A quasi-relevant, connexive 4-valued implicative expansion of Belnap-Dunn logic defining material connexive logic **MC**

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Abstract

In this paper, a connexive 4-valued implicative expansion of Belnap-Dunn logic we have dubbed $\mathbf{LMI4^C}$ is defined. It is a quasi relevant logic in the sense that it enjoys the "quasi relevance property". Also, $\mathbf{LMI4^C}$ defines "material connexive logic" **MC**. The fact that $\mathbf{LMI4^C}$ defines classical positive logic C_+ is used to provide it with a Hilbert-style formulation presenting $\mathbf{LMI4^C}$ as an expansion of C_+. Said formulation is obtained by using a Belnap-Dunn "two-valued semantics".

Keywords: Belnap-Dunn logic; connexive logics; quasi relevant logics; material connexive logic **MC**; Belnap-Dunn semantics.

Preamble

As Gemma Robles has commented in the introduction to her contribution to this special issue, we have been working on topics originally defined by Ross T. Brady since some time ago. The paper I present is on connexive logic, a topic he also has contributed to in [6]. I surely know that my paper is a poor token of my gratitude to Professor Brady's work on relevant logics and akin topics. But *lo que cuenta es la intención*, as we sometimes say in Spanish ("it is the intention that counts"). Thus, I can only hope that my intention has been hinted above.

1 Introduction

The name "connexive logic" was introduced by Storrs McCall and seems to suggest that there is some sort of connection between the antecedent and the consequent of valid implications in systems of connexive logic (cf. [29] for a clear exposition of the state of the art in the topic). In this sense, connexive logics and relevant logics pursue a similar aim, although with very different methodologies and results. Particularly, and contrary to what the case is with relevant logics (cf., e.g., [16]), connexive logics are *contraclassical logics* (cf. [15]): they are neither subsystems nor extensions of classical logic. And nevertheless, in this paper we are going to build a bridge between the two families of logics by presenting a connexive logic enjoying the quasi relevance property (cf. Proposition 4.9).

Connexive logics can be in short characterized by having as theorems Aristotle's theses (AT, AT′) and Boethius' theses (BT, BT′): (AT) $\sim(A \to \sim A)$; (AT′) $\sim(\sim A \to A)$; (BT) $(A \to B) \to \sim(A \to \sim B)$; (BT′) $(A \to \sim B) \to \sim(A \to B)$ (cf. Definition 2.5).

Well then, initially, the plan was to define an implicative expansion of Belnap-Dunn logic (cf. Definition 2.3) resulting in an extension of some weak relevant logic. But just at the outset we faced a problem: as proved in [20], it suffices to add AT to Routley and Meyer's basic relevant logic **B** for deriving $\sim(A \to B)$, hardly acceptable as a theorem of any logic whatsoever. Consequently, we focused on restrictions of **B**, in particular, to that obtained when the rules Suffixing (Suf) and Prefixing (Pref) are replaced by the transitivity rule. Later, it developed that the transitivity rule could be strengthened to the corresponding thesis. Thus, $\mathbf{B}'$ (**B** without Suf and Pref but with the transitivity axiom —cf. Definition 2.9) is the weak relevant logic we are going to extend to a 4-valued connexive logic. It has to be noted that $\mathbf{B}'$ is not included and neither includes **B**. It is, in fact, a restriction of Brady's **DJ** (cf. [7, 8]): roughly, it is **DJ** without Suf, Pref and the contraposition axiom (notice, however, that $\mathbf{B}'$ has the contraposition rule).

The connexive logic defined in the present paper and that we have dubbed $\mathbf{LMI4^C}$ is, we think, a strong logic with interesting properties, some of which are briefly discussed at the end of section 4. One of them is the definability of material connexive logic **MC** (cf. [29, §4.5.3]), a logic with a material conditional located at the antipodes of the relevant logics province and containing every possible classical instance of what relevant logicians have dubbed 'positive paradox', i.e., $A \supset (B \supset A)$.

The fact that a material conditional is definable in $\mathbf{LMI4^C}$ will be used in order to provide a Hilbert-style formulation (H-formulation) for it. This H-formulation is obtained by using Belnap-Dunn two-valued semantics (cf. [3, 4, 10, 11, 12]), according to a strategy devised by Brady in [5] (cf. section 4).

The paper is organized as follows. In section 2, we display some preliminary notions and results. In section 3, the matrix $\mathrm{MI4^C}$ and the logic $\mathbf{LMI4^C}$ it determines are defined. Also, we remark some definable connectives and, in particular **MC**-material conditional. In section 5, $\mathbf{LMI4^C}$ is given the H-formulation $\mathrm{HLMI4^C}$. Then some significant properties of both $\mathbf{LMI4^C}$ and $\mathrm{HLMI4^C}$ are briefly discussed. Finally, in section 5, we note some remarks on the results obtained and on possible future work in the same line. The paper is ended with an appendix and some technical results referred to along it.

2 Preliminary notions and results

In this section, we define some prior concepts and results that will prove useful in the sequel.

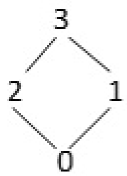
Remark 2.1 (Some preliminary notions)**.** The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ...$, and some or all of the following connectives: $\rightarrow$ (conditional or implication[1]), $\wedge$ (conjunction), $\vee$ (disjunction) and $\sim$ (negation). The biconditional ($\leftrightarrow$) and the set of formulas (wffs) are defined in the customary way. A, B, C, etc. are metalinguistic variables. Then the ensuing concepts are understood in a fairly standard sense: logical matrix M, M-interpretation, M-consequence and M-validity. Also, the following notions: functions definable in a matrix, functional inclusion and functional equivalence (cf., e.g., [24, §2] or [25]).

Remark 2.2 (Logics)**.** In this paper, logics are primarily viewed as M-determined structures, i.e., as structures of the type $(\mathcal{L}, \vDash_{\mathrm{M}})$ where $\mathcal{L}$ is a propositional language and $\vDash_{\mathrm{M}}$ is a (consequence) relation defined in $\mathcal{L}$ according to the logical matrix M as follows: for any set of wffs Γ and wff A, $\Gamma \vDash_{\mathrm{M}} A$ iff $I(A) \in D$ whenever $I(\Gamma) \in D$ for all M-interpretations I ($I(\Gamma) \in D$ iff $I(A) \in D$ for all $A \in \Gamma$; D is the set of designated values in M). Thus, from this viewpoint, we can safely travel back and forth from matrices to logics, given the aims of this paper.

Nevertheless, logics are sometimes defined as Hilbert-type axiomatic systems, the notions of "theorem" and "proof from premises" being the usual ones. Furthermore, in a derived or secondary sense, we can regard an M-determined logic as a, say, Hilbert-type system (or a natural deduction system or a Gentzen-type system) L such that $\Gamma \vdash_{\mathrm{L}} A$ iff $\Gamma \vDash_{\mathrm{M}} A$, where $\vDash_{\mathrm{M}}$ is the consequence relation defined above and $\Gamma \vdash_{\mathrm{L}} A$ means "A is provable from Γ in L".

[1] We follow Anderson and Belnap's "Grammatical Propaedeutic", Appendix to [1]: "The principal aim of this piece is to convince the reader that it is philosophically respectable to "confuse" implication and entailment with the conditional, and indeed philosophically suspect to harp on the dangers of such "confusion"" ([1, p. 473].

Definition 2.3 (Belnap and Dunn's matrix $\mathcal{FOUR}$)**.** The propositional language consists of the connectives $\wedge, \vee$ and $\sim$. Belnap and Dunn's matrix $\mathcal{FOUR}$ is the structure $(\mathcal{V}, D, \mathtt{F})$ where (1) $\mathcal{V}$ is $\{0, 1, 2, 3\}$ and is partially ordered as shown in the following lattice:



(2) $D = \{2, 3\}$; $\mathtt{F} = \{f_\wedge, f_\vee, f_\sim\}$ where $f_\wedge$ and $f_\vee$ are defined as the glb (or lattice meet) and the lub (or lattice join), respectively. Finally, $f_\sim$ is an involution with $f_\sim(0) = 3, f_\sim(3) = 0, f_\sim(1) = 1, f_\sim(2) = 2$ (cf. [3, 4, 10, 11, 12]). We display the tables for $\wedge, \vee$ and $\sim$:

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

	$\sim$
0	3
1	1
2	2
3	0

Remark 2.4 (On the symbols for referring to the four truth-values)**.** It is customary to use f, n, b and t instead of $0, 1, 2$ and 3, respectively (cf., e.g., [21]). The former stand for false only, neither true nor false, both true and false and true only, respectively. The latter have been chosen in order to use the tester in [14], in case one is needed and to put in connection the results in the present paper with previous work by us.

Definition 2.5 (Connexive logics)**.** Let $\mathcal{L}$ be a language with connectives $\rightarrow$ and $\sim$ and $\mathcal{L}'$ be an expansion of $\mathcal{L}$. A logic L built upon $\mathcal{L}'$ is a connexive logic if the theses AT, AT′, BT and BT′ are L-theorems but $(A \rightarrow B) \rightarrow (B \rightarrow A)$ fails to be an L-theorem (that is, implication is non-symmetric). As noted in the introduction, AT (resp., BT) abbreviate 'Aristotle's thesis' (resp., 'Boethius' thesis') which can be given in two versions (cf. [29]):

$$\text{AT. } \sim(A \rightarrow \sim A)$$
$$\text{AT}'. \sim(\sim A \rightarrow A)$$
$$\text{BT. } (A \rightarrow B) \rightarrow \sim(A \rightarrow \sim B)$$
$$\text{BT}'. (A \rightarrow \sim B) \rightarrow \sim(A \rightarrow B)$$

Definition 2.6 (The logic **B**)**.** Routley and Meyer's basic logic **B** can be defined with the following axioms and rules of inference (cf. [28, Chapter 4]; $A_1, ..., A_n \Rightarrow B$ means "if $A_1, ..., A_n$, then B").

Axioms:

a1. $A \rightarrow A$
a2. $(A \wedge B) \rightarrow A$; $(A \wedge B) \rightarrow B$
a3. $A \rightarrow (A \vee B)$; $B \rightarrow (A \vee B)$
a4. $[(A \rightarrow B) \wedge (A \rightarrow C)] \rightarrow [A \rightarrow (B \wedge C)]$
a5. $[(A \rightarrow C) \wedge (B \rightarrow C)] \rightarrow [(A \vee B) \rightarrow C]$
a6. $[A \wedge (B \vee C)] \rightarrow [(A \wedge B) \vee (A \wedge C)]$
a7. $A \rightarrow \sim\sim A$
a8. $\sim\sim A \rightarrow A$

Rules of inference:

r1 (Adj). $A, B \Rightarrow A \wedge B$
r2 (MP). $A \rightarrow B, A \Rightarrow B$
r3 (Suf). $A \rightarrow B \Rightarrow (B \rightarrow C) \rightarrow (A \rightarrow C)$
r4 (Pref). $B \rightarrow C \Rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$
r5 (Con). $A \rightarrow B \Rightarrow \sim B \rightarrow \sim A$

Adj, MP, Suf, Pref and Con abbreviate "adjunction", "modus ponens", "suffixing", "prefixing" and "contraposition", respectively.

Proposition 2.7 (No acceptable connexive extensions verifying **B**)**.** *The wff* $\sim(A \rightarrow B)$ *is derivable from* ***B*** *plus* $\sim(A \rightarrow \sim A)$ *(AT).*

Proof. In [20], it is proved that $\sim(A \rightarrow B)$ follows from Anderson and Belnap's *logic of entailment* **E** (cf. [1]) plus AT. But the proof holds in fact if **E** is restricted to **B** (cf. the first two lines of the proof of Theorem 2 in [20] —notice that the first occurrence of B in line 2 should be actually an A). The proof (abbreviated) could run as follows. (1) $(A \rightarrow \sim B) \rightarrow [A \rightarrow \sim(A \wedge B)]$ by a2, Con and Pref. (2) $[A \rightarrow \sim(A \wedge B)] \rightarrow [(A \wedge B) \rightarrow \sim(A \wedge B)]$ by a2 and Suf. (3) $(A \rightarrow \sim B) \rightarrow [(A \wedge B) \rightarrow \sim(A \wedge B)]$ by (1), (2) and transitivity of $\rightarrow$. (4) $\sim[(A \wedge B) \rightarrow \sim(A \wedge B)] \rightarrow \sim(A \rightarrow \sim B)$, by (3) and Con. (5) $\sim(A \rightarrow \sim B)$, by (4) and AT. Finally, (6) $\sim(A \rightarrow B)$, by (5), a7, Pref and Con[2]. □

Moreover, a similar unacceptable result can be obtained without appealing to the entire strength of **B**.

[2]Here and throughout the paper, MP is occasionally used without explicit reference to it.

Proposition 2.8 (Derivability of $\sim(A \to \sim B)$)**.** *Let* L *be a logic containing at least a2, Con and Suf plus BT′. Then* $\sim(A \to \sim B)$ *is provable in* L.

Proof. (Abbreviated) (1) $(A \to \sim B) \to [(A \wedge B) \to \sim B]$, by a2 and Suf. (2) $(A \to \sim B) \to \sim[(A \wedge B) \to B]$, by 1, BT′ and transitivity of $\to$. (3) $\sim(A \to \sim B)$ by (2), a2 and Con (notice that Con is used in the form $(A \to \sim B) \Rightarrow (B \to \sim A)$). □

Now, $\sim(A \to \sim B)$ seems hardly more acceptable than $\sim(A \to B)$, whence it follows that there are not acceptable connexive expansions of **B** minus Pref. Well then, although the result in Proposition 2.8 does not follow if Suf is replaced by the Prefixing axiom (cf. matrix M1 in the appendix), we propose to consider extensions of the logic **B′** defined below. Anyway, the connexive logic introduced in this paper is an extension of **B′**, which, in addition, will have Pref.

Definition 2.9 (The logic **B′**)**.** The logic **B′** is axiomatized exactly as **B** except that Suf and Pref are replaced by the transitivity axiom

$$\text{Trans. } [(A \to B) \wedge (B \to C)] \to (A \to C)$$

Notice that **B′** is not contained in **B**. In fact, it is contained in Brady's **DJ** (cf. [7, 8]) (**B′** is, roughly, **DJ** minus Pref, Suf and the contraposition axiom but with the rule Con).

In the next section, we define the matrix MI4$^{\mathrm{C}}$ and the logic **LMI4$^{\mathbf{C}}$**.

3 The matrix MI4$^{\mathrm{C}}$ and the logic determined by it

The matrix MI4$^{\mathrm{C}}$ (the label intends to abbreviate "a connexive implicative expansion of $\mathcal{FOUR}$") can be defined as follows.

Definition 3.1 (The matrix MI4$^{\mathrm{C}}$)**.** The matrix MI4$^{\mathrm{C}}$ is the structure $(\mathcal{V}, D, \mathtt{F})$, where $\mathcal{V}$, D and $\mathtt{F}$ are defined exactly as in $\mathcal{FOUR}$, except for the addition of $f_\to$ interpreted according to the following truth-table:

$\to$	0	1	2	3
0	2	2	2	2
1	0	2	0	2
2	0	1	2	3
3	0	1	0	3

Definition 3.2 (The logic **LMI4$^{\mathbf{C}}$**)**.** The logic **LMI4$^{\mathbf{C}}$** is the one determined by the matrix MI4$^{\mathrm{C}}$ in the sense explained in Remark 2.2: for any set of wffs Γ and wff A, $\Gamma \vDash_{\mathbf{LMI4^C}} A$ iff $I(A) \in \{2,3\}$ whenever $I(\Gamma) \in \{2,3\}$ for all MI4$^{\mathrm{C}}$-interpretations I ($I(\Gamma) \in \{2,3\}$ iff $I(A) \in \{2,3\}$ for all $A \in \Gamma$).

Next, we observe a remark on some useful conventions. Then we note some definable connectives.

Remark 3.3 (Functions and truth-tables. On displaying proofs of definability)**.** Let f_* be a function defined in $\mathcal{V} = \{0,1,2,3\}$. In this paper, f_* is usually represented by means of a truth-table t_* (or simply $*$), as for instance, it is the case with $\wedge$, $\vee$ and $\sim$ in $\mathcal{FOUR}$ (Definition 2.3). In addition, by k_* (or simply $*$) we refer to the connective defined by t_*. Now, let M be $\mathcal{FOUR}$ or an expansion of it. The proof that a given unary or binary function f_* is definable in M is easily visualized by using the connectives corresponding to the functions in M needed in the proof in question. In general, proofs provided below are simplified as just indicated (A, B refer to any wffs —cf. Remark 2.1).

Proposition 3.4 (Four unary connectives)**.** *Consider the following negation connectives given by the ensuing truth-tables*

	$\neg_1$
0	3
1	3
2	2
3	0

	$\neg_2$
0	3
1	2
2	2
3	0

	$\neg_3$
0	3
1	0
2	2
3	0

and the unary connective given by the following truth-table:

	$\circ$
0	0
1	1
2	2
3	2

These connectives are definable in $\mathbf{LMI4^C}$.

Proof. We set (cf. Remark 2.1) $\neg_1 A =_{\mathrm{df}} (A \to \sim A) \vee \sim A$; $\neg_2 A =_{\mathrm{df}} \neg_1 A \wedge \sim(\sim A \to A)$; $\neg_3 A =_{\mathrm{df}} \neg_2 A \wedge \sim A$; $\circ A =_{\mathrm{df}} \sim(\neg_2 A \to \neg_3 A) \wedge A$. □

In addition to the unary connectives defined above, we note that acceptable truth-functional necessity and possibility modal connectives are definable in $\mathrm{MI4^C}$ (we use Łukasiewicz's symbols L and M for the necessity and possibility operators, respectively —cf. [13, notes 2 and 3]).

Proposition 3.5 (Necessity and possibility connectives)**.** *Consider the modal connectives given by the ensuing truth-tables:*

	L
0	0
1	0
2	2
3	3

	M
0	0
1	3
2	2
3	3

These connectives are definable in MI4^{C}.

Proof. We set $LA =_{\mathrm{df}} \sim\neg_1 A$; $MA =_{\mathrm{df}} \sim L\sim A$. □

These modal connectives have been studied in the context of Brady's important 4-valued logic **BN4** (cf. [19] and references therein). In the appendix, we show how to interpret them in Belnap-Dunn semantics, independently of the 4-valued extension or expansion of $\mathcal{FOUR}$ (if any) they have been defined in or added to. Also, we describe the modal logic these connectives define in the case of MI4$^{\mathrm{C}}$ which will immediately facilitate a, so to speak, 'descriptive' comparison with standard modal systems such as, for example, Lewis' strong logics **S4** and **S5** and those in the vicinity of them.

Proposition 3.6 (A conditional connective)**.** *Consider the conditional connective given by the ensuing truth-table:*

$\overset{\circ}{\to}$	0	1	2	3
0	2	3	2	3
1	2	2	2	2
2	0	1	2	3
3	0	1	2	3

This connective is definable in **LMI4$^{\mathbf{C}}$**

Proof. We set $A \overset{\circ}{\to} B =_{\mathrm{df}} [(\neg_3 A \to B) \wedge (\neg_2 A \vee B)] \vee (A \to B)$. □

Moreover, let MI4° be the matrix obtained when replacing the $f_\to$-table in Definition 3.1 by that for $f_{\overset{\circ}{\to}}$ in Proposition 3.6. Then MI4° defines a "material conditional" in the sense that, together with modus ponens for $\overset{\circ}{\to}$, A1, A2 and A3 listed below are verified when replacing $*$ by $\overset{\circ}{\to}$.

A1. $A * (B * A)$
A2. $[A * (B * C)] * [(A * B) * (A * C)]$
A3. $[(A * B) * A] * A$

This fact will be used for giving a Hilbert-style formulation of **LMI4$^{\mathbf{C}}$**.

Finally, it is shown that **MC**-logic (cf. [29, §4.5.3] and references therein), a system of material connexive logic, is definable in **LMI4$^{\mathbf{C}}$**.

Proposition 3.7 (Definability of **MC**-logic)**.** *Material connexive logic* ***MC*** *is definable in* ***LMI4^{C}***.

Proof. It suffices to define the **MC**-conditional table since the **MC**-tables for conjunction, disjunction and negation are the same as those in $\mathcal{FOUR}$. The **MC**-conditional table is the following one:

$\overset{\mathbf{MC}}{\to}$	0	1	2	3
0	2	2	2	2
1	2	2	2	2
2	0	1	2	3
3	0	1	2	3

Well then, we have $A \overset{\mathbf{MC}}{\to} B =_{\text{df}} \circ(A \overset{\circ}{\to} B) \vee (A \to B)$ (cf. Propositions 3.4 and 3.6). □

The section is ended by showing that $\mathbf{LMI4^C}$ is an acceptable connexive logic. In the following section, we note some other properties of this logic.

Proposition 3.8 ($\mathbf{LMI4^C}$ is an acceptable connexive logic)**.** *The logic* $\boldsymbol{LMI4^C}$ *is an acceptable connexive logic in the sense that it has the theses AT, AT′, BT and BT′, a non-symmetric conditional, but it is free from* $\sim(A \to B)$

Proof. Matrix MI4^{C} verifies $\sim(A \to \sim A)$, $\sim(\sim A \to A)$, $(A \to B) \to \sim(A \to \sim B)$, $(A \to \sim B) \to \sim(A \to B)$, but falsifies $(A \to B) \to (B \to A)$ and $\sim(A \to B)$ (in fact, $\sim(A \to A)$). (In case a tester is needed, the one in [14] can be used.) □

4 A Hilbert-style formulation of the logic $\mathbf{LMI4^C}$

In order to give a Hilbert-style formulation (H-formulation) for $\mathbf{LMI4^C}$, we rely upon a strategy based upon Belnap-Dunn two-valued semantics (cf. [3, 4, 10, 11, 12]) introduced by Brady in [5] (cf. also [7, 8, 28]) as applied in some papers as [17, 18] or [19]).

This strategy has been meticulously explained, step by step, in [27, §5]. So here it will suffice to provide the H-formulation HLMI4^{C} of $\mathbf{LMI4^C}$, a Belnap-Dunn semantics for $\mathbf{LMI4^C}$ and then prove soundness and completeness of HLMI4^{C} w.r.t. said semantics.

Definition 4.1 (The system HLMI4^{C})**.** The system HLMI4^{C} can be formulated as follows:

Axioms:

A1. $A \supset (B \supset A)$

A2. $[A \supset (B \supset C)] \supset [(A \supset B) \supset (A \supset C)]$

A3. $(A \wedge B) \to A$; $(A \wedge B) \to B$

A4. $A \supset [B \supset (A \wedge B)]$
A5. $[(A \supset C) \wedge (B \supset C)] \supset [(A \vee B) \supset C]$
A6. $[(A \to B) \wedge (B \to C)] \to (A \to C)$
A7. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
A8. $(A \to B) \supset (A \supset B)$
A9. $A \to {\sim}{\sim}A$
A10. ${\sim}{\sim}A \to A$
A11. $(A \to B) \supset ({\sim}B \to {\sim}A)$
A12. ${\sim}A \supset [A \vee (A \to B)]$
A13. $B \supset [{\sim}B \vee (A \to B)]$
A14. $(A \vee {\sim}B) \vee (A \to B)$
A15. $({\sim}A \wedge B) \supset (A \to B)$
A16. $(A \wedge {\sim}B) \supset {\sim}(A \to B)$
A17. $(A \vee {\sim}A) \vee {\sim}(A \to B)$
A18. ${\sim}A \supset [A \vee {\sim}(A \to B)]$
A19. $[{\sim}(A \to B) \wedge A] \supset {\sim}B$

Rule of inference:

$$\text{MP}_{\supset}.\ A \supset B, A \Rightarrow B$$

Definitions:

$$\text{DF}_{\supset}.\ A \supset B =_{\text{df}} A \overset{\circ}{\to} B \text{ (cf. Proposition 3.6)}$$
$$\text{DF}_{\vee}.\ A \vee B =_{\text{df}} {\sim}({\sim}A \wedge {\sim}B)$$
$$\text{DF}_{\leftrightarrow}.\ A \leftrightarrow B =_{\text{df}} (A \to B) \wedge (B \to A)$$

Notice that A18 is "contraclassical" (cf. [15]). That is, ${\sim}A \Rightarrow [A \vee {\sim}(A \to B)]$ is not valid when $\to, \wedge, \vee$ and $\sim$ are understood as the corresponding classical connectives.

Next, we note some theorems and rules of HLMI4$^{\text{C}}$ that will prove useful in the completeness proof.

Proposition 4.2 (Some theorems and rules of HLMI4$^{\text{C}}$). *The following are provable in HLMI4^{C}: (1) Adj: $A, B \Rightarrow A \wedge B$; (2) MP$_{\to}$: $A \to B, A \Rightarrow B$; (3) E$\wedge$: $A \wedge B \Rightarrow A, B$; (4) deduction theorem for $\supset$ (DT$_{\supset}$): if $\Gamma, A \vdash_{HLMI4^C} B$, then $\Gamma \vdash_{HLMI4^C} A \supset B$; (5) $(A \to {\sim}B) \supset (B \to {\sim}A)$; $({\sim}A \to B) \supset ({\sim}B \to A)$; $({\sim}A \to {\sim}B) \supset (A \to B)$; (6) $(A \wedge B) \supset A$, $(A \wedge B) \supset B$; (7) $A \to (A \vee B)$,*

$B \to (A \vee B)$; (8) $A \supset (A \vee B)$, $B \supset (A \vee B)$; (9) if A is an intuitionistic positive propositional tautology, then $\vdash_{HLMI4^C} A$; (10) $[(A \to C) \wedge (B \to C)] \supset [(A \vee B) \to C]$; (11) $\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$, $\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$; (12) modus tollens (MT): $A \to B, \sim B \Rightarrow \sim A$.

Proof. (1) A4, $\text{MP}_{\supset}$. (2) A8, $\text{MP}_{\supset}$. (3) A3, $\text{MP}_{\to}$. (4) A1, A2 and $\text{MP}_{\supset}$. (5) A6, A9, A10, A11. (6) A3, A8, $\text{MP}_{\supset}$. (7) A3, the items in (5) and DF$\vee$. (8) By the items in (7) and A8. (9) A1, A2, A4, A5, the items in (6), (8) and $\text{MP}_{\supset}$ (of course, when $\supset$ is replaced by the intuitionistic conditional). (10) A7, the items in (5) and DF$\vee$. (11) A6, A7, A9, A10, DF$\vee$ and DF$\leftrightarrow$. (12) A11, $\text{MP}_{\to}$ and $\text{MP}_{\supset}$. □

As a second step of the strategy commented upon above, we define a Belnap-Dunn semantics for $\mathbf{LMI4^C}$ (BDLMI4$^{\text{C}}$-semantics). The key notions are 'BDLMI4$^{\text{C}}$-model' and the accompanying ones 'BDLMI4$^{\text{C}}$-consequence' and 'BDLMI4$^{\text{C}}$-validity'.

Definition 4.3 (BDLMI4$^{\text{C}}$-models)**.** A BDLMI4$^{\text{C}}$-model is a structure (K, I) where (i) $K = \{\{T\}, \{F\}, \{T, F\}, \emptyset\}$, and (ii) I is a BDLMI4$^{\text{C}}$-interpretation from the set of all wffs to K, this notion being defined according to the following conditions ('clauses')[3] for each propositional variable p and wffs A, B:

1. $I(p) \in K$
2a. $T \in I(\sim A)$ iff $F \in I(A)$
2b. $F \in I(\sim A)$ iff $T \in I(A)$
3a. $T \in I(A \wedge B)$ iff $T \in I(A)$ & $T \in I(B)$
3b. $F \in I(A \wedge B)$ iff $F \in I(A)$ or $F \in I(B)$
4a. $T \in I(A \vee B)$ iff $T \in I(A)$ or $T \in I(B)$
4b. $F \in I(A \vee B)$ iff $F \in I(A)$ & $F \in I(B)$
5a. $T \in I(A \to B)$ iff $[T \notin I(A)$ & $F \in I(A)]$ or $[T \in I(B)$ & $F \notin I(B)]$ or $[T \notin I(A)$ & $F \notin I(B)]$ or $[F \in I(A)$ & $T \in I(B)]$
5b. $F \in I(A \to B)$ iff $[T \notin I(A)$ & $F \in I(A)]$ or $[T \notin I(A)$ & $F \notin I(A)]$ or $[T \in I(A)$ & $F \in I(B)]$

Definition 4.4 (BDLMI4$^{\text{C}}$-consequence, BDLMI4$^{\text{C}}$-validity)**.** Let M be a BDLMI4$^{\text{C}}$-model. For any set of wffs Γ and wff A:

1. $\Gamma \vDash_{\text{M}} A$ (A is a consequence of Γ in M) iff $T \in I(A)$ whenever $T \in I(\Gamma)$ ($T \in I(\Gamma)$ iff $\forall A \in \Gamma(T \in I(A))$; $F \in I(\Gamma)$ iff $\exists A \in \Gamma(F \in I(A))$.)

[3] A referee of the AJL notes that clause (5b) can be simplified as follows: $F \in I(A \to B)$ iff $T \notin I(A)$ or $[T \in I(A)$ & $F \in I(B)]$.

2. $\Gamma \vDash_{\text{BDLMI4}^{\text{C}}} A$ (A is a consequence of Γ in BDLMI4$^{\text{C}}$-semantics) iff $\Gamma \vDash_{\text{M}} A$ for each BDLMI4$^{\text{C}}$-model M.

3. In particular, $\vDash_{\text{BDLMI4}^{\text{C}}} A$ (A is valid in BDLMI4$^{\text{C}}$-semantics) iff $\vDash_{\text{M}} A$ for each BDLMI4$^{\text{C}}$-model M (i.e., iff $T \in I(A)$ for each BDLMI4$^{\text{C}}$-model M).

By $\vDash_{\text{BDLMI4}^{\text{C}}}$ we shall refer to the relation just defined.

Proposition 4.5 ($\vDash_{\text{BDLMI4}^{\text{C}}}$ and $\vDash_{\mathbf{LMI4^C}}$ are equivalent)**.** *For each set of wffs Γ and wff A, $\Gamma \vDash_{BDLMI4^C} A$ iff $\Gamma \vDash_{\mathbf{LMI4^C}} A$.*

Proof. Recall that $\vDash_{\text{BDLMI4}^{\text{C}}}$ (resp., $\vDash_{\mathbf{LMI4^C}}$) is the consequence relation defined in Definition 4.4 (resp., Definition 3.2). Then the proof is easy: cf., e.g, the proof of Theorem 8 in [5] or that of Proposition 4.4 in [18], where the simple proof procedure is exemplified in the cases of the logics **BN4** and **Sm4**, respectively. □

Notwithstanding its simplicity, Proposition 4.5 is very useful since it gives us the choice of proving soundness and completeness w.r.t. no matter which of the two equivalent consequence relations. Well then, concerning soundness, the easiest way is to prove that MI4$^{\text{C}}$ verifies all axioms of HLMI4$^{\text{C}}$ and the rule MP$_{\supset}$; as regards completeness, we prove it by a canonical model construction.

Thus, after noting a couple of definitions, we prove the soundness and completeness theorem.

Definition 4.6 (Theories)**.** An HLMI4$^{\text{C}}$-theory (theory, for short) is a set of wffs containing all HLMI4$^{\text{C}}$-theorems and closed under MP$_{\supset}$. Then, a theory t is prime if $A \in t$ or $B \in t$ whenever $A \vee B \in t$.

Definition 4.7 (Canonical models)**.** Let t be a prime HLMI4$^{\text{C}}$-theory. A canonical HLMI4$^{\text{C}}$-model (canonical model, for short) is a structure (K_{c}, I_t) where K_{c} is defined as in Definition 4.3 and I_t is a function from the set of all wffs to K_c defined as follows: for each wff A, $T \in I(A)$ iff $A \in t$ and $F \in I(A)$ iff $\sim A \in t$.

Now, in order to prove the completeness of HLMI4$^{\text{C}}$ w.r.t. $\vDash_{\text{BDLMI4}^{\text{C}}}$, we need to prove the two ensuing facts.

Fact 1. A theory without a given wff can be extended to a prime theory without the same wff (primeness).

Fact 2. Canonical models are indeed models, which is proved when showing that the canonical translations of clauses (1) through (5b) are provable in any prime theory.

Proof. Fact 1 is easily proved by using positive intuitionistic logic (cf. Proposition 4.2(9)), as, for example, in [26, Lemma 5.9].

Regarding fact 2, the canonical translations of clauses (2b) through (5b) are proved as follows (clauses (1) and (2a) are trivial). Clause (2b): A9 and A10; clause (3a): by using A4 and the theorems $(A \wedge B) \supset A$, $(A \wedge B) \supset B$; clause (3b): by the theorem $\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$; clause (4a): by primeness of t and the theorems $A \supset (A \vee B)$, $B \supset (A \vee B)$; clause (4b): by the theorem $\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$ (cf. Proposition 4.2 on the quoted theorems); clause (5a): A12, A13, A14, A15, $\mathrm{MP}_{\rightarrow}$ and $\mathrm{MT}_{\rightarrow}$; clause (5b): A16, A17, A18 and A19. □

Finally, given the argument developed so far, we have:

Theorem 4.8 (Soundness and completeness of HLMI4$^{\mathrm{C}}$)**.** *For any set of wffs* Γ *and wff* A, *(1)* $\Gamma \vDash_{BDLMI4^C} A$ *iff* $\Gamma \vdash_{HLMI4^C} A$*; (2)* $\Gamma \vDash_{\boldsymbol{LMI4^C}} A$ *iff* $\Gamma \vdash_{HLMI4^C} A$.

Proof. (a) Soundness: Immediate. Given Proposition 4.5, it suffices to prove that the axioms of HLMI4$^{\mathrm{C}}$ are **LMI4$^{\mathbf{C}}$**-valid and that its rules preserves **LMI4$^{\mathbf{C}}$**-validity. (b) Completeness: (1) Suppose $\Gamma \nvdash_{\mathrm{HLMI4^C}} A$, that is, that A does not belong to the set of consequences derivable in HLMI4$^{\mathrm{C}}$ from Γ (in symbols, $A \notin Cn\Gamma[\mathrm{HLMI4^C}]$). Then $Cn\Gamma[\mathrm{HLMI4^C}]$ is extended to a prime HLMI4$^{\mathrm{C}}$-theory t such that $A \notin t$. Next, the canonical HLMI4$^{\mathrm{C}}$-model $\mathrm{M_c} = (K_{\mathrm{c}}, I_t)$ based upon t is defined, and we have $\Gamma \nvDash_{\mathrm{M_c}} A$ since $T \in I_t(\Gamma)$ (as $T \in I_t(Cn\Gamma[\mathrm{HLMI4^C}])$) but $T \notin I_t(A)$, whence $\Gamma \nvDash_{\mathbf{LMI4^C}} A$ (by Definition 4.3 and 4.4), as was to be proved. (2) It is immediate by (1) and Proposition 4.5. □

In the appendix, we have noted a few theorems and rules of HLMI4$^{\mathrm{C}}$, as well as several non-provable wffs of HLMI4$^{\mathrm{C}}$. In what follows, we highlight some interesting properties of this logic.

1. The H-formulation of HLMI4$^{\mathrm{C}}$ is not more complex than those of, say, certain well-known strong 3-valued logics (cf. [2] and references therein).

2. HLMI4$^{\mathrm{C}}$ enjoys the replacement property (RP): $A \leftrightarrow B \Rightarrow \mathrm{C}[A] \leftrightarrow \mathrm{C}[A/B]$, where $\mathrm{C}[A]$ is a wff in which A appears and $\mathrm{C}[A/B]$ is the result of replacing A by B in one or more places where A occurs. This fact is immediately provable (by induction on the length of $\mathrm{C}[A]$), leaning upon the rules about the biconditional appearing in the list of theorems and rules of HLMI4$^{\mathrm{C}}$ in the appendix.

3. Given that HLMI4$^{\mathrm{C}}$ has the rules MP, transitivity, replacement and the self-identity axiom ($A \rightarrow A$), it fulfills all the conditions defining 'implicative logics' in the classical Polish tradition (cf., e.g., [22, pp. 179-180] or [30, p. 228]), except for the rule 'verum e quodlibet' (Veq: $A \Rightarrow B \rightarrow A$) that does

not hold. In this sense, could we speak of 'weak implicative logics' to refer to logics with the four properties mentioned above but without Veq?[4]

4. $\mathbf{LMI4^C}$ defines material connexive logic **MC** (cf. [29, §4.5.3] and references therein). So notice that everything that **MC** can do can be accomplished with $\mathbf{LMI4^C}$. Also, notice, by the way, that the $\mathrm{HLMI4^C}$-conditional contraposes, but the rule contraposition is *inadmissible* in **MC**: $(A \vee {\sim}A) \to (A \to A)$ is an **MC**-theorem but ${\sim}(A \to A) \to {\sim}(A \vee {\sim}A)$ is not.

5. In [20, Theorem 1], it is proved that **B** (cf. Definition 2.6) plus AT is inconsistent. In fact, it is easy to see that the proof holds for the logic resulting from $\mathbf{B'}$ (cf. Definition 2.9) by restricting the transitivity axiom to its rule form. Consequently, $\mathbf{LMI4^C}$ is inconsistent. Nevertheless, it is not trivial: not every wff is provable in $\mathrm{HLMI4^C}$; moreover, for any A, the rule Ecq, $(A \wedge {\sim}A) \Rightarrow B$, does not hold. Thus, $\mathbf{LMI4^C}$ is paraconsistent. Moreover, it is paracomplete too: $A \vee {\sim}A$ is not provable.

6. As shown above (cf. Proposition 3.8), $\mathbf{LMI4^C}$ is an acceptable connexive logic.

7. $\mathbf{LMI4^C}$ is interpretable in the important Belnap-Dunn semantics.

8. And, sorry for the tag, last but not least, $\mathrm{HLMI4^C}$ has what Meyer named "the correct form of the relevance property for the intermediate logic **RM**" ([1, p. 417] —**RM** is the logic R-Mingle; cf.[1, Chap. IV, §29]) and it is dubbed the "quasi relevance property" in [23].

   Proposition 4.9 (QRP)**.** *$HLMI4^C$ has the 'quasi relevance property' (QRP), that is, if $A \to B$ is an $HLMI4^C$-theorem, then (1) A and B share at least a propositional variable or (2) both ${\sim}A$ and B are $HLMI4^C$-theorems.*

   Proof. By following the strategy in [23, Proposition 8.5], where it is proved that **E4** has the QRP. (In fact, the proof is quite similar.) □

5 Concluding remarks

In this paper, it is defined a 4-valued implicative expansion of Belnap-Dunn logic which is a connexive logic and enjoys the 'quasi relevance property': if $A \to B$ is a theorem, then either A and B share at least one propositional variable or both ${\sim}A$ and B are theorems.

[4]Cf. [9, Definition 2.8.1] (I owe this remark to a referee of the AJL).

It has been proved that this logic we have dubbed $\mathbf{LMI4^C}$ is a strong logic with considerable expressive power and interesting properties as the following ones. It is inconsistent but not trivial, paracomplete, enjoys the replacement property and defines material connexive logic **MC**, which means that everything you can do with **MC** is feasible with $\mathbf{LMI4^C}$. On the other hand, following a strategy devised by Brady in [5], $\mathbf{LMI4^C}$ is given a Hilbert-style formulation presenting $\mathbf{LMI4^C}$ as an expansion of positive classical propositional logic.

In what respects further work in the same line, we limit ourselves to note two paths.

1. We wonder whether the matrix $\mathrm{MI4^C}$ is but one of a class with similar properties. In this sense, consider the matrix $\mathrm{MI4_2^C}$ which is defined when replacing the conditional table in $\mathrm{MI4^C}$ by the following one:

$\to$	0	1	2	3
0	2	2	2	2
1	0	2	0	2
2	0	1	2	3
3	0	0	0	3

   It is not difficult to prove that the logic $\mathbf{LMI4_2^C}$ determined by this matrix is a connexive implicative expansion of Belnap-Dunn logic with similar properties to those enjoyed by $\mathbf{LMI4^C}$, including the QRP and definability of **MC**.

2. The logic $\mathbf{LMI4^C}$ is a connexive logic with the QRP, a characteristic property of the logic R-Mingle (**RM**). Now, as Meyer puts it "sometimes one doesn't need the whole relevance principle, and, on these occasions, **RM** is good enough, when some relevance is desirable" ([1, p. 393]). Fair enough. But we wonder whether it is possible to define interesting connexive implicative expansions of Belnap-Dunn logic with "the whole relevance principle", that is, with the "variable sharing property".

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Appendix

Matrix M1

Consider the 6-element matrix defined by using the following truth-tables (designated values are starred):

$\to$	0	1	2	3	4	5	$\sim$
0	2	3	2	3	2	3	5
1	0	5	0	5	0	5	4
$*2$	0	1	2	3	2	3	3
$*3$	0	1	0	5	0	5	2
$*4$	0	1	0	1	2	3	1
$*5$	0	1	0	1	0	5	0

$\wedge$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	0	1	0	1
$*2$	0	0	2	2	2	2
$*3$	0	1	2	3	2	3
$*4$	0	0	2	2	4	4
$*5$	0	1	2	3	4	5

$\vee$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	3	3	5	5
$*2$	2	3	2	3	4	5
$*3$	3	3	3	3	5	5
$*4$	4	5	4	5	4	5
$*5$	5	5	5	5	5	5

M1 verifies $\mathbf{B'}$ and BT plus the prefixing axiom, $(B \to C) \to [(A \to B) \to (A \to C)]$, and in addition, $[(A \to B) \wedge A] \to B$, $[A \to (A \to B)] \to (A \to B)$, $A \vee \sim A$ and $(A \to \sim A) \to \sim A$ but falsifies $\sim(A \to A)$ (let I be some M1-interpretation such that for some propositional variable p, $I(p) = 1$; then $I(\sim(p \to p)) = 0$).

A brief 'syntactical' descriptive view of HLMI4$^{\mathrm{C}}$

The following are provable in HLMI4$^{\mathrm{C}}$: (a) $[(A \to B) \wedge A] \to B$; (b) $[A \to (A \to B)] \to (A \to B)$; (c) $\sim(A \to A) \to (B \to B)$; (d) $A \to [(A \to A) \to [(A \to A) \to A]]$; (e) $(A \leftrightarrow B) \Rightarrow (\sim B \to \sim A)$; $(A \leftrightarrow B) \Rightarrow (\sim A \leftrightarrow \sim B)$; $(A \leftrightarrow B) \Rightarrow (A \to C) \leftrightarrow (B \to C)$; $(A \leftrightarrow B) \Rightarrow (C \to A) \leftrightarrow (C \to B)$; $[(A \leftrightarrow B) \wedge (B \leftrightarrow C)] \to (A \leftrightarrow C)$; $A \leftrightarrow B \Rightarrow (A \wedge C) \leftrightarrow (B \wedge C)$, $(C \wedge A) \leftrightarrow (C \wedge B)$; $A \leftrightarrow B \Rightarrow (A \vee B) \leftrightarrow (B \vee C)$, $(C \vee A) \leftrightarrow (C \vee B)$.

We note (1) the rules in (e) can be used in the proof (by induction) of the replacement theorem (cf. section 4); (2) the thesis (c) proves that HLMI4$^{\mathrm{C}}$ lacks the *variable sharing property* ("antecedent and consequent share at least a propositional variable in each conditional theorem"); (3) the thesis in (d) proves that HLMI4$^{\mathrm{C}}$ lacks the *Ackermann property* ("A contains at least a conditional connective in each theorem of the form $A \to (B \to C)$").

On the other hand, $\mathrm{HLMI4^C}$ does not prove, for example, $A \to (A \to A)$ (the 'mingle' axiom), $(A \to A) \to (B \to B)$, $\sim(A \to \sim B) \Rightarrow A \to B$ or Suf.

The modal logic in $\mathbf{LMI4^C}$

1. The clauses for the two modal operators (cf. Proposition 3.5) are:

   - $T \in I(LA)$ iff $T \in I(A)$; $F \in I(LA)$ iff $T \notin I(A)$ or $F \in I(A)$.
   - $T \in I(MA)$ iff $T \in I(\sim L \sim A)$; $F \in I(MA)$ iff $F \in I(A)$.

   These clauses can be introduced in any BD-semantics independently of the definability of the connectives they here interpret in the case of $\mathbf{LMI4^C}$.

   In order for completeness to be proved, the logic these connectives extend or expand needs to have the ensuing theses: $LA \to A$, $A \vee \sim LA$, $(LA \wedge \sim LA) \to \sim A$, $\sim A \to \sim LA$ and the rule $A \Rightarrow LA$ (Nec).

2. Anyway, the connectives L and M can alternatively be introduced according to the ensuing tables (not definable in $\mathbf{LMI4^C}$):

	L
0	0
1	0
2	0
3	3

	M
0	0
1	3
2	3
3	3

   The clauses are:

   - $T \in I(LA)$ iff $T \in I(A)$ and $F \notin I(A)$; $F \in I(LA)$ iff $T \notin I(LA)$.
   - $T \in I(MA)$ iff $T \in I(\sim L \sim A)$; $F \in I(MA)$ iff $T \notin I(MA)$.

   In order to prove completeness the required wffs are $LA \to A$, $\sim A \to \sim LA$, $A \to (\sim A \vee LA)$, $(LA \wedge \sim LA) \to B$, $(\sim LA \wedge A) \to \sim A$ and $LA \vee \sim LA$.

Finally, we note that both modal logics sketched work in very much the same way as the corresponding ones w.r.t. Brady's **BN4** treated in [19].

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