to interpret OT as a theory shouldn't divert attention away from the tight conceptual framework that OT provides for recognizing and reconciling two non-equivalent definitions of Routley star.

Indeed, if you look at how situations are defined in (2) and at how the Routley star operation is defined in (15) and (44), one might even suggest that the star operation is a logical one. Propositions are axiomatized as 0-ary relations and can be considered part of logic. Situations are defined in (2) as abstract objects that encode only propositional properties. And the * operation is then defined on situations in terms of the notions *the*, *truth in* (which is in turn defined in terms of the *encoding* mode of predication), *every* and *some*, *if and only if*, and *not*. If the star operation is logical, then we can explain why some have thought that the uses to which Routley star has been put in the literature helps us to capture semantically a more general and flexible logical concept of negation.[21]

Finally, we've shown that the basic principles governing Routley star need not be stipulated but can be derived from its definition. This integrates Routley star into a more general theory of (partial) situations that has been shown, in previous work, to ground the theory of both possible worlds and impossible worlds. This analysis of the Routley star operation clarifies our understanding of the Routley-Meyer ternary relation R (Routley-Meyer 1972, 1973) on 'setups', by systematically validating many of the assumptions of situation theory used in Mares' (2004) motivation and justification for R. But we shall not attempt to further explore the various definitions of the ternary relation R in this paper. It is sufficient to have shown how different groups of situations (e.g., those defined in (2), *HypeStates*, or others) can constitute a proper subdomain for ternary R. I take those subdomains to be consistent with all of the various attempts at understanding that relation.

[21] I'm indebted to Hannes Leitgeb for suggesting this point.

Appendix: Proofs of the Theorems

(8)[22] If we eliminate the restricted variable, then the theorem we have to prove becomes:

$$\exists x(Situation(x) \,\&\, \forall p(x \models p \equiv \varphi)), \text{ provided } x \text{ isn't free in } \varphi$$

So let φ be any formula in which x doesn't occur free. (Note that the variable p may or may not be free in φ.) Now, pick a property variable that doesn't occur free in φ. Without loss of generality, suppose it is G. Then let ψ be the formula $\exists p(\varphi \,\&\, G = [\lambda z\, p])$. Clearly, since x doesn't occur free in ψ, the following is a schematic instance of of the comprehension schema for abstract objects, stated in the text as (1):

$$\exists x(A!x \,\&\, \forall G(xG \equiv \psi))$$

But given our choice of ψ, this amounts to:

$$\exists x\big(A!x \,\&\, \forall G(xG \equiv \exists p(\varphi \,\&\, G = [\lambda z\, p]))\big)$$

Let a be such an object, so that we know both $A!a$ and:

(A) $\forall G(aG \equiv \exists p(\varphi \,\&\, G = [\lambda z\, p]))$

It follows *a fortiori* that $\forall G(aG \rightarrow \exists p(G = [\lambda z\, p]))$. Hence $Situation(a)$, by definition (2). So it remains to show $\forall p(a \models p \equiv \varphi)$. By GEN, it suffices to show $a \models p \equiv \varphi$, since we've made no special assumptions about p.

To prove this biconditional, we'll rely on the fact that $a \models p$ is defined as $a[\lambda z\, p]$, by (3), given that a is a situation. We'll therefore want to instantiate $a[\lambda z\, p]$ into (A). But there is a clash of variables and, to avoid this, we use the following alphabetic variant of (A), where q is a variable that is substitutable for p, and doesn't occur free, in φ:

(A′) $\forall G(aG \equiv \exists q(\varphi^q_p \,\&\, G = [\lambda z\, q]))$

Now we can properly instantiate $[\lambda z\, p]$ into (A′), and if we remember that G doesn't occur free in φ, we obtain:[23]

(B) $a[\lambda z\, p] \equiv \exists q(\varphi^q_p \,\&\, [\lambda z\, p] = [\lambda z\, q])$

With these facts we can prove $a \models p \equiv \varphi$.

($\rightarrow$) Assume $a \models p$, to show φ. Then $a[\lambda z p]$, by (3). So by (B), it follows that:

$$\exists q(\varphi^q_p \,\&\, [\lambda z\, p] = [\lambda z\, q])$$

Now suppose q_1 is such a proposition, so that we know:

[22] I'm indebted to Uri Nodelman for spotting a flaw in the original proof of this theorem.

[23] Strictly speaking, when we instantiate $[\lambda z\, p]$ into (A′), we obtain:

$$a[\lambda z\, p] \equiv \exists q((\varphi^q_p)^{[\lambda z\, p]}_G \,\&\, [\lambda z\, p] = [\lambda z\, q])$$

But since G isn't free in φ, $(\varphi^q_p)^{[\lambda z\, p]}_G$ is just φ^q_p.

(C) $(\varphi^q_p)^{q_1}_q \,\&\, [\lambda z\, p] = [\lambda z\, q_1]$

In OT, propositions are identical whenever the propositional properties constructed from them are identical (Zalta 1993, 409). So by the second conjunct of (C), it follows that $p = q_1$. Hence, by the first conjunct of (C), it follows that $(\varphi^q_p)^p_q$. But since the conditions of the Re-replacement Lemma are met (Enderton 2001, 130), this latter is just φ.

(←) Assume φ. Then $\varphi \,\&\, [\lambda z p] = [\lambda z p]$, by the reflexivity of identity. Hence, by existential introduction:

$$\exists q(\varphi^q_p \,\&\, [\lambda z\, p] = [\lambda z\, q])$$

Then by (B), $a[\lambda z\, p]$. So by (3) and the fact that a is a situation, $a \models p$. ⋈

(9) This is Theorem 2 in Zalta 1993. The proof was given in Zalta 1991 (Appendix A), which served as a precursor to Zalta 1993.

(10) This follows from (8) and (9) by the standard definition of the uniqueness quantifier $\exists! s\psi$.

(12) Suppose y is substitutable for x in φ and assume $y = \imath x\varphi$. Then by axiom (11), $\forall x(\mathcal{A}\varphi \equiv x = y)$. But since y is substitutable for x in φ, we can instantiate this last fact to y and we obtain $\mathcal{A}\varphi^y_x \equiv y = y$. So by the reflexivity of identity, $\mathcal{A}\varphi^y_x$. ⋈

(13) By hypothesis, φ is modally collapsed and y is substitutable for x in φ. Now assume $y = \imath x\varphi$, to show φ^y_x. It follows from this assumption by theorem (12) that $\mathcal{A}\varphi^y_x$. But since φ is modally collapsed, there is a proof of $\Box(\varphi \to \Box\varphi)$. Since this latter is a theorem, it follows by GEN that $\forall x\Box(\varphi \to \Box\varphi)$. Instantiating to y it follows that $\Box(\varphi^y_x \to \Box\varphi^y_x)$. But as we saw in footnote 12, a formula of this form implies $\mathcal{A}\varphi^y_x \equiv \varphi^y_x$. Hence, φ^y_x. ⋈

(14) Suppose s' isn't free in φ and φ is modally collapsed. To show:

$$s = \imath s'\forall p(s' \models p \equiv \varphi) \to \forall p(s \models p \equiv \varphi)$$

it suffices to show that the formula $\forall p(s' \models p \equiv \varphi)$ is modally collapsed, for then our theorem becomes an instance of (13). So we have to prove:

$$\Box\big(\forall p(s' \models p \equiv \varphi) \to \Box\forall p(s' \models p \equiv \varphi)\big)$$

By the Rule of Necessitation, it suffices to prove:

$$\forall p(s' \models p \equiv \varphi) \to \Box\forall p(s' \models p \equiv \varphi)$$

So assume $\forall p(s' \models p \equiv \varphi)$, to show $\Box\forall p(s' \models p \equiv \varphi)$. By the Barcan Formula, it suffices to show $\forall p\Box(s' \models p \equiv p)$. Since p isn't free in our assumption, it remains, by GEN, to show $\Box(s' \models p \equiv p)$. So p is a fixed, but arbitrary proposition, and so our assumption that $\forall p(s' \models p \equiv \varphi)$ implies:

(A) $s' \models p \equiv \varphi$

By hypothesis, φ is modally collapsed, and so we know that the following is a theorem:

(B) $\Box(\varphi \rightarrow \Box\varphi)$

But independently, note that $s' \models p$ is defined in (3) as $s'[\lambda y\, p]$, and so it is a formula of the form xF. Since the modal logic of encoding is expressed by the principle $xF \rightarrow \Box xF$ (Zalta 1993, 403), it follows by the Rule of Necessitation that $\Box(xF \rightarrow \Box xF)$. Hence as an instance, we know:

(C) $\Box(s' \models p \rightarrow \Box s' \models p)$

But it is a theorem of modal logic that if formulas ψ and χ necessarily imply their own necessity, then the material equivalence of ψ and χ necessarily implies their necessary equivalence:

$$(\Box(\psi \rightarrow \Box\psi) \,\&\, \Box(\chi \rightarrow \Box\chi)) \rightarrow \Box((\psi \equiv \chi) \rightarrow \Box(\psi \equiv \chi))$$

Given this theorem and setting ψ to $s \models p$ and χ to φ, (C) and (B) jointly imply:

$$\Box((s' \models p \equiv \varphi) \rightarrow \Box(s' \models p \equiv \varphi))$$

So by the T schema,

$$(s' \models p \equiv \varphi) \rightarrow \Box(s' \models p \equiv \varphi)$$

Hence, by (A), $\Box(s' \models p \equiv \varphi)$, which is what it remained to show. ⋈

(**16**) First, we show that $\neg s \models \overline{p}$ is a modally collapsed formula:

Lemma: $\Box(\neg s \models \overline{p} \rightarrow \Box\neg s \models \overline{p})$

Proof. By the Rule of Necessitation, it suffices to prove $\neg s \models \overline{p} \rightarrow \Box\neg s \models \overline{p}$. So assume $\neg s \models \overline{p}$, to show $\Box\neg s \models \overline{p}$. Now, as previously noted in the text, the modal logic of encoding is $xF \rightarrow \Box xF$. So, by the T schema and the Rule of Necessitation, we know $\Box(xF \equiv \Box xF)$. This implies $\Box(\Diamond xF \equiv xF)$. As an instance of this latter, $\Box(\Diamond s \models \overline{p} \equiv s \models \overline{p})$. Then by the T schema, $\Diamond s \models \overline{p} \equiv s \models \overline{p}$. So, negating both sides, $\neg\Diamond s \models \overline{p} \equiv \neg s \models \overline{p}$. Then by our assumption, it follows that $\neg\Diamond s \models \overline{p}$, which is equivalent to $\Box\neg s \models \overline{p}$, which is what we had to show.

Now note that we can apply GEN to (14), since s is a free variable, to conclude:

$\forall s\big(s = \imath s'\forall p(s' \models p \equiv \varphi) \rightarrow \forall p(s \models p \equiv \varphi)\big)$,
provided s' isn't free in φ and φ is modally collapsed

Now since s' isn't free in $\neg s \models \overline{p}$ and this formula is modally collapsed, we can let φ be $\neg s \models \overline{p}$, so that as an instance of the foregoing, we know:

$$\forall s\big(s = \imath s'\forall p(s' \models p \equiv \neg s \models \overline{p}) \rightarrow \forall p(s \models p \equiv \neg s \models \overline{p})\big)$$

So we may instantiate s^* into this universal claim and the result is:

$$s^* = \imath s' \forall p(s' \models p \equiv \neg s \models \overline{p}) \to \forall p(s^* \models p \equiv \neg s \models \overline{p})$$

So by definition (15), $\forall p(s^* \models p \equiv \neg s \models \overline{p})$. ⋈

(17) By (16) we know:

(A) $\forall p(s^* \models p \equiv \neg s \models \overline{p})$

Since $\varphi \equiv \neg\psi$ is necessarily equivalent to $\neg\varphi \equiv \psi$, it follows from (A) by the Rule of Substitution that:

(B) $\forall p(\neg s^* \models p \equiv s \models \overline{p})$

And since $\varphi \equiv \psi$ is necessarily equivalent to $\psi \equiv \varphi$, it follows from (B) by the Rule of Substitution that:

$$\forall p(s \models \overline{p} \equiv \neg s^* \models p) \qquad ⋈$$

(20) Take the following as a global assumption:

(A) $s = s^{**}$

We want to prove that if $GlutOn(s, p)$, then $GapOn(s^*, p)$. So assume $GlutOn(s, p)$, i.e., by (18), that:

(B) $s \models p \;\&\; s \models \overline{p}$

To show $GapOn(s^*, p)$, we have to show both (a) $\neg s^* \models p$ and (b) $\neg s^* \models \overline{p}$, by (19).
(a) If we instantiate (17) to s and p, we obtain:

$$s \models \overline{p} \equiv \neg s^* \models p$$

So by the 2nd conjunct of (B), $\neg s^* \models p$.
(b) If we instantiate (17) to s^* and p, we obtain:

(C) $s^* \models \overline{p} \;\equiv\; \neg s^{**} \models p$

But the 1st conjunct of (B) implies, under our global assumption $s = s^{**}$ (A), that $s^{**} \models p$. But this fact and (C) jointly imply $\neg s^* \models \overline{p}$. ⋈

(21) Take the following as a global assumption:

(A) $s = s^{**}$

We want to prove that if $GapOn(s, p)$, then $GlutOn(s^*, p)$. So assume $GapOn(s, p)$, i.e., by (19), that:

(B) $\neg s \models p \;\&\; \neg s \models \overline{p}$

Then to show $GlutOn(s^*, p)$, we show both (a) $s^* \models p$ and (b) $s^* \models \overline{p}$, by (18).
(a) If we instantiate (16) to s and p, we obtain:

$$s^* \models p \equiv \neg s \models \overline{p}$$

This result and the second conjunct of (B) imply $s^* \models p$.

(b) If we instantiate (17) to s^* and p, we obtain:

(C) $s^* \models \overline{p} \equiv \neg s^{**} \models p$

But given our global assumption (A) that $s = s^{**}$, it follows from the first conjunct of (B) that $\neg s^{**} \models p$. But from this fact and (C), it follows that $s^* \models \overline{p}$. $\bowtie$

(22) Assume both $\neg GlutOn(s,p)$ and $\neg GapOn(s,p)$. Then by definitions (18) and (19), we know:

$$\neg(s \models p \;\&\; s \models \overline{p})$$

$$\neg(\neg s \models p \;\&\; \neg s \models \overline{p})$$

These are, respectively, equivalent to:

(A) $\neg s \models p \lor \neg s \models \overline{p}$

(B) $s \models p \lor s \models \overline{p}$

We may then prove both directions of $s^* \models p \equiv s \models p$. ($\rightarrow$) Assume $s^* \models p$. Then by (16), $\neg s \models \overline{p}$. It follows from this and (B) that $s \models p$. ($\leftarrow$) Assume $s \models p$. This and (A) imply $\neg s \models \overline{p}$. So by (16), $s^* \models p$. $\bowtie$

(23) Assume:

$$\forall p(\neg GlutOn(s,p) \;\&\; \neg GapOn(s,p))$$

To show $s^* = s$, we have to show $\forall p(s^* \models p \equiv \neg s \models p)$, by (9). By GEN, we show $s^* \models p \equiv s \models p$. But if we instantiate our assumption to p, we obtain $\neg GlutOn(s,p) \;\&\; \neg GapOn(s,p)$, and so $s^* \models p \equiv s \models p$ follows by (22). $\bowtie$

(24) ($\rightarrow$) Our (global) assumption is:

$$\forall p(\neg GlutOn(s,p) \;\&\; \neg GapOn(s,p))$$

We want to show $\forall p(s \models p \equiv \neg s \models \overline{p})$. By GEN, it suffices to show $s \models p \equiv \neg s \models \overline{p}$. But it is an immediate consequence of our global assumption that:

(A) $\neg GlutOn(s,p) \;\&\; \neg GapOn(s,p)$

We use this to prove both directions of our biconditional:

($\rightarrow$) Assume (locally) $s \models p$. The first conjunct of (A) and definition (18) imply $\neg(s \models p \;\&\; s \models \overline{p})$, i.e., $\neg s \models p \lor \neg s \models \overline{p}$. This last fact and our local assumption jointly imply $\neg s \models \overline{p}$.

($\leftarrow$) Assume (locally) $\neg s \models \overline{p}$. The second conjunct of (A) and definition (19) imply $\neg(\neg s \models p \;\&\; \neg s \models \overline{p})$, i.e., $s \models p \lor s \models \overline{p}$. But his last fact and our local assumption jointly imply $s \models p$.

($\leftarrow$) Our (global) assumption is:

$$\forall p(s \models p \equiv \neg s \models \overline{p})$$

To show $\forall p(\neg GlutOn(s,p) \,\&\, \neg GapOn(s,p))$, it suffices by &I and GEN and to show both (a) $\neg GlutOn(s,p)$ and (b) $\neg GapOn(s,p)$. But it is an immediate consequence of our global assumption that:

(B) $s \models p \equiv \neg s \models \overline{p}$

We use this to show both directions of our biconditional:

($\rightarrow$) Assume, for reductio, that $GlutOn(s,p)$. Then by definition (18), we know both $s \models p$ and $s \models \overline{p}$. But the former implies the negation of the latter, by (B). Contradiction.

($\leftarrow$) Assume, for reductio, that $GapOn(s,p)$. Then by (19), we know both $\neg s \models p$ and $\neg s \models \overline{p}$. But again, the former implies the negation of the latter, by (B). Contradiction. $\bowtie$

(27) Take as our global assumption that $\forall s(s^{**} = s)$. From definition (25) and the fact that the condition $p \neq p$ is modally collapsed (by the necessity of identity), it follows that $\forall p(\boldsymbol{s}_{\emptyset} \models p \equiv p \neq p)$, by (14). But since no proposition fails to be self-identical, it follows from this last fact that $\neg\exists p(\boldsymbol{s}_{\emptyset} \models p)$. This implies $\forall p \neg(\boldsymbol{s}_{\emptyset} \models p)$. Now let q be an arbitrarily chosen proposition, so that we know both $\neg \boldsymbol{s}_{\emptyset} \models q$ and $\neg \boldsymbol{s}_{\emptyset} \models \overline{q}$. Then by definition (19), $GapOn(\boldsymbol{s}_{\emptyset}, q)$. But given our global assumption, we know $\boldsymbol{s}_{\emptyset}{}^{**} = \boldsymbol{s}_{\emptyset}$. So by the relevant instance of (21), it follows from $GapOn(\boldsymbol{s}_{\emptyset}, q)$ that $GlutOn(\boldsymbol{s}_{\emptyset}{}^{*}, q)$. From this, it follows *a fortiori* by definition (18) that $\boldsymbol{s}_{\emptyset}{}^{*} \models q$. Since q was arbitrary, we have established:

(A) $\forall p(\boldsymbol{s}_{\emptyset}{}^{*} \models p)$

But, independently, we also know, given definition (26) and the fact that the condition $p = p$ is modally collapsed (by the necessity of identity), that $\forall p(\boldsymbol{s}_{\boldsymbol{V}} \models p \equiv p = p)$. Since every proposition is self-identical, it follows from this last fact that:

(B) $\forall p(\boldsymbol{s}_{\boldsymbol{V}} \models p)$

Now $\forall p\varphi \,\&\, \forall p\psi$ implies $\forall p(\varphi \equiv \psi)$. So we may conclude from (A) and (B) that:

$$\forall p(\boldsymbol{s}_{\emptyset}{}^{*} \models p \equiv \boldsymbol{s}_{\boldsymbol{V}} \models p)$$

Since $\boldsymbol{s}_{\emptyset}{}^{*}$ and $\boldsymbol{s}_{\boldsymbol{V}}$ are situations that make the same propositions true, it follows by (9) that $\boldsymbol{s}_{\emptyset}{}^{*} = \boldsymbol{s}_{\boldsymbol{V}}$. $\bowtie$

(28) (Exercise)

(29) ($\rightarrow$) Assume $s^{**} = s$. By GEN, it suffices to show $s \models p \equiv s \models \overline{\overline{p}}$. The identity of s^{**} and s implies, by (9), that $\forall p(s^{**} \models p \equiv s \models p)$. Hence $s^{**} \models p \equiv s \models p$, which commutes to:

(A) $s \models p \equiv s^{**} \models p$

Now, independently, if we instantiate (16) to s^* and p, we also know:

(B) $s^{**} \models p \equiv \neg s^* \models \overline{p}$

Moreover, independently, we know $s^* \models \overline{p} \equiv \neg s \models \overline{\overline{p}}$, by instantiating (16) to s and $\overline{p}$. By negating both sides and eliminating the double negation, we have:

(C) $\neg s^* \models \overline{p} \equiv s \models \overline{\overline{p}}$

So $s \models p \equiv s \models \overline{\overline{p}}$, by biconditional syllogism from (A), (B), and (C).

($\leftarrow$) Assume:

(D) $\forall p(s \models p \equiv s \models \overline{\overline{p}})$

To establish $s^{**} = s$, we appeal to (9) and show $\forall p(s^{**} \models p \equiv s \models p)$. By GEN, it suffices to show $s^{**} \models p \equiv s \models p$. First note that, by GEN, (16) holds for all s and so if we instantiate the resulting universal claim to s^* and p, we obtain:

(E) $s^{**} \models p \equiv \neg s^* \models \overline{p}$

Independently, if we instantiate (17) to s and $\overline{p}$ and commute the result, we obtain:

(F) $\neg s^* \models \overline{p} \equiv s \models \overline{\overline{p}}$

Moreover, if instantiate (D) to p and commute the result, we know:

(G) $s \models \overline{\overline{p}} \equiv s \models p$

But now, (E), (F), and (G) jointly imply:

$$s^{**} \models p \equiv s \models p \qquad \bowtie$$

(31) Assume $Hype(p)$. Then by (30), $p = \overline{\overline{p}}$. So we may substitute $\overline{\overline{p}}$ for the first occurrence of p in the identity $\overline{p} = \overline{p}$, to obtain $\overline{\overline{\overline{p}}} = \overline{p}$. So by definition (30), $Hype(\overline{p})$. $\bowtie$

(34) By reasoning analogous to (8).

(35) By (34) and the definition of identity for situations (9).

(37) This is a consequence of (36) and (14), and the fact that $s \models p \vee s' \models p$ is modally collapsed. $\bowtie$

(39) We prove both directions.

($\rightarrow$) Assume $s \trianglelefteq s'$. It follows that $\forall p(s \models p \rightarrow s' \models p)$, by definition (38). Now to show $s \oplus s' = s'$, we have to show that $s \oplus s'$ and s' make the same propositions true, by (9). That is, we have to show, for an arbitrary p, that $s \oplus s' \models p \equiv s' \models p$. But both directions of this biconditional hold. If $s \oplus s' \models p$ then either $s \models p$ or $s' \models p$, by (37). But in either case, $s' \models p$, given that every proposition true in

s is true in s'. And if $s' \models p$, then clearly, by a fact about $\oplus$ (37), it follows that $s \oplus s' \models p$.

($\leftarrow$) Assume $s \oplus s' = s'$. It follows by (9) that $s \oplus s'$ and s' make the same propositions true. Now to show $s \trianglelefteq s'$, we need to show, for an arbitrary proposition p, that $s \models p \rightarrow s' \models p$. So assume $s \models p$, to show $s' \models p$. But since $s \oplus s'$ and s' make the same propositions true, it suffices to show $s \oplus s' \models p$. But this follows from our assumption that $s \models p$, by (37). $\bowtie$

(40) The idempotence, commutativity, and associativity of $\oplus$ with respect to situations and, *a fortiori*, *HypeStates*, follows from (37) and the the facts that $\vee$ is idempotent, commutative, and associative. $\bowtie$

(42) This follows from the definition of $s\,!\,s'$ (41) once it is instantiated when to *HypeStates* s and s'.

(43) Assume $\mathsf{s}\,!\,\mathsf{s}'$. Then by definition (41), we know $\exists \mathsf{p}(\mathsf{s} \models \mathsf{p}\ \&\ \mathsf{s}' \models \overline{\mathsf{p}})$. Suppose p_1 is such a proposition, so that we know $\mathsf{s} \models \mathsf{p}_1$ and $\mathsf{s}' \models \overline{\mathsf{p}_1}$. But since $\mathsf{s} \models \mathsf{p}_1$, so does $\mathsf{s} \oplus \mathsf{s}''$, by theorem (37). And by that same theorem, since $\mathsf{s}' \models \overline{\mathsf{p}_1}$, so does $\mathsf{s}' \oplus \mathsf{s}'''$. Hence:

$$\exists \mathsf{p}\big((\mathsf{s} \oplus \mathsf{s}'' \models \mathsf{p})\ \&\ (\mathsf{s}' \oplus \mathsf{s}''' \models \overline{\mathsf{p}})\big)$$

So by definition (41), $(\mathsf{s} \oplus \mathsf{s}'')\,!\,(\mathsf{s}' \oplus \mathsf{s}''')$. $\bowtie$

(45) (Exercise)

(46) By reasoning analogous to the proof of (ω) in Section 3, though stated in terms of *Hype*-propositions and *HypeStates*. $\bowtie$

(47) This follows from (45) by (46) and the Rule of Substitution. $\bowtie$

(48) (Exercise)

(49) Assume *GlutOn*(s,p), i.e., by (18) that:

(A) $\mathsf{s} \models \mathsf{p}$

(B) $\mathsf{s} \models \overline{\mathsf{p}}$

We want to to show *GapOn*$(\mathsf{s}^*,\mathsf{p})$, i.e., by (19), that both (a) $\neg\mathsf{s}^* \models \mathsf{p}$ and (b) $\neg\mathsf{s}^* \models \overline{\mathsf{p}}$. (a) This follows from (B) by (48). (b) If we instantiate (47) to $\overline{\mathsf{p}}$, we have $\mathsf{s}^* \models \overline{\mathsf{p}} \equiv \neg\mathsf{s} \models \overline{\overline{\mathsf{p}}}$. But this is equivalent to $\neg\mathsf{s}^* \models \overline{\mathsf{p}} \equiv \mathsf{s} \models \overline{\overline{\mathsf{p}}}$. Since *Hype*-propositions are identical to their double-negations (30), it follows that $\neg\mathsf{s}^* \models \overline{\mathsf{p}} \equiv \mathsf{s} \models \mathsf{p}$. Then by (A), we may infer $\neg\mathsf{s}^* \models \overline{\mathsf{p}}$. $\bowtie$

(50) Assume *GapOn*(s,p), i.e., by (19):

(A) $\neg\mathsf{s} \models \mathsf{p}$

(B) $\neg\mathsf{s} \models \overline{\mathsf{p}}$

We want to show *GlutOn*(s^*, p), i.e., by (18), that both (a) $s^* \models p$ and (b) $s^* \models \overline{p}$. (a) This follows from (B) by (47). (b) If we instantiate (47) to $\overline{p}$, we have $s^* \models \overline{p} \equiv \neg s \models \overline{\overline{p}}$. Since *Hype*-propositions are identical to their double-negations (30), it follows that $s^* \models \overline{p} \equiv \neg s \models p$. From this and (A) it follows that $s^* \models \overline{p}$. ⋈

(51) This follows by applying the reasoning in (22) to *HypeStates* and *Hype*-propositions. ⋈

(52) To establish $s^{**} = s$, we note that since *HypeStates* encode only *Hype*-propositions (33), it suffices by (9) to show $\forall p(s^{**} \models p \equiv s \models p)$. By GEN, it then suffices to show $s^{**} \models p \equiv s \models p$. Now if we instantiate (47) to s^*, we obtain:

(A) $s^{**} \models p \equiv \neg s^* \models \overline{p}$

Independently, if instantiate (48) to $\overline{p}$, we obtain $s \models \overline{\overline{p}} \equiv \neg s^* \models \overline{p}$, which by the commutativity of the biconditional, implies:

$$\neg s^* \models \overline{p} \equiv s \models \overline{\overline{p}}$$

And since *Hype*-propositions are identical with their double negations, it follows from this last result that:

(B) $\neg s^* \models \overline{p} \equiv s \models p$

But (A) and (B) imply $s^{**} \models p \equiv s \models p$. ⋈

(53) Assume, for reductio, that $s\,!\,s^*$. So by definition (41), $\exists p(s \models p \,\&\, s^* \models \overline{p})$. Let q_1 be such a proposition, so that we know $s \models q_1$ and $s^* \models \overline{q_1}$. By a key fact about s^* (47), the latter implies $\neg s \models \overline{\overline{q_1}}$. But since *Hype*-propositions are identical with their double negations, it follows that $\neg s \models q_1$. Contradiction. ⋈

(54) Assume $\neg s\,!\,s'$. So by definition (41):

(A) $\neg\exists p(s \models p \,\&\, s' \models \overline{p})$

We want to show $s' \oplus s^* = s^*$. By (9) and the fact that *HypeStates* encode only *Hype*-propositions (33), it suffices to show that $\forall p((s' \oplus s^*) \models p \equiv s^* \models p)$. So, by GEN, we show $(s' \oplus s^*) \models p \equiv s^* \models p$.

($\rightarrow$) Assume $(s' \oplus s^*) \models p$. Independently, by (37), we know:

$$\forall p((s' \oplus s^*) \models p \equiv s' \models p \vee s^* \models p)$$

Hence, $s' \models p \vee s^* \models p$. Assume, for reductio, that $\neg s^* \models p$. Then $s' \models p$, and since *Hype*-propositions are identical to their double negations (30), we know $s' \models \overline{\overline{p}}$. But it also follows from our reductio assumption, by (48), that $s \models \overline{p}$. So we've established $s \models \overline{p} \,\&\, s' \models \overline{\overline{p}}$. Existentially generalizing on $\overline{p}$, it follows that $\exists q(s \models q \,\&\, s' \models \overline{q})$, which contradicts (A).

($\leftarrow$) Exercise. ⋈

(55) Assume $s \trianglelefteq s'$. Since theorem (39) holds for any situations, it holds for *HypeStates*. So it follows that:

(A) $\mathsf{s} \oplus \mathsf{s}' = \mathsf{s}'$

Now independently, by (53), we know that s' is not incompatible with its Routley star image s'^*, i.e., $\neg \mathsf{s}' \,!\, \mathsf{s}'^*$. From this and (A), it follows that the fusion of s and s' is not incompatible with with the Routley star image of s', i.e., that:

(B) $\neg(\mathsf{s} \oplus \mathsf{s}') \,!\, \mathsf{s}'^*$

Now consider the following Lemma, which holds for any situations s, s', and s'':

> *Lemma*: $\neg(s \oplus s') \,!\, s'' \rightarrow \neg s \,!\, s''$
> *Proof*: Assume $\neg(s \oplus s') \,!\, s''$. Then by definition of ! (41), $\neg \exists p((s \oplus s') \models p \;\&\; s'' \models \overline{p})$. Now suppose, for reductio, that $s \,!\, s''$. Then $\exists p(s \models p \;\&\; s'' \models \overline{p})$. Suppose q_1 is such a proposition, so that we know both $s \models q_1$ and $s'' \models \overline{q_1}$. But the former implies $s \oplus s' \models q_1$, by definition of $s \oplus s'$ (36). So we know $(s \oplus s') \models q_1 \;\&\; s'' \models \overline{q_1}$. Hence, $\exists p((s \oplus s') \models p \;\&\; s'' \models \overline{p})$. Contradiction.

Given this Lemma, it follows from (B) that s is not incompatible with s'^*, i.e., $\neg \mathsf{s} \,!\, \mathsf{s}'^*$. But by (54), this last result implies $\mathsf{s}'^* \oplus \mathsf{s}^* = \mathsf{s}^*$. Hence, by (39), $\mathsf{s}'^* \trianglelefteq \mathsf{s}^*$. ⋈

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