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A variety of De Morgan negations in relevant logics

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Abstract

The present paper is inspired by Sylvan and Plumwood's logic $\mathrm{B_M}$ defined in "Non-normal relevant logics" and by their treatment of negation with the $*$-operator in "The semantics of first-degree entailment". Given a positive logic L including Routley and Meyer's basic positive logic and included in either the positive fragment of E or in that of RW, we investigate the essential De Morgan negation expansions of L and determine all the deductive relations they maintain to each other. A Routley-Meyer semantics is provided for each logic defined in the paper.

Keywords: Minimal De Morgan logic $\mathrm{B_M}$; $*$-operator; De Morgan type negation; relevant logics; Routley-Meyer ternary relational semantics.

1 Introduction

The present paper is inspired by two papers Valerie Plumwood (*née* Morelli, also named Routley) wrote with Richard Sylvan (*né* Routley), "The semantics of first degree entailment" (cf. [13]) and "Non-normal relevant logics" (cf. [15]).

In the first one of these papers, their authors used what is named in [2] the "$*$-operator", in order to represent negation in the system of first degree entailment. Afterwards, the $*$-operator has generally been used to modelize negation in relevant logics, as it is done in Chapter 4 of [12], the fundamental treatise on the Routley-Meyer semantics (RM-semantics) for relevant and indeed non-classical logics in general, also co-authored by Plumwood. In that respect, it has to be remarked that the $*$-operator is adequate to represent De Morgan negations and extensions thereof, not only in relevant logics but, as just pointed out, in non-classical logics by and large, and both in the context of a binary, as well as a ternary, relational semantics for interpreting the conditional or implication (cf. [9, 11] and references therein).

In the second one of the papers quoted above, (relevant) De Morgan minimal logic $\mathrm{B_M}$ is introduced. Although it is Routley and Meyer's basic logic B the logic usually taken as the fundamental basis for building up the main relevant logics (cf. Chapter 4 of [12]), it is actually $\mathrm{B_M}$, which is the minimal or basic logic interpretable with the RM-semantics in the sense that any logic defined in the language of $\mathrm{B_M}$ strictly included in it is not representable in said semantics.

In the second paper ([15]), extensions of $\mathrm{B_M}$ defined by using both the $*$-operator (cf. §8 below) and a falsity constant are introduced. We have conceived the present paper as a continuation of the results in [15] about extensions of $\mathrm{B_M}$ defined with the $*$-operator (as regards those with a falsity constant, cf. [10]). In particular, we define the taxonomy of expansions with the $*$-operator of any positive logic including Routley and Meyer's basic logic $\mathrm{B_+}$ and included in either $\mathrm{E_+}$ (the positive fragment of the logic of entailment E) or in $\mathrm{RW_+}$ (the positive fragment of contractionless logic of relevant implication R). In some cases, the positive spectrum is extended to contain logics equivalent to, or included in, $\mathrm{RM3_+}$, the positive fragment of the quasi-relevant logic RM3, the 3-valued extension of the logic RM, R-Mingle (cf. Definitions 2.3 and 3.1 below).

The De Morgan negation expansions to be defined in the sequel are built up by using the double negation axioms and the contraposition and reductio axioms and rules (cf. Definition 3.1). As a whole, it is shown that there are 33 different De Morgan negation expansions of any logic $\mathrm{L_+}$ including $\mathrm{B_+}$ and included in either $\mathrm{E_+}$ or in $\mathrm{RW_+}$. We determine all deductive relations these 33 expansions of $\mathrm{L_+}$ maintain to each other. Sylvan and Plumwood's logic $\mathrm{B_M}$ will play a fundamental role in the determination of said relations. as it will be apparent below.

In [15, p. 10], Sylvan and Plumwood note that "System $\mathrm{B_M}$ and various of its extensions are of especial interest, among other things, in the formalization of dialectical logics". It is to be hoped that some of the wealth of systems defined below will be interesting in this sense, but also in others as well.

The structure of the paper is as follows. We define some of the main positive relevant logics in §2 and some of the main full ones in §3. In §4, a variety of De Morgan negations for any logic L_+ including B_+ and included in $\mathrm{RM3}_+$ is defined by using the double negation axioms and the contraposition axioms and rules. By using, in addition, the 'constructive' reductio axioms and rule, in §5, a variety of De Morgan negations for any logic including B_+ and included in E_+ or RW_+ is defined. In §6, the 'non-constructive' axioms and rules as well as the 'constructive' ones are used along with the double negation axioms and the contraposition axioms and rule. In each one of these sections (§4, 5, 6), all the deductive relations the expansions obtained maintain to each other are defined. In §7, we provide an RM-semantics for each one of the logics defined in the paper. In §8, the paper is ended with some concluding remarks. We have added an appendix displaying some sets of truth-tables used in establishing the deductive relations referred to above. Most of these sets have were obtained with Slaney's MaGIC (cf. [14]).

2 Main positive relevant logics

Below, we recall some of the main positive (i.e., negationless) relevant logics. They can be taken as exemplary instances of the wealth of positive logics we are going to show how to expand with a variety of De Morgan negations. Firstly, we set some preliminary notions. Then Routley and Meyer's basic positive logic B_+ is defined.

Definition 2.1 (Preliminary notions)**.** The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ...$, and the following connectives: $\rightarrow$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction) and $\sim$ (negation). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B, C, etc. are metalinguistic variables. Then logics are formulated as Hilbert-type axiomatic systems, the notions of 'theorem' and 'proof from a set of premises' being the usual ones, as well as those of extension and expansion of a given logic.

Definition 2.2 (The logic B_+)**.** The logic B_+ can be formulated with the following axioms and rules of inference (cf., e.g., [12, Chapter 4]; $A_{1,}..., A_n \Rightarrow B$ means 'if $A_1, ..., A_n$, then B'):

Axioms:

a1. $A \to A$
a2. $(A \wedge B) \to A$; $(A \wedge B) \to B$
a3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
a4. $A \to (A \vee B)$; $B \to (A \vee B)$
a5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
a6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

Rules of inference:

Adjunction (Adj). $A, B \Rightarrow A \wedge B$
Modus ponens (MP). $A \to B, A \Rightarrow B$
Prefixing (Pref). $B \to C \Rightarrow (A \to B) \to (A \to C)$
Suffixing (Suf). $A \to B \Rightarrow (B \to C) \to (A \to C)$

Definition 2.3 (Main positive relevant logics)**.** Consider the following axioms:

b1. $[(A \to B) \wedge (B \to C)] \to (A \to C)$
b2. $(A \to B) \to [(B \to C) \to (A \to C)]$
b3. $(B \to C) \to [(A \to B) \to (A \to C)]$
b4. $[(A \to B) \wedge A] \to B$
b5. $[A \to (A \to B)] \to (A \to B)$
b6. $[[(A \to A) \wedge (B \to B)] \to C] \to C$
b7. $A \to [(A \to B) \to B]$
b8. $A \to (A \to A)$
b9. $A \vee (A \to B)$

Some of the main positive relevant logics are Brady's DJ_+ (cf. [5]); C_+ (cf. [12, p. 186]); Anderson and Belnap's *Ticket Entailment* (T_+), *Entailment* (E_+) and *Relevance* (R_+) (cf. [1]); RMO_+ (cf. [1, §8.15]) and $RM3_+$ (cf. [1, §29.3, 29.4]). The logic DJ_+ (equivalent to DL_+) is the result of adding b1 to B_+. Then the rest of the logics mentioned above are axiomatized with Adj, MP, a1-a6 and the following axioms:

- C_+: b2, b3, b4.
- T_+: b2, b3, b5.
- E_+: T_+ plus b6.
- R_+: b2, b5, b7.
- RMO_+: R_+ plus b8.
- $RM3_+$: RMO_+ plus b9.

Finally, we note the logic RW_+, which is important in the sequel:

- RW_+: b2, b7.

That is, RW_+ is the result of dropping the contraction axiom b5 from the formulation of R_+ (notice that b3 is provable from b2 and b7).

Concerning RMO_+ and $RM3_+$, it has to be remarked that their standard negation expansions are not relevant logics in the sense that they lack the *variable-sharing property*, although they enjoy the *weak relevant property* (cf. [1, §29]), while some authors consider them in the "family of relevance logics" anyway (cf., e.g., [3, p. 276]).

The logics in Definition 2.3 are related to each other as summarized in the following diagram (for any logics L, L′, L → L′ means that L′ is a proper extension of L).

$$\mathbf{B}_+ \rightarrow \mathbf{DJ}_+ \rightarrow \mathbf{C}_+ \rightarrow \mathbf{T}_+ \rightarrow \mathbf{E}_+ \rightarrow \mathbf{R}_+ \rightarrow \mathbf{RMO}_+ \rightarrow \mathbf{RM3}_+$$

Figure 1

We remark that RW_+ includes C_+, it is included in R_+ but does not include nor is it included in T_+ and E_+.

3 Main relevant logics

In this section, we recall some of the main relevant logics.

Definition 3.1 (Main relevant logics)**.** Consider the ensuing axioms and rules:

$$
\begin{aligned}
\text{A1.}\ & (\sim A \wedge \sim B) \to \sim(A \vee B) \\
\text{A2.}\ & \sim(A \wedge B) \to (\sim A \vee \sim B) \\
\text{A3.}\ & (A \to B) \to (\sim B \to \sim A) \\
\text{A4.}\ & A \to \sim\sim A \\
\text{A5.}\ & \sim\sim A \to A \\
\text{A6.}\ & (A \to B) \to \sim(A \wedge \sim B) \\
\text{Contraposition (con).}\ & A \to B \Rightarrow \sim B \to \sim A \\
\text{Reductio (r).}\ & A \to \sim A \Rightarrow \sim A
\end{aligned}
$$

Some of the main relevant logics are negation expansions of the positive logics defined in Definition 2.3. In particular, the logics B, DJ, DL, C, T, E, R, RM and RM3, whose relations mirror those between the corresponding positive logics (cf. Figure 1). In addition, the fundamental Sylvan and Plumwood's minimal De Morgan logic $\mathrm{B_M}$ has to be mentioned (cf. [15]). These logics are axiomatized as follows (cf. [5, 4]):

- $\mathrm{B_M}$: $\mathrm{B_+}$, A1, A2, con.
- B: $\mathrm{B_+}$, A4, A5, con.
- DJ: $\mathrm{DJ_+}$, A3, A4, A5.
- DL: DJ, A6.
- C: $\mathrm{C_+}$, A3, A4, A5.
- T: $\mathrm{T_+}$, A3, A4, A5, A6.
- E: $\mathrm{E_+}$, A3, A4, A5, A6.
- R: $\mathrm{R_+}$, A3, A4, A5.
- RM: $\mathrm{RMO_+}$, A3, A4, A5.
- RM3: $\mathrm{RM3_+}$, A3, A4, A5.

Remark 3.2 (The De Morgan laws and other questions). Notice that the De Morgan laws $\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$ and $\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$ are provable in $\mathrm{B_M}$ (by A1, A2, a2-a5 and con). In addition, it has to be noted that Sylvan and Plumwood's $\mathrm{B_M}$ is the minimal logic definable in the context of the Routley-Meyer ternary relational semantics, as remarked in the introduction (then, the De Morgan laws are provable in all logics referred to above). Also, that A6 (so, r) is a theorem of R (so, of RM and RM3). (A6 is usually rendered in the form $(A \to \sim A) \to \sim A$ —cf. [1]—. But this thesis and A6 in its present form are equivalent w.r.t. B_M plus A4 —cf. §8.) Lastly, it has to be mentioned that RMO_+ is not the positive fragment of RM3 (cf. [1, §8.1]) and that RM and RM3 enjoy the weak relevant property though they lack the variable-sharing property, as pointed out above (cf. [1, §29]).

4 Variety of De Morgan negations I. Double negation and contraposition

In this section, unless otherwise explicitly stated, by L_+, we refer to a positive logic including B_+ and included in $RM3_+$ (cf. Definition 2.3). Below, it is shown how to expand L_+ by using the double negation axioms A4, A5, the contraposition rule con and its corresponding axiom A3.

Definition 4.1 (Variety of minimal De Morgan negations)**.** There are essentially four ways of expanding L_+ with a minimal De Morgan negation (the labels 'm', 'i' and 'e' abbreviate 'minimal', 'introduction of double negation' and 'elimination of double negation').

1. Lm: L_+ plus A1, A2 and con.
2. Lmi: Lm plus A4.
3. Lme: Lm plus A5.
4. Lmie: Lm plus A4 and A5.

Definition 4.2 (Variety of basic De Morgan negations)**.** There are essentially four ways of expanding L_+ with a basic De Morgan negation (the label 'b' abbreviates 'basic'; the labels 'i' and 'e' are understood as in Definition 4.1).

5. Lb: L_+ plus A1, A2 and A3.
6. Lbi: Lb plus A4.
7. Lbe: Lb plus A5.
8. Lbie: Lb plus A4 and A5.

Remark 4.3 (On the contraposition axioms and rules). Consider the following contraposition axioms and rules:

$$\begin{aligned}
&\text{con1. } A \to {\sim}B \Rightarrow B \to {\sim}A\\
&\text{con2. } {\sim}A \to B \Rightarrow {\sim}B \to A\\
&\text{con3. } {\sim}A \to {\sim}B \Rightarrow B \to A\\
&\text{A}3_1.\ (A \to {\sim}B) \to (B \to {\sim}A)\\
&\text{A}3_2.\ ({\sim}A \to B) \to ({\sim}B \to A)\\
&\text{A}3_3.\ ({\sim}A \to {\sim}B) \to (B \to A)
\end{aligned}$$

We have (the easy proof is left to the reader): Lmi $\vdash$ con1; Lme $\vdash$ con2; Lmie $\vdash$ con1, con2, con3; Lbi $\vdash$ A3_1; Lbe $\vdash$ A3_2; Lbie $\vdash$ A3_1, A3_2, A3_3 (of course, $\vdash$ means 'is provable from').

The proposition that follows displays the relations the different De Morgan expansions just introduced in Definitions 4.1 and 4.2 maintain to each other.

Proposition 4.4 (Relations between the expansions in Definitions 4.1 and 4.2)**.** *Consider the eight expansions of L_+ introduced in Definitions 4.1 and 4.2. The relations the said expansions maintain to each other are summarized in the following diagram, where the arrow is interpreted as in Figure 1.*

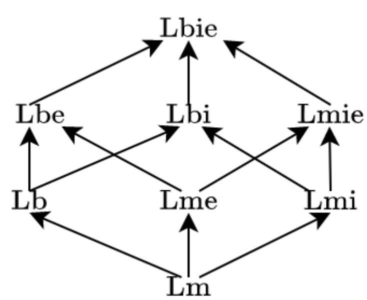

Figure 2

Proof. The inclusion relations are immediate. Then concerning the non-inclusion claims, it suffices to use the following facts. (1) A3 does not follow from RM3_+ plus A1, A2, A4 and A5. (2) A4 is not provable from RM3_+ plus A1, A2, A3 and A5. (3) A5 is not derivable from RM3_+ plus A1, A2, A3 and A4. Now, (1), (2) and (3) are a consequence of tables t1, t2 and t3, respectively (cf. the Appendix). ■

It is a result of Proposition 4.4 that there are, so far, eight different to each other ways of expanding L_+ with a De Morgan negation. For instance, let L_+ be B_+ (cf. Definition 2.2). Then Lm is Sylvan and Plumwood's minimal De Morgan logic B_M; Lmie is Routley and Meyer's basic logic B (cf. [12, Chapter 4]), and Lbie is the logic DW (cf. [12, Chapter 4]). But many more important relevant logics are included in Definitions 4.1 and 4.2. For instance, let L_+ be R_+. Then Lbi (included in Jankov's KC —cf. [7]) includes the basic 'constructive' logic $\mathcal{R}B_c$ defined in [10], while Lbie is Anderson and Belnap's R. A last example. Lbie is RM (resp., RM3) if L_+ is RMO_+ (resp., $RM3_+$).

Nevertheless, most of the logics defined in Definitions 4.1 and 4.2, even strong ones, lack the reductio axiom A6 or even the reductio rule r. For example, if L_+ is E_+, then r is not provable in Lbie. But, as we have seen above (cf. Definition 3.1), the reductio axiom A6 is instrumental in the characterization of some strong relevant logics. Moreover, the axiom A6 or the reductio rule r characterize some important weak relevant logics such as G (i.e., B plus r —cf. [12, Chapter 4]), TWR and EWR, which are the result of dropping b5 from T and R, respectively, while maintaining A6 (cf. Definition 3.1; notice that RWR and R are equivalent systems —cf. [12, Chapter 4]).

The reductio axioms and/or rules are crucial in a number of relevant logics weaker than R. So, we discuss them in the two following sections.

5 Variety of De Morgan negations II. The reductio rules and axioms

In this section, it is shown how to use the reductio rule r and the reductio axiom A6, in addition to the double negation axioms and the contraposition rule and axiom, for defining De Morgan negation expansions of positive relevant logics. Firstly, we note some theses and rules deductively equivalent (from now on, simply 'equivalent') to r. (In what follows, ai-ak where $i<k$ and $i, k \in \{7, 8, ..., 17\}$ abbreviates ai through ak.)

Proposition 5.1 (Theses and rules equivalent to r). *Consider the following* *rules and theses:*

$$
\begin{aligned}
&a7.\ A \to B, A \to \sim B \Rightarrow \sim A\\
&a8.\ A \to (B \wedge \sim B) \Rightarrow \sim A\\
&a9.\ A \to B \Rightarrow \sim(A \wedge \sim B)\\
&a10.\ A \to \sim B \Rightarrow \sim(A \wedge B)\\
&a11.\ \sim(A \wedge \sim A)\\
&a12.\ \sim A \vee \sim\sim A
\end{aligned}
$$

Given Sylvan and Plumwood's logic B_M (cf. Definition 3.1), we have that a7-a12 are equivalent to rule r ($A \to \sim A \Rightarrow \sim A$).

Proof. It is easy and is left to the reader. ■

Regarding the relations between the logics in Definitions 4.1 and 4.2 and the rules and theses in Proposition 5.1 just proved, we have the ensuing fundamental fact.

Proposition 5.2 (On the reductio axioms and rules)**.** *Let $L1_+$ and $L2_+$ be logics including B_+ the former being included in E_+ and the latter in RW_+ (cf. Definition 2.3). We prove: (1) the rule r (so a7-a12) is neither derivable from L1bie nor from L2bie; (2) A6 is neither derivable from L1bie plus r nor from L2bie plus r (cf. Definitions 4.1 and 4.2).*

Proof. We use the tables in the Appendix. (1) L1bie: t5; L2bie: t4. (2) L1bie: t6; L2bie: t7. ■

As a consequence of Proposition 5.2, we can state the following definitions:

Definition 5.3 (Variety of De Morgan negations with r)**.** Let L_+ be a logic including B_+ and included in either E_+ or RW_+ (cf. Definition 2.3). Then, in addition to the eight different De Morgan expansions of L_+ described in Definitions 4.1 and 4.2, L_+ can be expanded as follows:

9. Lmr: L_+ plus A1, A2, con and r.
10. Lmir: Lmr plus A4.
11. Lmer: Lmr plus A5.
12. Lmier: Lmr plus A4 and A5.
13. Lbr: L_+ plus A1, A2, A3 and r.

14. Lbir: Lbr plus A4.
15. Lber: Lbr plus A5.
16. Lbier: Lbr plus A4 and A5.

Definition 5.4 (Variety of De Morgan negations with A6)**.** Let L_+ be as in Definition 5.3. The logics LmR, LmiR, LmeR, LmieR, LbR, LbiR, LbeR and LbieR are defined as Lmr through Lmier, respectively, except that the rule r (or a7-a12) is replaced by A6.

Concerning the relations the logics introduced in Definitions 5.3 and 5.4 maintain to each other, we prove the ensuing proposition.

Proposition 5.5 (Relations between the logics with reductio)**.** *The relations the logics in Definitions 5.3 and 5.4 maintain to each other mirror those relating the reductioless logics to each other, as they have been summarized in Figure 2 (ρ stands uniformly for either r or else R; LM stands uniformly for Lm or else Lb, where L_+ is as in Definitions 5.3 and 5.4). Cf. Figures 3, 4 and 5:*

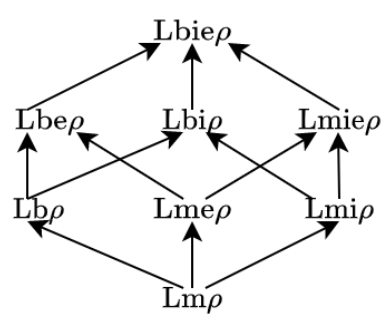

Figure 3

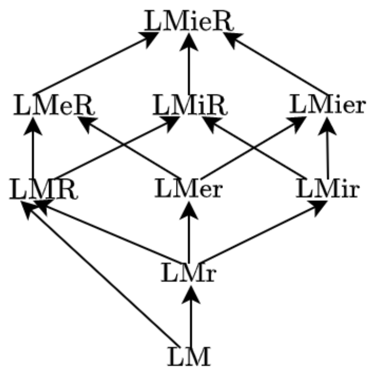

Figure 4

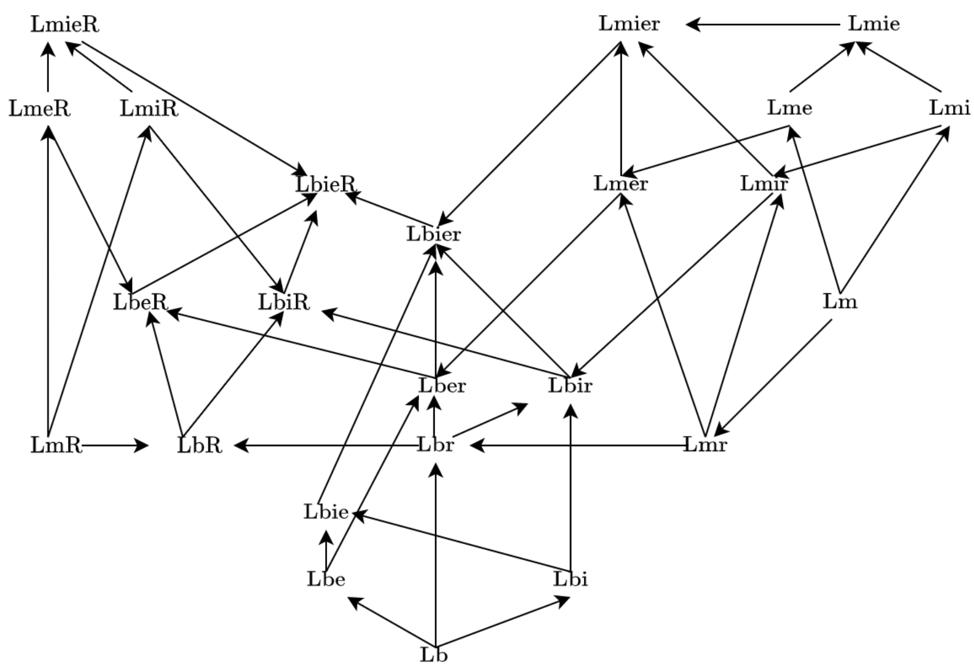

Figure 5

Proof. It follows from Propositions 4.4 and 5.2, once having noted that tables t1, t2 and t3 verify A6. ■

6 More on the reductio axioms and rules

In this section, we elaborate on the question of extending minimal or basic De Morgan logics (as defined in Definitions 4.1 and 4.2) with the reductio axioms and rules. We begin by introducing some additional reductio rules.

Proposition 6.1 (Some reductio rules equivalent to PEM). *Consider the following reductio rules and thesis.*

rbis. $\sim A \to A \Rightarrow A$
a13. $A \to B, \sim A \to B \Rightarrow B$
a14. $(A \vee \sim A) \to B \Rightarrow B$
a15. $A \to B \Rightarrow \sim A \vee B$
a16. $\sim A \to B \Rightarrow A \vee B$
a17. $A \vee \sim A$

Given Sylvan and Plumwood's logic B_M (cf. Definition 3.1), rbis and a13-a17 are equivalent to each other.

Proof. It is easy and is left to the reader (remark that a17 is the "Principle of Excluded Middle", PEM). ■

Proposition 6.2 (On the relations between r1 and r2). *Let us refer by r1 (resp., r2) to the set of rules and theses a7-a12 (resp., rbis and a13-a17). Given the logic B_M, we prove that r1 is derivable from r2.*

Proof. It suffices to show that the rule r, $A \to \sim A \Rightarrow \sim A$, is provable from rbis, which is immediate. ■

But there is more to be said on the relations between r1 and r2.

Proposition 6.3 (On the unprovability of r2). *Let L_+ be a logic included in $RM3_+$. Then r2 is unprovable from Lbi plus r1.*

Proof. By table t8 in the Appendix. ■

Proposition 6.4 (On the provability of r2). *Given B_M plus A5 ($\sim\sim A \to A$) and r1, r2 is derivable.*

Proof. It is immediate: rbis ($\sim A \to A \Rightarrow A$) follows from the rules r, con and re ($\sim\sim A \Rightarrow A$). ■

Remark 6.5 (On the extensions of Lm and Lb with r1 and r2). Let L_+ be a logic included in $RM3_+$ and Lm and Lb be its expansions with minimal and basic De Morgan negation, respectively (cf. Definitions 2.3, 4.1 and 4.2). From Propositions 6.2, 6.3 and 6.4, it follows that the result of extending Lme (so Lbe) with either r1 or r2 are equivalent systems. Nevertheless, Lm (resp., Lmi, Lb, Lbi) plus r1 is included in (but does not include) Lm (resp., Lmi, Lb, Lbi) plus r2.

The facts just remarked are contained in the ensuing proposition.

Proposition 6.6 (On the rel. between Lm- & Lb-extensions with r1 & r2). *Let L_+ be a logic including B_+ and included in either E_+ or in RW_+. The relations Lm and Lb (cf. Definitions 4.1 and 4.2) maintain with each other when extended with r1 and r2 are summarized in the following diagram (recall that, for instance, Lmer1 and Lmer2 are equivalent):*

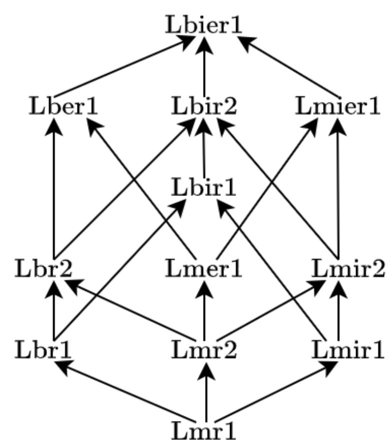

Figure 6

Proof. Given Propositions 6.1, 6.2, 6.3 and 6.4, the proof follows once having remarked that all items in r1 and r2 are verified by t1, t2 and t3. ■

Now, let us consider extensions of Lm and Lb (in the sense of the preceding proposition) by using the following alternative, A6bis, to A6:

$$\text{A6bis. } (A \to B) \to ({\sim}A \vee B)$$

Notice that r, a7-a11 and A6 are 'constructive' in the sense that they are provable in intuitionistic logic IPC. On the other hand, a12 (the characteristic axiom of KC —cf. [7]), rbis and a13-a17 and A6bis are 'non-constructive', since they are not derivable in IPC.

Well then, in order to maintain the parallelism with extensions by the set of rules r1 and r2, by R1 (resp, R2), let us refer to extensions of Lm and Lb by using A6 (resp., A6bis). Next, we investigate the relations the two types of extensions maintain to each other.

Proposition 6.7 (On the unprovability of R2)**.** *Let L_+ be a logic included in $RM3_+$. Then R2 is not derivable from Lbi plus R1.*

Proof. By t8 in the Appendix. ∎

Proposition 6.8 (On the provability of R2)**.** *Given Bme (B_M plus A5) and R1, R2 is derivable.*

Proof. By using A6, De Morgan laws and the rule con, we get $(\sim A \to B) \to \sim\sim(A \vee B)$. Then, by A5, we prove $(\sim A \to B) \to (A \vee B)$, whence A6bis is immediate by A5. ∎

Turning now to provability (or unprovability) of R1 from R2, we note the following facts.

Proposition 6.9 (On the unprovability of R1)**.** *Let L_+ be a logic included in $RM3_+$. Then R1 is not derivable from Lme and R2.*

Proof. By t9. ∎

Proposition 6.10 (On the provability of R1)**.** *R1 is provable given R2 and (1) Bmi (B_M plus A4) or (2) Bb (B_M plus A3).*

Proof. (1) Similarly as in Proposition 6.8, we get $(A \to \sim B) \to \sim(A \wedge B)$, whence A6 follows by A4. (2) An easy way is the following. $(A \to \sim A) \to \sim A$ is provable from B_M plus A6bis. Then A6 follows from Bb and $(A \to \sim A) \to \sim A$ ∎

Remark 6.11 (On extensions of Lm and Lb with R1 and R2)*.* Let L_+ be a logic included in $RM3_+$ and Lm and Lb be understood as in Remark 6.5. From Propositions 6.7, 6.8, 6.9 and 6.10, it follows that the result of extending Lbe (or Lmie) with either R1 or R2 are equivalent systems. Nevertheless, LbR1 (resp., LbiR1, LmiR1) is included in (but does not include) LbR2 (resp., LbiR2, LmiR2), while LmeR2 is included in (but does not include) LmeR1. Finally, LmR1 ad LmR2 are systems independent from each other.

The facts just remarked are contained in the ensuing proposition.

Proposition 6.12 (On the rel. between Lm- & Lb-extensions with R1 & R2)**.** *Let L_+ be a logic including B_+ and included in either E_+ or RW_+. The relations Lm and Lb (cf. Definitions 4.1 and 4.2) maintain to each other when extended with R1 and R2 are summarized in the following diagram (recall that LbeR1 —resp.,LmieR1— and LbeR2 —resp., LmieR2— are equivalent systems):*

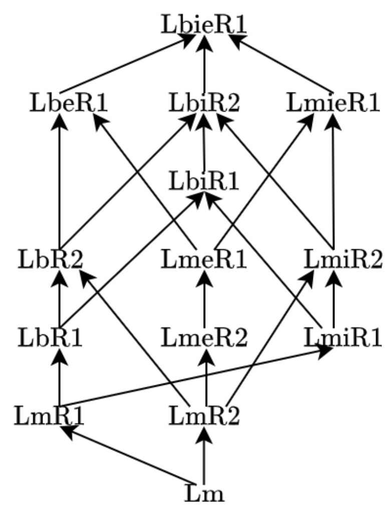

Figure 7

Proof. Similar to that of Proposition 6.6: given Propositions 6.7, 6.8, 6.9 and 6.10. the proof follows as R1 and R2 are verified by t1, t2 and t3. ■

Now, let us note the ensuing remark.

Remark 6.13 (On the reductio axioms and rules II)*.* The facts proved in Proposition 5.2 w.r.t. r1 and A6 are also provable for r2 and A6bis. Let L1bie and L2bie be as in said proposition. Then (1) rbis (so a13-a17) is neither derivable from L1bie nor from L2bie; (2) A6bis is neither derivable from L1bie plus rbis nor from L2bie plus rbis, since t4 nd t5 falsify rbis, whereas t6 and t7 verify rbis but falsify A6bis.

The section is ended with some concluding remarks on the results obtained in this and the previous two sections.

In §4, we saw that there are 24 different De Morgan negation expansions of any logic L_+ including B_+ and included in $RM3_+$, when built up with the double negation axioms and the contraposition axiom and rule. From the facts discussed in this and the preceding section, it follows that there are more possibilities if we consider positive logics including B_+, as before, but now included in either E_+ or RW_+ and extended with the reductio axioms and rules. In particular, we have 24 different extensions when using only the 'constructive' reductio rules and axiom A6, but 33 if the 'non-constructive' rules and axiom A6bis are addded. The 24 extensions with the constructive rules and axiom are related to each other as shown in Figure 5; the relations that result when both the constructive and non-constructive axioms and rules are added are summarized in Figures 6 and 7.

7 Semantics

In this section, a Routley-Meyer semantics (RM-semantics) is provided for each logic defined in the paper. We begin by defining EB_M-models for extensions of Sylvan and Plumwood's logic B_M.

Definition 7.1 (EB_M-models)**.** An EB_M-model, M, is a structure with at least the following items: (a) a set K and a subset of it, O; (b) a ternary relation R and a unary operation $*$ defined on K subject at least to the following definitions and postulates for all $a, b, c, d \in K$:

$$
\begin{aligned}
&\text{d1.}\ a \leq b =_{\text{df}} \exists x \in O\ Rxab\\
&\text{d1}'.\ a = b =_{\text{df}} a \leq b\ \ \&\ \ b \leq a\\
&\text{d2.}\ R^2abcd =_{\text{df}} \exists x \in K(Rabx\ \ \&\ \ Rxcd)\\
&\text{P1.}\ a \leq a\\
&\text{P2a.}\ (a \leq b\ \ \&\ \ Rbcd) \Rightarrow Racd\\
&\text{P2b.}\ (a \leq b\ \ \&\ \ b \leq c) \Rightarrow a \leq c\\
&\text{P2c.}\ (d \leq b\ \ \&\ \ Rabc) \Rightarrow Radc\\
&\text{P2d.}\ (c \leq d\ \ \&\ \ Rabc) \Rightarrow Rabd\\
&\text{P3.}\ a \leq b \Rightarrow b^* \leq a^*
\end{aligned}
$$

(c) a valuation relation $\vDash$ from K to the set of all formulas such that the following conditions (clauses) are satisfied for every propositional variable p, formulas A, B and $a \in K$:

(i). $(a \leq b \ \ \& \ \ a \vDash p) \Rightarrow b \vDash p$

(ii). $a \vDash A \wedge B$ iff $a \vDash A \ \ \& \ \ a \vDash B$

(iii). $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$

(iv). $a \vDash A \to B$ iff for all $b, c \in K$, $(Rabc \ \ \& \ \ b \vDash A) \Rightarrow c \vDash B$

(v). $a \vDash\sim A$ iff $a^* \nvDash A$

Additional elements of M are a set of semantical postulates $\mathrm{Pj}_1, ..., \mathrm{Pj}_n$.

Structures of the form $(O, K, R, *, \vDash)$ satisfying just d1, d1′, d2, P1, P2a, P2b, P2c, P2d, P3 and clauses (i), (ii), (iii), (iv) and (v) are the basic structures and in fact characterize the logic $\mathrm{B_M}$ (they are labelled $\mathrm{B_M}$-models). Introduction of additional postulates serve to determine extensions of $\mathrm{B_M}$ interpretable in RM-semantics.

Definition 7.2 (Truth in a class of $\mathrm{EB_M}$-models)**.** Let a class of $\mathrm{EB_M}$-models $\mathcal{M}$ be defined and $\mathrm{M} \in \mathcal{M}$. A formula A is true in M (in symbols, $\vDash_{\mathrm{M}} A$) iff $x \vDash A$ for all $x \in O$.

Definition 7.3 (Validity in a class of $\mathrm{EB_M}$-models)**.** Let a class of $\mathrm{EB_M}$-models $\mathcal{M}$ be defined and $\mathrm{M} \in \mathcal{M}$. A formula A is valid in $\mathcal{M}$ (in symbols, $\vDash_{\mathcal{M}} A$) iff A is true in every $\mathrm{M} \in \mathcal{M}$.

Now, $\mathrm{B_M}$ is sound and complete w.r.t. $\mathrm{B_M}$-models (cf. [15]). Then, in order to define RM-models characterizing extensions of $\mathrm{B_M}$, the basic notion is "corresponding postulate" (cp) (cf. [12, Chapter 4]). Consequently, below, a series of cp is listed, in order to provide RM-semantics characterizing each one of he logics defined throughout the paper.

Definition 7.4 (Corresponding postulates)**.** Given $\mathrm{B_M}$-models, we provide corresponding postulates (cp) to each one of the theses b1-b8, A3-A5, rule r, rule rbis, A6 and A6bis (unless otherwise stated, quantifiers range over K).

Pb1. $Rabc \Rightarrow \exists x[Rabx \ \ \& \ \ Raxc]$

Pb2. $R^2abcd \Rightarrow \exists x[Racx \ \ \& \ \ Rbxd]$

Pb3. $R^2abcd \Rightarrow \exists x[Rbcx \ \ \& \ \ Raxd]$

Pb4. $Raaa$

Pb5. $Rabc \Rightarrow R^2abbc$

Pb6. $\exists x \in ZRaxa[Za$ iff for all $b, c \in K, Rabc \Rightarrow \exists x \in ORxbc]$

Pb7. $Rabc \Rightarrow Rbac$

Pb8. $Rabc \Rightarrow (a \leq c$ or $b \leq c)$

Pb9. $(Rabc \ \ \& \ \ a \in O) \Rightarrow b \leq a$

PA3. $Rabc \Rightarrow Rac^*b^*$

PA4. $a \leq a^{**}$

PA5. $a^{**} \leq a$

Pr. $a \in O \Rightarrow a^* \leq a^{**}$

Prbis. $a \in O \Rightarrow a^{**} \leq a$

PA6. Raa^*a^{**}

PA6bis. Raa^*a

Remark 7.5 (On the cp in Definition 7.4). Concerning the cp for the positive axioms, cf. [10, 12] and references therein. Then Prbis and PA6bis can be found in [12, Chapter 4, pp. 288-289] as cp to $A \vee {\sim}A$ (equivalently $A \to B \Rightarrow {\sim}A \vee B$) and $(A \to {\sim}A) \to {\sim}A$ (equivalently, $(A \to B) \to ({\sim}A \vee B)$); Pr is immediate from Prbis and, finally, PA6 appears here maybe for the first time (as regards the equivalence between $(A \to {\sim}A) \to {\sim}A$ and $(A \to B) \to ({\sim}A \vee B)$; cf. the concluding remarks to the paper in §8).

Now, taking A6bis and PA6bis as an example, that the postulates in Definition 7.1 are in fact the cp to their respective axioms or rules means (1) given $\mathrm{B_M}$-models, A6bis is proved valid with PA6bis (that is, in any $\mathrm{B_M}$-model in which PA6 holds); (2) given canonical $\mathrm{B_M}$-models (cf. [15, 12]), PA6bis is proved canonically valid with A6bis.

As we have seen in the preceding sections, there are 33 different De Morgan negation expansions of a relevant positive logic L_+ including B_+ and included in E_+ or RW+. The fact that, given Sylvan and Plumwood's $\mathrm{B_M}$, the postulates in Definition 7.4 are the cp to their respective axioms and rules immediately provides us with an RM-semantics for any of the 33 possible De Morgan expansions of L_+. Suppose, for instance, that L_+ is C_+ (cf. Definition 2.3) and consider the logic CbeR2 (that is, B_+ plus b2, b3, b4, A3, A5 and A6bis). Well then, CbeR2 is sound and complete w.r.t. B_M-models in which Pb2, Pb3, Pb4, PA3, PA5 and PA6bis hold.

Let us end this section by noting that b1-b9 define but only a limited subset of the set of positive extensions of B_+, as well as that the meaning of reductio in relevant logics can be interpreted with (no necessarily equivalent) different rules and/or axioms from r, rbis, A6 and A6bis, as discussed in the following section.

The paper is ended with some concluding remarks.

8 Concluding remarks

In [15, p. 11], Sylvan and Plumwood note: "In the case of extensions of B_M, the modelling conditions corresponding to positive schemes are exactly the same as those for extensions of B and any of the positive schemes may be adjoined", where they refer to the extensions of positive logic B_+ defined in Chapter 4 of [12]. Then, they provide 'corresponding postulates' (cp) to the double negation axioms A4 and A5, the contraposition axiom A3 and the theses $A \vee {\sim}A$ and ${\sim}A \vee {\sim}{\sim}A$ (cf. Definition 3.1, Proposition 5.1 and Proposition 6.1 above). These cp are the same we have used in this paper (cf. §7 above). Additionally, they present cp to the contraposition axioms $(A \to {\sim}B) \to (B \to {\sim}A)$ and $({\sim}A \to B) \to ({\sim}B \to A)$ (cf. Remark 4.3 above), which are composed of the cp for A3 and A4, and the cp for A3 and A5, respectively. Also, they propose the cp we have dubbed PA6bis (cf. §7) as the cp to the specialized reductio axiom, sr, $(A \to {\sim}A) \to {\sim}A$. Next, they establish the soundness and completeness for any extension of B_M defined by using any of the schemata or theses just mentioned w.r.t. B_M-models strengthened with the cp the said theses agree with. Well then, this result does not seem correct as regards extensions of B_M with the specialized reductio axiom, sr, since, although the cp we have labeled PA6bis suffices to show the validity of sr, on the contrary, this thesis seems insufficient to prove the canonical validity of PA6bis (we would need, for example, A3 and A5, in addition to sr —notice that, for instance, A6bis is unprovable from $RM3_+$ plus A3, A4 and sr; cf. t8 in the Appendix). Indeed, this fact explains that we have chosen A6 and A6bis as the *basic* reductio axioms from the

RM-semantics point of view instead of sr or its 'non-constructive' companion srbis $(\sim A \to A) \to A$ (notice that above, in §7, it is claimed that PA6bis is the cp to A6bis).

On the other hand, the authors of [15] do not investigate the deductive relations the expansions of $\mathrm{B_M}$ they suggest maintain to each other. Conversely, in the present paper, we have established the relations the 33 different extensions of L_+ maintain to each other when L_+ is a logic including B_+ and included in either E_+ or RW_+.

In [15, p. 10], it is noted that a negation is *(relevantly) non-classical* if it lacks any of the double negation axioms A4 and A5 or any of the contraposition axioms A3, $\mathrm{A3}_1$, $\mathrm{A3}_2$ and $\mathrm{A3}_3$ (cf. Definition 3.1 and Remark 4.3). Well then, there is still much to be investigated about the reductio axioms and rules in systems with a relevantly non-classical negation. We limit ourselves to point out three comments to end the paper.

1. (a) A6 is not derivable from $\mathrm{B_M}$ extended with the assertion rule, $A \Rightarrow (A \to B) \to B$, the contraction axiom, $[(A \to (A \to B)] \to (A \to B)$, sr and srbis (cf. t10 in the Appendix).
   (b) A6bis is not derivable from $\mathrm{B_M}$ extended with the assertion rule, rule r, the contraction rule, $A \to (A \to B) \Rightarrow A \to B$, the rule ri, $A \Rightarrow \sim\sim A$ and the thesis srbis (cf. t11 in the Appendix).
2. (a) Given $\mathrm{B_M}$ and A4 $(A \to \sim\sim A)$, the ensuing theses and rules are equivalent to A6: $(A \to \sim A) \to \sim A$; $A \to \sim(A \to \sim A)$; $(A \wedge \sim B) \to \sim(A \to B)$; $(A \to \sim B) \to \sim(A \wedge B)$; $(A \wedge B) \to \sim(A \to \sim B)$; $A \to B \Rightarrow (A \to \sim B) \to \sim A$; $A \to \sim B \Rightarrow (A \to B) \to \sim A$.
   (b) Given $\mathrm{B_M}$ and A5 $(\sim\sim A \to A)$, the ensuing theses and rules are equivalent to A6bis: $(\sim A \to A) \to A$; $\sim A \to \sim(\sim A \to A)$; $(\sim A \to B) \to (A \vee B)$; $A \to B \Rightarrow (\sim A \to B) \to B$; $\sim A \to B \Rightarrow (A \to B) \to B$.
3. The strong reductio axioms, sra, are the following theses: $(A \to B) \to [(A \to \sim B) \to \sim A]$; $(A \to \sim B) \to [(A \to B) \to \sim A]$; $(\sim A \to B) \to [(\sim A \to \sim B) \to A]$; $(\sim A \to \sim B) \to [(\sim A \to B) \to A]$; $(A \to B) \to [(\sim A \to B) \to B]$; $(\sim A \to B) \to [(A \to B) \to B]$.

   The logic DW is the result of extending the basic logic B with the contraposition axiom A3.

   We have:

(a) The sra are not derivable from DW plus the contraction axiom, the assertion rule (cf. 1(a) above) and A6 (consequently, all theses and rules in 2(a) and 2(b) above). (Cf. t12 in the Apppendix.)

(b) The sra are not derivable from DW plus the assertion axiom $A \to [(A \to B) \to B]$ and A6 (consequently all the theses and rules is 2(a) and 2(b) above). (Cf. t13 in the Appendix.)

However, we note that cp to each one of the sra can be defined in any system containing DW plus either the axiom suffixing or prefixing: $(A \to B) \to [(B \to C) \to (A \to C)]$, $(B \to C) \to [(A \to B) \to (A \to C)]$, respectively (cf. [8]).

A Appendix

The following sets of truth-tables t1-t13 are used to prove some claims made throughout the paper (designated values are starred). Let L be a logic defined upon the language $\mathcal{L}$ (cf. Definition 2.1), Γ a set of wffs and A a wff of $\mathcal{L}$. On the other hand, let t be a set of truth-tables and v an assignment to the propositional variables of $\mathcal{L}$ built upon t. v verifies A if it assigns a designated value to A; and v verifies the rule $\Gamma \Rightarrow A$ if it assigns a designated value to A, provided it assigns a designated value to each $B \in \Gamma$. Then, t verifes L if every assignment v verifies all axioms and rules of L. Most of the sets have been found by using MaGIC (cf. [14]; each set of tables is the simpler one justifying the respective claim). (In case a tester is needed, the reader can use that in [6].) In what follows, p, q and r are distinct propositional variables.

t1.

$\to$	0	1	2	3	$\sim$
0	3	3	3	3	3
*1	0	1	2	3	2
*2	0	0	2	3	1
*3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
*1	0	1	1	1
*2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
*1	1	1	2	3
*2	2	2	2	3
*3	3	3	3	3

This set verifies all axioms and rules of $RM3_+$ plus A1, A2, A4, A5, a1-a17 and A6, A6bis but falsifies A3: $v[(p \to q) \to (\sim q \to \sim p)] = 0$ for any assignment v such that $v(p) = v(q) = 2$.

t2.

$\to$	0	1	2
0	2	2	2
*1	0	1	2
*2	0	0	2

$\wedge$	0	1	2
0	0	0	0
*1	0	1	1
*2	0	1	2

$\vee$	0	1	2
0	0	1	2
*1	1	1	2
*2	2	2	2

	$\sim$
0	2
*1	2
*2	0

This set verifies all axioms and rules of RM3_+ plus A1, A2, A3, A5, a1-a17 and A6, A6bis but falsifies A4: $v(p \to \sim\sim p) = 0$ for any assignment v such that $v(p) = 1$.

t3.

	$\sim$
0	2
*1	0
*2	0

The tables for $\to$, $\wedge$ and $\vee$ are as in t2. This set verifies all axioms and rules of RM3_+ plus A1, A2, A3, A4, a7-a17 and A6, A6bis but falsifies A5: $v(\sim\sim p \to p) = 0$ for any assignment v such that $v(p) = 1$.

t4.

$\to$	0	1	2	$\sim$
0	2	2	2	2
1	1	2	2	1
*2	0	1	2	0

The tables for $\wedge$ and $\vee$ are the same as in t2. This set verifies all axioms and rules of RW_+ (cf. Definition 2.3) plus A1-A5 but falsifies a7-a17 (this set verifies all —and only all— theses and rules provable in Łukasiewicz's 3-valued logic Ł3). It suffices to falsify the rule r. Well then, $p \to \sim p \Rightarrow \sim p$ is falsified for any assignment v such that $v(p) = 1$.

t5.

$\to$	0	1	2
0	2	2	2
1	0	2	2
*2	0	0	2

The tables for $\wedge$ $\vee$ and $\sim$ are the same as in t4. This set verifies all axioms and rules of E_+ (cf. Definition 2.3) plus A1-A5 but falsifies a7-a17. It suffices to falsify the rule r. Now, $p \to \sim p \Rightarrow \sim p$ is falsified for any assignment v such that $v(p) = 1$.

t6. The tables for $\to$, $\wedge$, $\vee$ and $\sim$ are the same as in t5 but now 2 and 3 are designated values. This set verifies all axioms and rules of E_+ (cf. Definition 2.3) plus A1-A5 and a7-a17 but falsifies A6 and A6bis. It suffices to falsify A6. Now, $(p \to q) \to \sim(p \wedge \sim q)$ is falsified for any assignment v such that $v(p) = v(q) = 1$.

t7.

$\to$	0	1	2	3	$\sim$
0	3	3	3	3	3
1	0	2	1	3	2
*2	0	1	2	3	1
*3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
*2	0	0	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
*2	2	3	2	3
*3	3	3	3	3

This set verifies all axioms and rules of RW_+ (cf. Definition 2.3) plus A1-A5 and a7-a17 but falsifies A6 and A6bis. It suffices to falsify A6bis: $v[(p \to q) \to (\sim p \vee q)] = 0$ for any assignment v such that $v(p) = 1$ and $v(q) = 2$.

t8.

$\to$	0	1	2
0	2	2	2
1	0	2	2
*2	0	1	2

The tables for $\wedge$ $\vee$ and $\sim$ are the same as in t3. This set verifies all axioms and rules of $\mathrm{RM3}_+$ (cf. Definition 2.3) plus A1-A4 and a7-a12 but falsifies a13-a17. It suffices to falsify rbis: $\sim p \to p \Rightarrow p$ is falsified for any assignment v such that $v(p) = 1$.

t9. The tables for $\to$, $\wedge$ and $\vee$ are as in t3, but the table for $\sim$ is as in t2. This set verifies $\mathrm{RM3}_+$, A1, A2, A5, A6bis and con but falsifies A6: $v[(p \to q) \to \sim(p \wedge \sim q)] = 0$ for any assignment v such that $v(p) = 2$ and $v(q) = 1$.

t10.

$\to$	0	1	2	$\sim$
0	1	1	2	1
*1	0	1	1	0
*2	0	0	1	0

The tables for $\wedge$ and $\vee$ are as in t2. This set verifies $\mathrm{B_M}$ plus a7-a17 and A6bis but falsifies A6: $v[(p \to q) \to \sim(p \wedge \sim q)] = 0$ for any assignment v such that $v(p) = 0$ and $v(q) = 2$.

t11.

$\to$	0	1	2	$\sim$
0	1	2	2	1
*1	0	1	1	1
*2	0	0	1	0

The tables for $\wedge$ and $\vee$ are as in t2. This set verifies B_M plus a7-a17 and srbis $((\sim A \to A) \to A)$ but falsifies A6bis: $v[(p \to q) \to (\sim p \vee q)] = 0$ for any assignment v such that $v(p) = 0$ and $v(q) = 1$.

t12. The tables for $\to$, $\wedge$ and $\vee$ are as in t10, but the table for $\sim$ is as in t4. This set verifies DW, the *contraction axiom* $([A \to (A \to B)] \to (A \to B))$, the *assertion rule* $(A \Rightarrow (A \to B) \to B)$ and the *special reductio axiom* (sr) $((A \to \sim A) \to \sim A)$, but falsifies the strong reductio axioms sra. For instance, $v[(p \to q) \to [(\sim p \to q) \to q]] = 0$ for any assignment v such that $v(p) = 0$ and $v(q) = 2$.

t13.

$\to$	0	1	2	3	4	$\sim$
0	4	4	4	4	4	4
1	0	2	3	3	4	3
*2	0	1	2	3	4	2
*3	0	0	1	2	4	1
*4	0	0	0	0	4	0

$\wedge$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
*2	0	1	2	2	2
*3	0	1	2	3	3
*4	0	1	2	3	4

$\vee$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	2	3	4
*2	2	2	2	3	4
*3	3	3	3	3	4
*4	4	4	4	4	4

This set verifies DW plus the *assertion axiom* $(A \to [(A \to B) \to B])$, the *contraction rule* $(A \to (A \to B) \Rightarrow A \to B)$ and the *special reductio axiom* (sr) $((A \to \sim A) \to \sim A)$ but falsifies the strong reductio axioms, sra. For example, $v[(p \to q) \to [(\sim p \to q) \to q]] = 0$ for any assignment v such that $v(p) = 2$ and $v(q) = 1$.

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