The Law of Excluded Middle and Berry's Paradox... Finally

Ross Brady
Department of Politics, Media and Philosophy, La Trobe University, Melbourne, Victoria, 3086, Australia
Ross.Brady@Latrobe.edu.au

Abstract

This is the culmination of a discussion on Berry's Paradox with Graham Priest, over an extended period from 1983 to 2019, the central point being whether the Paradox can be avoided or not by removal of the Law of Excluded Middle (LEM). Priest is of the view that a form of the Paradox can be derived without the LEM, whilst Brady disputes this. We start by conceptualizing negation in the logic MC of meaning containment and introduce the LEM as part of the classical recapture. We then examine the usage of the LEM in some other paradoxes and see that it is applied to cases of self-reference. In relation to Priest's [2019] paper, we go on to find a similar use of the LEM in Priest's derivation of Berry's Paradox. However, it is found to be deeper and trickier than other paradoxes, requiring a special effort to untangle the relationships between the LEM, self-reference and meta-theoretic influence. We then examine Brady's previous formalization of Berry's Paradox, considering Brady's most recent view of restricted quantification and his recursive account of the least number satisfying a property. We show that neither of these methods can be used to formalize the paradox.

1 Introduction.

This paper intends to be the culmination of a discussion between Priest and Brady on Berry's Paradox, over many years. This was started with Priest's [1983], in which he presents a derivation of Berry's Paradox in which he claimed that there is no usage of the Law of Excluded Middle, $A \vee \sim A$. This was in response to Brady [1983], where it was proved that the logic TWQ,[1] which does not include the Law of Excluded Middle

[1] We consider here the logic TWQ, which is a familiar subsystem of Brady [1983], rather than the unintuitive CSQ. TWQ is the logic MCQ, presented in §2, with the removal of the axiom, $(A \to B)\&(B \to C) \to .A \to C$, the rule, $A \to B, C \to D \Rightarrow B \to C \to .A \to D$ and both the meta-rules MR1 and QMR1, and with the addition of the axioms, $A \to B \to .B \to C \to .A \to C$, $A \to B \to .C \to A \to .C \to B$, $A\&(B \vee C) \to .(A\&B) \vee (A\&C)$, $\forall x(A \vee B) \to A \vee \forall xB$, and $A\&\exists xB \to \exists x(A\&B)$.

(henceforth abbreviated as LEM) can be used to prove the simple consistency of naïve set theory. Note also that in Brady [1989] (proved earlier in 1979), it was shown that the paradoxes can be non-trivially derived as contradictions in a logic which includes the LEM. These two results, regarding simple inconsistency and consistency, have the general effect of showing that the presence or absence of the LEM determines whether the set-theoretic paradoxes are respectively derivable or not, with their two respective systems not leading to the trivialization of the whole system due to the absence of Ex Falso Quodlibet ($A, \sim A \Rightarrow B$) in these logics. Indeed, this key role of the LEM was precisely shown somewhat later in Brady [2006] where two logics are used, one with the LEM included and the same one but with the LEM removed. Further, the derivation of many of the semantic paradoxes also use similar logical steps, with Berry's Paradox being among them. Indeed, in Brady [2006], it is proved, using the same logic and methodology, that the standard semantic paradoxes such as the Liar and Heterologicality can also be shown to be simply consistent, using a logic without the LEM. However, Priest claimed in his [1983] that the Berry's Paradox was an exception to all this in that it, together with the similar Richard's and König's Paradoxes, is derivable without the use of the LEM. Note here that Brady's two results in [2006] take place in logics without key contraction principles such as the rule, $A \rightarrow (A \rightarrow B) \Rightarrow A \rightarrow B$, which would have had the effect of trivializing the respective systems, with or without the LEM. The same applies to the background logic used in Priest's two papers.

As a response to Priest's [1983], Brady in [1984] showed that Berry's Paradox does use the LEM in its derivation with both an informal argument and a formalization, the latter to be discussed in §8. The informal argument shows up the use of the LEM in Priest's derivation in [1983]. This turns on Priest's move from the lack of deducibility of the denotation of a natural number by an expression of limited finite length to the deducibility of the negation of the denotation of such a natural number by such an expression. Such a negation is guaranteed by the LEM as this negation holds on account of the failure of the unnegated form, as this is essentially what the LEM says. Moreover, the holding of this negation uses the priming property for the LEM, which was argued initially in Brady [2019] for such theorem-instances but further in Brady [2022] for theorems of a metacomplete logic. In saying this, we note that a disjunction $A \vee B$ is prime when the priming property (if $A \vee B$ is a theorem then either A is a theorem or B is a theorem) holds. A whole formal system is prime when the priming property applies to all of its disjunctive theorems. We also note that metavaluations and metacompleteness were introduced in Meyer [1976], with further study in Slaney [1984] and [1987], and summed up in Brady [2017b]. A metavaluation on a system uses formula induction on its proofs and metacompleteness occurs when the metavaluation aligns exactly with the theorems of the system. Properties of the formula induction then apply to the theorems, yielding such properties as primeness and simple consistency.

Further, such a negation provides the essential negative half of the Berry's Paradox. This same point can be made for Priest's derivation of Berry's Paradox in his recent paper

[2019], as we will show in §6 in more detail. However, since the usage of the LEM in Priest's derivation is quite opaque, it is worth examining the conceptualization of negation in §2, and that of the LEM in §3 in the broader context of the classical recapture, together with its use in other paradox derivations in §4, before embarking on the informal Berry's Paradox in §5 and Priest's formalization of it in §6. We include a brief discussion on implication and entailment in §7, as it relates to Priest's formalizations. We re-assess Brady's formalization of Berry's Paradox in §8. We then discuss the Substitution of Identity in §9, as this was raised in Priest [2019].

2 Negation in the Context of the Logic of Meaning Containment.

As meaning is the key concept of logic, providing the yardstick for its axioms and rules to satisfy, we start by setting out the axiomatization of the logics MC and MCQ of meaning containment, as set out in Brady [2022]. (Its relationship with the concept of truth is also examined in [2022].) The bracketing convention follows that of Anderson and Belnap [1975].

MC

Primitives: $\sim, \&, \vee, \rightarrow$.
Definition: $A \leftrightarrow B \ =_{df} \ (A \rightarrow B)\&(B \rightarrow A)$.

Axioms.
A1. $A \rightarrow A$.
A2. $A\&B \rightarrow A$.
A3. $A\&B \rightarrow B$.
A4. $(A \rightarrow B)\&(A \rightarrow C) \rightarrow .A \rightarrow B\&C$.
A5. $A \rightarrow A \vee B$.
A6. $B \rightarrow A \vee B$.
A7. $(A \rightarrow C)\&(B \rightarrow C) \rightarrow .A \vee B \rightarrow C$.
A8. $\sim\sim A \rightarrow A$.
A9. $A \rightarrow \sim B \rightarrow .B \rightarrow \sim A$.
A10. $(A \rightarrow B)\&(B \rightarrow C) \rightarrow .A \rightarrow C$.

Rules.
R1. $A, A \rightarrow B \Rightarrow B$.
R2. $A, B \Rightarrow A\&B$.
R3. $A \rightarrow B,\ C \rightarrow D \Rightarrow B \rightarrow C \rightarrow .A \rightarrow D$.
Meta-Rule.
MR1. If $A \vee B$ then $C \vee A \Rightarrow C \vee B$.

MCQ.

Primitives: $\forall, \exists$,
$a, b, c, \ldots$ (free variables)
$x, y, z, \ldots$ (bound variables)

Axioms.
QA1. $\forall xA \to Aa/x$.
QA2. $\forall x(A \to B) \to .A \to \forall xB$.
QA3. $Aa/x \to \exists xA$.
QA4. $\forall x(A \to B) \to .\exists xA \to B$.

Rule.
QR1. $Aa/x \Rightarrow \forall xA$, where a does not occur in A.
Meta-Rule.
QMR1. If $Aa/x \Rightarrow Ba/x$ then $\forall xA \Rightarrow \forall xB$, where a does not occur in A or B, and QR1 does not generalize on any free variable in the premise Aa/x of the derivation $Aa/x \Rightarrow Ba/x$. The same condition then applies to the premise A of the derivation $A \Rightarrow B$ of the meta-rule MR1 of MC.

MC and MCQ are the result of tweaking over time. The axiom-forms of distribution in MC and MCQ were dropped in Brady and Meinander [2013] from the earlier version DJ^d and DJ^dQ of Brady [2006], the rule-forms being dropped along with it. Later, in Brady [2015], the rule-forms of distribution were reinstated by replacing the single-premise rules of MR1 and QMR1 by two-premise rules. However, it was only in Brady [2022] that the rule-form of distribution was finally dropped, by maintaining the meta-rules of Brady [2006] as applying to the single-premise rules $A \Rightarrow B$ and $Aa/x \Rightarrow Ba/x$, instead of the two-premise rules of [2015]. The problem is that distribution in both of these forms would require the conjunction of two formulae from its antecedent or premise which are based on differing assumptions.

Negation, as determined by A8 and A9 in the logic MC above, is clearly De Morgan in that it includes both double negation laws and all entailment forms of contraposition, all of which can then be applied to conjunction and disjunction, yielding their standard De Morgan properties. Such a negation can also be characterized by a cancellation concept in Brady [2008], whereby, using metavaluational trees, negations can be shown to cancel each other out, as occurs in double negation. Brady [2008] also shows that the single negation properties as occur in the LEM and the Disjunctive Syllogism Rule, $\sim A, A \vee B \Rightarrow B$ (henceforth abbreviated as DS), cannot be proved in such a metacomplete logic as MC. All this yields an incomplete concept of negation in that these concepts, whilst determining how negation applies to itself and to conjunction and disjunction, do not specify how negation applies to atoms and entailments. Nevertheless, this incomplete concept needs a deeper understanding as to how and why it functions as it does. If it is completed to a full Boolean negation, it would then be on a par with the other connectives which are all understood as complete concepts within the logic.

Whilst the two-place connectives relate their two components and create a composite concept, negation cannot have such a composite concept since it is single place. One must search for its meaning more widely, especially as it represents a concept external to that of its unnegated form. It would be wrong to limit its reach, as in the case of a dichotomy between two specific alternatives, which would require some assumption as a premise to achieve such a dichotomy within the application of the logic. Without such an assumption, its reach would have to be the whole system that one is formalizing through the medium of proof. So, we need to examine the whole system and its meaning in order to determine negation, rather than restrict ourselves to a part of it.

Formal logical systems are set up by people trying to capture a concept or concepts as best as they can. So, whilst such concepts can be ideal, the axiomatization of a formal system may differ from its goal in a number of ways. Positive theorems are meant to assert truths of the concept and negative theorems are meant to assert falsehoods of the concept. If the concept is such as to be completely conceptualized, its truths and falsities would be mutually exclusive and exhaustive, with Boolean negation applying. However, my reference to Boolean negation being the "intended negation" on p.28 of Brady [2019a] seems now to the author to be a throwback to his introduction to logic which was based entirely on truth and falsity.

Nevertheless, the positive and negative formulae A and $\sim A$, whilst most likely to be exclusive, will often not be exhaustive. Non-exclusivity would mean that the system would be simply inconsistent and exclusivity for all formulae would require the simple consistency of the whole system. As in Brady [2019a], simple inconsistency indicates an overdetermination of a concept or concepts, also referred to as conceptual overreach. Such overdetermination can arise through conceptual misunderstanding or a conceptual clash of some sort and so we could reasonably assume that the positive and negative formulae A and $\sim A$ are not both provable in the system, based on well-understood concepts. However, non-exhaustivity is quite common as people often just record salient features of a concept and are usually not in a position to say all there is to know about a concept, and indeed concepts are quite often vague as they are formed from a combination of positive and negative instances without concern for a fuller range of possibilities. Indeed, this is how children learn concepts in the first place. This creates what is often called a value gap, that is, where neither A nor $\sim A$ hold. This is discussed in Brady [2019a], where the four values are introduced, which correspond to the four deductive outcomes in accordance with the presence or absence of A and $\sim A$, and can thus be thought of as proof-theoretic values rather than the usual truth-theoretic values.

In this context, since the LEM is not a logical law with its negation being unconstrained and a value gap being quite likely, we can conclude that the LEM is only provable through one of its disjuncts using A5 or A6. That is, such a case of the LEM would be prime. (There is more on this point in the context of the classical recapture in §3.) So, the name of the

LEM, as the Law of Excluded Middle, does not apply here, but we will maintain its name here because our familiarity with its classical history. Priest in [2019] quite rightly calls it the Principle of Excluded Middle (PEM).

Getting back to conceptualizing the De Morgan negation of A8 and A9, we see that double negation and contraposition are both properties of negation that are driven by entailments. So, whatever negative formulae are in the formal system, these two properties can propagate such negations in accordance with the meaning relationships of these entailments. Double negation not only creates a mirror-image picture of negation but also ensures that all four forms of contraposition are present, so that they can participate in completing this picture. One must be wary, however, not to use an entailment property of the form $A \to \sim A \to \sim A$, as this implies the LEM, obtained by substituting $A\&\sim A$ for A and applying De Morgan properties. So, of course, such an entailment is not in MC.

3 The Law of Excluded Middle in the Context of the Classical Recapture.

We see some independent but still relevant value in examining the LEM in the broader context of the classical recapture which we set up to circumvent the above value gaps. (We use the term 'the' here as the classical recapture to be introduced will follow readily from logics MC and MCQ, in which it is immersed.) As above in §2, the Law of Excluded Middle and the Disjunctive Syllogism Rule are not derivable from the De Morgan negation, as set out in the logic MC of meaning containment. By the metacompleteness of MC and MCQ, these logics are easily shown to be prime and simply consistent, and this extends to the metacomplete arithmetic $MC^{\#}$, as axiomatized in Brady [2012] and [2021]. (See Brady [2017b] for these properties of metacompleteness, noting that simple consistency, expressed as 'if v(A) = T then v*(A) = T', for metavaluations v and v^*, is straight-forwardly proved by formula induction on A.) Since the DS straightforwardly preserves theoremhood, given the primeness and simple consistency properties, it is an admissible rule, which we will then add as a rule of the system just as in Brady [2012] and [2021] for formal arithmetic. And, it is formal arithmetic that is pertinent for our discussion of Berry's Paradox. So, the classical recapture is left to focus on the LEM. Also, by primeness, the LEM can only follow from one of its disjuncts by application of A5 or A6. Thus, there are no instances of the LEM where neither of its disjuncts hold, in the logics MC and MCQ, and in the above formal arithmetic.

It should be pointed out that priming can fail, but this generally occurs for specific non-logical dichotomies, such as the two-slit experiment, Heisenberg's Uncertainty Principle, and for the car facing a fork in the road. In the two-slit experiment, light travels through two slits to create a diffraction pattern, leaving it completely undetermined as to which slit a particular particle of light would have passed through. Heisenberg's Uncertainty Principle states, in particular, that it cannot be determined as to whether an electron spin is up or down, given its position. (This is discussed in Brady [2022], regarding distribution.) And, it is possible for a car to go down one fork or the other of the road and arrive at its common destination, without someone (possibly the absent-minded driver) knowing which fork was chosen and thus be unable to assert a particular fork, leaving the disjunction open-ended. In these examples, the respective disjunction is left without a supporting disjunct.

However, such failure of priming could not happen for the LEM, even where there is no value gap. This is essentially because the LEM is purely logical in that it just involves negation, whilst the priming failures, as exemplified above, are all non-logical with the disjunction being supported by specific physical situations. Thus, for the LEM, negation concerns the reach of the interpreted formal system, unconstrained by any specific circumstance as occurs in the above examples. So, the LEM would not hold without some supporting argument for one or other of its disjuncts, thus satisfying the Priming Property. We will use this argument in our critique of the derivation of the paradoxes in §4 and §5.

We proceed by introducing the classical recapture for the logic MC and then subsequently for MCQ. Such a classical recapture will require that both the LEM and the DS hold for a subset of formulae, for which classical logic applies.[2] We set up this classical recapture by forming the set CR as the set of theorems A, but with all their external negations removed. That is, A would be $\sim ... \sim B$, where B is a conjunction, disjunction or an entailment, preceded by a number of negations or indeed none at all. So, the formula B would either be a theorem, in the event of the number of external negations of A being an even number or zero, or its negation $\sim B$ would be a theorem, in the event of the number of external negations of A being an odd number. This forms a proper subset of the formulae in total and, as exactly one of B and $\sim B$ is provable, for each formula B in CR, with use of the double negation rule of MC. This subset forms what we shall call a meta-theoretic classical recapture, as the meta-theoretic properties of priming and simple consistency (both proved above using metacompleteness) are used to establish it, allowing its statement to be put in meta-theoretic terms. Certainly, any formula B in CR satisfies the LEM, $B \vee \sim B$. Also, the set CR is maximal in that any formula not in it is neither a positive nor a negative theorem and thus fails to satisfy the LEM, given the priming property. The DS, however, holds across the whole logic, due to priming and simple consistency, leaving just the LEM to be proved to establish the recapture, as we will see in the following.

We start, for the logic MC, with atoms for which the LEM holds, then show that the LEM holds for any formulae built up using the classical connectives, negation, conjunction and disjunction, from these atoms. This is done by induction on formulae, by proving $A \vee \sim A \Rightarrow \sim A \vee \sim\sim A$, using double negation, and $A \vee \sim A, B \vee \sim B \Rightarrow (A \& B) \vee \sim(A \& B)$ and $A \vee \sim A, B \vee \sim B \Rightarrow (A \vee B) \vee \sim(A \vee B)$, using De Morgan's Laws and the rule-

[2] An earlier classical recapture for the strong relevant logic R by Meyer and Dunn [1969] was established by proving that Ackermann's rule γ was an admissible rule of R, given that R contains the LEM. Note that γ and the DS are deductively equivalent and that they proved it for E as well, which also contains the LEM. However, Meyer was unable to extend this admissibility of γ to arithmetic. (See Brady [2021] on this point.)

form of distribution, all within the logic MC. However, the rule-form of distribution, $A\&(B \vee C) \Rightarrow (AB) \vee (AC)$, is an admissible rule of MC, given priming for $B \vee C$, which does add a meta-theoretic element to the process. In Brady [2022], it was pointed out that the priming (applied to $B \vee C$) allows either of the two conjunctions in the conclusion of the distribution rule, $A, B \vee C \Rightarrow (A\&B) \vee (A\&C)$, to be formed with both its conjuncts being formed from premises, without the further assumption of one of the disjuncts B or C being made in accordance with a disjunction elimination rule. This enables distribution in rule-form to be admissible. Without priming, the distribution rule can fail on account of each one of these conjuncts B or C being a disjunctive assumption whilst the premise A is not. So, we add the distribution rule to the logic, on this basis, as we did for the DS. This enables the LEM, together with the DS, to be extended to all formulae constructed from these atoms. So, for any set of atoms in CR, all formulae constructed from them using classical connectives are also in CR.

A full proof-theoretic classical recapture would allow all the classical tautologies to be provable in the logic, but this is not possible for logics like MC which are without the LEM. (See Note 2 for such logics with the LEM.) What we can do here is to start with a set of classical atoms, where the LEM holds, together with the DS, and then prove in MC all the tautologies that can be built up from these classical atoms. This can be done by taking such a tautology and reducing it to conjunctive normal form, which can be done using double negation, De Morgan's Laws and the rule-form of distribution, all within the logic MC, with the inclusion of rule-distribution. Such a conjunctive normal form would consist of a conjunction of disjunctions of such atoms and their negations. Such disjunctions of atoms must each consist of at least one occurrence of an LEM, for the sake of the tautology, and are thus all provable. We then reverse the process, reconstructing the tautology within the logic MC. This also shows that the elements of the set CR not only individually satisfy the LEM, but these elements, taken as a whole, satisfy all the tautologies constructed from them as atoms. This provides the classical recapture at the sentential level, and we now move onto the predicate calculus.

The corresponding classical recapture for the classical predicate logic uses the quantificational axioms and rules, as there are no suitable normal forms to provide a comparative proof. Just as in the sentential case, we first show that the LEM holds for any formulae built up using the classical universal and existential quantifiers, from atoms, possibly with free variables, for which the LEM holds. As above, this is done by induction on formulae. That is, by proving $Aa/x \vee {\sim}Aa/x \Rightarrow \forall xA \vee {\sim}\forall xA$ and $Aa/x \vee {\sim}Aa/x \Rightarrow \exists xA \vee {\sim}\exists xA$, with a not occurring in A, where the two proofs principally use QA3, QR1 and the universal distribution rule ($\forall x(A \vee B) \Rightarrow A \vee \forall xB$), together with the respective uses of $\exists x{\sim}A \rightarrow {\sim}\forall xA$ and $\forall x{\sim}A \rightarrow {\sim}\exists xA$, these last two being proved principally using QA1, A9, QR1 and QA4, all within the logic MCQ. However, the universal distribution rule is an admissible rule of MCQ, given priming for $A \vee Ba/x$, for a free variable a not occurring in B. So, as for the corresponding distribution rule for the sentential logic MC, we add this rule to the quantified logic, on this basis. This enables the LEM, together with the DS, to be extended to all quantificational formulae constructed from these atoms. So, for any set of atoms in CR, all quantificational formulae constructed from them using classical connectives and quantifiers are also in CR.

We assume that the LEM and the DS hold for atoms of the predicate calculus, possibly with free variables, for the purpose of showing that all the theorems of the classical predicate calculus, constructed from these atoms, are provable in MCQ. We axiomatize predicate calculus as follows, in a similar manner to that of MCQ, with the addition of terms:

Primitives: $\forall, \exists$,
$a, b, c, \ldots$ (free variables)
$x, y, z, \ldots$ (bound variables)
$s, t, u, ...$ (terms, which include constants as well as variables)

Definitions: $A \supset B =_{df} {\sim}A \vee B$, $A \equiv B\ =_{df}\ (A \supset B)\&(B \supset A)$.

Axioms.
CQA1. $\forall xA \supset At/x$.
CQA2. $\forall x(AB) \supset .A \supset \forall xB$.
CQA3. $At/x\exists xA$.
CQA4. $\forall x(A \supset B).\exists xA \supset B$.

Rule:
CQR1. $Aa/x \Rightarrow \forall xA$, where a does not occur in A.
Note that CQA2 and CQA4 can be replaced by their distributive forms CQA2′ and CQA4′, using properties of MCQ:
CQA2′. $\forall x(A \vee B) \supset A \supset \forall xB$.
CQA4′. $A\&\exists xB \supset \exists x(A\&B)$.
We will use these distributive forms below.

We need to prove each classical quantificational axiom and rule (called CQA1, etc.) using MCQ, for classical formulae satisfying the LEM and the DS. Key items in proofs are bracketed.
CQA1. ${\sim}\forall xA \vee \forall xA \to {\sim}\forall xA \vee At/x$. [QA1]
$\forall xAAt/x$. [LEM]
CQA2′. $\forall x(A \vee B) \Rightarrow A \vee \forall xB$. [Universal Distribution Rule]
${\sim}\forall x(A \vee B) \vee \forall x(A \vee B) \Rightarrow {\sim}\forall x(A \vee B) \vee (A \vee \forall xB)$. [MR1]
$\forall x(A \vee B) \supset A \vee \forall xB$. [LEM]
noindent CQA3. ${\sim}At/x \vee At/x \to {\sim}At/x \vee \exists xA$. [QA3]
$At/x \supset \exists xA$ [LEM]
CQA4′. $A\&\exists xB \Rightarrow \exists x(AB)$. [Existential Distribution Rule (proved below)]
${\sim}(A\&\exists xB) \vee (A\&\exists xB) \Rightarrow {\sim}(A\exists xB) \vee \exists x(A\&B)$. [MR1]

$A \& \exists x B \supset \exists x(AB)$. [LEM]
CQR1. As for MCQ.
The Existential Distribution Rule is established by Existential Instantiation, if $\exists x A$ then At/x, for some constant t, which follows from the metacompleteness of MCQ. (See Brady [2017b] for this metacompleteness result.) Thus, the Universal and Existential Distribution Rules are both admissible rules and we add them to the logic, on this basis, as we did for the sentential distribution rule.

Thus, the above axioms of predicate calculus are all provable which, together with the two above rules, yield all its theorems, with formulae constructed from the classical atoms, as we required.

We next consider the classical recapture for set theory and arithmetic. In Brady [2022], it is argued that classical logic applies to extensional sets, this being because such sets are collections of individuated objects, where there is clarity as to what objects are in the collection and what objects are not. There is no room for failure of either the LEM or the DS here, as each object is either in such a set or not in it and there are no objects that are both in such a set and not in it. Thus, this is an ideal structure for which classical logic can be applied, thus extending the recapture to extensional sets. Taking this further, Brady [2022] goes on to say that this application would be restricted to recursive sets, because there needs to be a process for determining whether an object is in a set or not and recursion provides such a process. Here, we rely on recursion to supply the proofs needed to establish the required instances of the LEM through one or other of its disjuncts. Further, by its very nature, we cannot determine all the individuated objects of a non-recursive set, even though some of its objects may be so determined. It should be noted however that there are instances of the use of the LEM in a non-recursive context. For example, the LEM can be established from theorems of an undecidable system. Specifically, any proof of the LEM must be established from theorems of a recursively enumerable system, as indeed proofs themselves need to be recursively enumerable, even if the system as a whole might be undecidable. Further, any proof must be made in a recursively enumerable system, as this is needed for the standard arguments by induction on proof steps. It is this fact that we will take up in §6 when examining Priest's formalization of Berry's Paradox.

It would initially seem that the above considerations that apply to MCQ would extend to recursive arithmetic. However, the Universal Distribution Rule, $\forall x(A \vee B) \Rightarrow A \vee \forall x B$, fails in the arithmetic of Brady [2012] because the metavaluation of the universal quantifier is expressed in such a way as to require each universal to be proved using mathematical induction. So, mathematical induction on the $\forall x(A \vee B)$ does not always carry over to the $\forall x B$ of $A \vee \forall x B$. See p.65 of Beall and Restall [2006] for a good example of the failure of the Universal Distribution Rule for recursive sets, also repeated in Brady [2012]. This failure prevents the simple consistency result of Brady [2012] from applying to the full classical arithmetic, contradicting Gödel's famous result on the inability to prove simple consistency of classical arithmetic using finitary methods, such as the above metavaluations. However, the Existential Distribution Rule is unaffected. Due to the metacompleteness of recursive arithmetic, as in Brady [2012] and [2021], priming and simple consistency are still maintained.

Nevertheless, both recursion and recursive enumerability break down with the inclusion of self-reference, as we will see in §5 when analysing Berry's Paradox. Meanwhile, we will examine some other paradoxes in §4 to help pinpoint the application of the LEM.

4 The Use of the LEM in Class-theoretic and Semantic Paradoxes More Generally.

We briefly examine four key class-theoretic and semantic paradoxes principally to show where the LEM is assumed in their derivations. In addition, we will focus on self-reference and the overlay between object and meta-theory, these two issues being important cognate concerns. So, to show up these issues, we will focus on the Liar Paradox, Russell's Paradox and the Extended Liar Paradox. However, we will add Yablo's Paradox, which does not involve explicit self-reference but still uses the LEM in its derivation. We do not include Curry's Paradox here as it uses the contraction rule, $A \to .A \to B \Rightarrow A \to B$, instead of the LEM, trivializing the class theory, and is thus of a different type.

(i) The Liar Paradox.

The Liar Paradox occurs by saying 'I am now lying'. Let this sentence be symbolized as L. Thus, L is defined: $L =_{df} L$ is false. We formalize this using classical logic in the object language as $L \equiv {\sim}L$, from which $L \& {\sim}L$ follows using the LEM. This can be easily seen from its two-way rule equivalent, $L \vee {\sim}L \supset L \& {\sim}L$. However, this should be formalized using MC, since a definition is a meaning equivalence and so a logic of meaning containment such as MC is ideal, as argued for in Brady [2017]. This would similarly yield $L \leftrightarrow {\sim}L$, with two-way rule equivalent $L \vee {\sim}L \to L \& {\sim}L$.

We make two points about this. The first is that neither L nor ${\sim}L$ can be shown, either of which would provide deductive support for the LEM, and so the LEM cannot be proved from one of its disjuncts. Indeed, if either one of L or ${\sim}L$ is proved, then, by the above equivalence, they are both provable. This cannot be the case because the Liar Paradox can be formalized in a simply consistent system, based on MC, as shown in Brady [2006] for the slightly stronger DJ^d. That is, any proof of either L or ${\sim}L$ to provide support for the LEM would enable the proof of the contradiction $L \& {\sim}L$ which is not derivable. The second point is that the LEM applies to L, which is self-referentially defined and so cannot occur within any recursive system or indeed within any recursively enumerable system. So, neither L nor ${\sim}L$ can be proved earlier as such a proof would be recursively enumerable. Here, the the LEM fails as it occurs at the point of self-reference.

An alternative presentation of the Liar Paradox, as commonly employed, uses the truth predicate rather than the truth operator, as occurs above. This takes the form $L =_{df} {\sim}Tr(`L')$, where 'L' denotes the sentence L. Thus, L is defined as: 'L' is not true, putting the 'not true' predicate into the meta-language and hence the defined L also, leaving the quoted sentence 'L' in the object language. This leads us to the following three scenarios. First, this attempts to avoid the paradox by separating these two languages, thus constituting a solution based on levels of language. Indeed, by self-reference, this develops into an infinite sequence of distinct languages. This is fine, but then there is no paradox to solve. Second, it is more likely that this is an attempt to merge the two languages into one combined language, but this takes us to the Extended Liar Paradox, which is dealt with in (iv) below. Third, in order to produce a paradox in the object language, one can replace the truth predicate by the truth operator, as dealt with in the above account of the Liar Paradox.

(ii) Russell's Paradox

Similar points can be made about Russell's Paradox which follows a similar pattern of argument, viz. $R \in R \equiv R \notin R$, derived from the contextual definition, $\forall x(x \in R \equiv x \notin x)$, within classical logic. Again, the LEM is used in proving $R \in R \& R \notin R$ using the two-way rule equivalent $(R \in R \vee {\sim} R \in R) \supset (R \in R \& {\sim} R \in R)$. As for the Liar Paradox, it should have been captured using $\rightarrow$ and $\leftrightarrow$ of MC, for the same reasons. The same two points apply, namely that the LEM was applied to the self-referential $R \in R$ and that neither $R \in R$ nor ${\sim} R \in R$ can be shown to support the LEM on pain of contradiction, which is shown not to be the case in Brady [2006] with a simple consistency proof.

Further, this shows that the Liar Paradox should not be solved by toying around with the concept of truth as the same deductive reasoning applies to both these paradoxes, which then shows that it is the logic that needs to be re-assessed rather than truth. Indeed, we have done this by using the logic MC of meaning containment, which is appropriate for formalizing definitions, as was argued in Brady [2017].

(iii) Yablo's Paradox

As stated in Brady [2017], Yablo's Paradox was introduced in Yablo [1993] as a paradox without self-reference or circularity but involving an infinite number of defined sentences. There are finite varieties as well, but these do involve circularity over the finite sequence of sentences and do reduce to the Liar Paradox. Yablo introduced the following infinite sequence of sentences, $S_1, S_2, S_3, \ldots$:

S_1: for all $k > 1$, S_k is untrue.
S_2: for all $k > 2$, S_k is untrue.
S_3: for all $k > 3$, S_k is untrue.
$\vdots$

In symbols, $S_n \equiv (\forall k > n){\sim} S_k$, for arbitrary n. Unpacking the right-hand side, $S_n \equiv {\sim} S_{n+1} \& (\forall k > n+1){\sim} S_k$ and hence, by definition, $S_n \equiv {\sim} S_{n+1} \& S_{n+1}$. By De Morgan's Law, ${\sim} Sn \equiv S_{n+1} \vee {\sim} S_{n+1}$. By the LEM, ${\sim} S_n$, and generalizing over the arbitrary n, $(\forall n){\sim} S_n$. In particular, $(\forall k > n){\sim} S_k$ and, by definition, S_n, for any n. This then contradicts ${\sim} S_n$. We note that definitions are used for each sentence S_n in the sequence, but here there is indeed no direct self-reference and no finite circularity. Nevertheless, meaning identity is applied and hence the use of the logic MC is appropriate, as for the above two paradoxes. We also note the essential use of the LEM in the above argument, which, as for the Liar above, overreaches the property of definition, as captured by the logic MC. Whilst it does not reduce to the Liar Paradox, it still uses the LEM, but in a different way in the argument to contradiction.

Indeed, the LEM is applied to S_{n+1}, for each n, and each S_n is defined in terms of S_k, for all $k > n$. So, one can consider $S_2, S_3, ...$ as one block to which the LEM is applied to each element. Further, self-reference is applied within this block, with S_2 referring to S_3 and beyond, and generally S_n referring to S_{n+1} and beyond. So, instead of considering the application of the LEM and self-reference to each single sentence, we can see that they both apply within the block. Thus, the conceptual relationship between the LEM and self-reference can still be maintained within this block of inter-related sentences.

(iv) The Extended Liar Paradox

As presented in Brady [2017], the Extended Liar Paradox is like the Liar Paradox, except that it replaces falsity with non-truth, where the 'non-' is understood as an external meta-linguistic negation. Here, the LEM can be used to convert the 'non-' to a 'not' by applying its priming property. The reason for introducing the meta-linguistic negation is to try to ensure that the LEM holds, even if it does not hold in the object theory, as for the Liar Paradox above. The reason why the LEM is used in the meta-theory is that it is normally taken to be classical. (However, there can be problems in insuring this because of an undecidable object theory, where what is proved in it is done so in a recursively enumerable way, but with this not applying to its non-theorems. See Brady [2019a] on this point, where values are determined by proof rather than truth.) We still need to address the use of 'non-true' in the Extended Liar Paradox. However, it refers to the proof rather than the semantics of the object theory, this being because a specific semantics is not usually referred to in ordinary speech and whatever is said can be immersed in a deductive system or other.

Nevertheless, however it is dealt with, 'non-truth' is still meta-theoretic. This means that the definition $E =_{df} \sim TE$, where E is the extended sentence 'this very sentence is non-true', is an attempted identity between E in the object language and $\sim TE$ in the meta-language, since '$\sim T$' in the present context is meta-theoretic. So, this attempt at a definition would represent what we would call an illicit identity, as the two supposedly identical objects are of different types. This means that the definition cannot be represented as a meaning identity, as we have done for the Liar Paradox.

Further, given our argument in §3 and above for the Liar Paradox, that the proof of the LEM should be recursively enumerable, and so no self-reference as occurs for E above should occur in such a proof. So, the Extended Liar Paradox exhibits all three concerns: the use of the LEM with neither of its disjuncts provable, self-reference which is outside the realm of the recursively enumerable, and illicit meaning identity across the object- and meta-languages.

5 The Use of the LEM in Berry's Paradox in Particular.

We start by examining the informal version of Berry's Paradox, before treating the formalized versions of Priest and Brady in §6 and §8, which are rather more specific. However, this examination of the informal version will inform our treatment of these formalized versions. As stated in Brady [2017], Berry's Paradox arises by considering the natural numbers that are denoted by English noun phrases of less than 100 letters. (Priest in [2019] uses 1000 words, but our earlier use of 100 letters will suffice equally well.) Since there are finitely many such numbers, there are denumerably many that are not so denoted. We then consider the least such number, but the expression 'the least number not denoted by an English noun phrase of less than 100 letters' is an English noun phrase of less than 100 letters and as such denotes a natural number. This yields a contradiction, which is called Berry's Paradox.

On the face of it, the LEM does not appear to be used in Priest's formalized derivation and we need to refer to the above sections §2, §3 and §4 for help in identifying its location. However, we proceed with the determination of the use of the LEM in the above informal version. With reference to Brady [1984], [2006] and [2017], there is some advantage in the semi-formal presentation as follows:

> For any natural number n, 'the least number not denoted by an English noun phrase with less than 100 letters' denotes n iff n is the least number not denoted by an English noun phrase with less than 100 letters.

The equivalence defines what the expression means in terms of its denotation, putting it on the same definitional footing as the above paradoxes. This is clearly a semantic paradox but concerning denotation of natural numbers instead of the truth of sentences, as occurs for the Liar and Extended Liar Paradoxes. Importantly, as we will see, this form, being stated as an equivalence, will leave it open as to whether the quoted expression denotes a natural number at all. Indeed, such a lack of denotation is needed if one is going to avoid the paradox at all.

We start by focussing on the origin of the word 'not' as it appears in the right-hand side of the above equivalence, i.e. in the expression 'n is the least number not denoted by an English noun phrase with less than 100 letters'. As argued in the informal version, this negation occurs as an alternative to the finite set of denotations of finite denoting expressions. This. version would normally use the LEM so that only the two options occur, i.e. that of being denoted and that of not being denoted. If the LEM fails, this would create a situation where neither of these two options apply. Indeed, each application of the LEM removes a case of neither of these two options occurring, replacing it by a case of not denoting. That is, the LEM replaces non-denoting by not denoting, which is standardly what it does. The reason 'not denoting' is needed is that the paradox is stated in the object language, whereas 'non-denoting' is meta-linguistic.

The key to Berry's Paradox is in the self-reference of the expression 'the least number not denoted by an English noun phrase with less than 100 letters' as being such an English noun phrase of less than 100 letters, which indeed does not denote some natural number, and so we can form the least such number. As argued above, the LEM is used in establishing this 'not denoting' and, due to this self-reference, the expression does denote a natural number, thus creating the contradiction which is Berry's Paradox. As for three of the above paradoxes of §4, this application of the LEM is made to a self-referential expression, whilst in the Yablo's Paradox the relationship occurs within a block of sentences.

Also, as for the paradoxes in §4, there is no proof given for either of the disjuncts of the LEM, viz. 'the least number not denoted by an English noun phrase with less than 100 letters' denotes the natural number n or does not denote n. The assumption of either of these disjuncts would yield a contradiction, since if the above expression denotes n then n is not so denoted, in accordance with the expression, and if it does not denote n then there is a least such number which is not so denoted, in which case the expression denotes that number. So, in order avoid Berry's Paradox, the above expression must non-denote every natural number, which requires the lack of LEM. However, unlike for the Liar Paradox and Russell's Paradox, the consistency proofs of Brady [2006], which are object-theoretic, do not extend to cover Berry's Paradox, as its self-referring expression has a meta-theoretic feature which includes a count of the number of letters in it. So, we do not have a consistency proof for a theory containing Berry's Paradox. Also, like the Extended Liar Paradox, Berry's Paradox is in part meta-theoretic. This is because Berry's Paradox attempts to employ the above expression in an object language but contains the meta-theoretic count of its numbers of letters, thus creating an illicit overlay of object and meta-theory.

We finish this section with an interesting deductive interplay between the three key concepts: the LEM, self-reference and meta-theory. Self-reference implies a lack of recursive enumerability, which prevents the proof of one of the disjuncts of the LEM, due to the need for such a proof to include a self-referential definition, which then leads to the failure of the LEM. This is an important conclusion in itself, which justifies our entire approach to the paradoxes. However, we can take the argument further. We start with LEM failure from which follows the usage of the meta-theoretic negation 'non', which conflicts with the usage of the object-theoretic 'not' of the above expression, creating a clash between object and meta-theory, which suffices to prevent the self-reference. Putting these two arguments together, self-reference implies a lack of self-reference. Hence, if self-reference satisfies the LEM then failure of self-reference follows, by applying the intuitive meta-rule MR1. So, the conclusion of all this argument is that self-reference fails, given that the LEM applies to the self-reference itself. And, this instance of the LEM is justifiable as this self-reference or its failure as such can be established one way or the other by a mechanical process. So, we conclude that self-reference fails and the reason for this would be the clash between object- and meta-theory, which is in turn induced by the lack of LEM.

6 The Use of the LEM in Priest's Formalization of Berry's Paradox.

With reference to §5 on the informal version of Berry's Paradox, we examine the negative component of Berry's Paradox in Priest's [2019] to show that the Law of Excluded Middle is applied to the same formula as was done in Priest's [1983] paper.

We consider Priest in [2019] as going through three stages in his deduction of Berry's Paradox, starting with a simple numerical argument, which introduces a finite set of natural numbers X and a partial function f with domain X, and concludes that the range of f cannot include all natural numbers. This follows because the range of f is finite and so there are infinitely many natural numbers outside its range. We consider this stage first as it neatly expresses the key components of Priest's argument for Berry's Paradox.

This case of being outside the range of the function f is the negative which will ultimately form the negative component of Berry's Paradox. If the set X and its range under f are recursively enumerable then the LEM applies here, as this recursion ensures that one of its disjuncts can be proved. Here, the LEM enables the negative to be expressed as an object-language negation, essential to establishing the contradiction of the paradox. Normally, finite sets are recursive and certainly are in the case of standard arithmetic, but not if there is self-reference in defining them as this would be a circular definition, where recursion and recursive enumerability would fail to be the case, as was argued in §5. This is the reason Berry's Paradox creates a special difficulty in comparison to other paradoxes and hence the time spent on its study has been worthwhile.

The second stage puts X as the finite set $\{x : A(x)\}$ and f as the function $B(x, y)$, where y is uniquely determined for each x satisfying $A(x)$. Consequently, there is a number y not satisfying $A(x)$ and $B(x, y)$, for any x, for the same reason as above. Again, if it takes place within recursive arithmetic the LEM applies, and if there is a circular definition the LEM will not apply.

However, the third stage puts $A(x)$ as $M(x)$, which says that x is a term of the language with less than 100 letters, and puts $B(x, y)$ as Dxy, which says that the term x denotes y. [Note that Priest uses 'has at most 1,000 words/symbols' instead of the author's 'has less than 100 letters', which is immaterial.] The conclusion $\exists y{\sim}\exists x(M(x)\&Dxy)$ is then drawn, where y is outside the range of Dxy, where x satisfies $M(x)$. That is, "something is not denoted by any x satisfying $M(x)$", as on p.43 of Priest [2019]. Priest then lets τ be the least y such that ${\sim}\exists x(M(x)\&Dxy)$ and hence ${\sim}\exists x(M(x)\&Dx\tau)$. This is then the negative half of the contradiction which constitutes Berry's Paradox.

Priest then proceeds with the positive half of Berry's Paradox, as follows. Since $M(x)$ says that x is a term of the language with less than 100 letters, there are only a finite number of terms x such that $M(x)$. By considering 'the least y such that $\sim\exists x(M(x)\&Dxy)$' as a term $\langle\tau\rangle$ of the language, $M(\langle\tau\rangle)$ holds. This step involves counting the letters (and symbols) in the above expression and, as such, is meta-theoretic and it is also the point of self-reference as this term refers to itself in the process. However, it is not the simple arithmetic truth referred to on p.43 of Priest [2019]. Proceeding with Priest's argument, $D(\langle\tau\rangle, \tau)$, by the meaning of denotation. Hence, $M(\langle\tau\rangle)\&D(\langle\tau\rangle, \tau)$ and $\exists x(M(x)\&Dx\tau)$, which is the positive half of Berry's Paradox, contradicting $\sim\exists x(M(x)\&Dx\tau)$, established above for the negative half of the paradox.

The use of the LEM, however, goes back to the negative half where the above self-reference is used in determining the finite set $\{x : M(x)\}$. Such self-reference prevents the proof of the LEM through one of its disjuncts, as such a proof would need to be made prior to this instance of the LEM as a recursively enumerable proof. Any such proof that uses the definition of $\{x : M(x)\}$ would involve the circularity, breaking its recursive enumerability. So, Priest, by assuming he is working in standard arithmetic, illicitly uses the LEM in establishing the number y outside the range of Dxy, where x satisfies $M(x)$. He considered this negative statement as part of the object-language and the LEM is needed to do this to prevent it being left as a meta-theoretic 'non-', as explained in §5.

This then covers everything important that needs to be said, but three small points remain, which we will cover in §7, §8 and §9.

7 Implication and Entailment.

It does not matter as far as Priest's general arguments in his [2019] are concerned, but we think some attention should be made to distinguish implication from entailment in their formalization. We have tried to put the above arguments in English as much as possible so that they follow in a general deductive fashion. Such deduction would generally be in the form of rules rather than entailments. As argued in §4 of Brady [2022], implication should be captured in rule-form rather than as a connective. The principal reason behind this is that implications do not have negative requirements on their antecedents. Note that material implication is always true when its antecedent is false. However, this is quite artificial and is one of the major fallacies of material implication. Note that we deliberately make no mention of material implication in the presentation of the classical recapture. Further, relevant implication has conceptual problems, as argued in Brady [1996] and [2022]. Rules also have no negative requirements on their premises and so it makes sense for implications to be cast as rules. Entailments, however, should be captured by a connective as they involve a meaning comparison between antecedent and consequent, with a meaning containment of consequent within antecedent being the ideal.

The cases in point occur on p.42 of Priest [2019]. The D-Schema, $D\langle t\rangle x \leftrightarrow x = t$, is fine, as $x = t$ captures the meaning of the denotation $D\langle t\rangle x$. However, the Description Schema should be a rule, as $A(\mu xA)$ follows, given $\exists xA(x)$, but not as a meaning containment.

(However, the converse $A(\mu x A) \rightarrow \exists x A(x)$ is fine as $\mu x A$ is an instantiation.) Further down the page, the two premises of the deduction should be a rule equivalence and a rule, instead of their $\leftrightarrow$ and $\rightarrow$ forms. In the first premise, $A(x) \leftrightarrow V(i \in I)x = n_i$, the disjunction of identities representing A's individual elements, using the finite index set I, provides an extension for the set $A(x)$, rather than capturing its meaning, and the identity in the second premise, $A(x)\&B(x,y)\&B(x,z) \rightarrow y = z$, which establishes the functionality of the relation B, could be extensional rather than intensional.

That has now covered all the critical material, and the author will now, apologetically, make two concessions in §8 and §9 respectively.

8 Brady's Formalization of Berry's Paradox.

In Brady [1984], there are both informal and formal examinations of Berry's Paradox, as presented by Priest in [1983]. Here, we examine the formalization, having discussed the informal examination briefly in §1 and much more fully in §5. The following semi-formal presentation from p.157 of Brady [1984] was given as a useful starting point in §5. For any natural number n, 'the least number not denoted by an English noun phrase with less than 100 letters' denotes n iff n is the least number not denoted by an English noun phrase with less than 100 letters.

As in Brady [1984], repeated in [2006], this equivalence is fully formalized as:

$$(\forall n)(Den \leftrightarrow (\forall x, Lx){\sim}Dxn \ \& \ (\forall m, (\forall x, Lx){\sim}Dxm)n \leqslant m), \quad \ldots \text{(E)}$$

where e stands for the expression 'the least number not denoted by an English noun phrase of less than 100 letters', D stands for the relation of denotation for English noun phrases, L stands for the predicate 'less than 100 letters', m and n range over natural numbers and x ranges over English noun phrases. The expressions '$(\forall x, Lx)$', '$(\forall m, (\forall x, Lx){\sim}Dxm)$' are restricted universal quantifiers, respectively restricting English noun phrases x to those with less than 100 letters, and restricting natural numbers m to those not denoted by an English noun phrase with less than 100 letters. Thus, (E) says that e denotes a natural number n iff n is not denoted by an English noun phrase of less than 100 letters and n is the least such number.

The main problem here is to do with the restricted quantifiers $(\forall x, Lx)$ and $(\forall m, (\forall x, Lx){\sim}Dxm)$, in relation to Brady's recent work on restricted quantification in his [2023]. Following this work, in order to ensure that the domains of the respective restricted quantifiers are non-empty, we must assume $\exists x Lx$ and $\exists m(\forall x, Lx){\sim}Dxm$. $\exists x Lx$ is not an issue since there is a noun phrase of less than 100 letters. However, whether such an expression does not denote a natural number is an issue, as this would then yield a contradiction, which is indeed Berry's Paradox, as was argued in §5 above. So, we cannot use restricted quantification to express the full formalization as in (E) above.

There is also the question of the relation to the axiomatic introduction of the least number $\mu x A(x)$, as set out in Brady [2021]. As quoted, we set out the following principles for the formula A to follow:

Least Number Principles.

1. $A(a) \vee \sim A(a)$.
2. $\exists x A(x) \Rightarrow A(\mu x A(x))$. [Note the use of '$\Rightarrow$' here.]
3. $\exists x A(x), m < \mu x A(x) \Rightarrow \sim A(m)$.

However, the LEM of Least Number Principle 1 does not hold for $A(x)$ in this context, which is 'x is a natural number not denoted by a noun phrase of less than 100 letters'. The Least Number Principles 2 and 3 provide a recursive introduction of the least number $\mu x A(x)$.

So, the conclusion is that the informal form of Berry's Paradox cannot be formalized, at least by these two sensible methods. One should realize however that, without the LEM, there would be numbers that are neither denoted nor not denoted by the expression 'x is a natural number not denoted by a noun phrase of less than 100 letters', indeed non-denoted by the expression. This would break down the recursive process, leaving the least number not so denoted as a meta-theoretic concept, which we think is the way it was meant to be understood, that is, allowing for non-denoting as well as not denoting. So, a formalization of Berry's Paradox in the one object-language is not possible, as with the Extended Liar Paradox. Note that the Liar and Russell's Paradoxes, together with Yablo's Paradox, are formalizable in the object-language, as occurs in §4.

As a final note, Priest in [2019]. does say that Berry's Paradox is still derivable if there is some number not so denoted, without having to determine the least such number, thus circumventing the above two methods of formalizing the least number by replacing the least number operator by an indefinite description operator. We agree with Priest on this point, but this does not change its lack of formalizability.

9 Substitution of Identity.

Lastly, we apply the distinction between intension and extension, discussed in Brady [2022], to determine the corresponding forms of substitution of identity principles, as this applies to Priest's derivation of Berry's Paradox in his [1983] paper. The form of substitution of identity appropriate for an extensional identity $a = b$, where a and b would represent the same objects, is one which $A(b)$ would preserve the truth of $A(a)$. Given the discussion in Brady [2022], it would take the shape: $a = b, A(a) \Rightarrow A(b)$, where $A(b)$ is $A(a)$ with b substituted for a, for an arbitrary formula A of the logical system. This can be reshaped as: $a = b \Rightarrow .A(a) \Rightarrow A(b)$, and, because of the symmetry of identity, $a = b \Rightarrow .A(a) \Leftrightarrow A(b)$. Assuming that the classical recapture applies, with the addition of the Law of Excluded Middle and the Disjunctive Syllogism for both $A(a)$ and $A(b)$, this becomes deductively equivalent to the classical form: $a = b \Rightarrow A(a) \equiv A(b)$.

Let us now consider the appropriate form of substitution for intensional identity. Given that such an identity requires an equivalence in meaning between a and b, this would generate a meaning equivalence between $A(a)$ and $A(b)$, which is represented in the logic by the entailment equivalence '$\leftrightarrow$'. Thus, the corresponding substitution of identity rule would be of the form: $a = b \Rightarrow A(a) \leftrightarrow A(b)$. Note that both these forms are in inductive shape, which is established by induction on their formulae. Lack of induction would create a problem at any particular step in the inductive process which would fail to apply.

Finally, we note that, as argued in Priest [2019], the form of substitution of identity used in his [1983], though of inappropriate form, is nevertheless not needed in his derivation of Berry's Paradox. We agree and accept his point on this matter.

10 In Conclusion.

We have shown that Priest in [2019] does use the LEM in the same place in the argument for Berry's Paradox as in his [1983], repeated in [1987], as argued in §5 and §6, viz. at the point of self-reference, where the denoting expression refers to itself through the counting of the number of letters used to express it. Note also that the LEM has the effect of replacing a 'non' of the meta-language by a 'not' of the object language.

Indeed, we have seen that Berry's Paradox cannot be formalized without illicitly putting the object and meta-theories together into one system in order to create a contradiction in the object language. This is so as a result of a meta-theoretic influence, viz. the count of the number of letters in the denoting expression, that cannot be absorbed into the object-language. This applies to Priest's formalizations in his [1983] and [2019] and Brady's formalization in [1984], repeated in [2006]. The lack of formalization also applies to the Extended Liar Paradox in §4, though not to the Liar, Russell's and Yablo's Paradoxes.

We have admitted to the lack of need for the Substitution of Identity in the Berry's Paradox argument, though we have maintained a need to separate the extensional and intensional versions and express them both in an inductive shape.

Acknowledgements

I would like to say a special thank you to the anonymous referee for a large number of detailed comments and suggestions, which have all helped to improve the general quality of this paper. I make special mention of the referee's help in specifying the set CR for the classical recapture and for help in tidying up my ideas about negation.

References

ANDERSON, A.R. and BELNAP, N.D. Jr.

1975: Entailment, The Logic of Relevance and Necessity, Vol. 1, Princeton U.P., Princeton.

BEALL, JC and RESTALL, G.

2006: Logical Pluralism, Oxford U.P., Oxford.

BRADY, R.T.

1983: ”The Simple Consistency of a Set Theory Based on the Logic CSQ”, *Notre Dame Journal of Formal Logic*, Vol. 24, pp. 431-449.

1984: ”Reply to Priest on Berry’s Paradox”, *Philosophical Quarterly*, Vol. 34, pp. 157-163.

1989: ”The Non-Triviality of Dialectical Set Theory” in *Paraconsistent Logic, Essays on the Inconsistent* (ed. by G. Priest, R. Routley and J. Norman), Philosophia Verlag, Munich, 1989, pp.437-471.

1996: ”Relevant Implication and the Case for a Weaker Logic”, *Journal of Philosophical Logic*, Vol. 25, pp. 151-183.

2003: (editor) *Relevant Logics and their Rivals*, Vol. 2, Ashgate, Aldershot, U.K.

2006: *Universal Logic*, CSLI Publications, Stanford.

2008: ”Negation in Metacomplete Relevant Logics”, *Logique et Analyse*, Vol. 51, pp.331-354.

2012: ”The Consistency of Arithmetic, Based on a Logic of Meaning Containment”, *Logique et Analyse*, Vol.55, pp.353-383.

2015: ”Logic - The Big Picture”, in *New Directions in Paraconsistent Logic*, ed. by Jean-Yves Beziau, Mihir Chakraborty and Soma Dutta, Springer, New Delhi, pp.353-373.

2017: ”The Use of Definitions and their Logical Representation in Paradox Derivation”, in a special issue of *Synthese* on Substructural Approaches to Paradox, ed. by Elia Zardini, with DOI: 10.1007/s11229-017-1362-7. Subsequently published in Vol. 199 (2021), pp.527-546.

2017b: ”Metavaluations”, *The Bulletin of Symbolic Logic*, Vol. 23, pp.296-323.

2019: ”On the Law of Excluded Middle”, in *Ultralogic as Universal?, The Sylvan Jungle -Vol. 4*, edited by Zach Weber, Springer, Switzerland, pp.161-183.

2019a: ”The Number of Logical Values”, in *Graham Priest on Dialetheism and Paraconsistency*, ed. by Can Baskent and Thomas Macauley Ferguson, Springer, Switzerland, pp.21-37.

2021: ”The Formalization of Arithmetic in a Logic of Meaning Containment”, in a special issue of the *Australasian Journal of Logic* devoted to Meyer’s work on relevant arithmetic, edited by Thomas Ferguson and Graham Priest, Vol. 18, pp. 447-472.

2022: ”Intension, Extension, Distribution and Decidability”, in *Relevance Logics and other Tools for Reasoning. Essays in Honor of J. Michael Dunn*, ed. by Katalin Bimbó, College Publications, London, UK, pp.128-149.

2023: ”A Conceptual Approach to Restricted Quantification in Relevant Logics”, to appear in *New Directions in Relevant Logics*, ed. Shawn Standefer, Springer.

BRADY, R.T. and MEINANDER, A.

2013: ”Distribution in the Logic of Meaning Containment and in Quantum Mechanics”, in *Paraconsistency: Logic and Applications*, ed. by Koji Tanaka, Francesco Berto, Edwin Mares and Francesco Paoli, Springer Publishing, Dordrecht, pp.223-255.

MEYER, R.K.

1976 ”Metacompleteness”, *Notre Dame Journal of Formal Logic*, Vol. 17, pp. 501-516.

MEYER, R.K. and DUNN, J.M.

1969: ”E, R and ”, *The Journal of Symbolic Logic*, Vol. 34, pp. 460-474.

PRIEST, G.

1983: ”Logical Paradoxes and the Law of Excluded Middle”, *Philosophical Quarterly*, Vol. 33, pp. 160-165.

1987: *In Contradiction*, Nijhoff, Dordrecht, 1st Edition

2006: *In Contradition*, Oxford University Press, 2nd Edition, Oxford.

2019: ”Berry’s Paradox... Again”, *Australasian Journal of Logic*, Vol. 16, pp. 41-48.

SLANEY, J.K.

1984: ”A Metacompleteness Theorem for Contraction-free Relevant Logics”, Studia Logica, Vol. 43, pp. 159-168.

1987: ”Reduced Models for Relevant Logics without WI”, *Notre Dame Journal of Formal Logic*, Vol. 28, pp. 395-407.

YABLO, S.

1993: “Paradox without Self-Reference”, *Analysis*, Vol.53, pp.251-252.