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Relevance through topical unconnectedness

Ackermann and Plumwood's motivational ideas on entailment

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Abstract

Ackermann's motivational spin on his theory of rigorous implication is analyzed and it is shown to contain en equivalent idea to Plumwood's notion of suppression freedom. The formal properties these ideas back turn out to be properly weaker than Belnap's variable sharing property, but it is shown that they can be strengthen in various ways. Some such strengthenings, it is shown, yield properties which are equivalent to Belnap's, and thus provide for new ways of motivating Belnap's fundamental relevance principle.

Keywords: criteria of relevance · entailment · enthymeme · formal relevance properties · relevant logics · suppression · topics

1 Introduction and background

This paper analysis two ideas related to the theory of entailment: one due to Wilhelm Ackermann and one to Valarie Plumwood. Both relate, then, to entailment as a broadly conceived *relevant* relation. Ackermann's highly influential paper *Begründung Einer Strengen Implikation* [1] is analysed and it is shown that it contains four different motivating ideas. Ackermann, like Lewis before him, wanted a theory of a *conditional* which could be interpreted as expressing entailment. This is a theme explicitly discussed in

Anderson and Belnap's *Grammatical Propaedeutic*—the infamous appendix to the first volume of their *magnum opus*, *Entailment: The Logic of Relevance and Necessity* [6]—which has been enormously influential within the tradition of relevant logics that Anderson and Belnap's work initiated. Ackermann's paper, on the other hand, had an enormous influence on Anderson and Belnap's search for their own theory of entailment, yet they only picked up two of Ackermann's ideas. Ackermann's fourth idea is here taken up for the first time and compared to other relevance criteria.

Although Anderson and Belnap's main motivation for their theory of entailment is to be found in their *use*- and *meaning-connection*-criteria, they also motivated their theory by pointing to the notion of an *enthymeme*—an, for want of a missing premise, *invalid* argument, yet in many context acceptable argument due to the missing premise being true and readily available in the context the argument is given in. Anderson and Belnap viewed the intuitionistic conditional, for instance, as one which allowed the suppession of *every* true premise, whereas the strict conditional of **S4** allowed the suppression of only *necessarily* true premises. Neither plain, necessary, nor even *logical* truths, however, can in general be suppressed in valid arguments, according Anderson and Belnap. Their view, then, was that the theory of entailment must be recast in such a way so as to uphold the distinction between a valid argument and that of a merely enthymematically valid one.

The suppession-take on the theory of entailment, however, stands in contrasts to their two main other strands of motivating their theory: Anderson and Belnap came up with, and stressed to importance of, formal properties to account for their two relevance criteria, but non to give precise content their notion of suppression. Valarie Plumwood's conference paper *Some False Laws of Logic* [19]—available for the first time in this issue of *Australasian Journal of Logic*—does set forth such properties, however, and as such is the first attempt within the tradition of relevant logics at giving precise content to the notion of suppression. Plumwood's analysis was further expanded upon in her joint work with Richard Sylvan starting with [22], and especially [21] where a lot of the material from Plumwood's conference paper was incorporated.

Anderson and Belnap claimed that a theory of entailment with a total ban on suppression would yield their logic **E**, and thus rule out principles such as the permutation law $(A \to (B \to C)) \to (B \to (A \to C))$, a law which their logic **R** does validate without violating the variable sharing property. Ackermann does not mention the notion of suppression, although one of his motivating ideas, we will see, is equivalent to one of Plumwood's "anti-suppression" principle. Even Ackermann mentions a weaker permutation principle, namely the *assertion* axiom $A \to ((A \to B) \to B)$—equivalent to the rule version of the above permutation law—as ruled out by the same considerations which is supposed to rule out more standard implicational paradoxes such as the "positive paradox" $A \to (B \to A)$. Plumwood and Sylvan claimed that their notion of suppression-freedom would yield the variable sharing property as a corollary. They furthermore claimed that even though **E** does manage to rule out the most abominable suppressive principles—manifest in the paradoxes of implication—it nevertheless does harbor many suppressive principles. Chief amongst these is the principle Plumwood and Sylvan for good measure calls *Suppression*, namely the **E**-axiom $((A \to A) \to B) \to B$ which is commonly categorized as a permutation principle.

Plumwood's formal suppression-principles—the *Anti-Suppression Principle* and the *Joint Force Principle*—were analyzed in [16] where it was shown that these formal principles are in fact properly *weaker* than the variable sharing property.[1] This paper shows first of all that Plumwood's suppression principles are equivalent to principles motivated by one of Ackermann's motivational spins on his notion of *rigorous implication*. Even though the formal suppression principles used in [16] turn out to be too weak to do the job they were thought to do, it could be that tweaking the principles slightly would be sufficient to rectify this. [16] did not explore any such tweaked principles. This paper shows forth ways of tweaking the Plumwoodian and Ackermannian principles so as to at least deliver on one of Plumwood and Sylvan's promise, namely to deliver the variable sharing property, and variant thereof, as a corollary. However, one does not obtain *stronger* principles by so tweaking, and so neither these formal relevance principles manage to categorize permutation principles as truly suppressive. In fact, the strongest tweaked principles that will be considered in this paper turn out to be *equivalent* to the variable sharing property.

The plan for the paper is as follows: Sect. 2 defines some of the logics that will be of interest in this paper as well as the consequence relation that will be used throughout. Sect. 3 discusses Ackermann's motivational ideas

[1][16] also discussed intensional variants of Plumwood and Sylvan's formal suppression principles and showed that also these fail to yield the variable sharing property. In this paper, however, I will only focus on extensional suppression.

for his theory of rigorous implication followed by sect. 4 where Anderson and Belnap's account of enthymemes and Plumwood's formal take on the notion of suppression is given. One way of strengthening Plumwood's principles is considered, but shown to allow implicational paradoxes to hold true. Sect. 5 introduces the notion of a topic and shows how to make use of the idea to strengthen Ackermann's and Plumwood's ideas so as to yield formal relevance properties on par with the variable sharing property. Sect. 6 gives a short summary. The appendix shows that a strong sublogic of **R** augmented with *Dummett's axiom*—$(A \to B) \vee (B \to A)$—satisfies the variable sharing property, despite, then, the fact that this formula is commonly regarded as an implicational paradox and the variable sharing property as a guard against such paradoxes.

2 Logics, logical consequence relations and entailment

Table 1 shows how some of the logics used in the paper can be pieced together. The minimal logic, unless otherwise stated, will be the weak relevant logic **BB**. The consequence relation used in this paper is throughout the Hilbertian one, defined as follows:

Definition 1 (Hilbert consequence)**.** *A Hilbert proof of a formula* A *from a set of formulas* Γ *in the logic* **L** *is defined to be a finite list* $A_1, \ldots, A_n$ *such that* $A_n = A$ *and every* $A_{i \leq n}$ *is either a member of* Γ*, a logical axiom of* **L***, or there is a set* $\Delta \subseteq \{A_j \mid j < i\}$ *such that* $\Delta \Vdash A_i$ *is an instance of a rule of* **L***. The existential claim that there is such a proof is is written* $\Gamma \vdash_{\mathbf{L}} A$*.*

BB	Ax1–Ax5, R1–R7
E	**BB** +: Ax6–Ax10, Ax12–Ax14; −: R3–R7
Π'	**E** +: R8
R	**E** +: Ax11; −: Ax14
RUE	**R** +: Ax15
RD	**R** +: Ax16
RM	**R** +: Ax17

Table 1: Some of the logics mentioned in this paper

Ax1	$A \to A$	
Ax2	$A \to A \vee B$ and $B \to A \vee B$	
Ax3	$A \wedge B \to A$ and $A \wedge B \to B$	
Ax4	$\sim\sim A \to A$	
Ax5	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$	
Ax6	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$	strong lattice $\wedge$
Ax7	$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$	strong lattice $\vee$
Ax8	$(A \to \sim B) \to (B \to \sim A)$	contraposition axiom
Ax9	$(A \to B) \to ((B \to C) \to (A \to C))$	suffixing axiom
Ax10	$(A \to B) \to ((C \to A) \to (C \to B))$	prefixing axiom
Ax11	$A \to ((A \to B) \to B)$	assertion axiom
Ax12	$(A \to \sim A) \to \sim A$	reductio
Ax13	$(A \to (A \to B)) \to (A \to B)$	contraction axiom
Ax14	$((A \to A) \wedge (B \to B) \to C) \to C$	**E**-axiom
Ax15	$A \wedge \sim A \to B \vee \sim B$	unrelated extremes
Ax16	$(A \to B) \vee (B \to A)$	Dummett's axiom
Ax17	$A \to (A \to A)$	mingle
R1	$\{A, B\} \Vdash A \wedge B$	adjunction
R2	$\{A, A \to B\} \Vdash B$	modus ponens
R3	$\{A \to B\} \Vdash (B \to C) \to (A \to C)$	suffixing rule
R4	$\{A \to B\} \Vdash (C \to A) \to (C \to B)$	prefixing rule
R5	$\{A \to \sim B\} \Vdash B \to \sim A$	contraposition rule
R6	$\{A \to B, A \to C\} \Vdash A \to B \wedge C$	lattice $\wedge$
R7	$\{A \to C, B \to C\} \Vdash A \vee B \to C$	lattice $\vee$
R8	$\{A, \sim A \vee B\} \Vdash B$	γ, disjunctive syllogism

We saw in the above section that one of the criteria for Anderson and Belnap's theory of relevant entailment was the meaning-relatedness criterion which [8] cashed out using the following formal property:

Definition 2. *A logic* **L** *has the* Variable Sharing Property (VSP) *just in case for every formula A and B, $\varnothing \vdash_{\mathbf{L}} A \to B$ only if A and B share a propositional variable.*

The only consequence relation that I will consider in this paper is the Hilbertian one. For conceptual clarity, however, note, that Anderson and Belnap's theory of entailment was intended as a theory of *logical consequence*, not of merely logically true conditionals. They specified their theory using a restriction on Hilbert proofs, as well as a Fitch calculus which would take us too far afield to go into.[2] For present purposes, however, it suffices to note that their notion of entailment can simply be defined as the following consequence relation:

Definition 3 (Entailment). $\Gamma \vdash^{e}_{\mathbf{L}} A =_{df}$ *there are* $\gamma_{i\leq n} \in \Gamma$ *such that* $\varnothing \vdash^{e}_{\mathbf{L}} (\gamma_1 \wedge \ldots \wedge \gamma_n) \to A$.

It is easily seen, then, that the variable sharing property naturally extends to entailment viewed as a consequence relation, and so if A does entail B given some logic which does satisfy this property, then these formulas are indeed meaning-related by way of sharing a propositional variable.

A last comment on notation: so as to cut back on unnecessary symbols, and since many of the results in this paper relate to entailment, and thus to logically true conditionals, I will usually write '$\vdash_{\mathbf{L}} A$' instead of the formally correct '$\varnothing \vdash_{\mathbf{L}} A$.'

3 The forgotten motivational criteria of Ackermann's *rigorous implication*

Ackermann's paper *Begründung einer strengen Implikation* has had an enormous influence on shaping the research program in relevant logics. Indeed, Anderson and Belnap dedicated the first volume of *Entailment* to Ackermann with the laudation "whose insights in *Begründung einer strengen Implikation* [...] provided the impetus to this enterprise" [6, p. *v*]. This is so even though the paper is merely 15 pages long, and, it must be admitted, rather vague with regards to the motivation for the logics it sets forth. This section gives a brief contextualisation of Ackermann's paper, and then analyzes one of the motivations that Ackermann states for his *rigorous implication*.[3] In the next section it will be shown how this connects up with Plumwood's suppression principles.

Russell had used "imply" to signify a merely true material conditional; he writes in his *Principles of Mathematics* that

[2]See [6, § 23] for these calculi.

[3]*Rigorous implication* is the term Anderson and Belnap used [cf. 4] to translate Ackermann's *strengen Implikation* so as to set it apart from Lewis' *strict implication.*

> the assertion that q is true or p false turns out to be strictly equivalent to "p implies q"; [...]. It follows from the above equivalence that of any two propositions there must be one which implies the other, that false propositions imply all propositions, and true propositions are implied by all propositions. [23, §16]

Both MacColl and Lewis reacted at this use of "imply." In response, MacColl, for instance, writes that

> It is surely an awkward assumption (or convention) that leads here to the conclusion that "either W implies E or else E implies W". *War* in Europe does not necessarily imply a disastrous *earthquake* the same year in Europe; nor does a disastrous *earthquake* in Europe necessarily imply a great *war* the same year in Europe. [15, p. 453]

Lewis responded similarly some years later claiming that the consequences of such a view of implication is preposterous:

> If 'p implies q' means only 'it is false that p is true and q false,' then the implication relation is far too ubiquitous to be of any use. If we ask for the consequences of any proposition, we are immediately confronted with all the truths we can think of. If we are so foolish as to make a condition contrary to fact, we must at once accept its own contradictory as the logical result. [12, p. 246]

MacColl's and Lewis' remedy was to replace the material conditional with the strict conditional as the analysis of implication. It is this tradition that [1] must be read. Indeed, it is quite evident from its short motivational introduction that Ackermann proposed his logic Π' of *rigorous implication* as a *competitor* to Lewis' logics of strict implication. Although different results, they share the theoretical aim of giving an account of the relation of logical consequence, or *entailment*, expressed using an object-language conditional. MacColl and Lewis claimed—and few, I think it fair to say, have ever disagreed—that the material conditional cannot be read this way: A proposition is not entailed by every proposition just because it is true, nor does a proposition logically imply every other just because it is false. One may debate how to best read the material conditional. Anderson and Belnap, for instance, famously claimed that it simply is no kind of conditional at all on account that modus ponens does not hold for it. It is argued in [18] that this view is not *forced* by Anderson and Belnap's selection criteria for a theory of entailment, but that, of course, does not provide an answer to the question of just what kind of conditionality—if any—the material conditional expresses.

Ackermann, Anderson and Belnap, as well as Plumwood, all agree with MacColl and Lewis's theoretical ambition of providing a theory of entailment, that is logical consequence, expressed using an object-language conditional. Where they disagree is over which logical laws hold true of such a conditional and which logical laws the theory of entailment itself is subject to. Lewis and Langford, for instance write that "$p \prec q$ has the property requisite to that relation which holds when q is deducible from p and does *not* hold when q is not deducible from p" [13, p. 245]. Note, however, that this is stated immediately after affirming that "whenever any truth-implication, pIq, expresses a tautology (is necessarily true) the relation $p \prec q$ holds."[4] Thus the *correctness*, so to speak, of B, *suffices* for any A entailing it. Similarly, the MacColl-Lewis analysis yields that if A is classically unsatisfiable, then $A \prec B$ holds for every B. Thus the *incorrectness* of A suffices for it entailing any B. These two consequences of the "strict" analysis of entailment—that entailment is identifiable as the strict conditional—are commonly known as the *paradoxes of strict implication.* The first class of authors—Ackermann, Anderson, Belnap and Plumwood—all agree that these features point to that the *strict* account is incorrect due to precisely this feature. Let's first look at Ackermann's account. Ackermann writes that

> The rigorous implication, expressed as $A \to B$, expresses that there exists a logical connection between A and B; that the content of B is part of the content of A, or how now best to put it. That there exists such a connection has nothing to do with the correctness or falsity of A and B. This is why one ought to reject the validity of the formula $A \to (B \to A)$; it expresses that $B \to A$ can be inferred from A while it is obvious that the correctness of A has no bearing on whether there is a logical connection between B and A. The same reason tells agains viewing any of $A \to (B \to A \mathbin{\&} B)$, $A \to (\overline{A} \to B)$ or $A \to ((A \to B) \to B)$ as

[4]From context it is clear that any formula $p \supset q$ which is a classical tautology would fit the bill here.

> universally valid. The same also holds for $B \to (A \to A)$, since the validity of $A \to A$ is independent of the correctness of B. My own rigorous implication differs from the strict one in that the latter formula,[5] as well as $(A \,\&\, \overline{A}) \to B$, is rejected as a universally valid formula on account of the fact that the concept of implication—understood as a logical connection between two statements—does not encompass statements which imply or are implied by every other. [1, p. 113][6]

There are four ideas related to the entailment- or rigorous implication conditional that can be glimpsed here:

1. A logical connection expressed by the conditional
2. Meaning-relatedness between the antecedent and consequent of the conditional
3. That entailment cannot be decided on the basis of truth or correctness criteria applying to the antecedent and the consequent.
4. Total weakening failure: That no statement entails or is entailed by every other.

Note, then, that the latter two ideas directly latch on to ideas of Lewis: Since Lewis allows for the necessary truth of B to *suffice* for A entailing B, Lewis-entailment is far too ubiquitous, according to Ackermann. Anderson and Belnap appealed to Ackermann's notion of a logical connection when arguing for their use-criterion of entailment,[7] whereas Belnap's variable sharing property is naturally seen as a way of giving precise content to Ackermann's second mereologically framed idea of meaning-relatedness. What we will look closer at here, however, is Ackermann's explication of the notion of rigorous implication using the fourth idea—the negative property that it *does not encompass statements which imply or are implied by every other*. This is an idea that, as far as I know, have not discussed before.

[5] As noted in [17, p. 6997], Ackermann seems to think that $B \prec (A \prec A)$ holds in Lewis' systems. This is true for *normal* modal logics such as **S4**, but it fails in Lewis' preferred systems **S2** and **S3**. Note, however, that Ackermann does not mention $B \prec (A \prec A)$ in the sequel-paper [2] wherein he compares his logic to **S2**.

[6] My own translation.

[7] See for instance [3] and [8] for two early examples of such appeals.

The problem, to quote Belnap, is to restate this condition in formal terms [cf. 8, p. 144]. As a straight-forward first attempt, consider the following formal property:

Definition 4. *A logic* **L** *has the property of* TOTAL WEAKENING FAILURE (TWF) *just in case for every formula* A *there is a formula* B *such that* $\nvdash_{\mathbf{L}} B \rightarrow A$.

(TWF) seems clearly covered by Ackermann's dictum in the above quote. That requirement seems to demand more, however, than merely the non-existence of a formula which is rigorously implied every other: there should neither be any which itself *implies* every other. It seems therefore evident that (TWF) ought to be strengthened to its "double" variant if it is to capture Ackermann's idea:

Definition 5. *A logic* **L** *has the property of* TOTAL DOUBLE WEAKENING FAILURE (TDWF) *just in case for every formula* A *there are formulas* B *and* C *such that both* $\nvdash_{\mathbf{L}} A \rightarrow C$ *and* $\nvdash_{\mathbf{L}} B \rightarrow A$.

In the presence of the contraposition rule (R5), however, it turns out that this is no strengthening at all:

Theorem 1. *(TDWF) co-entails (TWF).*

Proof.

$\Rightarrow$ Trivial

$\Leftarrow$ Assume (TWF) and for contradiction that (TDWF) fails. Then there is some A such that for all B's and C's, either $\vdash_{\mathbf{L}} B \rightarrow A$, or $\vdash_{\mathbf{L}} A \rightarrow C$. Thus for some A, either $\vdash_{\mathbf{L}} B \rightarrow A$ for all B's or $\vdash_{\mathbf{L}} A \rightarrow C$ for all C's. Since (TWF) holds, however, there is a D such that $\nvdash_{\mathbf{L}} D \rightarrow A$, so $\vdash_{\mathbf{L}} A \rightarrow C$ for all C's. By using contraposition one gets that $\vdash_{\mathbf{L}} C \rightarrow {\sim}A$ for all C's. However, since (TWF) is assumed to hold, there is a B_1 such that $\vdash_{\mathbf{L}} B_1 \rightarrow {\sim}A$. We thus arrive at a contradiction which, then, ends the proof.

□

Theorem 2. *(TWF) rules out all of the following "implicational paradoxes":*

(1) $A \rightarrow (B \rightarrow A)$	(2) $A \rightarrow ({\sim}A \rightarrow B)$	(3) $A \rightarrow (B \rightarrow B)$
(3) $A \wedge {\sim}A \rightarrow B$	(4) $A \rightarrow B \vee {\sim}B$	(5) $A \rightarrow (B \rightarrow A \wedge B)$

Proof. The proofs are almost trivial and are left for the reader. □

Note, then, that (TWF) rules out every logical law mentioned by Ackermann, except for $A \to ((A \to B) \to B)$—the *assertion* axiom. Anderson and Belnap agreed with Ackermann that this ought not to be a valid entailment, but their reason for this was due to modal considerations, not to relevance [cf. 6, §. 28].

We have seen that Ackermann formulated four different motivating ideas for his notion of rigorous implication—an object language conditional intended to express entailment. One of these is shun light on for the first time here and it was shown that it can be formalized so as to yield a principle which rules out many of the standard implicational paradoxes. Even though Ackermann's idea hasn't directly been discussed before, the issue of how weakening thwarts relevance has. For instance, in Standefer's recent discussion of what is to count as a properly relevant propositional connective, [26] appeals to so-called *ubiquitously true/false* formulas. A formula is defined to be ubiquitosly true (false) relative to a Routley-Meyer model just in case it is true (false) in every point/world in the model. Such formulas, it is pointed out, will allow a kind of weakening to hold in the model since if A is ubiquitously true, then so will $B \to A$ for any formula B. Standefer's ubiquitous-lesson, then, is that if a connective generates formulas which are ubiquitously true (false) in ever model, then it will engender violations of variable sharing which goes against the motivating idea that every formula is non-trivially satisfiable: every formula fails somewhere [cf. 26, § 7]. Ackermann's third motivational idea—that entailment cannot be decided on the basis of truth or correctness criteria—is evidently connected to Standefer's notion of freedom from ubiquitous formulas. Note, then, that the model-theoretic equivalent of (TDWF) is that no formula is either ubiquitously true or ubiquitously false: every formula A holds somewhere and fails to hold somewhere. Thus Ackermann's fourth motivational idea of entailment is evidently also connected to Standefer's notion of freedom from ubiquitous formulas.

Ackermann's totally total weakening failure-idea can be strengthened in various ways. I will get back to this after having presented Plumwood's ideas for logical theory choice.

4 Plumwood's formal account of Suppression

The arguably most central motivational idea behind relevant logics is that if B is entailed by A, then these sentences must be *meaning*-related somehow. This is an idea that we've seen voiced by Ackermann, and which Anderson and Belnap cashed out as the by now well-known *variable sharing property*—that if $A \to B$ is to be logically true, then A and B must share a propositional variable. Ackermann's notion of a "logical connection" was furthermore recast as the requirement that B must be connected to A by way of a notion of *premise use*—it cannot be the case that B follows *from* A if A need not be used in obtaining B from the assumption A. I will not go into details of this *from-nes* aspect, save to comment that their *Entailment Theorem* is to the effect that A does indeed entail B according to their theory, just in case $A \to B$ is to be logically true. The two criteria put together, then, yield that if A entails B, according to their theory of entailment, then A and B share a propositional variable and thus are, arguably, meaning-related.[8]

In addition to motivating their theory using the notions of a connection of meaning and the proper use of premises, Anderson and Belnap gave a radically different motivation for their logic **E** even after having established that the variable sharing property and the Entailment Theorem hold true of **E**, namely as the—and I should emphasize that singular particle here—logic which allows one to distinguish between valid entailments and *enthymemes* in which premises—be they merely true, necessarily or even logically so—have been suppressed:

> If we are *very* careful, and always put down all the premises we need (i.e., if we argue *logically*), then we arrive *precisely* at the formal system E of logical implication (without quotes, this time), or *entailment*. [5, p. 722]

The heart of [5]'s argument is that a theory of entailment must be able to account for the difference between a valid argument and an enthymeme which they identify rather vaguely as a valid argument had it not been for a required premise which happens to be true, necessarily true, or logically true. To exemplify their view, they showed forth a two-premised argument in which the minor premise is true, yet where the argument without this

[8]For a more in depth discussion of Anderson and Belnap's relevance criteria and selection criteria for logical theory choice more generally, see [18].

premise is not valid:[9]

(M)	All bodies moving in elliptic orbits are subject to the law of gravitation;
(m)	Comets move in elliptic orbits;
(C)	Therefore comets are subject to the law of gravitation.

From this valid argument, however, there corresponds a logically true conditional proposition, namely that expressed by "if M and m, then C." Since, however, m is true, so is "if M, then C" if the conditional expressed in these propositions are that of either the material conditional or the intuitionistic one. A similar example shows, then, that the same also holds true for the strict conditional where the example is but tweaked so that the minor premise be necessarily true. But M without m does not entail C, and so, or so goes Anderson and Belnap's argument, these conditionals can at best be taken to express enthymematic relations of implication where certain premises are indeed suppressible, but not the relation of entailment itself.

Unlike their other two motivating features, however, Anderson and Belnap never tried to identify a formal criterion of suppression-freedom and therewith substantiate their claim that one would reach their logic **E** by adhering to it. This is where Plumwood comes in. Plumwood's conference paper *Some False Laws of Logic* explicitly acknowledges its commonality with [5] [cf. 19, fn. 3]. Contra Lewis and Langford's claim that "Socrates is a man" entails "Socrates is mortal," provided that "all men are mortal" is necessarily true and thus that necessary truths may be suppressed from valid argument [cf. 13, p. 165], Anderson and Belnap restrict themselves to stating that they "believe, rather, that [. . .] Necessary premises are just as necessary as premises that are not necessary" [5, p. 713]. Plumwood does not merely state what she believes, but directly *argues* that premises, whether merely, necessarily or logically true, cannot simply be dropped in valid arguments. The purpose of this paper is not to evaluate her arguments, but rather her notion of suppression and how it relates to the meaning-connectivity thesis. Now Ackermann's idea in so regard seems to have been that since entailment requires logical connectivity and meaning relatedness, it cannot be as ubiquitous—to reuse Lewis' idea—as either Russell or Lewis thought. Plumwood, on the other hand, thought of the logical connectivity of true entail-

[9]The argument is to be found in [5, p. 713] and is therein a quote from Jevons' *Elementary Lessons in Logic*.

ment as involving meaning-connectivity, and that suppression may take one from a true entailment wherein meaning-connectivity does obtain to one in which it doesn't:

> But a most important objection to the deductive suppression derives from deducibility as a meaning relation between propositions. q should be deducible from p only if there is a connection of meaning between p and q. But this connection may be destroyed if suppression is allowed; for the suppressed proposition, which although used no longer appears in the premiss set p, may be just what originally made the meaning connection between p and q. Once this used proposition has been dropped off, p and q may no longer have the right connection of meaning (e.g. inclusion), or worse still, may have no connection at all. [19, § I]

After pointing to various features of logical consequence, Plumwood's tentative conclusion is as follows:

> All these features of deducibility, then, provide reasons for saying that every proposition sometimes occurs essentially and has its own bit to add. This leads to the Suppression Principle: *for every proposition p there is some proposition q such that the consequences of q are a proper subset of the joint consequences of p and q. There is no priviliged class of propositions which are generally suppressible.* [19, §I]

This principle of suppression was restated in [21, p. 146]—with reference to Plumwood's paper as its source—but therein renamed as the *Anti-Supression Principle*.

Not only, then, did Plumwood argue for her conclusion, but she also provided a quasi-formal principle with which one can evaluate the content of her notion of suppression. Even though Anderson and Belnap claimed that the formal use-condition for relevance was both necessary and sufficient, but that the variable sharing property was only necessary [cf. 6, §. 5.1], they never tried to *derive* the latter from the former. Similarly, then, with their notion of suppression: if a total ban on suppression does yield the logic **E**, then one would expect that it could be formulated in such a way as to yield the variable sharing property as a consequence. [5] was reprinted—with only minor changes—over 30 years later in [7, § 36]. It is worth noting, then, that the conclusion of the first enthymeme-study was retained in the latter, namely that by not allowing any suppression one would thereby arrive "*precisely* at the formal system E of logical implication [...] or *entailment*" [7, § 36]. The reader, however, is left without any indication as to how this will come to be, nor even how to derive at least a necessary criteria of entailment such as the variable sharing property using this notion of suppression-freedom. Although Anderson and Belnap had already motivated their theory of entailment by using the traditional notion of an enthymeme, it was Plumwood who first came up with formal properties—the *Anti-Supression Principle* and the *Joint Force Principle* which we'll get to shortly—which one then can use to evaluate whether or not a logic can distinguish between valid and merely enthymematically valid arguments, and whether or not such a principle does yield **E** or not.

Plumwood's ideas were further expanded upon by her joint work with Richard Sylvan. Plumwood and Sylvan explicitly claimed that the variable sharing property is *derivable* given their account of entailment [cf. 21, p. 3]. [16] showed that Plumwood's two principles are in fact properly weaker than the variable sharing property, in fact properly weaker than the so-called *quasi* version of that property. In the remainder of this section I will show that Plumwood's principles are equivalent to Ackermannian principles in the spirit of the weakening-failure principle of the previous section. I will then show forth some straight forward ways of strengthening Plumwood's principle and show that this does not suffice for in fact deriving the variable sharing property as Plumwood and Sylvan thought should be the case given a "a good sufficiency relation" [21, p. 3]. In the next section I will then show how to in fact strengthen these principle so as to make the derivation hold true.

4.1 Comparing formal relevance properties: initial results

It was mentioned in [16, fn. 8] that Ackermann's fourth motivational idea for his concept of rigorous implication is in fact equivalent to Plumwood's first suppression principle. The first task of this subsection is to prove that this is in fact the case.

Definition 6. *A logic* **L** *satisfies the* ANTI-SUPPRESSION PRINCIPLE (ASP) *just in case for every formula A, there exist formulas B and C such that*

$\vdash_{\mathbf{L}} A \wedge B \to C$, *but* $\nvdash_{\mathbf{L}} B \to C$.

Theorem 3. *(ASP) co-entails (TWF).*

Proof.

⇒ Assume that (ASP) holds, but that (TWF) fails. Since (TWF) fails, there is a formula A such that for every D, $\vdash D \to A$. According to (ASP), however, there are B's and C's such that $\vdash A \wedge B \to C$, but $\nvdash B \to C$. Since $\vdash B \to A$, however, we also get that $\vdash B \to A \wedge B$, and therefore $\vdash B \to C$. Hence, (TWF) can't fail if (ASP) holds.

⇐ Assume (TWF). Let A be any formula. According to (TWF) there is a formula B such that $B \to A$ is not a logical theorem. However, $\vdash A \wedge B \to A$. Hence, for every A there is a B and a C such that $\vdash A \wedge B \to C$, but $\nvdash B \to C$.

□

Plumwood's first suppression principle, however, seems not to quite cover the notion of suppression as it allows for there being consequences r which q does not yield, but p together with q does, but for the simple reason, then, that p yields r on its own. Plumwood must have realised this and does provide a principle to guard against this, namely the *Joint Force Principle*: "for every proposition p there is some other q such that p and q are jointly sufficient for r but neither p nor q on its own is sufficient for r" [19, § II].[10]

Plumwood mentioned the Joint Force Principle as one which is properly stronger than the Anti-Suppression Principle.[11] Plumwood stated this principle formally as

$$(\forall p)(\exists q)(\exists r)(p \mathbin{\&} q \Rightarrow r \mathbin{\&} {\sim}(p \Rightarrow r) \mathbin{\&} {\sim}(q \Rightarrow r)),$$

where '⇒' is Plumwood's object-language arrow of entailment. Although open to interpretation, it was argued in [16, fn. 10] that this ought to be stated as follows:

[10] Both these principles re-occur in [21]. See [16] for details.

[11] "Exportation also violates another principle which implies, but is not implied by, the Suppression Principle—the Joint Force Principle" [19, § II]. It is evidently at least as strong as, but I have not been able to find an interesting logic which satisfies the latter but not the former.

Definition 7. *A logic* **L** *satisfies the* Joint Force Principle (JFP) *just in case for every formula A, there exist formulas B and C such that $\vdash_{\mathbf{L}} A \wedge B \to C$, but $\nvdash_{\mathbf{L}} A \to C$ and $\nvdash_{\mathbf{L}} B \to C$.*

We have seen that (ASP) is equivalent to the Ackermannian principle (TWF), which in turn was shown to be equivalent to (TDWF). Is there, then, a way of strengthening the latter so as to obtain a principle equivalent to (JFP)? The answer is in the affirmative: by making it *uniform*:

Definition 8. *A logic* **L** *has the property of* Uniform Total Double Weakening Failure (UTDWF) *just in case for every formula A there is a formula B such that both $\nvdash_{\mathbf{L}} A \to B$ and $\nvdash_{\mathbf{L}} B \to A$.*

Theorem 4. *(UTDWF) co-entails (JFP)*

Proof. Assume first that (UTDWF) holds, but that (JFP) does not. Then there is a A such that for all B's and C's, if $\vdash_{\mathbf{L}} A \wedge B \to C$, then $\vdash_{\mathbf{L}} A \to C$ or $\vdash_{\mathbf{L}} B \to C$. However, since (UTDWF) holds, there is a D such that $\nvdash_{\mathbf{L}} D \to A$ and $\nvdash_{\mathbf{L}} A \to D$. Furthermore, since $\vdash_{\mathbf{L}} A \wedge D \to A \wedge D$, either $\vdash_{\mathbf{L}} A \to A \wedge D$, and therefore $\vdash_{\mathbf{L}} A \to D$, or $\vdash_{\mathbf{L}} D \to A \wedge D$, and therefore $\vdash_{\mathbf{L}} D \to A$. Both disjunct yield a contradiction, which therefore ends the proof.

Assume now that (JFP) holds, but that (UTDWF) does not. It follows, then, that there is some formula A such that for every D, $\vdash_{\mathbf{L}} A \to D$ or $\vdash_{\mathbf{L}} D \to A$. From (JFP) it follows that there is some B and some C such that $\vdash_{\mathbf{L}} A \wedge B \to C$, but $\nvdash_{\mathbf{L}} A \to C$ and $\nvdash_{\mathbf{L}} B \to C$. Now either $\vdash_{\mathbf{L}} A \to B$ or $\vdash_{\mathbf{L}} B \to A$. If the first, then $\vdash_{\mathbf{L}} A \to A \wedge B$, and so $\vdash_{\mathbf{L}} A \to C$ by transitivity. If the latter, then $\vdash_{\mathbf{L}} B \to A \wedge B$, and so $\vdash_{\mathbf{L}} B \to C$ by transitivity. Both options lead to a contradiction which, then, ends the proof. □

(UTDWF), it seems to me, could legitimately be regarded as implied by Ackermann's considerations. In terms of variable sharing, however, it remains rather weak: It was shown in [16, thm. 1f] that its equivalent (JFP) is in fact properly weaker than Meyer's *quasi* variable sharing property:

Definition 9. *A logic* **L** *has the* Quasi Variable Sharing Property (QVSP) *just in case for every formula A and B, $\vdash_{\mathbf{L}} A \to B$ only if either A and B share a propositional parameter, or both $\vdash_{\mathbf{L}} {\sim}A$ and $\vdash_{\mathbf{L}} B$.*

One idea for how to strengthen Plumwood's suppression idea would be to take note of the fact that strong logics such as **R** satisfy a certain deduction theorem to the effect that $\{A\} \vdash_{\mathbf{R}} B$ just in case $\varnothing \vdash_{\mathbf{R}} A \wedge \theta \to B$, where $\varnothing \vdash_{\mathbf{R}} \theta$.[12] Such a logical theorem θ will in many cases be insuppressible. For instance $\{A \to B\} \vdash_{\mathbf{R}} A \to A \wedge B$, and by letting θ be $A \to A$, we have as an instance of (Ax6) that $\varnothing \vdash_{\mathbf{R}} (A \to B) \wedge \theta \to (A \to A \wedge B)$, yet $\varnothing \nvdash_{\mathbf{R}} (A \to B) \to (A \to A \wedge B)$ and $\varnothing \nvdash_{\mathbf{R}} \theta \to (A \to A \wedge B)$. One idea, then, would be to try to strengthen Plumwood's ideas of anti-suppression and joint forces so as to require, respectively, that every formula be insuppressible in the context of a logical theorem and have non-reducible joint consequences with some logical theorem:

Definition 10. *A logic* **L** *satisfies the* Strong Anti-Suppression Principle (SASP) *just in case for every formula A, there exist formulas B and C such that $\vdash_{\mathbf{L}} A \wedge B \to C$ and $\vdash_{\mathbf{L}} B$, but $\nvdash_{\mathbf{L}} B \to C$.*

Definition 11. *A logic* **L** *satisfies the* Strong Joint Force Principle (SJFP) *just in case for every formula A, there exist formulas B and C such that $\vdash_{\mathbf{L}} A \wedge B \to C$ and $\vdash_{\mathbf{L}} B$, but $\nvdash_{\mathbf{L}} A \to C$ and $\nvdash_{\mathbf{L}} B \to C$.*

Note first of all that (SJFP) is easily shown to be equivalent to the similarly strengthened version of the Ackermannian principle (UTDWF) and similarly for (SASP) vs. (TWF). Let's briefly look at (SASP) first.

Theorem 5. *(QVSP) implies (SASP) provided the logic in question is theorem-wise consistent.*

Proof. Let A be any formula. Let B be any logical theorem which does not share any propositional variables with A. Lastly, let C simply be $A \wedge B$. Since the logic is theorem-vise consistent it follows that $\nvdash_{\mathbf{L}} {\sim}B$ since we have assumed that $\vdash_{\mathbf{L}} B$. Since A and B do not share any propositional variable and $\nvdash_{\mathbf{L}} {\sim}B$, it follows that $\nvdash_{\mathbf{L}} B \to A$, and therefore also that $\nvdash_{\mathbf{L}} B \to C$ which ends the proof. □

RM satisfies (QVSP) [cf. 6, p. 417], and since it is theorem-wise consistent, it follows from the above theorem that it also satisfies the beef-up version of Plumwood's anti-suppression principle. It does not, however, satisfy the strengthened principle of joint forces. Before I show this, note that

[12] See [10, § 1.5] and references therein for various deduction theorems for relevant logics.

the proof used in [16, thm. 2] to show that (JFP) is properly weaker than (QVSP) also shows that Π' augmented with the axiom $(A \to A) \to {\sim}({\sim}B \to B) \vee {\sim}(B \to {\sim}B)$ satisfies (SASP) without, then, satisfying (QVSP). We therefore have the following corollary:

Corollary 1. *(QVSP) is properly stronger than (SASP).*

Thus strengthening Plumwood's anti-suppression principle so as to have every formula insupppressible even in the context of a logical theorem does not seem to yield a substantially stronger relevance principle. The corresponding strengthening of the joint forces idea, however, does. I will first show that (SJFP) is not implied by (QVSP). Afterwards, however, it will be shown that it is too is properly weaker than the variable sharing property itself. To get going we first need a lemma which relies on that the *Kleene axiom* which in relevant contexts sometimes goes by the name *Unrelated Extremes*, namely

$$\text{(UE)} \quad A \wedge {\sim}A \to B \vee {\sim}B,$$

is a theorem of **RM** [cf. 24].

Lemma 1. $\vdash_{\mathbf{RM}} {\sim}A \;\&\; \vdash_{\mathbf{RM}} B \Longrightarrow\; \vdash_{\mathbf{RM}} A \to B$

Proof. Let ${\sim}A$, and B be any logical **RM**-theorem. Because of the Kleene axiom, we have that (1) $\vdash_{\mathbf{RM}} A \wedge {\sim}A \to B \vee {\sim}B$. Since both $B \to (B \to B)$ and ${\sim}B \to (B \to B)$ are logical theorems of **RM**, it follows that (2) $\vdash_{\mathbf{RM}} B \vee {\sim}B \to (B \to B)$. Furthermore, $B \to ((B \to B) \to B)$ is a logical theorem of **RM**, and since B was assumed to be so as well, it follows that (3) $\vdash_{\mathbf{RM}} (B \to B) \to B$. From (1), (2) and (3) it follows by transitivity that (4) $\vdash_{\mathbf{RM}} A \wedge {\sim}A \to B$. Now ${\sim}A \to (({\sim}A \to {\sim}A) \to {\sim}A)$ is an axiom of **RM**, and since ${\sim}A$ is an assumed theorem, so is $({\sim}A \to {\sim}A) \to {\sim}A$. The negation axioms then easily yield that $\vdash_{\mathbf{RM}} (A \to A) \to {\sim}A$ which coupled with the mingle axiom yield that $\vdash_{\mathbf{RM}} A \to {\sim}A$, and therefore that $\vdash_{\mathbf{RM}} A \to A \wedge {\sim}A$ which together with (4) yield $\vdash_{\mathbf{RM}} A \to B$. □

Let A be any formula and p any propositional variable which does not occur in A. Since **RM** satisfies (QVSP) and neither p nor ${\sim}p$ are logical theorems, it follows that both $p \to A$ and $A \to p$ fail to be theorems of **RM**. It follows then, that **RM** does not have *ubiquitous* formulas in the sense of [26] which we looked at in sect. 3—in the Routley-Meyer semantics for the logic there is for every formula A a model in which A holds true at some point, and a model in which A fails to hold true at some point since for every formula A. Despite this, however, the above lemma shows that **RM** does validate "mild" instances of weakening: it validates the admissibility-version of the weakening rule with regards to the negation of any other theorem. This cannot be the case if a logic is to validate the strengthened version of Plumwood's joint forces principle:

Corollary 2. ***RM*** *does not satisfy (SJFP).*

Proof. Let $\sim A$ be any logical theorem of **RM**. According to (SJFP) there must be a logical theorem B and a formula C such that $\vdash_{\mathbf{RM}} A \wedge B \to C$, but $\nvdash_{\mathbf{RM}} A \to C$. From Lem. 1, however, it follows that $\vdash_{\mathbf{RM}} A \to B$, and therefore that $\vdash_{\mathbf{RM}} A \to A \wedge B$ which by transitivity yields that $\vdash_{\mathbf{RM}} A \to C$. It follows, then, that **RM** does not satisfy (SJFP). □

Theorem 6. *(VSP) implies (SJFP).*

Proof. For (SJFP) to fail to hold, there must be a formula A such that for every formula B and C, if $\vdash_{\mathbf{L}} A \wedge B \to C$ and $\vdash_{\mathbf{L}} B$, then either $\vdash_{\mathbf{L}} A \to C$ or $\vdash_{\mathbf{L}} B \to C$. Let B be $p \to p$ for some propositional variable not occurring in A and let C be $A \wedge B$. $\vdash_{\mathbf{L}} A \to C$ or $\vdash_{\mathbf{L}} B \to C$ implies that $\vdash_{\mathbf{L}} A \to B$ or $\vdash_{\mathbf{L}} B \to A$, and since A and B do not share a propositional variable, it follows that **L** cannot satisfy (VSP). □

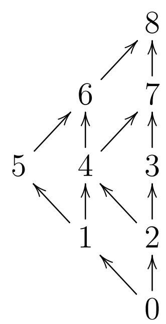
Thus Plumwood's principle can naturally strengthened so as to yield a stronger principle than here principle of joint forces. (SJFP) isn't strong enough, however; at least not strong enough to yield the variable sharing property as derivable, which Plumwood and Sylvan thought it ought to be from "a good sufficiency relation" [21, p. 3]. Quite the contrary, in fact. That (VSP) is *properly* stronger than (SJFP) follows by noting as the next lemma does that **RUE**—**R** augmented by unrelated extremes axiom (UE) above—satisfies this stronger property without—obviously—satisfying the variable sharing property.

Lemma 2. ***RUE*** *satisfies (SJFP).*

Proof. Let A be any formula. We must show that there is a **RUE**-theorem B and a formula C such that $A \wedge B \to C$ is a logical theorem while $A \to C$ and $B \to C$ are not. Let B be $p \to p$ where p is some propositional variable which does not occur in A, and let C be $A \wedge B$. It suffices, then, to show that there is a model in which both $A \to B$ and $B \to A$ fails to hold.

With that in mind, note that the model in Fig. 1 is a model for **RUE**.[13] Now assign to every propositional variable occurring in A the value 4. It is easy to check that $\{4\}$ is closed under every propositional function, and so A will be evaluated to 4. By assigning p to 5, B will be evaluated to 5 as well. Inspecting the implicational matrix shows, then, that both $A \to B$ and $B \to A$ fails to hold in this model. □

$\mathcal{T} = \{1, 4, 5, 6, 7, 8\}$



$\to$	0	1	2	3	4	5	6	7	8	$\sim$
0	8	8	8	8	8	8	8	8	8	8
1	0	1	2	3	4	5	6	7	8	7
2	0	0	1	5	4	0	4	6	8	6
3	0	0	0	5	0	0	0	5	8	5
4	0	0	0	0	4	0	4	4	8	4
5	0	0	0	3	0	5	5	3	8	3
6	0	0	0	0	0	0	1	2	8	2
7	0	0	0	0	0	0	0	1	8	1
8	0	0	0	0	0	0	0	0	8	0

Figure 1: A model for **RUE**

Corollary 3. *(VSP) is* properly *stronger than (SJFP).*

Proof. Immediate from Thm. 6 and Lem. 2. □

We saw above that (QVSP) does not imply (SJFP). It seems rather unlikely that the latter should imply the former, but it is not at all evident what kind of logic would satisfies (SJFP), but not (QVSP). On that note I would like to point out that it is unsettled whether **RUE** and its stronger sibling **R** augmented by Dummett's axiom—both of which are sublogics of **RM** [cf. 24, § 3]—satisfy (QVSP) or not. If, then, it should turn out that **RUE** fails to satisfy this property, it would follow that (SJFP) and (QVSP) are truly incomparable properties.

[13] All models displayed in this paper have been found with the aid of `MaGIC`—an acronym for *Matrix Generator for Implication Connectives*—which is an open source computer program created by John K. Slaney [25].

There are other ways to strengthen Ackermannian and Plumwoodian principles that lead to stronger properties. One idea for strengthening (SJFP) would be to require that C also be a logical theorem. Another would be to take note of Plumwood's claim that the logical law $(A \wedge B \to C) \to (A \to C) \vee (B \to C)$ "which for every proposition denies the Joint Force Principle" [19, § II][14] and then restate the joint force principles as having $\nvdash_{\mathbf{L}} (A \to C) \vee (B \to C)$ instead of the conjunctive clause that $\nvdash_{\mathbf{L}} A \to C$ and $\nvdash_{\mathbf{L}} B \to C$. Similarly, since Dummett's axiom—as the MacColl war-earthquake quote testifies to—was already at the time acknowledge as an implicational paradox, it is not unlikely that Ackermann's fourth idea was rather intended as claiming that there for every formula A must be a formula B such that $\nvdash_{\mathbf{L}} (A \to B) \vee (B \to A)$.[15] Or one might even strengthen Ackermann's idea to yield that every formula should be *logically independent* from some formula, i.e. that

$$\nvdash_{\mathbf{L}} (A \to B) \vee (A \to {\sim}B) \vee (B \to A) \vee ({\sim}B \to A)$$

should rather hold. It is easily verified, however, that the model in Fig. 1 is such as to verify such stronger properties as well, and so these ways of strengthening the Ackermannian and Plumwoodian principles also fail to yield the variable sharing property as a corollary. The next section, however, shows forth more successful such strengthenings.

This section has shown that Plumwood's idea of suppression is equivalent to that of Ackermann which by itself is nicely surprising fact. Plumwood and Sylvan thought of logical consequence as a sufficiency relation which, then, couldn't validate any suppression principles. Indeed such a relation would, according to them, yield the variable sharing property as a "derivable feature" They furthermore claimed that one by eliminating suppression would thereby also eliminate the implicational paradoxes [cf. 21, p. 359]. Neither of these claims are verified by cashing out suppression freedom using either the original, or the strengthened formal properties obtained from Ackermann's and Plumwood's motivational ideas of entailment: we have seen that these properties can hold true yet allow $A \wedge {\sim}A$ to entail $B \vee {\sim}B$. The next section tries out a different idea of how to strengthen the Ackermannian and

[14]The claim is also to be found in [21, p. 269, fn. 1].

[15]The variable sharing property does somewhat surprisingly not rule out Dummett's axiom. See the appendix for a strong sublogic of $\mathbf{R}$ augmented by Dummett's axiom which satisfies the variable sharing property.

Plumwoodian principles which will be shown to yield properties equivalent with Belnap's variable sharing property.

5 Insuppressible topics

The reason, according to Ackermann, a well-formed formula such as $A \rightarrow (B \rightarrow B)$ should not be a theorem of the theory of rigorous implication is that A and B can express unrelated propositions. We saw that Ackermann appealed to a mereological part-whole relation on propositional content to give some content to the idea that rigorous implication involves a non-trivial meaning-relation of some sort. I will suggest, however, that a better way of obtaining an intuitive motivation for the theory of rigorous implication, and theories in its vicinity, is by appealing to the notion of a *topic* or *subject-matter*. It is because p and q can be taken to express propositions *about* different topics that the relation of rigorous implication cannot obtain between $p \rightarrow p$ and $q \rightarrow q$. For instance, the subject-matter-overlap of "Socrates is a man" and that of "that 'comets move in elliptical orbits' entails that 'comets move in elliptical orbits'" What happens, then, if we take the Ackermannian dictum and update it to the requirement that the theory of rigorous implication does not encompass formulas implicationally related to formulas expressing different topics? Or if we update Plumwood's idea of joint forces to the effect that every formula have joint irreducible consequences with topically unrelated formulas? The problem, to again quote Belnap, is to restate this condition in formal terms [cf. 8, p. 144].

Now theories of topics diverge quite radically on just *what* topics are. [14], for instance, cashes out topics as equivalence classes of possible worlds. The metaphysical side to topics will not concern us in this paper, nor will I attempt a general discussion of theories of topics.[16] I will follow the recent and topically related [9] in regarding the topic of a formula as the fusion of the topics of its atomic subformulas. If A and B, then, are about non-overlapping topics, then so will A and any subformula C of B. This, then, is the guiding idea behind the definitions and results in this section.

Definition 12.

$$\begin{aligned} \mathbf{Var}(A) &=_{df} \textit{the set of propositional variables occurring in } A \\ \mathbf{WFF}(A) &=_{df} \textit{the set of well-formed formulas generable from } \mathbf{Var}(A) \end{aligned}$$

[16]See [11] for a good discussion and overview over theories of topics.

Now if $C \in \mathbf{WFF}(B)$, C must be about some suptopic of B. If, then, some such C entails A, if, that is, $C \to A$ is a logical truth, then B must be topically connected to A somehow. The guiding idea in the following is that it should also be the case that if *all* such C's fail to yield A—if $C \to A$ fails to be a logical truth for every $C \in \mathbf{WFF}(B)$—then B is topically unconnected to A.[17]

Definition 13 (Topically Unconnected)**.** *B is* TOPICALLY UNCONNECTED *to A given the logic $\boldsymbol{L}$,*

$$TopUCon_{\mathbf{L}}(B, A) =_{df} \forall C(C \in \boldsymbol{WFF}(B) \Rightarrow \nvdash_{\mathbf{L}} C \to A)$$

This property is easily seen to be a strengthened version of the Ackermannian property (TWF) to the effect that ever formula fails to be rigorously implied by some formula. It's topical sibling is then as follows:

Definition 14. *A logic* **L** *has the property of* TOPICAL WEAKENING FAILURE (TOPWF) *just in case*

$$\forall A \exists B(TopUCon_{\mathbf{L}}(B, A)).$$

Theorem 7. *(TopWF) implies (VSP) for any logic* $\boldsymbol{L}$ *for which the meta-rule of uniform substitution—that if* $\vdash_{\mathbf{L}} A$*, then also* $\vdash_{\mathbf{L}} A[p/q]$*, where* $A[p/q]$ *is the formula A in which the propositional variable p is everywhere substituted by q—holds.*

Proof. Assume that **L** satisfies (TopWF), and for contradiction that **L** does not satisfy (VSP). Then there are formulas A and B such that $\mathbf{Var}(A) \cap \mathbf{Var}(B) = \varnothing$ and $\vdash_{\mathbf{L}} A \to B$. Since **L** satisfies (TopWF) there is a formula C such that $\nvdash_{\mathbf{L}} D \to B$ for every $D \in \mathbf{WFF}(C)$.

Let q by some propositional variable such that $q \in \mathbf{Var}(C)$, and assume that $\mathbf{Var}(A) = \{p_{a_1}, \ldots, p_{a_n}\}$. For any formula E, let $E^q =_{df} E[p_{a_1}/q, \ldots, p_{a_n}/q]$ where this latter formula, then, is simply D where each propositional variable occurring in A has been replaced by q if it occurs in E. It follows that

[17] If *unconnected* seems too strong a term to cover this notion, the reader is of course welcome to substitute it for some other. One could, however, define topically unconnectedness as

$$\forall A' \forall B'((A' \in \mathbf{WFF}(A) \;\&\; B' \in \mathbf{WFF}(B)) \Rightarrow \nvdash_{\mathbf{L}} B' \to A')$$

which would do the same job as the definition of topically unconnectedness used here.

$A^q \in \mathbf{WFF}(C)$, and since $\mathbf{Var}(A) \cap \mathbf{Var}(B) = \varnothing$, that $(A \to B)^q = A^q \to B$. Since $\mathbf{L}$ is assumed to satisfy the meta-rule of uniform substitution, it follows from the assumption that $\vdash_{\mathbf{L}} A \to B$, that also $\vdash_{\mathbf{L}} A^q \to B$ which contradicts the fact that $\nvdash_{\mathbf{L}} A^q \to B$ since $A^q \in \mathbf{WFF}(C)$. □

Theorem 8. *(VSP) implies (TopWF).*

Proof. Assume that $\mathbf{L}$ satisfies (VSP). Let A be any formula, and let B be any formula with no propositional variables in common with A. Since $\mathbf{L}$ satisfies (VSP) it follows that $\nvdash_{\mathbf{L}} B \to A$. Let C be any formula in $\mathbf{WFF}(B)$. Since the propositional variables occurring in C are amongst $\mathbf{Var}(B)$, it follows that neither C shares any propositional variables with B. Since $\mathbf{L}$ satisfies (VSP) it follows, then, that also $\nvdash_{\mathbf{L}} C \to A$, and therefore that $\mathbf{L}$ satisfies (TopWF). □

Satisfying uniform substitution is a reasonable requirement on being a logic to begin with, and so it follows that one may for extensional purposes identify the topical weakening failure property and the well-known variable sharing property. How about, then, Plumwood's suppression ideas? One way to strengthen the joint forces idea would be as the claim that every formula have non-reducible joint consequences with some topically unconnected formula:

Definition 15. *A logic* $\mathbf{L}$ *has* Topically Unconnected Forces (TUF) $=_{df}$

$$\forall A \exists B \exists C (TopUCon_{\mathbf{L}}(B, A)\ \& \\ \vdash_{\mathbf{L}} A \wedge B \to C \quad \& \nvdash_{\mathbf{L}} A \to C \quad \& \nvdash_{\mathbf{L}} B \to C)$$

Lemma 3. *(TUF) implies (TopWF).*

Proof. Trivial. □

To show that (TUF) is in fact equivalent to the variable sharing property, I will show that it is implied by the topical version of the Ackermannian property (UTDWF)—that for every formula A there is some formula B such that neither does A entail B, nor does B entail A. The variable sharing property, it will be shown, implies the topical version of this Ackermannian property, which by transitivity, then, yields that it implies (TUF).

Definition 16 (Topically Unrelated)**.** *A and B are* TOPICALLY UNRELATED *given the logic* $\boldsymbol{L}$*,*

$$TopURel_{\mathbf{L}}(A, B) =_{df} TopUCon_{\mathbf{L}}(A, B) \ \& \ TopUCon_{\mathbf{L}}(B, A).$$

Definition 17. *A logic* **L** *has the property of* TOPICAL ACKERMANNIAN UNRELATEDNESS (TAU) *just in case*

$$\forall A \exists B(TopURel_{\mathbf{L}}(A, B)).$$

Lemma 4. *(TAU) implies (TUF).*

Proof. Let A be any formula. From (TAU) it follows then that there is some formula B such that both $TopUCon_{\mathbf{L}}(A, B)$ and $TopUCon_{\mathbf{L}}(B, A)$. We must show that there is some C such that all of $\vdash_{\mathbf{L}} A \wedge B \rightarrow C \nvdash_{\mathbf{L}} A \rightarrow C$, and $\nvdash_{\mathbf{L}} B \rightarrow C$) hold true.

Let C be $A \wedge C$. Since $A \wedge B \rightarrow C$ is a logical axiom, the the first task is done. If $\vdash_{\mathbf{L}} A \rightarrow C$, then also $\vdash_{\mathbf{L}} A \rightarrow B$, which cannot be the case since $A \in \mathbf{WFF}(A)$ and $TopUCon_{\mathbf{L}}(A, B)$ this cannot be the case. Similarly, if $\vdash_{\mathbf{L}} B \rightarrow C$, then also $\vdash_{\mathbf{L}} B \rightarrow A$ which is ruled out since $TopUCon_{\mathbf{L}}(B, A)$. □

Lemma 5. *(VAR) implies (TAU).*

Proof. Similar to Thm. 8. □

Corollary 4. *(VAR), (TAU), (TUF) and (TopWF) are all equivalent properties.*

Proof.

- (VAR) implies (TAU) (Lem. 5).
- (TAU) implies (TUF) (Lem. 4).
- (TUF) implies (TopWF) (Lem. 3).
- (TopWF) implies (VAR) (Thm. 7).

□

This, then, goes to show that Ackermann's and Plumwood's early motivational ideas of relevance can in fact be used to carve out the same formal notion of relevance that Belnap's variable sharing property does. This goes some way towards backing up Plumwood and Sylvan's claim that the variable sharing property is but a *derivative feature* of a truly non-suppressive account of entailment [cf. 21, p. 3]. It should be noted, however, that the real oomph of (TUF), for instances, comes from the topically unconnectivity requirement, not the joint forces part of the property. Of course, it may be possible to come up with other generalizations of these properties with a more "active" suppression-part.

What these relevance properties all fail to do, however, is to, as Anderson and Belnap had hoped, allow one to "arrive *precisely* at the formal system E of logical implication [...] or *entailment*" [5, p. 722], or to show that permutation laws are incorrect as Ackermann though, and *suppressive* as Plumwood and Sylvan thought. They are, as was evident from the get-go with regards to the variable sharing property, at best necessary criteria for a theory of entailment for which the premises must be content-related—be it cashed out in terms of meaning or in terms of topicology—to the conclusion. What they do, however, is to provide a different context in which to ground the variable sharing property, namely the theory of topics.

6 Summary

This paper has shed light on two different ways of motivating a conditional theory of entailment in the tradition of relevant logics. The first goes back to Ackermann's essay—*Begründung Einer Strengen Implikation*—in which Ackermann motivates his *rigorous implication*-conditional as one which does not encompass statements which rigorously imply or are implied by every other. This is but one of four features Ackermann mentions to motivate his theory. The other three are that the conditional expresses some sort of logical connection, non-reducible to the "correctness or falsity" of its antecedent and consequent, and lastly that the content of the latter be part of the content of the former. These two latter ideas were picked up by Anderson and Belnap. The latter was specified as the requirement that a conditional expressing entailment must be such as to ensure that the meaning of the antecedent is connected to the meaning of the consequent, given, that is, that the former does indeed entail the latter. Belnap, then, suggested the variable sharing property as a formal property to ensure such meaning connectedness. The logical connectedness-feature was rebranded as the "use-condition", that A must be *used* in obtaining B for A to entail B. Anderson and Belnap, however, never touched on Ackermann's fourth explication of the concept of rigorous implication. They did, however, provide a motivation for their theory—the theory of *entailment* as they preferred—which they did not trace back to Ackermann, namely as *the* theory which could uphold the difference between a valid argument and enthymemes. Enthymematically valid arguments generally allow one to suppress certain true premises, but Anderson and Belnap thought that truths in general—be they merely, necessarily or even logically true—are generally *not* suppressible. They then speculated that their account—the logic **E**—of entailment would turn out to be *the* account which could differentiate between valid argument and enthymemes for the three truth-modes of suppression.

In discussing the selection criteria for the theory of entailment, Anderson and Belnap were keen to stress the importance of *formal* properties. However, they never tried to come up with a formal criteria to identify suppression. This is one of the important contributions of Plumwood. In her conference paper *Some false laws of logic*, Plumwood presents two properties meant as adequacy-criteria for a theory of entailment for which suppression of premises is argued to be lead to false entailment-claims. Plumwood's two criteria turn out to be equivalent to two criteria extracted using Ackermann's fourth explication of his rigorous implication.

These Ackermannian and Plumwoodian principles turn out to be properly weaker than Belnap's variable sharing property. Taking inspiration from the theory of topics, it was shown, however, that it is possible to strengthen these principles. The resultant properties were shown to be equivalent to the variable sharing property. Even though, then, they fail to deliver what Anderson and Belnap had hoped an account of suppression would deliver—namely the logic **E**—and furthermore fail to show up permutation laws as fundamentally suppressive principles which Plumwood together with Sylvan claimed, these principles do show that it is possible to motivate the property in a slightly different way than what Belnap did, namely as a consequence of the inexistence of a sentence topically related to ever other.

Appendix: The variable sharing property for a strong logic with Dummett's axiom

One of the classical examples of an implicational paradox is that, to re-quote Russell, "that of any two propositions there must be one which implies the other"[23, §16]. We have already seen that MacColl thought this to be false—war in Europe, according to [15], does not imply a disastrous earthquake happening the same year, nor does such an earthquake imply war. Thus MacColl would deny the truth of one of the logical theorems of **RM**, namely *Dummett's axiom*:

$$(\mathbf{D}) \quad (A \to B) \vee (B \to A).$$

According to Anderson and Belnap, Dummett's axiom brings the implicational theory of **RM** so close to the dreaded material "implication" [cf. 6, p. 429] that, despite it's nice formal properties, has no claim to capture the "if. . . then__"-locution. Beyond these rather inconclusive reasons, Anderson and Belnap never seem to have given something closer to an argument as to why Dummett's axiom should fail.

Anderson and Belnap thought of the variable sharing property as a safeguard against implicational paradoxes in the strong sense that if a logic has that property, then it will not validate any implicational paradoxes. But is this true? Now Dummett's axiom does imply Kleene's axiom—$A \wedge {\sim}A \to B \vee {\sim}B$—and so even though it itself does not directly violate the requirement of the variable sharing property, it seems to entail the existence of such a violation. An argument in line with Anderson and Belnap's view, then, for why Dummett's axiom cannot hold true is that it entails such a violation. And indeed it does:

Theorem 9. *Any logic extending* ***BBD****—****BB*** *augmented by Dummett's axiom—for which the meta-rule of reasoning by cases holds, has the Kleene axiom as a logical theorem.*

Proof. For any formulas A and B, it is easy to derive $A \wedge {\sim}A \to B \vee {\sim}B$ from $B \vee {\sim}B \to A \wedge {\sim}A$. Thus the Kleene axiom follows using reasoning by cases and the contraposition rule together with the following instance of Dummett's axiom

$$(A \wedge {\sim}A \to B \vee {\sim}B) \vee (B \vee {\sim}B \to A \wedge {\sim}A).$$

The interesting part of the proof goes as follows:

(1)	$B \vee \sim B \to A \wedge \sim A$	assumption
(2)	$B \to B \vee \sim B$	axiom
(3)	$A \wedge \sim A \to \sim A$	axiom
(4)	$B \to \sim A$	1–3 transitivity
(5)	$A \to \sim B$	4, contraposition rule
(6)	$A \wedge \sim A \to A$	axiom
(7)	$\sim B \to B \vee \sim B$	axiom
(8)	$A \wedge \sim A \to B \vee \sim B$	5–7 transitivity

□

The purpose of this section is to show that the variable sharing property can hold true in quite strong logics, both in terms of implicational validity, but also in terms of negation laws, despite validating Dummett's axiom and the meta-rule of reasoning by cases. That Dummett's axiom is compatible with the variable sharing property was to my knowledge first shown in [20, prop. 6.19], but therein for a rather weak logic. The following model shows that one may add Dummett's axiom to even **R** if, basically, the contraposition axiom is but weakened slightly.

Let $\mathbf{RD^{t+}}$ be the positive fragment of the logic **R**, but augmented by Dummett's axiom. $\mathbf{RD^{t+}}$ is defined to have the truth-constant known as the *Ackermann constant* which in the current case we can axiomatize using the two axiom $A \to (\mathbf{t} \to A)$ and $(\mathbf{t} \to A) \to A$. This truth constant is added so as to have the negation-laws statable as *axioms* and not as mere rules. The model displayed in Fig. 2 is a model for $\mathbf{RD^{t+}}$. To see that the model can be used to show that $\mathbf{RD^{+t}}$ has the variable sharing property, let A and B be any two negation- and **t**-free formulas which share no propositional variables. Assign every propositional variable in A to 3 and every propositional variable in B to 2. Then it is easy to verify that A will be evaluated to 3 and B to 2, and therefore that $[\![A \to B]\!] = 0$. Thus $A \to B$ is not a logical theorem of $\mathbf{RD^{t+}}$. Now the model displays two different matrices for how to evaluate $\sim$. Using the $\sim_1$-matrix we obtain a model for $\mathbf{RD^{t}}$—**R** augmented by Dummett's axiom as well as the Ackermann constant. As such, however, it also validates $p \wedge \sim_1 p \to q \vee \sim_1 q$. This formula is in fact a theorem of **RD** [cf. 24, § 3], and so this logic obviously fails to satisfy the variable sharing property. However, if we rather use the $\sim_2$-matrix to evaluate the negation, this is no longer the case. In fact, since both $\sim_2 3 = 3$ and $\sim_2 2 = 2$ the proof of the variable sharing property extends to formulas with negation. The question, then, is which negation laws hold true in the model. The following formulas are easily verified to hold in the model.

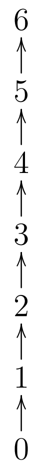
$$\begin{array}{l}
A \leftrightarrow B =_{df} (A \to B) \land (B \to A) \\
A \mapsto B =_{df} A \land \mathbf{t} \to B \\
A \leftrightsquigarrow B =_{df} (A \mapsto B) \land (B \mapsto A) \\
\hline\hline
A \leftrightarrow {\sim}{\sim}A \\
\hline
{\sim}(A \land B) \leftrightsquigarrow ({\sim}A \lor {\sim}B) \\
{\sim}(A \lor B) \leftrightsquigarrow ({\sim}A \land {\sim}B) \\
\hline
(A \to A) \mapsto ({\sim}A \to {\sim}A) \\
(A \to B) \mapsto ({\sim}B \mapsto {\sim}A) \\
(A \to {\sim}B) \mapsto (B \mapsto {\sim}A) \\
({\sim}A \to B) \mapsto ({\sim}B \mapsto A) \\
({\sim}A \to {\sim}B) \mapsto (B \mapsto A) \\
\hline
A \lor {\sim}A \\
(A \to {\sim}A) \to {\sim}A \\
A \mapsto {\sim}(A \to {\sim}A) \\
(A \to B) \to ({\sim}A \lor B) \\
(A \land {\sim}B) \mapsto {\sim}(A \to B) \\
\hline
{\sim}A \leftrightsquigarrow (A \to {\sim}\mathbf{t})
\end{array}$$

Note that $\mapsto$ is weaker than $\to$: $(A \to B) \to (A \mapsto B)$ holds, but the antecedent can hold true without the antecedent: $2 \mapsto 1$ holds true in the displayed model, but $2 \to 1$ does not. The *enthymematic* conditional $\mapsto$ can be viewed as a sort of object-language representative of the Hilbert consequence relation, seeing as the *enthymematical* deduction theorem—-that $\Gamma, A \vdash B \iff \Gamma \vdash A \mapsto B$—also holds for $\mathbf{RD^{t+}}$ augmented by any of the negation laws above.[18]

All the negation axioms listed above hold true in **R** in the stronger form where $\mapsto$ is everywhere replaced by $\to$. In all these axioms, however, it can be verified that the model does not hold if $\mapsto$ is replaced by $\to$. Even so, the list of negation principles which do hold true in the model is quite extensive as it includes both double negation, distribution of negation over conjunction and disjunction, contraposition axioms, excluded middle and reductio-axioms as well as the interderivability of the negation of A and it implying the negated

[18]The proof is a simply induction on the length of proof, and so I leave the proof to the reader. Using this theorem it is easily shown that reasoning by cases also holds true.

Ackermann constant. Thus $\mathbf{RD}^{\mathbf{t}+}$ can be augmented by quite strong negation laws while still validating the variable sharing property.



$\mathcal{T} = \{1, 2, 3, 4, 5, 6\}$
$[\![\mathbf{t}]\!] = 1$

$\to$	0	1	2	3	4	5	6	$\sim_1$	$\sim_2$
0	6	6	6	6	6	6	6	6	6
1	0	1	2	3	4	5	6	5	5
2	0	0	2	3	4	4	6	4	2
3	0	0	0	3	3	3	6	3	3
4	0	0	0	0	2	2	6	2	4
5	0	0	0	0	0	1	6	1	1
6	0	0	0	0	0	0	6	0	0

Figure 2: A model for $\mathbf{RD}^{\mathbf{t}}$

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