Some False Laws of Logic

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This paper argues that some widely used laws of implication are false, and arguments based upon them invalid.[a] These laws are *Exportation*, *Commutation*, (as well as various restricted forms of these), *Exported Syllogism* and *Disjunctive Syllogism*. All these laws are false for the same reason – that they license the suppression or replacement in some position of some class of propositions which cannot legitimately be suppressed or replaced. These laws fail to preserve the property of sufficiency of premiss set for conclusion. They are false, and can be seen to be false, independently of their responsibility for the paradoxes. Hence the main 'independent' argument for the paradoxes – that they follow from an allegedly immaculate set of laws – is undermined. Counterexamples to all these laws are produced.

In order to assess any logical law as true or false, it must be considered with respect to some assumed interpretation, otherwise the ways in which it may be assessed are very limited indeed. 'Assumed interpretation' does not mean 'formal semantics'. These laws have properties incompatible with those of the assumed interpretation, whereas a system cannot have properties incompatible with its formal[b] semantics.

The assumed interpretation, both that generally assumed, and that assumed here, is that p entails q, symbolized '$p \Rightarrow q$', holds iff q is deducible from p. The feature of this interpretation with which these laws are incompatible is that if q is deducible from p, *p must be sufficient for q*. Another important interpretation which requires sufficiency, and with which these laws are therefore also incompatible, is the so-called 'nomological' or lawlike

[a]Footnotes numbered alphabetically are editorial notes indicating changes or points of interest in the text, and numerical footnotes are Plumwood's. Typesetting and editorial notes by Andrew Tedder. Many thanks are due to Tore Fjetland Øgaard for detailed comments and typo-spotting in a previous version.

[b]This word is "formed" in the manuscript.

implication, for the logical feature which characterises such implications is that the antecedent states a *sufficient* condition for the consequent – one which is contingently sufficient in the case of natural laws. Deductive implication or entailment is then related to lawlike implication $p \to q$ such that p states a sufficient condition for q, as the special case where it is logically necessary that p states a sufficient condition for q i.e. $p \Rightarrow q =_{Df} \Box(p \to q)$. Only the deducibility interpretation is considered in detail, but since the salient feature of all the counterexamples is lack of sufficiency, some parallel counterexamples can be constructed to these laws interpreted as holding for lawlike implication.

Almost every major system of implication has adopted deducibility as the assumed interpretation.[1] Only Anderson and Belnap have clearly committed themselves to the interpretation of $p \Rightarrow q$ as p is sufficient for the deduction of q, while the Intuitionists [e.g [9] p.99 & 102.] specifically disclaim sufficiency. But sufficiency of premiss for conclusion is, and always has been, an *essential* feature of deducibility, without which it would be unable to perform its essential and defining roles. It is a traditional and perfectly correct criticism of someone's claim that q follows from p, *that q does not follow from p alone*, that in proving q we needed also to assume some further proposition r which is not stated. In showing this, we would have shown that the claim that q follows from p was false. In a deductive system, the axioms should be sufficient for the theorems; it is a common criticism of the Euclidean system, for example, that the axioms of continuity, which are needed for the proofs of some theorems, are not included in the axiom set, the assumption being that the theorems do not follow from the insufficient axiom set. Similarly, it is a common criticism of logicism that mathematics is not deducible from logic because logic on its own is not sufficient. A basic principle of traditional deductive theory and practice has always been that all propositions which are used in arriving at the conclusion must be stated in the premiss set.

If we used, or needed to use, a proposition which was true but which did not appear in the premiss set, what we had was called, at least by some writers later than Aristotle, an enthymeme, and enthymemes are not valid

[1]The deducibility interpretation is assumed in the following: Whitehead & Russell [23, Sect. A, p. 94], Lewis & Langford [14], Johansson [10], Hilbert [8], Heyting [9], Ackermann [1], Anderson & Belnap [3].

arguments,[2] and have traditionally been distinguished from valid arguments.[3] Aristotle says of a valid syllogism [[5] 24^a 18–21, my italics] "A syllogism is discourse in which, certain things being stated, something other than what is stated follows of necessity *from their being so.* I mean by the last phrase that *they produce the consequence,* and by this, *that no further term is required from without in order to make the consequence necessary.*"

Aristotle explains his otherwise odd statement that the premisses *produce* the consequence as a way of saying that they are *necessarily sufficient* for the consequence. The premisses of a valid argument are sufficient for the consequence just as the cause is sufficient for the effect it produces. For just as we refuse to say that some cause produces some effect if something *further* is needed to produce it, so we refuse to say that some conclusion is validly deducible from a premiss if something further is needed as a premiss.

The traditionally-backed prohibition of the suppression of needed premisses is at the same time a prohibition of insufficiency of the premiss set for the conclusion. The proposition or class of propositions which, if added to the premiss set, would make it sufficient for its conclusion is just the class which is said to be (illegitimately) suppressed.

The standard terminology leaves room for a trivial ambiguity. Suppression is sometimes used to mean the mere omission of some premiss from the premiss set, sometimes its illegitimate suppression i.e. the omission of some premiss so that the argument becomes invalid, or the premiss set insufficient for the conclusion. In the first sense there are legitimate forms of suppression, (for clearly some premisses may be omitted without the argument becoming invalid as with q in $p \;\&\; q \Rightarrow p$), while in the second there are no legitimate forms of suppression. This second sense of 'suppression' is adopted here, the first being called 'omission'. The suppression of some proposition always results in insufficiency of the premiss set for the conclusion. Suppression is always illegitimate suppression.

It is, of course, analytic that unnecessary premisses may be omitted, but they are not then suppressed. A premiss is in fact suppressed if it is in fact

[2]Except that some irrelevant traditional disagreement over whether enthymemes were valid derived from ambiguous use of the word 'enthymeme' to cover both the invalid, incomplete argument and the valid one corresponding to it – a tradition which is preserved in the acceptance of arguments involving suppression as 'elliptical versions' of those without suppression.

[3]As Anderson & Belnap have rightly pointed out in [2]. Many of the distinctions of this paper are similar to distinctions pointed out by Anderson & Belnap.

needed, as well as omitted. Unnecessary premisses are unnecessary for the sufficiency of the premiss set for its conclusion, and their omission leaves the premiss set sufficient, so they are not suppressed.

The use of the expression 'needed' may seem to have unfortunate psychologistic overtones, to assume uniqueness. Although some traditional ways of explaining suppression have been psychologistic, making reference to the obviousness of the suppressed premiss, this is not needed here, nor is the general notion of suppression thereby discredited. Nor is the uniqueness of the suppressed premiss here assumed, although it sometimes has been.[4] *Any* proposition P which will, if added to a premiss set A insufficient for conclusion C, result in a new set A & P becoming sufficient for C, is here regarded as being suppressed in A for conclusion C. In view of *Conjunctive Simplication,* there will always be an infinite number of such propositions, and we say of one of them that it is needed because *some* such proposition is needed, and this one will fill the bill. In practice the field is often narrowed by the necessity of stopping the original set becoming redundant or other such requirements.

Formally then, P_i is conjunctively omissible under condition K in system S in a conjunction of premisses P_i & $P_{i+1} \dots P_n$ for a conclusion C iff

$$\vdash_S P_i \,\&\, P_{i+1} \dots P_n \Rightarrow_S C \text{ provided } K\ (P_i)$$

and

$$\vdash_S P_{i+1} \,\&\, \dots P_n \Rightarrow_S C.$$

Where '$\Rightarrow_S$' is the '$\Rightarrow$' of S, '$\Rightarrow_S$' allows the omission of P_i in the above situation.

Suppression is omission which is not permitted in the assumed interpretation of S, called M_S, (Roughly, M is the structure which S is intended to formalise). We can measure the correctness of S according as it has or lacks the important features of M_S. In our case the assumed model of $p \Rightarrow q$ is that given by 'p is sufficient for the deduction of q' and the features of this

[4] For example in W.E. Johnson [11, p. 100] "Now in an enthymeme there is one, and only one, proposition which could be introduced to render the corresponding syllogism valid." If *Conjunctive Simplification* is correct, either Johnson is wrong or there are no enthymemes.

relation, taking account of the usual sense of these expressions insofaras they are established by the traditional roles of deducibility.

Then '$\Rightarrow_S$' allows the illegitimate conjunctive suppression of P_i in the conjunction P_i & ... & P_n for conclusion C iff

$$\vdash_S P_i \mathbin{\&} P_{i+1} \mathbin{\&} \ldots P_n \Rightarrow_S C$$

$$\text{and } \vdash_S P_{i+1} \mathbin{\&} \ldots P_n \Rightarrow_S C$$

provided $n \neq 0$

i.e. if P_i is conjunctively omissible in S, but

$$\vdash_{M_S} P_i \mathbin{\&} P_{i+1} \mathbin{\&} \ldots P_n \Rightarrow_M C$$

$$\text{and } \vdash_{M_S} P_{i+1} \mathbin{\&} \ldots P_n \not\Rightarrow_M C$$

provided $n \neq 0$

i.e. P_i is not conjunctively omissible in the model.

To put the definition informally, an implication of a system S allows illegitimate suppression if it allows the omission in a certain premiss set for a certain conclusion of some proposition which is in fact needed to make the argument from these premisses to that conclusion valid, i.e. to make the premiss set given sufficient for that conclusion.

Hence the requirement that $p \Rightarrow_S q$ iff p is sufficient for q, is satisfied iff '$\Rightarrow_S$' does not license illegitimate suppression. The requirement that there be no suppression is for deducibility the same as the requirement that we put down all premisses used, or a sufficient set of premisses.

The notion is not really a formal one, for we have had to make reference to the assumed interpretation of S to define it. It is defined here merely to free it from the psychologistic overtones usually associated with it.

Why is suppression bad? It defeats some of the essential features of deducibility.

The first, but not the most important feature, is that an argument with a suppressed premiss is much less readily assessable than one in which all the premisses are explicitly stated. Because a very large number of propositions could fill the gap, there is a loss of clarity and precision. It is only if all used premisses are explicitly stated that they can be properly scrutinised. A more important and related feature concerns the role of deductive logic in carrying conviction – the ‘compulsion’ feature. It is only if *all* the premisses used in the argument are stated and *accepted* that we are compelled to accept the conclusion. If some premiss is used and not stated, then in accepting the incompletely stated premisses we do not accept a set *completely* sufficient for the conclusion; and if we should demur at the further needed premiss, as we may do even if it is necessarily true, we are not compelled to accept the conclusion. (Those who think that necessary truths are generally accepted or obvious should look at some of the disputes in modal logic.). No proposition is so unquestionable, so protected from rejection, that we are simply entitled to drop it, without notice, from valid argument. But conviction cannot be the whole story; for even if the suppressed proposition were one we were not inclined to question, such as the law of identity, we might still want to know whether it was used to obtain a particular conclusion. We may, and often do, want to know not merely whether the conclusion is true, but whether it can be obtained from *this* set of premisses. We expect a correct assessment of deductive responsibility.

Deductive logic has never been concerned with the willy-nilly churning out of true propositions implied by *some* true set of premisses, we care not which. A major function has always been the correct assigning of responsibility for those conclusions, and with the converse relation, assessing exactly what is involved in asserting some set of propositions. Hence its important traditional uses in criticising what someone says as insufficient for the conclusions he draws, and criticising what he says by looking at its consequences. It is not peripheral, but essential, to deduction that it should be able to fill these roles and both presuppose correct assigning of responsibility for conclusions drawn. In terms of this notion of valid argument, the modern systemic notion which claims to have replaced it – that of the churning out of theorems from some of the axioms, we care not which – is of much more restricted applicability.

But suppression has a disastrous effect on all these functions. By omitting some premiss without which the deduction of some conclusion is not valid, it misrepresents the premiss from which this conclusion is obtained, and hence responsibility for the conclusion. To agree to accept partial responsibility as good enough here is like agreeing to say that somebody was responsible for the dinner when he peeled potatoes and the cook did the rest. The first statement cannot be accepted as an elliptical, but allowable, way of making the second statement. And similarly suppression enables us to obtain as causally responsible a partially sufficient rather than a fully sufficient causal condition.

Likewise we cannot assess exactly what is involved in asserting that p if we are unable, for the purpose of drawing some conclusion r, to separate p from some other proposition q, and to separate p's consequences from q's. But if q is suppressible when it occurs conjoined with premiss p for some conclusion r, this is just what we are unable to do. We have assessed, not what is involved in p, but what is involved in p and q, which may be a very different matter indeed.

But a most important objection to the deductive suppression derives from deducibility as a *meaning* relation between propositions. q should be deducible from p only if there is a connection of meaning between p and q. But this connection may be destroyed if suppression is allowed; for the suppressed proposition, which although used no longer appears in the premiss set p, may be just what originally made the meaning connection between p and q. Once this used proposition has been dropped off, p and q may no longer have the right connection of meaning (e.g. inclusion), or worse still, may have no connection at all. We find the latter situation where the suppressed proposition provided the *only* link between premisses and conclusion, as, for example, p may do in some substitution instances of $p \mathbin{\&} q \Rightarrow p$. By just such a use of suppression the paradoxes are produced, as is argued in the last section.

If the meaning connection is taken specifically to be one of *inclusion of content*, any suppression will lead to misrepresentation of that relation. For, by definition, suppression is incompatible with the sufficiency of premiss for conclusion. But sufficiency of premiss p for conclusion q is a necessary condition on p's containing or including q. For something A contains something B only if A *alone* contains B; if we have to add something further to a to obtain B then A simply does not contain B. p contains q only if we do not need to go beyond p to obtain q i.e. if p is sufficient for q, q's consequences are subset of p's consequences. But if the logical content of a proposition is given by its logical consequences, the logical content of q is contained in that of p.[5] In this way suppression leads to the misrepresentation not merely of the logical consequences of a proposition, but also of its meaning. For we are then unable to differentiate it, in terms of its logical consequences, from other different propositions. When we suppress a proposition we discount its meaning, by treating it as adding nothing. No distinct proposition can al-

[5]Similarly a law such as $p \mathbin{\&} q \Rightarrow p$ satisfies both the inclusion and the sufficiency models for the same reason – that if the premiss p is sufficient for the conclusion p (as it is), it must remain so upon the addition of further premisses, just as it must continue to *include* the conclusion upon the addition of further premisses.

ways be treated in this way. For suppose that some proposition p is generally omissible. Then, using *Conjunctive Simplification*, $p \,\&\, q \Rightarrow p$, since p is generally omissible, (by hypothesis), q implies p i.e. any proposition is sufficient for p. Therefore p is included in the content of every proposition. But it is reasonable to suppose that some propositions are entirely disjoint in content; therefore this generally omissible proposition can have *no* content. Hence it is not a distinct proposition, and, contraposing, no proposition should be generally omissible.

All these features of deducibility, then, provide reasons for saying that every proposition sometimes occurs essentially and has its own bit to add. This leads to the Suppression Principle: *for every proposition p there is some proposition q such that the consequences of q are a proper subset of the joint consequences of p and q. There is no privileged*[6] *class of propositions which are generally suppressible.*

II

The most general and blatant conjunctive suppression law of classical logic is *Exportation*, $p \,\&\, q \Rightarrow r \Rightarrow .p \Rightarrow (q \Rightarrow r)$. *Exportation* permits the wholesale suppression of truth propositions whenever they occur in a conjunction of premisses. For if we have $\vdash_S p \,\&\, q \Rightarrow r$, we have also by Exportation, $\vdash_S p \Rightarrow (q \Rightarrow r)$, and hence by detachment, $\vdash_S q \Rightarrow r$ provided p is true.

Applying the definition of omission, p is omissible when p is true, according to *Exportation*, given only detachment.

Under the sufficiency interpretation, *Exportation* allows us to say that whenever we have two propositions p and q, which jointly imply some conclusion r, q alone is sufficient for r if p is true, and p alone is sufficient for r if q is true. But if q is sufficient for r, p is inessential. Hence true propositions are, according to *Exportation*, always inessential whenever they occur as conjoined premisses in a deduction. *Exportation* is inconsistent with the Suppression Principle.

The omission of true premisses allowed in *Exportation* is a genuine case of suppression, because it is not permitted in the model. If p and q jointly imply some proposition r, p may have to be used essentially in getting to

[6]Or perhaps we should say *underprivileged*. For while the generally suppressible proposition is privileged in being protected from rejection, it is underprivileged in that its meaning is discounted.

r. But since we have $\vdash_S q \Rightarrow r$, q must be assumed to be sufficient for r. But if p is used essentially q could not be sufficient for r. *Exportation* also violates another principle which implies, but is not implied by, the Suppression Principle – the Joint Force Principle. This is that two propositions may, taken together, have consequences which neither proposition on its own has: for every proposition p there is some other q such that p and q are jointly sufficient for r but neither p nor q on its own is sufficient for r. Formally, the Joint Force Principle says:

$$(p)(\exists q)(\exists r)(p \,\&\, q \Rightarrow r \,\&\, \sim (p \Rightarrow r) \,\&\, \sim (q \Rightarrow r))$$

It tells us that the *joint* consequences of propositons may be more than the sum of the consequences of each.[7] Such joint consequences are not included in the meaning of either of the premisses separately, but are obtained by the deductive *interaction* of the premisses – as with the generation of theorems from an axiom set.

Given the truth either of the Joint Force Principle, or of the Suppression Principle, *Exportation* is false. We can construct generalised counterexamples such that it violates modus ponens.

According to both the Joint Force Principle and the Suppression Principle, p and q may jointly imply some proposition r, both p and q being true, and it may often be false that q on its own is sufficient for r i.e. that $q \not\Rightarrow r$.[c] Hence we have for *Exportation* the assignments:

$$\begin{array}{cccccc} p \,\&\, q & \Rightarrow r & \Rightarrow . & p & \Rightarrow & (q \Rightarrow r) \\ & 1 & 0 & 1 & 0 & 0 \end{array}$$

This generalised counterexample enables us to construct a host of specific counterexamples to *Exportation*, based on picking as p some true premiss which is used essentially in some argument, or, more naturally, on cases where *both* premisses of an argument are used.

[7] *Exportation*, together with *Conjunctive Syllogism*, *Contraposition*, *Rule Factor* $A \Rightarrow B \longrightarrow A \,\&\, C \Rightarrow B \,\&\, C$ and *Rule Syllogism* $A \Rightarrow B, B \Rightarrow C \longrightarrow A \Rightarrow C$ implies the classical law

$$p \,\&\, q \Rightarrow r. \Rightarrow .p \Rightarrow r \vee q \Rightarrow r$$

which for *every* proposition denies the Joint Force Principle. Names of laws are taken from [23], except that a further distinction is made between exported and conjunctive forms of certain laws.

[c] Have used $\not\Rightarrow$ here where the manuscript has $\Rightarrow$.

According to Russell [21]

$$(\exists x)(fx \ \& \ gx) \ \& \ (\exists \iota x)(fx) \text{ jointly imply } g(\iota x fx)$$

but neither of the conjoined propositions on their own implies the conclusion. But picking the first predicate 'f' as 'is an Emperor of Ethiopia', the second predicate 'g' as 'keeps giant tortoises in his garden', we obtain from this the very plausible implication:

> There exists an emperor of Ethiopia who keeps giant tortoises in his garden (p) & There exists exactly one emperor of Ethiopia (q) $\Rightarrow$. The emperor of Ethiopia keeps giant tortoises in his garden (r).

But applying *Exportation*, since $p \ \& \ q \Rightarrow r, p \Rightarrow (q \Rightarrow r)$. Since p is in fact true, and since '$\Rightarrow$' is truth-preserving, it must be true that $q \Rightarrow r$ i.e. that

> There exists exactly one emperor of Ethiopia $\Rightarrow$ The emperor of Ethiopia keeps giant tortoises in his garden.

But surely Russell is right. q on its own does not imply r; r is the consequence of both p and q, and not of either one separately.

This example illustrates how *Exportation* (and suppression) repudiates deducibility as a meaning relation, by refusing to distinguish between a proposition's being true and its being stated as a premiss. But it is only if p is actually *stated* as a premiss that it is able to supply the needed meaning connection between q and r. The mere truth of p cannot supply such a meaning connection. The truth of p does not compensate for its absence as a premiss because p itself must be explicitly stated if the premiss is to state sufficient ground for the conclusion.

Exportation often allows the replacement of a premiss which is fully sufficient for the conclusion by one which is only partially sufficient. It therefore has a disastrous collapsing effect on lawlike implication. As in the following example, a statement which is initially lawlike may cease to be lawlike upon the application of *Exportation*. An instance of a classic example of catalysis is: The contents of a smooth-welled jar at s_1t_1 are brought into contact with a rough surface (p) & The smooth-walled jar at s_1t_1 contains hydrogen peroxide ($2H_2O_2$) (q) $\rightarrow$. The contents of the smooth-walled jar will decompose to $2H_2O + O_2$ (r). p and q are jointly sufficient for the result r i.e. p & $q \rightarrow r$. Applying *Exportation*, however, we find that $p \rightarrow (q \rightarrow r)$. Consider a case where p is true; then, as before, $q \rightarrow r$ must be true. But that the smooth-walled jar contains hydrogen peroxide is certainly not sufficient for the decomposition of the hydrogen peroxide, for hydrogen peroxide in a smooth-walled jar will keepy almost indefinitely. And this resulting statement, $q \rightarrow r$, similarly is no longer lawlike. Further counterexamples to *Exportation* interpreted as a law of lawlike implication may be obtained by choosing conditions which are only jointly sufficient for some result, not individually sufficient.

The defects in *Exportation* exposed by such examples cannot be ascribed to the fact that the truths suppressed are *contingent*, so that a violation of the modal law $p \Rightarrow q. \Rightarrow \Box p \Rightarrow \Box q$ is obtainable. First, although in these examples it was contingent truths that were suppressed, the resulting defective implication did not violate any such modal law. Second, since the trouble with *Exportation* is that it allows the suppression of some proposition essential to the argument, we cannot remedy this defect just by restricting the class of suppressible propositions to necessary ones, since necessary propositions may be used essentially. This is the solution embodied, however, in strict implication above $S1$, since a full form of *Necessity-Restricted Exportation* p & $q \Rightarrow r \Rightarrow .\Box p \Rightarrow (q \Rightarrow r)$ is found in $S3, S4, S5$ and T, while the rule form[8] is found in $S2$. *Necessity-Restricted Exportation* allows the general suppression of necessary truths.

The view that necessary premisses may be suppressed derives in part from the mistaken view that necessary truths are trivial, add nothing, and so can always be omitted in any argument. But it derives, too, from looking only at examples where necessary truths are instantiated, or used as principles of inference – when indeed they may be omitted without the resulting argument becoming invalid. When some logical law, such as $(x)(fx \Rightarrow gx)$, is instantiated as in the argument

$$(x)(fx \Rightarrow gx) \;\&\; fa \Rightarrow .ga,$$

then since the instantiation $fa \Rightarrow ga$ holds, the omissions of $(x)(fx \Rightarrow gx)$ does not render the remaining premiss fa insufficient for the conclusion ga,

[8]Where the rule form of an implicational law is obtained by replacing the main connective by a rule analogue, assumed to be at least truth-preserving. Hence the counterexamples given here to laws are also counterexamples to rule forms.

so that the instantiated proposition is in such a case omissible. This, and not the general suppressibility of necessary truths, explains why we do not need to add to the premisses of an argument the logical law which represents the argument form used. We use such laws as *methods* of argument, and the particular instances of them in the argument hold in virtue of exemplifying the pattern the logical laws claim to be correct. But not all uses of logical laws are of this instantiation sort. Logical laws may be argued *from* not merely *by*, used as premisses in the argument, not merely as the method of argument, and when this is so they are no more suppressible than are contingent propositions. When we argue that

> Arthur believes (with good ground) that p & $p \Rightarrow$ Arthur knows p,

p is essential to the argument, whether we substitute for it '$p \Rightarrow \Diamond p$' or 'the world is round'. For even if it is true that $p \Rightarrow \Diamond p$, it is false that

> Arthur believes (with good ground) that $p \Rightarrow \Diamond p \Rightarrow .$ Arthur knows that $p \Rightarrow \Diamond p$.

The case is even worse when the suppressed necessary proposition is the *only* one used in the argument, as it is in

> The world is round & $p \Rightarrow p \Rightarrow .p \Rightarrow p$.

The suppression of the necessary proposition $p \Rightarrow p$ leaves us with

> The world is round $\Rightarrow .p \Rightarrow p$.

The necessary proposition we have dropped is the only thing which formed the connecting link between the original premiss and the conclusion. Its removal leads to utter irrelevance.

III

The fact that there are cases of this sort, where a logical law is used essentially in some argument, can also be used against *Exported Syllogism*, $(p \Rightarrow q) \Rightarrow .(q \Rightarrow r) \Rightarrow (p \Rightarrow r)$, (found in every major entailment system except $S2$ and subsystems, and in rule form in $S2$). For *Exported Syllogism* allows the suppression of $p \Rightarrow q$ in this argument. This is best brought out by a comparison with *Conjunctive Syllogism*, $(p \Rightarrow q)$ & $(q \Rightarrow r) \Rightarrow .(p \Rightarrow r)$, according to which the conclusion $p \Rightarrow r$ is a joint consequence of the two premises $p \Rightarrow q$ and $q \Rightarrow r$, which we treated as having equal status and as equally essential, as they are in the traditional Syllogism in *Barbara*. Just as with *Exportation*, *Exported Syllogism* and detachment allow us to drop off one of these premisses, $p \Rightarrow q$, if it is true, yielding the other $q \Rightarrow r$ as sufficient for the conclusion $p \Rightarrow r$. But it is not. It is not true that any two propositions p and q are so related that $q \Rightarrow r$ states sufficient ground for concluding that $p \Rightarrow r$. Hence we must make essential use of the special relation between p and q, that $p \Rightarrow q$, to obtain $p \Rightarrow r$ given $q \Rightarrow r$; and so $p \Rightarrow q$, even if true, must be retained as a premiss if the premisses are to state sufficient ground for the conclusion.[9]

Without doubt *Exported Syllogism* allows the omission of this relation $p \Rightarrow q$, when it occurs in conjunction with $q \Rightarrow r$ for conclusion $p \Rightarrow r$, while *Conjunctive Syllogism* does not. But because of[d] the fact that $p \Rightarrow q$ sometimes forms an essential part of this argument this omission cannot be correct. Moreover if this omission were correct, examples of the following sort would also be correct: Let $A \Rightarrow B$ be any true entailment. Then by *Exported Syllogism*

$$A \Rightarrow B \Rightarrow .B \Rightarrow B \Rightarrow .A \Rightarrow B$$

and by truth-preservation of '$\Rightarrow$', or detachment,

$$B \Rightarrow B \Rightarrow .A \Rightarrow B \text{ where } A \Rightarrow B \text{ is true.}$$

But $(p)(p \Rightarrow p) \Rightarrow .B \Rightarrow B$ hence by *Exported Syllogism*, $(p)(p \Rightarrow p) \Rightarrow .A \Rightarrow B$ i.e. the general law of identity is sufficient for any true entailment. This is indeed Leibnitz's view, but it is usually regarded as discredited, particularly by the modern axiomatic method. The result is obtained by the suppression of the true entailment in the premiss.

If the omission of $p \Rightarrow q$ allowed by *Exported Syllogism* were correct, we could correctly argue in the following way. Let James be some theorem

[9] It is surely just this sort of objection that Lewis puts forward, although rather half-heartedly, in [14], p. 496. A similar objection is implicit in Moore [17], e.g. p. 317, p. 289.

[d] Added this word.

implied by some set of axioms S. Let Joan be some further theorem which is deducible from the theorem James.

Then by *Exported Syllogism*

$$S \Rightarrow \text{ James } \Rightarrow . \text{ James } \Rightarrow \text{ Joan } \Rightarrow .S \Rightarrow \text{ Joan.}$$

If James is in fact a theorem of S, then by detachment or the truth-preservation of '$\Rightarrow$',

$$\text{James } \Rightarrow \text{ Joan } \Rightarrow .S \Rightarrow \text{ Joan .}$$

But that Joan is deducible from James is not sufficient to show that Joan is a theorem of S. The fact that James is a theorem of S is crucial to this conclusion, for if James were *not* a theorem it might be true that Joan was deducible from James and false that Joan was a theorem of S. But our premiss set, supposedly adequate, now tells us nothing at all about the status of James. And it is just this crucial fact, that James is a lemma, that *Exported Syllogism* allows us to omit, but the correct *Conjunctive Syllogism* forces us to retain as a premiss.

As against this, Anderson and Belnap [3] have claimed that 'the mathematician is involved in no ellipsis in arguing that "if the lemma is deducible from the axioms, then this entails that the deducibility of the theorem from the axioms is entailed by the deducibility of the theorem from the lemma." '. But this sample mathematician's argument is not clearly a case of *Exported* rather than[e] *Conjunctive Syllogism*. For precisely the difference between the two laws lies in our ability to drop from the premiss set, in the one case but not in the other, the implication $p \Rightarrow q$, in this case the information that the lemma is deducible from the axioms. But we are prevented from dropping this implication in the example by the use of the word "lemma", for "lemma" *means* something which is deducible from the axioms. Once this implication, present in the meaning of the word "lemma", is explicitly inserted the consequent becomes, not the consequent of *Exported Syllogism*, but a case of *Conjunctive Syllogism*. And if we restate the example in neutral terminology which does allow this implication to be dropped, such as by the use of proper names, we arrive at the counterexample to *Exported Syllogism* just discussed.

Another defence of *Exported Syllogism* similarly depending upon neglecting its difference from *Conjunctive Syllogism*, pretends that the detached

[e]Added this word.

premiss is still there somehow, haunting the scene. We can't really consider *Exported Syllogism* as allowing the dropping off of the premiss $p \Rightarrow q$, leaving the remaining premiss $q \Rightarrow r$ as sufficient for the conclusion $p \Rightarrow r$, for we must already have proved the detached premiss $p \Rightarrow q$ to have ever got as far as obtaining the consequent $q \Rightarrow r \Rightarrow .p \Rightarrow r$. Thus, because it must have been proved already, it can't really be considered as absent when detached; the needed condition on the relation between p and q, that $p \Rightarrow q$, has already been imposed by the conditions for detachment.

This objection amounts to covertly abandoning the sufficiency interpretation – the properties of which are just the relevant issue – leaving us of course unable to distinguish between *Conjunctive Syllogism* and *Exported Syllogism*. For if we have to refer back in this way to something not stated as a premiss to justify the implication between the stated premiss and the conclusion, then the stated premiss cannot be sufficient for the conclusion. But if deducibility of conclusion from premiss involves the premiss being sufficient for the conclusion, the correctness of *Exported Syllogism* for some enthymematic notion has no relevance to its correctness for deducibility. Furthermore, the same manoeuvre justifies *Exportation* and the general suppression of truths.[10] The detached premiss is claimed to be "present", and the condition that $p \Rightarrow q$ "imposed", simply because it happens to be true; the condition that $p \Rightarrow q$, which is vital to the argument, has not been imposed in the relevant sense that it is *stated* as a condition. This argument, like the suppression of truths, ignores the distinction between being true and being stated as a premiss.

Another move which relies upon confusing these two importantly different laws, attempts to discredit rejection of *Exported Syllogism* by branding it as rejection of "transitivity of entailment".[11] But *Conjunctive Syllogism* is sufficient for entailment to be transitive. *Exported Syllogism* is transitivity *plus* suppression, and it is quite feasible to reject its suppression features without rejecting transitivity. Similarly we are told [Smiley [22], p. 242; Pollock [18], p. 191] that we cannot do without *Exported Syllogism* because it provides the difference, so essential to deducibility, between "p implies q" and "p obviously implies q". But this also *Conjunctive Syllogism*, and indeed the operations of the Joint Force Principle of which *Conjunctive Syllogism* provides an example, is sufficient to explain. Of course *Exported Syllogism* and suppression in general, will produce a difference in obviousness between

[10] As well as the paradoxes of material implication; see e.g. Pollock [18].

[11] For example Prior [19].

premiss and conclusion, by suppressing some proposition needed to make the meaning connection between them. But the difference in obviousness *should* be obtained, not by suppression and the resulting loss of meaning connection, but by the deductive interaction of two propositions, neither of which implies the conclusion on its own. The conclusion differs in meaning from each of the premisses (as in *Conjunctive Syllogism*) and therefore is not obviously implied by the conjunction. This is the way in which an axiom set contrives to generate interesting and sometimes surprising theorems.

A further effect of *Exported Syllogism* is worth noting because it is important in paradox production, as will emerge in Section V. The suppression features of *Exported Syllogism* enable us to make illegitimate replacements inside implicational formulae – replacements which are not allowed by *Conjunctive Syllogism*. We may say that a law allows the replacement of a variable p in formula $A(p)$ by a variable q if the law allows us to obtain $A(q)$ given $A(p)$. Such replacement is illegitimate if in some cases $A(p)$ is true in the model and the resulting formula $A(q)$ becomes false in the model. Illegitimate replacement is a more general notion than suppression – suppression is the special case where a conjunction of premisses is illegitimately replaced by a sub-conjunct.

Exported Syllogism allows replacement of the sub-implicants of an implicant, and herein lies its great formal power. For in formula $s \Rightarrow (q \Rightarrow r)$, *Exported Syllogism* allows replacement of q by any p such that $p \Rightarrow q$, and replacement of r by any t such that $r \Rightarrow t$; and in $(p \Rightarrow t) \Rightarrow s$ *Exported Syllogism* allows replacement of p by any q such that $p \Rightarrow q$ and of t by any r such that $r \Rightarrow t$. To prove just one case, that $(p \Rightarrow q) \Rightarrow .(p \Rightarrow r) \Rightarrow s \Rightarrow .(q \Rightarrow r) \Rightarrow s$ only *Exported Syllogism*, substitution and detachment are needed.

From $(p \Rightarrow q) \Rightarrow .(q \Rightarrow r) \Rightarrow (p \Rightarrow r)$ we obtain by substitution of $q \Rightarrow r$ for p, $p \Rightarrow r$ for q, and s for r, $(q \Rightarrow r) \Rightarrow (p \Rightarrow r) \Rightarrow .(p \Rightarrow r) \Rightarrow s \Rightarrow .(q \Rightarrow r) \Rightarrow s$. By *Exported Syllogism* the result follows.

The remaining replacements, represented by the laws

$$r \Rightarrow t \Rightarrow .(p \Rightarrow t) \Rightarrow s \Rightarrow .(p \Rightarrow r) \Rightarrow s$$

$$r \Rightarrow t \Rightarrow .s \Rightarrow (q \Rightarrow r) \Rightarrow .s \Rightarrow (q \Rightarrow t)$$

$$p \Rightarrow q \Rightarrow .s \Rightarrow (q \Rightarrow r) \Rightarrow .s \Rightarrow (p \Rightarrow r)$$

are similarly easily obtained using just substitution, detachment and permitted forms of *Exported Syllogism* which stand and fall together with *Exported*

Syllogism since they are obtainable from it using just negation laws.[12]

The replacement results are clearly due to *Exported Syllogism*'s suppression of the connecting implication, but because this has been done repeatedly these replacements cannot be legitimised merely by conjoining this suppressed condition to the antecedent of the consequent i.e. by importing. Repeated application of *Exported Syllogism* results in the meaning connection becoming increasingly tenuous. No results of this sort at all are obtainable if only *Conjunctive Syllogism* is available – not even the imported forms of these laws.

Illegitimate replacement is yet another way of obtaining as valid arguments in which the premisses are insufficient for the conclusions drawn. A classical law which although not directly a suppression law allows illegitimate replacement is *Commutation*, $p \Rightarrow (q \Rightarrow r) \Rightarrow .q \Rightarrow (p \Rightarrow r)$. *Commutation*, like *Exportation*, presents difficulties for strict implication, since it also violates the principle that a necessary proposition may not imply a contingent one; consequently a similarly necessity-restricted version of it $p \Rightarrow (q \Rightarrow r) \Rightarrow .\Box q \Rightarrow (p \Rightarrow r)$ was adopted for strict implication systems including T or $S3$, and a rule version is found in $S2$. But as with *Exportation*, this is neither a complete nor a correct remedy for its problems. Once the folly of *Exportation* has been seen, the main justification for *Commutation*, that it simply represents reversibility of premisses, is undermined; for nested implications can no longer represent commutable conjunctions, and only conjunctions can represent the traditional concept of interacting premisses of equal status. Once a clear distinction is made between nested implications and conjoined premisses, it becomes plausible to say that a proposition p may imply that some *other* proposition q is sufficient for r, without p *itself* being sufficient for r. But given only that q is true, such a case yields a generalised counterexample to *Commutation*, for the antecedent $p \Rightarrow (q \Rightarrow r)$ is true (by hypothesis), while the consequent of the consequent, $p \Rightarrow r$, is false, and hence the whole consequent $q \Rightarrow (p \Rightarrow r)$ is false if q is true. *Commutation* allows illegitimate replacement of q by p in the consequent of $p \Rightarrow (q \Rightarrow r)$.

Instantiation arguments provide an obvious class of propositions satisfying this condition. If for example, we take 'a' as 'Arthur', 'f' as 'is a man' and 'g' as 'is an animal', then it is true that $(x)(fx \Rightarrow gx) \Rightarrow .fa \Rightarrow ga$, since this is instantiation. But what we obtain by commuting, that $fa \Rightarrow .(x)(fx \Rightarrow gx) \Rightarrow ga$ is not true, for it is not true that $(x)(fx \Rightarrow gx)$ is *itself*

[12]For details of the proofs see Angell [4].

sufficient for ga, although it does imply that *something else* viz. fa, is sufficient for ga. This was the sort of example which lead to *Necessity-Restricted Commutation* being regarded as the correct repair, for $(x)(fx \Rightarrow gx)$ is necessary, while what it is supposed to imply once commuted, ga, is contingent. But if we require that fa be necessary, ga cannot be contingent, since implied by fa, so such trouble cannot arise. So the story goes; but the modal moral is the wrong one. For in fact counterexamples can be produced at least as damaging to *Commutation* as these supposedly modal ones, but without modal fallacy. For example

> Arthur knows $(p \Rightarrow p) \Rightarrow .(p \Rightarrow p)$ is a paradigmatic entailment

But the result of applying *Commutation*,

> $p \Rightarrow .$ Arthur knows $(p \Rightarrow p) \Rightarrow p$

is false. For consider the antecedent. Arthur's knowing the law of identity for some p, say, 'the world is round', cannot be sufficient for p, for of course one may know the law of identity for a great many propositions which are false. Now is there a valid argument leading from 'Arthur knows that the world is round $\Rightarrow$ the world is round' as premiss, to ' the world is round' as conclusion; we could not properly deduce that the world is round from the fact that Arthur knows that it entails itself. But since p, which is true, implies the false statement that p *is* deducible from Arthur's knowing the law of identity for it, the main implication, and hence the whole statement which resulted from *Commutation*, must be counted false.

The example works perfectly well against *Commutation* if p is contingent, but no modal fallacy can then explain the falsity of the consequent, that Arthur knows $(p \Rightarrow p) \Rightarrow p$, for both its antecedent and its consequent are then contingent.

Furthermore, the same counterexample can easily be extended against *Necessity-Restricted Commutation* itself, by picking p as a necessary proposition, e.g. as
$p \Rightarrow \sim\sim p$. For although it is true that

$$(\text{ Arthur knows } p \Rightarrow \sim\sim p \Rightarrow .p \Rightarrow \sim\sim p) \Rightarrow .p \Rightarrow \sim\sim p \Rightarrow .p \Rightarrow \sim\sim p$$

and[f]

$$p \Rightarrow \sim\sim p \Rightarrow .p \Rightarrow \sim\sim p$$

it is not true that

$$p \Rightarrow \sim\sim p \Rightarrow .(\text{ Arthur knows } p \Rightarrow \sim\sim p \Rightarrow .p \Rightarrow \sim\sim p) \Rightarrow p \Rightarrow \sim\sim p \Rightarrow p \Rightarrow \sim\sim p$$

The consequent of this is just as false as was the consequent of the previous example, and it is false for the same reason. Someone's knowing the law of identity for some proposition is not sufficient for the deducibility of that proposition, whether the proposition be contingent, necessary – or an entailment, as required by the implication-restricted versions of *Commutation* adopted by Anderson & Belnap for E [see [2, 3] and [6]] and Storrs McCall, for $CC1$, [see [15]]

$$p \Rightarrow (r \Rightarrow s) \Rightarrow q) \Rightarrow .(r \Rightarrow s) \Rightarrow (p \Rightarrow q)$$

to which this latter example also applies. Since the method of arguing licensed by *Commutation* is basically an invalid one, leading as it does to premisses which are quite inadequate for their conclusions, we can't remove this invalidity by restricting its applications to a progressively smaller class of propositions.

Anyone arguing by the method licensed by *Commutation*, as well as in some of the ways licensed by *Exportation* and *Exported Syllogism*, would be arguing invalidly, for he is able to obtain from unexceptional premisses e.g. that $(x \text{ knows } (p \Rightarrow p) \Rightarrow .(p \Rightarrow p))$, a false conclusion e.g. that $p \Rightarrow (x \text{ knows } (p \Rightarrow p) \Rightarrow p)$. No connective for which these laws hold can be adequate to represent the notion of valid argument of sufficiency of premiss set for conclusion.

But how can any truth-falsity counterexample (i.e. where the antecedent is true and conclusion false) to any of these laws be possible, when these are laws of material implication, $\supset$, which is, as everyone knows, truth-preserving? The truth-falsity counterexamples are possible because $p \supset q$ is only a *necessary*, and not a *sufficient*, condition for $p \Rightarrow q$. The truth-preserving properties of material implication mean that one cannot find a truth-falsity counterexample to $p \Rightarrow q$ when $p \supset q$ holds; but if $p \supset q$ is only

[f] This word added.

a necessary condition for $p \Rightarrow q$, we can find a true-false counterexample to the main connective in $p \supset (q \Rightarrow r)$ when $p \supset (q \supset r)$ holds. If $q \supset r$ is only a *necessary*, and not a sufficient, condition for $q \Rightarrow r, q \Rightarrow r$ may be false when $q \supset r$ is true. Hence we have only to find one of these cases to obtain a higher-degree truth-falsity counterexample to the main connective in $p \supset (q \Rightarrow r)$ when p is true and $p \supset (q \supset r)$ holds.

In the light of this, the common view[13], which might be summed up by the slogan 'You can't go wrong with material implication', that because $\supset$ is truth-preserving it is safe to use for valid argument, must be defective. First, the truth-preservation property of $\supset$ does not show that it is safe unless it is assumed quite circularly that the *only* way an implication can be wrong is by having its premiss true and conclusion false. Second, $\supset$ is not truth-preserving in the right sense to guarantee safety for valid argument. For it is not truth-preserving in the sense that if we substitute 'there is a valid argument from ... to ....' or '.... is deducible from ...' for *all* occurrences of '$\supset$' in the laws of $\supset$, what we obtain will continue to be true where the original $\supset$ law was true. Similarly, once such a substitution is made, $\supset$ *can* lead from true premisses to a false conclusion, as the counterexamples show. Admittedly all the false conclusions are themselves *about* valid argument, but failure of this sort cannot be discounted given the uses of deducibility in assessing responsibility. Nor is it an adequate defence against these counterexamples to claim that the supposedly false conclusions are always about valid argument and that since truth-preservation, as represented by $\supset$, is all that is required for deducibility, they must not be false at all, but true, even if surprising. Such a defence is quite circular, for we can only decide whether a particular connective, say $\supset$, *is*, as claimed, adequate for valid argument, by substituting 'there is a valid argument from ... to ....' for '$\cdots \supset \ldots$.' and seeing whether the results continue to hold; hence we could not without circularity decide these results on the basis of what holds for the connective itself. (A similar objection applies to the claim that the material implication paradoxes are not paradoxical for deducibility because they follow from the truth-table for $\supset$.)

A connective is only safe to use for valid argument if upon substitution of 'there is a valid argument from... to ....' for all occurrences of the connective

[13]Well represented by Lemmon [12, p. 60] 'While admitting that this discrepancy exists, we may continue safely to adopt '$p \supset q$' as a rendering of 'if p then q' serviceable for reasoning purposes, since, as will emerge... our rules at least have the property that they will never lead from true assumptions to a false conclusion.'

in its laws, the result continues to be true i.e. if every law of the connective is also a correct law of valid argument. It must license no methods which are not *also* methods of valid argument; but the laws discussed, which are $\supset$ laws, do license methods which are unacceptable as methods of proof or valid argument.

IV

But to accept *Disjunctive Syllogism* $A \,\&\, A \supset B \Rightarrow B$, is just to accept these methods as acceptable deductive methods, at least sufficient to establish a conclusion deductively. For according to *Disjunctive Syllogism*, a 'proof' of B from premiss A by these unsatisfactory $\supset$-methods, and premiss A itself, is sufficient for a proper deduction of B. In other words, we regard A, and an *invalid* argument from A to B, as yielding a *valid* argument to B! Starting from premiss A and a derivation of B from A by $\supset$-methods we regard ourselves as having obtained a *proper* proof of B. But we could only accept this as a proper deduction of B if all the methods used in arriving at B from A are acceptable deductive methods. Because the main connective is '$\Rightarrow$', it is not merely a question of the conclusion's being true, but of establishing this truth from the premisses stated by *adequate deductive methods*. If the methods used in deriving B from A include such deductively unacceptable ones as asking a mathematical oracle, we cannot claim to have a deductive proof of B from[g] these premisses. The essence of proof lies in the methods used, not merely the results obtained. But $\supset$-methods are just as much not included in deductive ones as asking a reliable mathematical oracle, as earlier counterexamples show.

Disjunctive Syllogism does not allow a direct reduction of the methods of '$\Rightarrow$' to the methods of '$\supset$'. It does not let $\supset$-methods in by the front door, as does the reduction law $A \supset B \supset .A \Rightarrow B$, but it does let them in by the back door, for we are able to obtain, over a restricted range of cases, exactly the same effects as if we had $\supset$-methods at our disposal as $\Rightarrow$-methods. For suppose $A \Rightarrow B$ is false, but $A \supset B$ is true. Then whenever $A \supset B$ is provable, it can, given adjunction, be adjoined to A, and B is now obtainable as $\Rightarrow$-proved from these, even if not from A. Although $A \Rightarrow B$ may fail, when $A \supset B$ is provable there is always a procedure for obtaining B as $\Rightarrow$-proved given A, for the additional premiss required, $A \supset B$, is automatically supplied

[g]Originally "for".

when $A \supset B$ is provable. Hence we obtain the same effect as if $\supset$-methods were $\Rightarrow$-methods, when $A \supset B$ is provable. This amounts to the acceptance of provable $\supset$-methods as adequate proof methods.

It is not surprising that given a normal battery of laws *Disjunctive Syllogism* has in fact the formal effect of reducing $\Rightarrow$-methods to the methods represented by the provable laws of '$\supset$'; for reduction to a strict implication system results.

If we do not regard $\supset$-methods as providing adequate deductive proof or $\Rightarrow$-methods, we cannot accept A and $A \supset B$ as providing an $\Rightarrow$-proof of B because then we should be accepting as an $\Rightarrow$-proof of B, in at least some cases, a derivation of B which made *essential* use of methods we did not recognise as acceptable $\Rightarrow$-proof methods. The use of these unacceptable methods is *essential* in at least some cases because we cannot in some cases replace the $\supset$-methods used by $\Rightarrow$-methods, since some $\supset$-methods are not $\Rightarrow$-methods. (But it is only if the provable laws of one implication are included in the laws of the other that the methods of the first are eliminable in favour of the second, and hence only then can we accept the use of the methods as allowable in providing a proof). Nor is there any direct proof by already accepted $\Rightarrow$-methods available from A & $A \supset B$ itself – otherwise *Disjunctive Syllogism* would not be, as it is, independent of $\Rightarrow$-methods. But in allowing a proof which makes essential use of non-deductive methods to be a deductive proof, by allowing a supposedly valid proof of B to depend essentially on an invalid proof of B, we have added these unsatisfactory proof methods to our original class of deductive proof methods, and hence we break down the distinction.

No one who wishes to distinguish $\supset$-methods and proper proof methods can consistently accept A and $A \supset B$ as providing a premiss set for a proper proof of B, as distinct from a $\supset$-proof or strict proof of B.[14] Imagine a constructivist who said: Here is a non-constructive ($\supset$) derivation of B from A, and I am going to accept this non-constructive derivation, and A itself, as yielding a constructive proof of B!. The Intuitionists, who also wish to claim that classical (i.e. $\supset$) methods do not yield proper (i.e. Intuitionistic)

[14]Hence the so-called 'independent' proofs of the paradoxes using this form of Disjunctive Syllogism are not really independent at all; for we have to accept $\supset$-methods as correct in order to accept *Disjunctive Syllogism*. Thus no one who objected to $\supset$-methods on the grounds that they licensed paradoxes should have accepted these arguments; for they should not have accepted *Disjunctive Syllogism*, which presupposes the correctness of such methods.

proof methods, do not accept *Disjunctive Syllogism* in the relevant sense.

$$p\ \&\ \sim (p\ \&\ \sim q) \longrightarrow_I q$$

fails in the Intuitionistic calculus where '$\longrightarrow_I$' stands for 'is an intuitionistically valid proof' (as can be shown using the matrices in Church [7] Exercise 26.10.).

But the Intuitionistic negation '$\sim$' and conjunction '&' are sufficient for the classical calculus (see [7] Exercise 26.16 p. 147.). Hence we can define '$p \supset q$' in the Intuitionist Calculus as '$\sim (p. \sim q)'$. Hence

$$p\ \&\ p \supset q \longrightarrow_I q$$

also fails in the Intuitionist Calculus.

The fact that[h]

$$p\ \&\ (\sim p \vee_I q) \longrightarrow_I q$$

holds is not relevant to the rejection of

$$p\ \&\ p \supset q \longrightarrow_I q$$

for the '$\vee_I$' of the Intuitionist Calculus is not the usual extensional '$\vee$' as given by the laws of classical logic, such that

$$\sim p \vee q \equiv \sim (p\ \&\ \sim q)$$

holds, and $\sim p \vee_I q$ is not adequate for a definition of '$\supset$'. In fact the intuitionistic '$\vee$' is stronger than the intuitionistic implication and

$$p \vee_I q \supset .p \longrightarrow_I q$$

Just as it is inconsistent for the Intuitionist, with his distinction between Intuitionistic and $\supset$-methods, to accept $p\ \&\ p \supset q \longrightarrow_I q$, so it is inconsistent for those who distinguish between enthymematic and deductive proof to accept *Disjunctive Syllogism*. Yet many of those who have accepted *Disjunctive Syllogism* have also wished to maintain a distinction between enthymematic, or imperfect, arguments, and non-enthymematic or perfect ones which the acceptance of *Disjunctive Syllogism* simply abolishes. For to accept *Disjunctive Syllogism* is to accept that a perfect argument may depend essentially

[h] In the below formula, the right conjunct in the antecedent was written $\sim (p \vee_I q)$.

on an imperfect one. If $p \supset q$ is an argument from p to q using suppression methods, there are some cases where we cannot replace this argument by one from this same premiss p to q which does not use suppression methods. Consequently the use of suppression methods is in some cases not eliminable in favour of an argument from the same premiss p not using suppression methods, and our supposedly 'perfect' proof of q from these premisses depends essentially on an imperfect proof.

For example, suppose $p \,\&\, r \Rightarrow q$ represents a valid syllogism with $p \,\&\, r$ as premisses and q as conclusion, in which r is true (e.g. one of the examples A1–3). Then $p \supset q$[i] holds, since *Exportation* holds for '$\supset$'. If we then take '$\supset$' as representing an argument from p to q, it is an imperfect one in every case where r is essential. In such cases there is no perfect argument from p to q, and hence the imperfect argument from p to q is not eliminable in favour of a perfect argument from p to q. But according to *Disjunctive Syllogism* we have a perfect argument from $p \,\&\, p \supset q$ to q – a perfect argument which depends essentially on an imperfect argument from $p \supset q$.

Disjunctive Syllogism then is a suppression law, but it does not suppress directly by stating a method for '$\Rightarrow$' which allows the dropping-off of a used premiss; rather it *passes on* suppression from the first connective '$\supset$' to the second '$\Rightarrow$'. *Disjunctive Syllogism* is a suppression law only because '$\supset$' allows methods which *are* suppression methods, and to accept *Disjunctive Syllogism* is to accept these methods as correct proof methods.

For this reason *Disjunctive Syllogism* on its own does not effect suppression in a formal system, but only in conjunction with some set of laws which either fix '$\supset$' as allowing suppression, or fixes the '&' and '$\vee$' of the disjunctive and conjunctive forms as truth-functional. While we may in informal talk assume '$\supset$', '&' and '$\vee$' to have a certain sense, in a formal system what this is is determined just by what laws hold for them, so that further laws than just *Disjunctive Syllogism* are needed before the sense of the connectives is fixed as that allowing suppression. The suppression methods of '$\supset$' must be formally avilable before they can be passed on.

One result of this indirectness of *Disjunctive Syllogism* is that the suppression in a proof using it may be quite subtle and difficult to detect, because it is spread over a number of laws. A further result is that it is easy to mistake *Disjunctive Syllogism* for a more harmless principle than it is because one is not aware that the methods being passed *are* suppression methods

[i]The connective was missing, but context suggests that $\supset$ is needed here.

in the absence of a clear understanding of these further laws. And in the absence of such laws *Disjunctive Syllogism* cannot be distinguished from the correct *Modus Ponens* $p \,\&\, p \Rightarrow q \Rightarrow .q$. These points weaken the historical argument in favour of *Disjunctive Syllogism*.

But the basic inconsistency of the historical treatment of *Disjunctive Syllogism* – the fact that *Disjunctive Syllogism* has apparently been accepted since the time of Boethius by many logicians who also accepted the prohibition of suppression – means anyway that no weight can be given to its historical acceptance as an argument in its favour. For even if it *were* clear that tradition accepted *the relevant sense* of *Disjunctive Syllogism*, why should we accept this tradition in favour of *Disjunctive Syllogism* rather than the even older and better established tradition prohibiting suppression? The view that *Disjunctive Syllogism* is a paradigmatic valid argument is also vitiated by this inconsistency. For this 'paradigm' conflicts with long-established practices such as the prohibition of suppression, which, stemming as it does from the basic uses of reasoning, must be considered equally paradigmatic.

A common argument in defence of *Disjunctive Syllogism* attempts to replace the main connective '$\Rightarrow$' by some weaker notion which allows suppression e.g. by strict implication, then claims that this weaker notion is sufficient for a valid argument form the premisses to the conclusion. But this fails to show that *Disjunctive Syllogism* is a valid argument in the relevant sense – that the premisses are sufficient for the conclusion. Consider to illustrate the claim that if the premisses of *Disjunctive Syllogism* are true, the conclusion of Disjunctive Syllogism cannot be false, and hence can be taken as proved. Such a notion however involves suppression, so that we cannot replace it by '$\Rightarrow$'; rather it embodies strict implication. For even if it is merely the case that stated premisses *together with* some truth, or better, some necessary truth, imply the conclusion in a suppression-free sense of 'imply', i.e. if there is a valid argument from the premisses and some necessary truth to the conclusion, the conclusion cannot be false *and* the premisses true; for the further (suppressed) premiss is *true*, and hence the conclusion could not be false and the stated premisses true unless the original argument form the stated premisses and the necessary truth to the conclusion violated truth-preservation. But it was, by hypothesis, an entailment. The conclusion could not be false and the premiss true, but that the conclusion is true nevertheless does not follow *from* the stated premisses. Similarly $\sim \Diamond$(premisses. $\sim$conclusion)[j] sometimes holds even when the premisses do not imply ($\Rightarrow$) the conclusion, because the premiss requires some other necessarily true proposition to yield a valid argument to the conclusion. Then, since it is necessarily true, it is not possible that the premisses are true and the conclusion false, on pain of supposing the valid argument from the full set of premisses to the conclusion to be such that the premisses are true and the conclusion false. Since such a weaker notion may hold where there is suppression, it cannot be adequate for valid argument.

A similar difficulty affects the argument for *Disjunctive Syllogism* that if the premisses are true, we can construct a proof of the conclusion. This again involves suppression, for while it is true that if the premisses of *Disjunctive Syllogism* are true, could construct a correct proof of the conclusion from *some* set of true premisses (which is, after all, merely to say that the conclusion must be true), it is not true that we can construct such a proof from just the *stated* set of premisses. But the claim that there is a valid argument from p to q is the claim that there is such an argument to q *from the stated premiss* p, not from some *other* premiss.

A more subtle version of the argument makes use of the notion of 'guaranteeing'. The truth of the conclusion of *Disjunctive Syllogism* is said to be guaranteed by the premisses of *Disjunctive Syllogism*, and this is regarded as sufficient for a valid argument. And so it would be if the conclusion were indeed guaranteed *by the premisses*, but it is not. The conclusion of *Disjunctive Syllogism* can be said to be guaranteed by the premisses only in the sense that it is guaranteed by the premiss *together with something else*. But when we say that something q is guaranteed *by* something p, we mean that it is guaranteed just by p, and not by something other than p; we mean to convey that p is *responsible* for that guarantee, or at the very best that it has some connection with it. But even this latter condition is eroded if we adopt such a weak sense of 'guarantee', for, as the paradoxes show, given only the normal possibility of having p as an inessential or passenger premiss, we may end by destroying all connection between what is guaranteed and what it is supposed to be guaranteed by. We end by saying that the premisses guarantee the conclusion merely because the conclusion *is* guaranteed (by something). In such a weak sense of 'guarantee' it is no longer true that a 'guarantee' of

[j] In the manuscript, this was written $\sim \Diamond$(premisses.conclusion), but the additional negation seems necessary.

the conclusion by the premisses is sufficient for a valid argument from the premisses to the conclusion. In the context of an interest in responsibility, such an extension of 'guarantee' cannot be regarded as harmless. Imagine a salesman who claims that *his company* guarantees a vacuum cleaner on the same ground as the logician claims that his conclusion is guaranteed by the premiss – that it *is* guaranteed (by somebody). But why should we allow the logician a laxity in the use of the word 'guarantee' which, in a similar context of interest in responsibility, amounts to dishonesty in a vacuum-cleaner salesman?

All these points apply equally well to the '&' and '$\vee$' forms of *Disjunctive Syllogism*, $p \mathbin{\&} (\sim p \vee q) \Rightarrow q$, $p \mathbin{\&} \sim (p \mathbin{\&} \sim q) \Rightarrow q$,[k] as to the form $p \mathbin{\&} (p \supset q) \Rightarrow . \, q$. For as the bracketed formulae are interdefinable and satisfy exactly the same set of laws, they allow exactly the same set of proof methods, so that objections in terms of these methods simply transfer to the '&' and '$\vee$' forms. But so strong is the magnetism of the '$\vee$' form, that it is worthwhile examining the direct objections to the disjunctive argument, in particular to considering it as an elimination argument.

Perhaps the most compelling argument in favour of $p \mathbin{\&} (\sim p \vee q) \Rightarrow q$ is that it represents the paradigmatic elimination of alternatives argument. It is true that the elimination argument is paradigmatic as a valid argument but it is *not* true that the elimination argument is correctly represented by $p \mathbin{\&} (\sim p \vee q) \Rightarrow q$.

The elimination of alternatives argument, generalised, proceeds by setting up a set of alternatives (thought to correspond to the disjunction $p \vee q$), eliminating all but one, leaving us with the conclusion as the only remaining alternative. This elimination argument cannot be correctly represented by $p \mathbin{\&} \sim p \vee q \Rightarrow q$, because if it were it could not be employed in the way in which it is typically employed. In the elimination argument as typically employed by the scientist, the detective, the geometer, or the dog, the disjunction, the set of alternatives, is set up *first*, before the elimination is carried out; that these represent the alternatives, and *all* the alternatives, is established first, before we actually know *which* alternatives hold i.e. what the truth-values of the individual components of the disjunction are. And indeed it is often only on the basis of knowing these alternatives that we can proceed to the next step – the elimination of some of them. It must therefore be possible to know what the alternative possibilities are without actually

[k]In the manuscript, this formula appears as $p \mathbin{\&} \sim (p \mathbin{\&} q) \Rightarrow q$.

knowing which ones hold. But this is impossible if the disjunction is merely truth-functional; for if the disjunctive $A \vee B$ is merely truth-functional i.e. no *non-truth-functional* disjunction $A \vee_1 B$ (implies $A \vee B$) *also* holds, the truth of $A \vee B$ can only be established from the truth-values of the components $A \vee B$. But in this case the argument *Disjunctive Syllogism* licenses as opposed to one with a non-truth-functional disjunction, is a particularly circular form of special pleading, since we must already know, and make use of, the truth-value of the conclusion B in order to establish the premisses from which we are supposed to argue for B. But the elimination argument, particularly when used in trial (of alternatives) and error (elimination of possibilities) fashion, is an outstanding example of a progressive argument, one where we do *not* need to know the truth of the conclusion to establish the premisses.

The elimination argument is progressive, *Disjunctive Syllogism* is not. Hence *Disjunctive Syllogism* is not the elimination argument.

The progressive nature of the elimination is explained thus. The disjunction is a statement which exhausts the deductively viable options. The disjunction may therefore be established independently of knowing *which* of these options hold and which do not, because it is itself the disjunctive analogue of an entailment. Which of these alternatives actually hold is established subsequently by the elimination, which leaves the conclusion as *the only remaining deductive option*, therefore established deductively. For the elimination argument to be a valid one it is not sufficient that the disjunction should merely exhaust the truth-functional alternatives. For after all, $p \;\&\; \sim p \vee q \Rightarrow q$, must, if previous arguments are correct, be distinct from $p \;\&\; \sim p \vee q \supset .q$. Any argument which tried to justify the former by considerations which only justified the latter would be question-begging (for it would assume that entailment is strict or material implication, which would be false if *Disjunctive Syllogism* were not correct). But these two are distinguished by the fact that whereas in the second it is merely necessary that the premiss never be true and the conclusion false (and this may be so even when it is only the stated premisses together with something *else* which is sufficient for the conclusion), the premisses of the first must state sufficient ground to establish the conclusion deductively. The remaining alternative must not merely be true, but deductively established on sufficient grounds by the stated premisses. For the conclusion to be *deductively* established by elimination, we must have considered all the *deductively viable options* to B and eliminated them. B must be obtained by elimination from a set which exhausts the deductively viable options, B being the only one left.

It seems reasonable to say that something B is still a *viable option* with respect to B if it is not ruled out by A. But then we must distinguish different ways of being a viable option, depending how we understand this notion of 'ruling out'. Accordingly we define a relative notion of 'viable option' or 'possible with respect to': A is $\longrightarrow_1$ possible with respect to B iff $\sim (A \longrightarrow_1 \sim B)$. Thus A is still deductively possible or is a deductively viable option with respect to B if it is not deductively ruled out by B i.e. $\sim (A \Rightarrow \sim B)$. If we distinguish, as according to previous argument we must, '$\supset$' and '$\Rightarrow$', we must accordingly distinguish ways of ruling out, hence ways of being an option to something. It follows from the relation between '$\supset$' and '$\Rightarrow$' that A may be deductively possible with respect to B, since $\sim (B \Rightarrow \sim A)$, but $\supset$-impossible with respect to B, because $B \supset \sim A$.

A pair of $\longrightarrow_1$ alternatives A and B is said to be exhaustive or to exhaust the $\longrightarrow_1$ possibilities, iff there is no *further* viable $\longrightarrow_1$ option, so that the elimination of the one (A) leaves the other B as the *only* viable $\longrightarrow_1$ option, and so $\longrightarrow_1$ establishes it i.e. $\sim A \longrightarrow_1 B$. A pair of exhaustive $\longrightarrow_1$ possibilities A and B provides a disjunction $A \vee_1 B$, since $A \vee_1 B$ is defined as $\sim A \longrightarrow_1 B$. The property of an exhaustive set of $\longrightarrow_1$ possibilities, that at least one of them must hold, follows if the relevant implication '$\longrightarrow_1$' is truth-preserving.

It follows from this definition that A and B may constitute an exhaustive set of $\supset$-possibilities, but not of $\Rightarrow$-possibilities. It is no more true that 'The moon is made of green cheese' and 'The potoroo is a marsupial' present an exhaustive set of deductively viable options than that the negation of the second is deducible from the first, although the negation of the first materially implies the second and they exhaust the $\supset$-possibilities.

The intuitive elimination argument eliminates on the basis of an *exhaustive set of possibilities*. The statement of possibilities must be exhaustive, because if there is some further possibilities, we are not entitled to arrive at the conclusion until this also has been eliminated. But the statement of alternatives must not merely be exhaustive of *some* set of possibilities – it must exhaust the *appropriate* set of possibilities for the conclusion. And a set which is exhaustive for one sort of alternatives may not be so for another. If the conclusion is to be established with $\longrightarrow_1$ strength, it must be left as the only remaining $\longrightarrow_1$ possibility, not the only remaining possibility of some *other* sort. To argue from some conclusion's being established as the only remaining $\longrightarrow_1$ possibility to its being $\longrightarrow_2$ established, would, (unless there were some special relation between '$\longrightarrow_1$' and '$\longrightarrow_2$' such as holds between material and strict implication), be like arguing that something was in room 2 on the basis of eliminating everything else from a set which exhausted the possibilities for being in a completely different room, room 1. But this is just how *Disjunctive Syllogism does* argue. For from a set of $\supset$-possibilities, which may be $\supset$-exhaustive but not $\Rightarrow$-exhaustive, we claim the conclusion to be left not just as the only remaining $\supset$-possibility, and therefore $\supset$-established, (i.e. as true), but as established with $\Rightarrow$-strength from these premisses. *Disjunctive Syllogism*, where the '$\vee$' is truth-functional, argues only from a set of $\supset$-possibilities; $\sim p \vee q$ holds when and only when $p \supset q$, and applying the definition of $\supset$-exhaustiveness, when and only when p and q are an $\supset$-exhaustive set of possibilities.

Since a pair of propositions which are $\supset$-exhaustive may not be $\Rightarrow$-exhaustive, an argument which from a merely truth-functionally exhaustive set of possibilities purports to establish the remaining possibility *deductively*, can be seen as incorrect, *either* because the $\supset$-exhaustive possibilities are not of the appropriate sort to establish the conclusion deductively, and can only establish it in some weaker way, *or* because the possibilities, although of the appropriate sort (i.e. $\Rightarrow$-possibilities), are not exhaustive. Whichever way we interpret such an argument, either as having premisses which are true but are too weak to support the conclusion, or as having premisses which are strong enough for this but are then false, the argument fails. Whichever way we look at such an argument, it involves suppression. For in terms of the deductively exhaustive set of options needed to establish the conclusion, a non-exhaustive set involves suppression. In terms of some exhaustive set of options $(A \vee B) \vee C$, a non-exhaustive set $A \vee B$ is one in which some elimination, the elimination of C, has already been carried out, *without, however, being explicitly stated*. The falsity of the neglected option C, or the truth of its negation $\sim C$, has been used in allowing us to drop C from the exhaustive set $(A \vee B) \vee C$, and $\sim C$ therefore becomes a suppressed premiss when C is dropped. A disjunctive reflection of the truth-suppression properties of '$\supset$' (illustrated by *Exportation*), is that truth-functional disjunction allows the neglecting of all false options. This follows from the definition of viable option, and the truth-table for '$\vee$' for B is a viable $\supset$-option to A iff $\sim (A \supset \sim B)$ i.e. $\sim (\sim A \vee \sim B)$, i.e. if A and B are both true. Falsehoods can always be ignored in a truth-functional disjunction without being explicitly eliminated, but falsehoods can still be deductively viable options, and cannot be so treated in an exhaustive set of deductive possibilities. Hence in the argument $A \,\&\, \sim A \vee B \Rightarrow B$ one can simply ignore, without explicitly eliminating, a large number of options which although false are still deductively viable; and the negations of these propositions are then suppressed premisses in the argument. This disjunctive form of *Disjunctive Syllogism* is like the implication form, a suppression law allowing the suppression of necessary truths through the ignoring of false options (which must in fact be *provably* false and therefore impossible to be dropped off) in the disjunction. This is revealed by the simple formal transformation of $p \,\&\, (\sim p \vee q) \Rightarrow q$ to (a) $(p \,\&\, \sim p) \vee (p \,\&\, q) \Rightarrow .q$ by an application of the distribution law $(p \,\&\, q) \vee (p \,\&\, r). \iff .p \,\&\, (q \vee r)$. Unless $p \,\&\, \sim p$ *does* entail q, (a) depends for its plausibility on the neglecting of the impossible disjunct $(p \,\&\, \sim p)$. But contraposing (a) and applying deMorgan laws it follows (b) $\sim q \Rightarrow \sim (p \,\&\, q) \,\&\, (p \vee \sim p)$. (b) clearly suppresses $(p \vee \sim p)$. Uncontroversial transformations on extensional connectives reveal *Disjunctive Syllogism* as a suppression law; but principles determining features of the extensional connectives have to be applied to penetrate *Disjunctive Syllogism*'s disguise. It is but a step, of course, from (b) to the paradox of $\sim q \Rightarrow p \vee \sim p$, by *Conjunctive Syllogism* and *Rule Conjunctive Syllogism*.

Since *Disjunctive Syllogism* allows the neglecting of any false proposition in its disjunctive premiss, it can represent neither a valid argument nor a practical elimination argument. For it is not true that in practical elimination *any* false proposition can be neglected, just as it is not true that a practical enthymeme allows us to assume and use *any* true proposition, or that *any* partially sufficient condition can be selected as a cause. Just as in any deductively sound argument we must consider all deductively viable options, so a sound practical argument allows us to neglect some of these deductive options only when they can *for practical purposes* be discounted. This class does not coincide with the class of propositions which are in fact false.

A practical elimination argument of this sort is the one some Stoics took to support *Disjunctive Syllogism*, claiming that even dogs recognised its validity; 'when chasing their prey to a point where the road forks, they will sniff along one fork, and if they catch no scent there, they will chase along the other without further sniffing.' [Prior [19]] Clearly in this argument a very large number of possibilities are not ruled out by there not being scent along one fork of the road. It is quite possible, for all that this shows, that the prey had[1]

[1]This word seems to be inked in in the original.

developed wings, or went off across the fields. Hence the prey's going down one fork is not sufficient for its going down the other, so that the elimination of this possibility is not sufficient to establish that he went down the other and the argument is not a valid one. If there is some *practical* reason for neglecting these deductively viable options, the dog has a sound practical argument, but it is not a valid one. But if the argument is not a valid one, its use cannot support the validity of *Disjunctive Syllogism.*

What formal argument does such a practical elimination argument, with a disjunction stating all practically viable alternatives, rest on and which does its widespread use therefore support? Can we take formal truth suppression as giving a formal rendering of, and as justified by, practical argument? The omission of truths allowed in practical argument no more supports formal truth suppression than the fact that people sometimes speak enthymematically about cause shows that the *logical* analysis of cause is not that of a sufficient condition, but that of a partially sufficient one. That omission is practised in practical argument (Aristotle's *Enthymeme*) does not mean that they rest on a less vigorous, more permissive sort of formal reasoning – one which allows suppression (Aristotle's imperfect argument). Such a practical argument is not a lower grade formal argument – it is not a formal argument at all. Practical omissions of truth are allowed when it is obvious in the context what sort of proposition has been omitted, and so clear how the argument could be made valid and what valid argument it purports to rest on. Because not all truths satisfy this condition in every argument, truths are not generally omissible and falsehoods neglectible in practical argument. But such practical argument depends for its formal justification on valid argument. Since the dog's argument is a practical version of *Modus Ponens* with a large number of possibilities practically neglected, its use does not support *Disjunctive Syllogism.*

Since *Antilogism* $(p \mathbin{\&} q) \Rightarrow r \Rightarrow .(p \mathbin{\&} \sim r) \Rightarrow \sim q$ yields *Disjunctive Syllogism*[15], using just *Identity*, *Antilogism* is also a suppression law. The justification of *Antilogism* lies in contraposing to obtain from $p \mathbin{\&} q \Rightarrow r$ that $\sim r \Rightarrow \sim (p \mathbin{\&} q)$ i.e. no both premisses are true, but then a *truth-functional* elimination argument is used on the premisses to establish the falsity of one. *Antilogism* is therefore vulnerable to all the difficulties of *Disjunctive Syllogism.*

[15]On the other hand it appears *Disjunctive Syllogism* only yields *Antilogism* given as well as *Contraposition* both *Factor* and *Exported Syllogism.*

V

The most striking manifestation of the presence of suppression and illegitimate replacement is the production of relevance-violating paradoxes. The mechanism of suppression or of illegitimate replacement can be traced in every classical 'independent' paradox proof i.e. proof of a relevance-violating law from premisses which are not themselves relevance violating.

Suppression and illegitimate replacement laws are not on their own sufficient for paradox production; rather they, together with certain other Addition laws, which are themselves quite harmless, are *jointly sufficient* for relevance violations. These Addition laws simply provide the conditions under which the *partial* loss of meaning connection normally resulting from suppression or illegitimate replacement is able to become *total* irrelevance or loss of meaning connection, much as some condition, itself perfectly normal and harmless, may yet provide the environment for the production of a pathological condition. The mechanism of paradox production is that these Addition laws allow the adding of an irrelevant proposition in some position, whereupon the use of a suppression or illegitimate replacement law allows the elimination of the relevant portion, yielding total irrelevance. The mechanism is most clearly visible in paradox proofs using the most general conjunctive suppression law, *Exportation*, and the Addition law *Conjunctive Simplification*, which allows the adding of an irrelevant variable to the antecendent or of an unneeded premiss to the premiss set. This added variable q is irrelevant to the conclusion, and to the premiss p, in the sense of the Anderson and Belnap relevance requirement – it fails to share a variable with p. Hence, since *no* conditions have been placed on q's relation to p, we simply can pick a q which has no meaning connection with p, so q is irrelevant to p in this sense also. It is also irrelevant in the further sense that it is *deductively irrelevant* – neither its presence nor its absence affects the validity of the deduction.

Starting from this possibility of adding an irrelevant premiss

$$S \,\&\, q \Rightarrow S$$

we obtain by *Exportation*,

$$S \Rightarrow (q \Rightarrow S).$$

This last move puts us in a position to obtain by detachment $q \Rightarrow S$ in the event that S is true. By the familiar suppression mechanism of *Exportation* we are enabled to drop off S, the only *relevant* premiss, leaving the completely irrelevant added premiss q as *sole premiss.* q was introduced in *Conjunctive Simplification* as a passenger in the premiss set, but since the abdication of p it has been forced to take over the whole job of the premiss – a job it is not fitted for. Because we have now dropped the variable S which caused the original premiss set S & p to share a variable with the conclusion S, the resulting premiss q is quite irrelevant to the conclusion.

The dropping-off manoeuvre permitted by *Exportation* is the same one which, in the earlier counterexamples where *both* premisses were used in obtaining the conclusion and bother were relevant to the deduction, yielded a premiss *partially* sufficient for the conclusion. It is the quite permissible situation allowed by *Conjunctive Simplification* where not both premisses are relevant, which gives *Exportation* the opportunity to produce irrelevance where it formerly produced only partial sufficiency.

Where a law allows the suppression of a *conjoined premiss*, as do *Exportation*, *Necessity-Restricted Exportation*, *Antilogism*, and *Disjunctive Syllogism* together with laws which pin down the disjunction as truth-functional, the crucial law for obtaining relevance-violating paradox is *Conjunctive Simplification*, allowing the adjoining of any proposition to the premiss set. Then we obtain paradoxes of the familiar form, any proposition $q \Rightarrow S$, where S is the class of propositions which the suppression law allows to be suppressed.

However, laws which only allow illegitimate replacement of a proposition in some position other than that of a conjoined premiss, naturally do not react just with *Conjunctive Simplification* but with Addition laws allowing the addition of irrelevant variables in the position where replacement is to occur. Chief of these is *Factor*

$$p \Rightarrow q \Rightarrow .p \,\&\, r \Rightarrow q \,\&\, r$$

In this law the variable r is added to each side of the consequent as a passenger – it is the occurrence of the variables p and q in the consequent which connects the antecendent and consequent of the consequent, as well as the whole antecedent and consequent. But the same mechanism of the elimination of the connecting factor or variable, this time by replacement, can be seen in paradox proofs from the illegitimate replacement law *Commutation*, and the Addition laws *Factor* and *Conjunctive Simplification.* Capital letters symbolise the elements of the replaceable class; thus in the case of *Necessity-Restricted Commutation* entailments etc. The mechanism is that, starting from *Factor*

$$(1)\ \ S \Rightarrow q \Rightarrow .S \,\&\, T \Rightarrow q \,\&\, T$$

where T may be quite irrelevant to, and share no variable with, S and q, we illegitimately commute out and then detach the antecedent of the consequent, $S \,\&\, T$, leaving

$$(2)\ \ S \Rightarrow q \Rightarrow q \,\&\, T,$$

which, by *Conjunctive Simplification* and *Rule Conjunctive Syllogism* yields

$$(3)\ \ S \Rightarrow q \Rightarrow . T$$

But since T was originally adjoined in such a way as to take no account of the relation to $S \Rightarrow q$, to S or to q, (for T could be any variable at all), 3) is relevance-violating; for if T is any variable from the commutable class, we can pick one which is irrelevant to $S \Rightarrow q$, or one which fails to share a variable with it. But although T is irrelevant to $S \Rightarrow q$, to S and to q, in (1), this does not mean that some relevance fault lies in (1). For T is still quite relevant in the whole consequent of (1) in which it occurs, because it is added to each side of the consequent. It is *Commutation* which allows us to convert this overall relevance of T in the consequent of (1) to irrelevance by removing the formula $S \,\&\, T$, which was just the connecting link which gave T relevance. From this point of view the mechanism involved is precisely the same as that involved in producing the earlier counterexamples to *Commutation*. The move from (2) to (3) is a mere formality, because by the time we have reached (2) the role of T has been completely altered. The moral is that it is alright to take on passengers, but they should not be allowed to drive the bus.

This mechanism can be seen in steps 6 and 7 of the following sketched paradox derivation (due to John Bacon but adapted R. Routley to avoid *Exported Syllogism*).

1. $Q \Rightarrow \sim T \Rightarrow .Q \Rightarrow \sim T$
2. $Q \Rightarrow .Q \Rightarrow \sim T \Rightarrow . \sim T$ — Applying Commutation
3. $Q \Rightarrow .T \Rightarrow .\overline{Q \Rightarrow \sim T}$ — Contraposition, Rule Syll.
4. $T \Rightarrow .\overline{Q \Rightarrow \sim T} \Rightarrow .T \,\&\, R \Rightarrow .\overline{Q \Rightarrow \sim T} \,\&\, R$ — Factor

5. $Q \Rightarrow .T \,\&\, R \Rightarrow .\overline{Q \Rightarrow\sim T} \,\&\, R$ 3, 4 Rule Syll.

6. $Q \Rightarrow \overline{Q \Rightarrow\sim T} \,\&\, R$ 5, by Commutation and Detachment provided $T \,\&\, R$ is true.

7. $Q \Rightarrow R$ Rule Syll., *Conjunctive Simplification*

Thie last more complicated proof enables us to show in the system R+Factor[16] that any proposition implies a true one, and in E that any entailment implies any true entailment. More refined restrictions on *Commutation* result in more refined paradoxes.

A very similar result is produced using slightly different means in paradox proofs using *Exported Syllogism*, *Factor*, and *Conjunctive Simplification*. Here we use the replacement properties of *Exported Syllogism*, proved in section II, rather than commuting, to remove the antecedent of a consequent, and similarly to erode the meaning connections.[17]

Starting with

(1) $p \Rightarrow q \Rightarrow .p \,\&\, p^* \Rightarrow p \,\&\, q^*$ Factor

we obtain

(2) $p \Rightarrow q \Rightarrow .p \Rightarrow p \,\&\, q$

by the replacement law

$(p \Rightarrow q) \Rightarrow .S \Rightarrow (q \Rightarrow r) \Rightarrow S \Rightarrow (p \Rightarrow r)$

and $p \,\&\, q \Rightarrow p$, we obtain from (2)

(3) $p \Rightarrow q \Rightarrow .p^* \Rightarrow p^*$

The starred formulae in (3) have replaced those in (1).

From a disjunctive form of *Factor* we have

(4) $q \Rightarrow q \Rightarrow .q \vee p^* \Rightarrow .q \vee p$

(5) $q \Rightarrow r \Rightarrow .p \Rightarrow^* q \vee p$ using again replacement and *Addition* $p \Rightarrow q \vee p$

(6) But $(p \Rightarrow .q \vee p) \Rightarrow p \Rightarrow p$ substituting $q \vee p$ for q in (3)

(7) Hence $q \Rightarrow q \Rightarrow .p \Rightarrow p$ from (5), (6) by *Rule (Conjunctive) Syllogism*

Iteration of similar steps leads to $r \Rightarrow p \Rightarrow . \, p \Rightarrow p$ and thence by the first counterexample to *Exported Syllogism* it follows that any entailment implies any true entailment. The application of illegitimate replacement to obtain (3) has resulted in a formula in which the antecedent $p \Rightarrow q$ is only partly sufficient for the consequent $p \Rightarrow p$, for we have by replacement *eliminated* the

[16]For details of R see Belnap [6]. R is a relevance-satisfying extension of E with full Commutation.

[17]The proof sketched here is due to R. Routley, as are many other things in this paper.

original (non-passenger) variables in the consequent p and q, which connected the antecedent and consequent of (1). The result is a damaged connection in (3). A similar damaged connection is made in (5), where in a similar way the variable q which served to connected the consequent $q \vee p \Rightarrow q \vee p$ with the antecedent $q \Rightarrow q$, is eliminated in the antecedent of the consequent. Now we use *Rule Conjunctive Syllogism* (transitivity), to accumulate these damaged connections to yield total irrelevance. One might compare a case wher colour shades are ordered so that each member of a series is similar in colour to but differing slightly from the next. The differences in colour accumulate so that the first and last members may be completely different in colour. But such an accumulation in differences due to *Rule Conjunctive Syllogism*, is not, as Smiley [22, p. 237, 242] claims, the inevitable result of the fact that premisses and conclusions differ in meaning, plus the accumulating effect of transitivity. For not all differences accumulate by transitivity e.g. no chain iteration of proper inclusions will lead away from proper inclusions, whereas an iteration of partial inclusion may lead to total disjointness. The accumulation here is due to the fact that illegitimate replacement allows us to obtain premisses which are only partially sufficient for their conclusions, and hence the difference between premiss and conclusion is of the wrong, non-inclusive, and therefore accumulative, sort.

The crucial roles played by the Addition laws, *Conjunctive Simplification* and *Factor*, should emerge from these proofs. Many who have noted these roles have seen in them a reason for dropping these Addition laws, thus blocking paradox proofs and relevance-violations. But it should be clear why this, the dropping of Addition laws while suppression and illegitimate replacement laws are retained, is not a satisfactory solution to the problem. For, as the earlier counterexamples show, the paradoxes are not the only bad result of suppression and illegitimate replacement laws; others, equally bad perhaps, although less obviously so, are obtainable directly from them without the need to use the Addition laws as well. To cut out relevance-violations by just removing the Addition laws would be to remove only an obvious but superficial symptom of illegitimate replacement, while the disease itself remained untouched.

Obviously then the satisfaction of a variable-sharing relevance requirement by a system is not on its own a guarantee of the correctness of that system. First because suppression or illegitimate replacement, although present, may not lead to relevance-violation if *Addition* laws are absent, since they are only jointly sufficient. (Hence also it would be incorrect to argue that because a relevance-satisfying system lost this property upon addition of some law, this law must itself contain a relevance fallacy.). Second, because even when all Addition laws are present, not all sorts of insufficiency of a premiss set for its conclusion are of the general sort that lead to relevance violations. We could add instances of suppression laws with constants, as in some of the earlier counterexamples, or suppression laws such as *Exportation* with all variables identified, without violating relevance.

In view of the mechanism involved in relevance violations, it would appear that there are the following main types of solutions to the paradoxes, or ways of avoiding relevance violations, within the framework of classical logic:

1. Keeping all suppression and illegitimate replacement laws, while rejecting Addition laws. For this it appears to be sufficient to reject *Conjunctive Simplification*.

2. Keeping replacement laws but rejecting suppression laws; keeping *Conjunctive Simplification* but rejecting *Factor*

3. Rejecting all illegitimate replacement and suppression laws, while keeping Addition laws *Conjunctive Simplification* and *Factor*.

A fourth obvious case, keeping suppression but rejecting replacement laws, will only yield a system which is a sub-system of that yielded by (1). Since suppression laws are kept, *Conjunctive Simplification* would have to be dropped to avoid paradox; but then replacement laws could be added without relevance-violations, since they do not appear to yield paradox in the absence of *Conjunctive Simplification*.

It is not claimed that these solutions are exhaustive. It may be possible to keep all laws suitably qualified and still satisfy relevance. But these types do not represent the main solutions short of qualification.

The first solution, the rejection of the Addition laws, particularly *Conjunctive Simplification*, offers a formal 'solution' which might have some superficial appeal as the most economical, since it involves the rejection of one law only. But solutions of this sort raise grave difficulties for deducibility interpretations.[18] Not only do they still contain the defective suppression and replacement laws and their bad consequences which themselves raise such

[18]For a consideration of some related difficulties in non-classical systems see R. Routley and H. Montgomery [20].

difficulty for a deducibility interpretation, but now they also *lack* some laws, the Addition laws, which are so essential for even an enthymematic deducibility interpretation. There is therefore a serious question as to whether the resulting implication could properly be interpreted as any sort of deducibility relation. But the paradoxes it was designed to solve are paradoxes for deducibility.

The second solution, accepting one sort of insufficiency but not the other, is a hybrid one. Paradoxes from replacement laws are avoided by dropping *Factor*. This is the solution represented by the relevance-satisfying systems E and R. Addition of *Factor* to these systems produces paradoxes, those of material implication in the case of R, and higher degree paradoxes in the case of E.

Since both of these solutions accept, although in varying degrees, laws leading to insufficiency of premiss set for conclusion, while rejecting instead Addition laws which do satisfy a sufficiency interpretation, neither can adopt a sufficiency interpretation. At best then, they can adopt the only alternative deducibility interpretation, an insufficiency or enthymematic one, such that p implies q iff p is *partially* sufficient for q, or if p and some other proposition r (satisfying condition C) implies q. But then they have undermined the case for having a solution to the paradoxes, indeed for even considering relevance-violations a problem. For under an account of deducibility in which the premiss need not be sufficient for the conclusion, but something else also may be used, some form of relevance-violation is correct and should be a consequence. We should not *expect* the premisses of an enthymematic deducibility relation to be connected in meaning with their conclusions, nor to share variables with them; for the missing but used proposition may be what makes the connection with the conclusion, or provides the variable which results in variable sharing. But since this used proposition need not be stated as a premiss, we may find ourselves violating relevance with the stated premiss and conclusion, quite correctly.

If for example we are prepared to say that p implies q when it is only true that p and some necessary proposition is sufficient for q then we should also be prepared to accept paradoxes of the form 'anything p implies a necessary truth'; for it is *true* that anything plus some necessary truth is sufficient for a necessary truth. The problem remains however when p is claimed to be sufficient for q although it is irrelevant to q and shares no variable with it; for how can p include in its meaning or state sufficient ground for something with which it can have no connection? The paradoxes are only a problem for a sufficiency interpretation of deducibility; we cannot both admit the problem, and, as a solution, abandon the interpretation which gives rise to it. The problem can only be resolved in the way independently recommended by the counterexamples – by the rejection of suppression and illegitimate replacement laws.

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