Understanding Christine Ladd-Franklin's Logic

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Abstract
In the late 19^{th} century Christine Ladd-Franklin proposed a new logical system in the algebraic tradition championed by Boole, Jevons, Schröder, and her teacher Charles Sanders Peirce. This new logic was at the time celebrated as providing a novel and complete characterization of the valid syllogisms, although Ladd-Franklin's work was largely forgotten until recently. Here we present a careful reconstuction of Ladd-Franklin's work, concentrating on her characterization of the valid syllogisms, and we clear up some earlier confusions regarding how this novel logical system works.

1 Introduction

In "On the Algebra of Logic" (Ladd 1883), Christine Ladd Franklin develops a new variant of the algebraic approach to logic pioneered by George Boole, Hermann Grassman, William Stanley Jevons, Hugh MacColl, Ernst Schröder, and Charles Sanders Peirce.[1] This logic allowed her, among other things, to provide an elegant characterization of the valid syllogisms (although, as we shall see, we will need to be careful regarding exactly what Ladd-Franklin meant by "syllogism" and how her understanding diverged from the Aristotelian tradition). This work was fairly well-known during her lifetime. For example, Eugene Shen concluded a survey of Ladd-Franklin's logic with the following comments:[2]

[1] "On the Algebra of Logic" was published shortly after Ladd-Franklin was married in 1882, and hence appeared under the name "Christine Ladd". In what follows I will reference this paper using the name under which it was published, but will refer to Ladd-Franklin herself as, well, Ladd-Franklin. Although we will for the most part be focusing on Ladd-Franklin's explication of the logic in (Ladd 1883), her novel formalization is further explored in (Ladd-Franklin 1889), (Ladd-Franklin 1890), (Ladd-Franklin 1912), and (Ladd-Franklin 1928).

[2] Ladd-Franklin called her **Theorem II** the *inconsistent triad* in earlier work, and the *antilogism* in later work. In order to avoid confusion, we shall just call it **Theorem II** throughout.

> The Antilogism was at first called by Dr. Ladd-Franklin the "inconsistent triad"; apropos of it the late Professor Josiah Royce of Harvard was in the habit of saying to his classes: "There is no reason why this should not be accepted as the definitive solution of the problem of the reduction of syllogisms. It is rather remarkable that the crowning activity in a field worked over since the days of Aristotle should be the achievement of an American woman." (Shen 1927, 60)

As the 20^{th} century progressed, however, Ladd-Franklin's work was largely forgotten, only re-entering discussions on the history of turn-of-the-century logic in the last two decades as a result of studies such as (Russinoff 1999) and (Uckelman 2021). The present essay is meant to build on this work, providing a detailed summary of Ladd-Franklin's logic and a careful analysis of exactly what Ladd-Franklin's logic was designed to do.

We will proceed as follows. In §2 we will look at the what we might call the "traditional" portion of Ladd-Franklin's system – that is, the operators, axioms, and rules that would be relatively familiar to her contemporaries and predecessors in the algebraic tradition. Then, in §3, we will explore the logic of her two new operators $\vee$ and $\overline{\vee}$ and the axioms and rules that govern them. Much of this material in these sections will be familiar from earlier studies (especially, as already mentioned, (Russinoff 1999) and (Uckelman 2021)). But we will also, as we go along, discuss some issues not addressed in earlier work (such as the fact that Ladd-Franklin takes propositions to be a special kind of class term), and we will also clarify some confusions in earlier discussions (such as the fact that Ladd-Franklin, following De Morgan, understands syllogisms to involve eight different types of "categorical" proposition, rather than four). In §4 we will then move into more novel territory, by reconstructing the proofs of Ladd-Franklin's two main theorems (previous studies merely claim that Ladd-Franklin had such a proof, but do not investigate how her rather cryptic sketch of the proof in question can be turned into a genuine deduction within her novel logical system). In this section we will also provide a more detailed and complete explanation of the way in which the second of these two theorems is used by Ladd-Franklin to provide not only a complete characterization of the valid syllogisms (on her wider understanding of this notion), but also a mechanical procedure for enumerating all of the valid syllogisms. Finally, in §5, some confusions in earlier literature – in particular, in (Uckelman 2021) and especially in (Russinoff 1999) – regarding Ladd-Franklin's logic will be cleared up.

2 The Traditional Part of Ladd-Franklin's Logic

The algebraic logics of Ladd-Franklin and her predecessors are often described as follows: The letters (A, B, C, etc.) and operations that apply to the values these letters take are systematically ambiguous: A and B can be used to represent classes (or, equivalently from a mathematical perspective, properties, attributes, or qualities), and they can also be used to represent propositions.[3] Operations like $^{-}$, $+$, and $\times$ are then used either to form more complex class terms (complement, union, and intersection respectively) or to form more complex propositions (negation, disjunction, and conjunction respectively). A *copula* of some sort ("=" in Boole's, Grassmann's, Jevons's, and Schröder's work, ":" in MacColl's work, "$\prec$" in Peirce's work, and "$\vee$" in Ladd-Franklin's work) is then used to form propositions about classes in the first instance, or propositions about propositions in the second.[4]

In the presentation of Ladd-Franklin's logic given in this essay, however, we will present the material solely in terms of the first, class/quality interpretation of the logic.[5] The reason is simple: Ladd-Franklin understood the propositional reading of letters in her logic as a special case of the class reading. In her discussion of the ∞ symbol she writes:[6]

> The symbol ∞ represents the universe of discourse. It may be the universe of conceivable things, or of actual things, or any limited portion of either. It may include non-Euclidean n-dimensional space, or it may be limited to the surface of the earth, or to the field of a microscope. It may exclude things and be restricted to qualities, or it may be made co-extensive with fictions of any kind. *In any proposition of formal logic, ∞ represents what is logically possible*; in a material proposition it represents what

[3]And, for some thinkers, such as Ladd-Franklin's mentor Peirce, the letters can also be used to represent probabilities.

[4]For details on these earlier copulas see (Boole 1854), (Grassmann 1872), (Jevons 1864), (Schröder 1877), (MacColl 1878), (Peirce 1867).

[5]Of course, interpreting letters as picking out classes, versus interpreting them as picking out qualities or properties or attributes, are quite different readings of the logic from a philosophical perspective, since, for example, these interpretations will likely involve different accounts of the metaphysics underlying the logic. The point here is that, from a purely *mathematical* perspective we can treat these interpretations as equivalent, since we can understand qualities or properties or attributes in terms of their extensions.

[6]It is also interesting to note that Ladd-Franklin is clearly describing a conception of logical calculi with varying domains.

> exists, (Peirce). The symbol 0 is the negative of the symbol ∞; it denotes either what is logically impossible, or what is non-existent in an actual universe of any degree of limitation. (Ladd 1883, 19, emphasis added)

And in her discussion of representing arguments in terms of her novel operators $\vee$ and $\overline{\vee}$ she writes:

> If a is a proposition, then $a\overline{\vee}$ states that the proposition is not true in the universe of discourse. For several propositions, $abc\overline{\vee}$ means that they are not all at the same times true; and the way in which they are stated to be not all at the same time true depends on the character of the universe. *If it be the universe of the logically possible, the* $p_1p_2\overline{c}$ *states that* p_1 *and* $_2$ *may be taken as the premises and c as the conclusion of a valid syllogism.* (Ladd 1883, 30, emphasis added)

As the italicized sentences makes clear, in a formula where the letters a and b are understood as propositions, and where the formula itself is intended to represent a valid argument (which will be the main case of interest in our analysis of Ladd-Franklin's account of syllogisms below), we are to understand a and b as denoting classes – namely, the class of logical possibilities where a holds and the class of logical possibilities where b holds respectively. Furthermore, we should understand any complex proposition (i.e., any formula governed by either $=$, $\vee$, or $\overline{\vee}$) as shorthand for a class term whose extension is a (possibly non-proper) subset of the collection of logical possibilities. Thus, all expressions in Ladd-Franklin's logic are, strictly speaking, class terms. In what follows, we will (following Ladd-Franklin) focus on the interpretation where the letters of the calculus denote classes (remembering that those classes can be classes of possibilities).

It is worth pausing here to clarify this aspect of Ladd-Franklin's logic a bit more. In a footnote in her analysis of Ladd-Franklin's dissertation, Russinoff writes that:[7]

[7]Landon Elkind and Richard Zach seem to fall prey to a similar misapprehension of Ladd-Franklin's understanding of universes of discourse when they write that:

> Ladd-Franklin (Ladd 1883, 25 – 26) used '$\vee$' and '$\overline{\vee}$' in her dissertation, but only for the copula, that is, for predication among classes *and not for propositional connectives.* So, Ladd says that '$A \vee B$' means "A is in part B", that is, "some A is B", and '$A\overline{\vee}B$' means "A is-not B", or "No A is B". The uses of '$\vee$' and '$\overline{\vee}$' correspond to, as Ladd-Franklin puts it, (Boolean class) "inclusions" and "exclusions" (Ladd 1883, 26). She deals, rather, with an

> Note that in Ladd-Franklin's system, symbols are used in various ways, and at times in the same formula. Here logical multiplication is used to conjoin propositions. We will see below that she uses her symbol $\overline{\vee}$ in two ways in her principle of the syllogism, for when a and b are classes, $a\overline{\vee}b$ means no a is b, and when a and b are propositions, $a\overline{\vee}b$ means *a and b are inconsistent.* (Russinoff 1999, 460, emphasis in original)

This, however, is a misunderstanding. It is not that the logical negation, logical addition, logical multiplication, and copula operators in Ladd-Franklin's logic are ambiguous – sometimes applying to class terms and sometimes applying to propositions. As we have just seen, propositions are, for Ladd-Franklin, just a special kind of class term, and hence the operators in the logic always apply to class terms. Rather, within a particular proposition, the universe of discourse relevant to the expression as a whole might be (in fact, often will be) different from the universe of discourse relevant to sub-expressions.

Sorting out exactly how the universe of discourse varies as we move from expression to expression, or in particular from an expression to one of its sub-expressions, is an important issue that deserves further attention. Here however, we have other fish to fry, so we note that for present purposes it is sufficient to observe that (i) propositions are, in Ladd-Franklin's logic, a special case of class terms which pick out possibilities, and, as a result, (ii) the relevant universe of discourse can vary as we shift our attention from an expression to one of its subexpressions. We (regretfully) must leave further examination of this fascinating aspect of Ladd-Franklin's logic for future work.

We now move on to Ladd-Franklin's primitive notation. The first important notion we need in order to formulate Ladd-Franklin's logic is the identity symbol (=). Ladd-Franklin writes that:

> The sign = is the sign of equality. $a = b$, a equals b, means that any logical expression a can be substituted for b, or b for a,

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> analogue of propositional disjunction from a Boolean perspective, where the symbol occurs between categorical terms or concepts rather than between truth-apt formulas. (Elkind & Zach 2022, 27, emphasis added)

While Elkind and Zach are correct that there seems to be no connnection between Ladd-Franklin's use of $\vee$ and $\overline{\vee}$ and the contemporary use of $\vee$ as a symbol for disjunction, it should be noted that, if we understand propositions as a special case of class terms, their (correct) observation that Ladd-Franklin's copula only applies to class-terms in no way implies that she does not treat these as (often but not always) propositional operators.

> without change of value. It is equivalent to the two propositions, "there is no a which is not b", and, "there is no b which is not a." (Ladd 1883, 17 – 18)

Thus, if A and B denote classes, then:

$$A = B$$

states that everything that is in A is in B (and vice versa). Equivalently (and perhaps more importantly), $A = B$ states that A can be interchanged *salve veritate* with B in any proposition (and again, vice versa).

Next up are three relatively familiar logical operators: logical negation ($^{-}$), logical addition ($+$), and logical multiplication ($\times$). Given a class term A, $\overline{A}$ denotes the class of things that are not A (relative to the chosen domain of discourse). Given class terms A and B, $A + B$ denotes the class of things that are either in A or in B, while $A \times B$ denotes the class of things that are both in A and in B. (Ladd 1883, 18) Following Ladd-Franklin, we will abbreviate $A \times B$ as AB. Additionally, we have two special class terms: ∞ and 0. ∞ denotes the class of all things (relative to our chosen domain of discourse), and 0 denotes its complement, the class containing no things.

Ladd Franklin provides the following principles governing these notions (Ladd 1883, 19):

$$
\begin{array}{rl}
(4') & aaa = a \\
(5') & abc = bca = cba \\
(6') & a(b + c) = ab + ac \\
(7') & a\overline{a} = 0 \\
(8') & a = a\infty = a(b + \overline{b})(c + \overline{c}) \dots \\
(9') & \infty = a + \infty = a + (b + \overline{b}) + \dots \\
(10') & ab + a\overline{b} + \overline{a}b + \overline{a}\overline{b} = (a + \overline{a})(b + \overline{b}) = \infty \\
(11') & a + ab + abc + \dots = a
\end{array}
$$

plus the following "duals" of the above list (Ladd 1883, 19):[8]

$$
\begin{array}{rl}
(4^\circ) & a + a + \cdots = a \\
(5^\circ) & a + b + c = b + c + a = c + b + a \\
(6^\circ) & a + bc = (a + b)(a + c) \\
(7^\circ) & a + \overline{a} = \infty \\
(8^\circ) & a = a + 0 = a + b\overline{b} + c\overline{c} \ldots \\
(9^\circ) & 0 = a0 = ab\overline{b}c\overline{c} \ldots \\
(10^\circ) & (\overline{a} + \overline{b})(\overline{a} + b)(a + \overline{b})(a + b) = a\overline{a} + b\overline{b} = 0 \\
(11^\circ) & a(a + b)(a + b + c) \cdots = a
\end{array}
$$

At this point, it is worth making some clarificatory comments regarding the role that the identity symbol plays in Ladd-Franklin's logic. As noted above, the identity symbol (or its negation) is the primary copula in the algebraic logics developed by Boole, Grassmann, Jevons, and Schröder. Ladd-Franklin argues at length that her copula $\vee$ (and its negation $\overline{\vee}$) is superior to the copulas mobilized by others. After presenting a list of problems associated with the various copulas used by Boole, Grassmann, Jevons, MacColl, Peirce, and Schröder (Ladd 1883, 24 – 5), Ladd-Franklin writes that:

> Another kind of copula is possible, – namely, one which is particular when positive, and universal when negative. [. . .] it will appear that this latter plan [Ladd-Franklin's $\vee$ and $\overline{\vee}$] has certain advantages. (Ladd 1883, 25)

In short, Ladd-Franklin takes her own logic, which she claims *replaces* the copulas used in previous work with $\vee$ (and $\overline{\vee}$) to be an improvement on those previous efforts (see, e.g., (Ladd 1883, 23 – 31)). If this is Ladd-Franklin's view, however, then the prominent role played by the identity symbol in the principles just listed might seem surprising, if not outright incoherent.

There is really no puzzle here, however – instead, we merely need to distinguish between *copulas* generally (which for Ladd-Franklin would include $=$ and both of $\vee$ and $\overline{\vee}$) and *the primary* copula(s),[9] where the latter is taken

[8] It is perhaps worth noting explicitly that in Ladd-Franklin's notation, $a + bc$ is to be read as $a + (b \times c)$, not as $(a + b) \times c$.

[9] Interestingly, as we shall see, Ladd-Franklin's addition of $\vee$ and $\overline{\vee}$ to her system allow her to avoid ever negating an identity. There seems to be no reason to think that:

$$\overline{A = B}$$

to be the operator(s) used to formulate logical principles such as validities, consistencies, and invalidities (see below). In Ladd-Franklin's theory, the primary copulas would be $\vee$ and $\overline{\vee}$. Thus, in Ladd-Franklin's logic, it is not the elimination of other copulas (such as identity) that is important, but rather the introduction of (and central role played by) the new copulas $\vee$ and $\overline{\vee}$ that distinguish it from other, (according to Ladd-Franklin in many ways inferior) systems.

We can now move on to one of the most powerful techniques used in "The Algebra of Logic" for generating new identities (and eventually, powerful theorems): the *complete development of* n *class terms*. Given n-many class terms $x_1, x_2, \ldots x_n$, the complete development of $x_1, x_2, \ldots x_n$ is the class term obtained by summing up the 2^n distinct class terms of the form (Ladd 1883, 20):

$$\alpha_1 \times \alpha_2 \times \cdots \times \alpha_n$$

where, for each k, $1 \leq k \leq n$, $\alpha_k = x_k$ or $\alpha_k = \overline{x_k}$. Thus:

$$AB + A\overline{B} + \overline{A}B + \overline{A}\,\overline{B}$$

is the complete development of the two class terms A and B. Ladd-Franklin uses the complete development of the class terms a and b to provide the first identities dealing with negations of complex class terms:

$$\text{(12)} \qquad \begin{aligned} \overline{ab} &= a\overline{b} + \overline{a}b + \overline{a}\overline{b} \\ \overline{ab + a\overline{b}} &= \overline{a}b + \overline{a}\overline{b} \\ \overline{ab + a\overline{b} + \overline{a}b} &= \overline{a}\overline{b} \end{aligned}$$

Put informally, an object is in the logical sum of a subsequence of the complete development of two terms (the right-hand-side) if and only if it is in the complement of the remaining elements from that complete development (the left-hand-side). While this provides us with three distinct identities, Ladd-Franklin is clear that they are all instances of a single process:

> ...the process is the same for the complete development of any number of terms. (Ladd 1883, 20)

The process in question can be described as follows (adapted from (Ladd 1883, 20)):

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is not well-formed. But no such negated identities appear anywhere in her work (other than in her discussion of the notations of her predecessors and contemporaries, of course).

Let:

$$\alpha_1 + \alpha_2 + \dots \alpha_{2^n}$$

be the complete development of n terms. Then, if:

$$\beta_1, \beta_2, \dots \beta_j$$
$$\delta_2, \delta_2, \dots \delta_k$$

are disjoint sequences of terms such that:

$$\{\beta_1, \beta_2, \dots \beta_j\} \cup \{\delta_2, \delta_2, \dots \delta_k\} = \{\alpha_1, \alpha_2, \dots \alpha_{2^n}\}$$

then:

$$\overline{\beta_1 + \beta_2 + \dots + \beta_j} = \delta_2 + \delta_2 + \dots + \delta_k$$

is a correct identity.

Interestingly, although Ladd-Franklin does not note this explicitly, the simplest instance of the complete development process provides us with a identity that is applied throughout (Ladd 1883) without comment: the elimination of double negations. Note that:

$$a + \overline{a}$$

is the complete development of a single class term a. Hence, we can correctly formulate a identity by placing $\overline{a}$ on the left of the identity symbol, within the scope of a second logical negation, and the remainder (i.e., a) on the right, obtaining:

$$\overline{\overline{a}} = a$$

Following Ladd-Franklin's lead, and in the interest of minimizing notational clutter, we will continue to apply this identity when appropriate (often, but not always, without comment).

The next set of identities that Ladd-Franklin provides for dealing with negations of complex class terms are more familiar, and Ladd-Franklin unsurprisingly attributes them to DeMorgan:

$$(13') \qquad \overline{ab} = \overline{a} + \overline{b}$$
$$\overline{a\overline{b}} = \overline{a} + b$$
$$(13^\circ) \qquad \overline{a + b} = \overline{a}\overline{b}$$
$$\overline{a + \overline{b}} = \overline{a}b$$

She also shows that these identities can be derived from the identities given earlier (including, crucially, the identities generated by the complete development method described above) (Ladd 1883, 21).

Finally, Ladd-Franklin provides the following identity:

$$(14) \quad \overline{pab + qa\bar{b} + r\bar{a}b + s\bar{a}\bar{b}} = \bar{p}ab + \bar{q}a\bar{b} + \bar{r}\,\bar{a}b + \bar{s}\,\bar{a}\bar{b}$$

She does not prove this from previous identities, but she does sketch how it can be justified in the same manner as the identities that we obtain from the complete development method.

This completes our presentation of the "traditional" part of Ladd-Franklin's logic. Over the next few pages of "On the Algebra of Logic" she discusses how the various notations used by other logicians in the algebraic tradition (e.g., Jevons, McColl, Peirce, and Venn) can be translated into her formalism. While interesting, this is not our main focus here, so we shall move on to the truly novel part of Ladd-Franklin's logic.

3 The Copula

To the above system, which would be relatively familiar to her contemporaries in the algebraic logical tradition of the time, Ladd-Franklin adds two new operators, which she defines as follows:

> The sign $\overline{\vee}$ is a wedge, sign of exclusion. $A\overline{\vee}B$ is to be read "A is-not B", or "A is excluded from B". The sign $\vee$ is an incompleted wedge, sign of incomplete exclusion. $A \vee B$ is to be read "A is in part B", or "A is not-wholly excluded from B". $\vee$ is made into $\overline{\vee}$ by the addition of the negative sign; what is not not wholly excluded from anything is wholly excluded from it. $A\overline{\vee}B$ and $A \vee B$ are contradictory propositions; each simply denies the other. (Ladd 1883, 25)

It is worth emphasizing that there is strictly speaking only one new operator being introduced here – the copula $\vee$. The negative version of this operator $\overline{\vee}$ is a *defined* notion – in short, we have:

$$A\overline{\vee}B =_{df} \overline{A \vee B}$$

Ladd-Franklin immediately notes that we can use the copula to express *eight* different propositions using class terms A and B, logical negation, and a single instance of either $\vee$ or $\overline{\vee}$ (Ladd 1883, 26):[10]

$A\overline{\vee}B$: A is-not B; no A is B.

$A \vee B$: A is in part B; some A is B.

$A\overline{\vee}\overline{B}$: A is-not not-B; all A is B.

$A \vee \overline{B}$: A is partly not-B; some A is not B.

$\overline{A}\overline{\vee}B$: What is not A is-not B; A includes all B.

$\overline{A} \vee B$: What is not A is in part B; A does not include all B.

$\overline{A}\overline{\vee}\overline{B}$: What is not A is-not not-B; there is nothing besides A and B.

$\overline{A} \vee \overline{B}$: What is not A is in part not-B; there is something besides A and B.

In short, Ladd-Franklin's copula allows us to express, not merely the four traditional "Aristotelian" propositions, but *eight* distinct forms of proposition that might appear in syllogisms. Expressed a bit more traditionally:

A :	$A\overline{\vee}\overline{B}$	All A is B.
∀ :	$\overline{A}\overline{\vee}B$	No not-A is B.
E :	$A\overline{\vee}B$	No A is B.
Ǝ :	$\overline{A}\overline{\vee}\overline{B}$	No not-A is not-B.
a :	$A \vee B$	Some A is B.
ɐ :	$\overline{A} \vee B$	Some not-A is B.
e :	$A \vee \overline{B}$	Some A is not-B.
ə :	$\overline{A} \vee \overline{B}$	Some not-A is not-B.

The labelling scheme using **A**, **E**, **a**, **e** and their upside-down variants is Ladd-Franklin's – see (Ladd-Franklin 1890, 79, 82). **a** is the proposition labelled **I** in traditional discussions of Aristotelian logic, and **e** is the proposition labelled **O** in such contexts.

Before moving on, it is important to note that, like most of her contemporaries in the 19^{th} century algebraic tradition, Ladd-Franklin did not take universal propositions (i.e., propositions of form **A**, **∀**, **E** and **Ǝ**) to have existential import. Thus, even if we restrict our attention to the traditional

[10]Ladd-Franklin notes that there is a sense in which there are only six distinct propositional forms here, since we can replace $\overline{A}\overline{\vee}B$ (line 5) with $B\overline{\vee}\overline{A}$ (an instance of line 3), and likewise we can replace $\overline{A}\vee B$ (line 6) with with $B\vee\overline{A}$ (an instance of line 4). Nevertheless, she uses all eight forms when characterizing the valid syllogisms – see §4 below.

Aristotelian propositions (that is, to **A**, **E**, **a**, and **e**), there are, as is now well-known, fifteen valid forms, not twenty-four.[11] This extension of the traditional Aristotelian system is important, since the theorem regarding classifications of syllogisms that Ladd-Franklin proves (and which we will discuss below) applies not just to traditional syllogisms involving propositions of the first four, traditional "Aristotelian" forms, but also applies to syllogisms involving propositions of these additional forms as well (see §4).

Ladd-Franklin then goes on to show how to represent validity and invalidity within her system. Given a premise p and a conclusion c, we can represent that claim the argument from p to q being valid as:

$$p\overline{\vee}\overline{c}$$

which "states that the premise and the denial of the conclusion can never go together" (Ladd 1883, 28), and we can express the claim that the argument is invalid as:

$$p \vee \overline{q}$$

which "states that the premise is sometimes accompanied by the falsity of the conclusion." (Ladd 1883, 28). Ladd goes on to emphasize the modal nature of the second claim, noting that it:

> . . . implies that both the premise and the negative of the conclusion *must*, at some time, be true. (Ladd 1883, 28, emphasis added)

The modal nature of propositions involving the copula (or its negation) is further emphasized in her formulation of *inconsistency* claims, which she introduces as follows:

> The argument $p\overline{\vee}c$ may be called an inconsistency. It is an argument into which the idea of succession does not enter; it simply denies the *possible* co-existence of two propositions. (Ladd 1883, 29, emphasis added)

[11] Ladd-Franklin does discuss developing syllogistic logic with existential import in her system (Ladd 1883, 43 – 45). The basic idea is to supplement conditionally valid syllogisms (those that depend on existential import) with an additional premise of the form:

$$aa\vee$$

In short, we supplement one of the universal claims regarding class term a with the claim that something is both a and a (and hence something is a). Detailed discussion of this approach (including a determination of the syllogisms including all *eight* categorical propositions that can be shown to be conditionally valid via an application of this trick) is, unfortunately, beyond the scope of the present essay.

At this point in the essay, Ladd-Franklin notes that we can (and thus often will) eliminate both 0 and ∞ from formulas of the calculus. In particular, she suggests that instead of writing:

$$x\overline{\vee}\infty$$

to express the claim that x cannot exist (or cannot be true), that we instead merely write $x\overline{\vee}$ (and similarly, we will write $x\vee$ instead of the more cumbersome $x \vee \infty$). We should understand these to be abbreviations, however, and not a genuine removal of 0 and ∞ from the system, since, as we shall see, both 0 and ∞ appear in intermediate steps when applying identities to arrive at theorems.

Next, Ladd-Franklin provides the following *associative* identities for $\overline{\vee}$ (Ladd 1883, 30):[12]

$$(17') \quad a\overline{\vee}b = ab\overline{\vee}$$
$$abc\overline{\vee} = a\overline{\vee}bc = ca\overline{\vee}b = \dots$$

and "dual" versions of these identities for $\vee$:

$$(17^\circ) \quad a \vee b = ab\vee$$
$$abc\vee = a \vee bc = ca \vee b = \dots$$

Simply put, these identities allow us to write any expression involving $\vee$ or $\overline{\vee}$ as "dominant" operator in terms of a single string of logically multiplied terms on the left (or on the right) of $\vee$ or $\overline{\vee}$.

In the next section of the text, Ladd-Franklin finally provides us with the tools to derive *theorems* in her logic (i.e., logical truths that do not involve the identity symbol). First, she notes that the following identity follows from her (informal) definition of identity (quoted in §2 above):

$$(18) \quad (a = b) = (a\overline{\vee}\overline{b})(\overline{a}\overline{\vee}b)$$

Combining this with the substitution instance obtained by uniformly replacing a with $\overline{a}$ and b with $\overline{b}$ (plus some rearrangement and a couple of double negations eliminations) provides us with:

$$(19) \quad (a = b) = (\overline{a} = \overline{b})$$

[12] Ladd-Franklin always takes $\vee$ and $\overline{\vee}$ to have the widest scope possible.

And from this, it follows that:[13]

$$(20) \quad (ab = 0) = (\overline{ab} = \infty) = (ab\overline{\vee}\infty)$$

Ladd-Franklin notes that this principle entails simple versions of the *law of non-contradiction* and the *law of excluded middle* (Ladd 1883, 32):[14]

$$(7') \quad a\bar{a}\overline{\vee}$$
$$(7^\circ) \quad \overline{a + \bar{a}}\overline{\vee}$$

which we can obtain by applying identity principle (20) to:[15]

$$a\bar{a} = 0$$
$$\overline{a + \bar{a}} = 0$$

Next up we have variants of the De Morgan Laws for the copula and its negation (Ladd 1883, 32):[16]

$$(21') \quad (a\overline{\vee})(b\overline{\vee}) = (a + b\overline{\vee})$$
$$(21^\circ) \quad (a\vee) + (b\vee) = (a + b\vee)$$

Ladd-Franklin points out that each of these follows immediately from the other, given the fact that the formula on the left side of (21°) is just the

[13]The typography in the middle identity is a bit unclear in (Ladd 1883, 31). In particular, the second sub-formula is ambiguous between:

$$\overline{ab} = \infty$$

and:

$$a\bar{b} = \infty$$

Only the former makes sense in this context, however.

[14]Note that the second formula, expressing the *law of excluded middle*, is not equivalent to:

$$a + \bar{a}\vee$$

since this simpler formula, loosely put, captures merely the claim that $a + \bar{a}$ is *sometimes* true.

[15]Ladd-Franklin labels these principles (7′) and (7°), understanding them to be *replacements* for the earlier principles with the same label respectively.

[16]It is interesting to note that these principles are $\overline{\vee}$-involving analogues of (13′) and (13°) respectively, in exactly the same way that the new versions of (7′) and (7°) are analogues of the original, identity-involving versions of (7), but Ladd-Franklin does not treat (21′) and (21°) as replacements of the earlier De Morgan principles, but as supplements to them, with their own labels.

negation of the formula on the left of $(21')$, via principle (13),and likewise for the formulas on the right side of each identity.

Ladd-Franklin then notes that we can get a powerful identity-free theorem via applying (20) to $(21')$ and (21°), obtaining the following pair of formulas (Ladd 1883, 33)

$$(21) \quad (a\overline{\vee})(b\overline{\vee})\overline{\vee}(a+b\vee)$$
$$(a\overline{\vee})+(b\overline{\vee})\overline{\vee}(a+b\overline{\vee})$$

Finally, Ladd-Franklin provides two principles for inserting and dropping terms within propositions (Ladd 1883, 33):

$$(22) \quad (a+b+c\overline{\vee})\overline{\vee}(a+b\vee)$$
$$(23) \quad (abc\vee)\overline{\vee}(ab\overline{\vee})$$

This completes our discussion of Ladd-Franklin's basic results regarding $\vee$ and $\overline{\vee}$. We now move on to the main theorems of (Ladd 1883).

4 Theorems I and II and the Syllogism

Ladd Franklin begins her treatment of the syllogism by proving the following theorem:

$$\textbf{I:}\ (a\overline{\vee}b)(c\overline{\vee}d)\overline{\vee}(ac\vee b+d)$$

Her "proof" of the theorem is as follows:

> As a particular case of both the inconsistencies (22) and (23) we have:
>
> $$(a\overline{\vee}b)(c\overline{\vee}d)\overline{\vee}(ac\vee b+d). \quad I.$$
>
> If into the expression which is affirmed not to exist, $ab+cd$, we introduce the factor $c+a$; and if from the product $acb+acd+ab+cd$, we drop the part of a sum, $ab+cd$, – there remains $ac(b+d)$, the existence of which is inconsistent with the non-existence of ab and cd. (Ladd 1883, 34)

Admittedly, exactly what construction Ladd-Franklin has in mind in this passage is far from crystal clear upon first reading. In addition, since Ladd-Franklin is often unclear regarding whether she is giving an informal justification for a new axiom, or sketching a formal proof of a principle based on previous axioms, one might suspect that there is no proper proof here at all, and instead **Theorem I** is merely one more additional axiom. But, with some work, we can reconstruct what Ladd-Franklin had in mind.

First, we need to prove the following lemma, which is the principle involved that allows us to "drop the part of a sum":

Lemma: For any class terms x, y, and z, if:

$$x(y\overline{\vee})\overline{\vee}(y + z\vee)$$

is a theorem, then:

$$x(y\overline{\vee})\overline{\vee}(z\vee)$$

is a theorem.

The proof is as follows. Assume that:

$$x(y\overline{\vee})\overline{\vee}(y + z\vee)$$

is a theorem. Then, by (20) and some rearranging, we have that:

$$x(y\overline{\vee})(y + z\vee) = 0$$

By (21°) this gives us:

$$x(y\overline{\vee})((y\vee) + (z\vee)) = 0$$

and hence:

$$x((y\overline{\vee})(y\vee) + (y\overline{\vee})(z\vee)) = 0$$

This is just:

$$x(0 + (y\overline{\vee})(z\vee)) = 0$$

that is:

$$x(y\overline{\vee})(z\vee) = 0$$

By (20) once again, we obtain:

$$x(y\overline{\vee})\overline{\vee}(z\vee)$$

Quod erat demonstrandum.

Using this lemma, we can now provide a simple proof of **Theorem I** in exactly the manner sketched by Ladd-Franklin. First, note that:[17]

$$(ab + cd\overline{\vee})\overline{\vee}((ab + cd)(c + a)\vee)$$

[17]Ladd-Franklin's discussion implies that we can prove **Theorem I** from *either* (22) *or* (23). Here we give the proof sketched in the quoted paragraph which uses principle (23). Reconstructing a proof based on (22) is left to the reader.

is an instance of (23). We also have:

$$(ab\bar{\vee})(cd\bar{\vee}) = (ab + cd\bar{\vee})$$

by (21′). Applying the obvious substitution on the left, and multiplying the terms on the right, we obtain:

$$(ab\bar{\vee})(cd\bar{\vee})\bar{\vee}(abc + acd + ab + cd\vee)$$

Two applications of our lemma provides:

$$(ab\bar{\vee})(cd\bar{\vee})\bar{\vee}(abc + acd\vee)$$

Finally, we factor the left (and rearrange):

$$(ab\bar{\vee})(cd\bar{\vee})\bar{\vee}(ac \vee b + d)$$

and the proof is complete.

Before moving on to examine how Ladd-Franklin uses this result to analyze syllogisms, it is worth noting that previous commentaries on Ladd-Franklin's result either do not recognize that **Theorem I** is a *theorem* – that is, it is a result that is proven from simpler principles (i.e., the numbered axioms discussed in the previous section) – or make no attempt to reconstruct the proof. For example, Russinoff writes that:

> Ladd-Franklin gives the following formula, which she interprets as a statement of inconsistency between two propositions: (Russinoff 1999, 461)

Uckelman does better, stating that:

> From these conditions we can derive the following thoerem (equivalent, Ladd notes, to Theorem I of (Peirce 1867), "if *a* is *b* and *c* is *d*, then *ac* is *bd*". (Uckelman 2021, 540)

Uckelman then states **Theorem I**, but gives no indication of how Ladd-Franklin's proof actually proceeds.

This objection to the lack of proofs of **Theorem I** in previous work on Ladd-Franklin's logic is not mere quibbling. In appreciating Ladd-Franklin's achievement, and in particular her analysis of syllogisms, it is important to note that she did not merely write down a complicated formula to which all valid syllogisms could be reduced. Instead, she *derived* this more complicated theorem from simpler, intuitive axioms.

Theorem I is powerful and general, providing a number of important results. For the purpose of this essay, however, the most important such result begins with us replacing d with $\bar{b}$, obtaining (Ladd 1883, 37):

$$(a\overline{\nabla}b)(c\overline{\nabla}\bar{b})\overline{\nabla}(ac \vee b + \bar{b})$$

hence:

$$(a\overline{\nabla}b)(c\overline{\nabla}\bar{b})\overline{\nabla}(ac \vee \infty)$$

thus:

$$(a\overline{\nabla}b)(c\overline{\nabla}\bar{b})\overline{\nabla}(ac\vee)$$

or:

$$\mathbf{II} : (a\overline{\nabla}b)(c\overline{\nabla}\bar{b})\overline{\nabla}(a \vee c)$$

Generalizing Ladd-Franklin's discussion of representing validity in terms of $\overline{\nabla}$ a bit, **Theorem II** expresses an inconsistency between three categorical propositions (in Ladd-Franklin's sense – that is, an inconsistency between three propositions, each of which is of one of the forms **A**, **∀**, **E**, **Ǝ**, **a**, **ɐ**, **e**, and **ə**). She then provides her characterization of validity for syllogisms:

> The argument of inconsistency,
>
> $$(a\overline{\nabla}b)(c\overline{\nabla}\bar{b})\overline{\nabla}(a \vee c) \quad \mathbf{II}$$
>
> is therefore the single form to which all the *ninety-six* valid syllogisms (both universal and particular) may be reduced. It is an affirmation of inconsistency between three propositions in three terms, – such that one of the propositions is particular, and the other two are universal; and such that the term common to the two universal propositions appears with unlike signs, and the other two terms appear with like signs. Any given syllogism is immediately reduced to this form by taking the contradictory of the conclusion, and by seeing that universal propositions are expressed with a negative copula and particular propositions with an affirmative copula. (Ladd 1883, 40, emphasis added)

She further fleshes out the use of **Theorem II** as a test for the validity of syllogisms in her *Rule of Syllogism*:

> Take the contradictory of the conclusion, and see that universal propositions are expressed with a negative copula and particular propositions with an affirmative copula. If two of the propositions are universal and the other particular, and if that term

> only which is common to the two universal propositions has unlike signs, then and only then, the syllogism is valid. (Ladd 1883, 41)

Note that the description of the proposition in question, obtained by negating the conclusion, is exactly the logical form of **Theorem II** (or perhaps a substitution instance obtained by uniformly replacing a class term with its negation).

Ladd-Franklin immediately works through a pair of examples to demonstrate how her *Rule of Syllogism* works, reducing valid syllogisms to instances of **Theorem II**. Using the translations given earlier, the valid syllogism she calls Baroko:[18]

All P is M.
Some S is not M.
$\therefore$ Some S is not P.

and the valid syllogism she calls Bokardo:[19]

Some M is not P.
All M is S.
$\therefore$ Some S is not P.

become the following in her system:

$$(P \overline{\vee} \overline{M})(S \vee \overline{M})(S \overline{\vee} \overline{P}) \overline{\vee}$$
$$(M \vee \overline{P})(M \overline{\vee} \overline{S})(S \overline{\vee} \overline{P}) \overline{\vee}$$

These are, of course, just instances of **Theorem II**. In addition, valid syllogisms such as:

Some M is not P.
No M is S.
$\therefore$ Some not-S is not-P.

which involve the "new" propositions introduced by De Morgan and incorporated into her analysis by Ladd-Franklin, also reduce to instances of **Theorem II**. In this case we obtain:

$$(M \vee \overline{P})(M \overline{\vee} S)(\overline{S} \overline{\vee} \overline{P}) \overline{\vee}$$

[18] As (Uckelman 2021, 550, footnote 11) notes, this would more traditionally be called "Baroco".

[19] As (Uckelman 2021, 550, footnote 12) notes, this would more traditionally be called "Bocardo".

Thus, **Theorem II** provides a complete characterization of the valid syllogisms, since all and only the valid syllogisms using propositions of form **A**, **Ɐ**, **E**, **Ǝ**, **a**, **ɐ**, **e**, and **ə** will reduce to instances of **Theorem II**. And, as Ladd-Franklin notes, there are exactly ninety-six such syllogisms. Hence, this analysis goes a good bit further than the reduction techniques used in the more limited context of traditional Aristotelian syllogisms, which are restricted to propositions of form **A**, **E**, **a**, and **e** (in Ladd-Franklin's terminology).

Theorem II provides more than merely a reduction technique, however. In addition, it allows us to *computationally generate* all ninety-six of the valid syllogisms. In order to so so, we first need to note that **Theorem II**, in combination with Ladd-Franklin's characterization of validity for syllogisms, actually gives us eight relevant instances. If we consider the result of uniformly replacing none, one, two or three of a, b, and c with its negation ($\bar{a}$, $\bar{b}$, and $\bar{c}$) in **Theorem II** (eliminating double negations when appropriate), we obtain:

$$[1]\ (a \triangledown b)(c \triangledown \bar{b}) \triangledown (a \vee c)$$
$$[2]\ (\bar{a} \triangledown b)(c \triangledown \bar{b}) \triangledown (\bar{a} \vee c)$$
$$[1^*]\ (a \triangledown \bar{b})(c \triangledown b) \triangledown (a \vee c)$$
$$[3]\ (a \triangledown b)(\bar{c} \triangledown \bar{b}) \triangledown (a \vee \bar{c})$$
$$[3^*]\ (\bar{a} \triangledown \bar{b})(c \triangledown b) \triangledown (\bar{a} \vee c)$$
$$[2^*]\ (a \triangledown \bar{b})(\bar{c} \triangledown b) \triangledown (a \vee \bar{c})$$
$$[4]\ (\bar{a} \triangledown b)(\bar{c} \triangledown \bar{b}) \triangledown (\bar{a} \vee \bar{c})$$
$$[4^*]\ (\bar{a} \triangledown \bar{b})(\bar{c} \triangledown b) \triangledown (\bar{a} \vee \bar{c})$$

As the numbering suggests, we only have four logically distinct theorems in this list, since for each n ($1 \leq n \leq 4$), $[n^*]$ is the result of switching a and c in $[n]$ and re-ordering the three sub-formulas in accordance with the associative identities given above. Thus, we have four genuinely distinct variants of **Theorem II** from which to choose.

$$[1]\ (a \triangledown b)(c \triangledown \bar{b}) \triangledown (a \vee c)$$
$$[2]\ (\bar{a} \triangledown b)(c \triangledown \bar{b}) \triangledown (\bar{a} \vee c)$$
$$[3]\ (a \triangledown b)(\bar{c} \triangledown \bar{b}) \triangledown (a \vee \bar{c})$$
$$[4]\ (\bar{a} \triangledown b)(\bar{c} \triangledown \bar{b}) \triangledown (\bar{a} \vee \bar{c})$$

In order to get some non-Aristotelian propositions into the mix, let us pick formula [3]. We then obtain a valid syllogism by taking two of the sub-formulas as premises, and the negation of the remaining subformula as conclusion. Applying this to formula [3] (and re-arranging premises so that the arguments are in standard form) we obtain the following valid syllogisms:

Ǝ	$(\bar{c}\,\overline{\vee}\,\bar{b})$	No not-C is not-B.
E	$\underline{(a\,\overline{\vee}\,b)}$	No A is B.
A	$(a\,\overline{\vee}\,\bar{c})$	All A is C.

E	$(a\,\overline{\vee}\,b)$	No A is B.
e	$\underline{(a \vee \bar{c})}$	Some A is not-C.
ə	$(\bar{c} \vee \bar{b})$	Some not-C is not-B.

Ǝ	$(\bar{c}\,\overline{\vee}\,\bar{b})$	No not-C is not-B.
e	$\underline{(a \vee \bar{c})}$	Some A is not-C.
a	$(a \vee b)$	Some A is B.

Thus, we obtain the following three valid syllogisms:[20]

Figure 1	Figure 2	Figure 3	Figure 4
Əea	**ƎEA**	**Eeə**	

Since we could repeat this process for any of the other three variants of **Theorem II**, this would net us twelve distinct valid syllogisms. But it is here that the symmetry of Ladd-Franklin's $\vee$ and $\overline{\vee}$ does its work. For any of the twelve syllogisms obtained via (i) choosing one of the four variants of **Theorem II** and then (ii) taking one of the three subformulas as the conclusion, we can actually generate eight distinct syllogisms via reversing

[20]Syllogisms are classified into *figures* based on the arrangement of the *major*, *minor*, and *middle* (where the minor term is the *subject* of the conclusion and the major term is the *predicate* of the conclusion) term when the syllogism is presented with the major term occurring in the first premise. Where S is the minor term, P is the major term, and M is the middle term, the four figures correspond to the following arrangements:

Figure 1		Figure 2		Figure 3		Figure 4	
M	P	P	M	M	P	P	M
S	M	S	M	M	S	M	S
S	P	S	P	S	P	S	P

the order of the arguments in none, one, two, or three of the propositions involved. For example, the second syllogism above **Eeə**, Figure 3) can be rewritten as any of:

E	$(a\overline{\vee}b)$	No A is B.
e	$\underline{(a \vee \bar{c})}$	Some A is not-C.
ə	$(\bar{c} \vee \bar{b})$	Some not-C is not-B.

e	$(a \vee \bar{c})$	Some A is not-C.
E	$\underline{(a\overline{\vee}b)}$	No A is B.
ə	$(\bar{b} \vee \bar{c})$	Some not-B is not-C.

E	$(a\overline{\vee}b)$	No A is B.
ɐ	$\underline{(\bar{c} \vee a)}$	Some not-C is A.
ə	$(\bar{c} \vee \bar{b})$	Some not-C is not-B.

E	$(b\overline{\vee}a)$	No B is A.
e	$\underline{(a \vee \bar{c})}$	Some A is not-C.
ə	$(\bar{c} \vee \bar{b})$	Some not-C is not-B.

E	$(b\overline{\vee}a)$	No B is A.
ɐ	$\underline{(\bar{c} \vee a)}$	Some not-C is A.
ə	$(\bar{c} \vee \bar{b})$	Some not-C is not-B.

e	$(a \vee \bar{c})$	Some A is not-C.
E	$\underline{(b\overline{\vee}a)}$	No B is A.
ə	$(\bar{b} \vee \bar{c})$	Some not-B is not-C.

ɐ	$(\bar{c} \vee a)$	Some not-C is A.
E	$\underline{(a\overline{\vee}b)}$	No A is B.
ə	$(\bar{b} \vee \bar{c})$	Some not-B is not-C.

ɐ $(\bar{c} \vee a)$ Some not-C is A.
E $(b \overline{\vee} a)$ No B is A.
ə $(\bar{b} \vee \bar{c})$ Some not-B is not-C.

This provides us with some additional entries in our table of valid syllogisms:

Figure 1	Figure 2	Figure 3	Figure 4
Ǝea	**ƎEA**	**Eeə**	**Eeə**
Eɐə	**Eɐə**	**eEə**	**ɐEə**
eEə	**ɐEə**		

To sum all of this up, we can generate a valid syllogism using **Theorem II** by implementing the following algorithm:

1. Select one of the *four* distinct substitution instances of **Theorem II**.
2. Select one of the *three* sub-formulas in that instance to serve and the conclusion, and negate it.
3. Perform one of the *eight* possible re-orderings on the premises and conclusion.

Each sequence of such choices will generate a unique valid syllogism expressed in terms of **A**, **∀**, **E**, **Ǝ**, **a**, **ɐ**, **e**, and **ə**. Hence, there are:

$$4 \times 3 \times 8 = 96$$

distinct valid syllogisms.

We will leave the identification of the remaining eighty-six valid syllogisms to the ambitious reader. But, before moving on, it is worth emphasizing that this aspect of Ladd-Franklin's work is completely new. She has provided, not only a reduction procedure for testing each syllogism to see if it is valid, but, in addition, she has given us a means for mechanically enumerating all the valid syllogisms.[21]

5 Clearing Up Some Confusions

We conclude by clearing up a few misunderstandings of Ladd-Franklin's work, focusing on the two most extensive and most prominent recent studies: (Russinoff 1999) and (Uckelman 2021).

[21] It is also worth noting the mathematically beautiful fact that the ninety-six valid syllogisms are evenly divided between the four syllogistic figures, with twenty-four valid syllogisms in each figure.

First, we address questions regarding the tradition within which Ladd-Franklin worked, and the extent to which that tradition was continuous with, or connected to, more traditional work on the Aristotelian syllogism. In "What Problem did Ladd-Franklin Think She Solved?" (Uckelman 2021), Sara Uckelman notes that many commentators, including Russinoff (about whom more below), characterize Ladd-Franklin's work as having solved a major outstanding problem in the Aristotelian tradition. For example, Russinoff writes that:

> In 1883, while a student of C.S. Peirce at John Hopkins University, Christine Ladd-Franklin published a paper titled *On the Algebra of Logic*, in which she develops an elegant and powerful test for the validity of syllogisms that constitutes the most significant advance in syllogistic logic in two thousand years. Sadly, her work has been all but forgotten by logicians and historians of logic. Ladd-Franklin's achievement has been overlooked, partly because it has been overshadowed by the work of other logicians of the nineteenth century renaissance in logic, but probably also because she was a woman. Though neglected, the significance of her contribution to the field of symbolic logic has not been diminished by subsequence achievements of others.
>
> In this paper, I bring to light the important work of Ladd-Franklin so that she is justly credited with having solved a problem over two millennia old. (Russinoff 1999, 451)

After surveying Ladd-Franklin's novel logical system, Uckelman attempts to identify the supposed two thousand year old problem that Ladd-Franklin solved, which is usually described in terms of Ladd-Franklin having reduced all of the valid syllogisms to a single form. For example, Ahti-Veikko Pietarinen characterizes Ladd-Franklin work as involving:

> . . . the ground-breaking discovery involving the reduction of Aristotelian syllogistics to a single formula (Pietarinen 2013, 3)

Uckelman argues convincingly that (i) Aristotle already had the rudiments of a reduction of the valid syllogisms to a single formulation, and (ii) Ladd-Franklin did not characterize her work as solving a problem due to Aristotle. Instead, Uckelman argues that Ladd-Franklin took herself to be solving a problem due to Jevons based on the following passage:

> It [**Theorem II**] contains the solution of what Mr. Jevons calls the "inverse logical problem". (Ladd 1883, 50)

Jevons explains the problem in question as follows:

> Three terms and their negatives may be combined . . . in eight different combinations, and the effect of laws or logical conditions is to destroy any one or more of these combinations. Now we may make selections from eight things in 2^8 or 256 ways, so that we have no less than 256 different cases to treat, and the complete solution is at least fifty times as troublesome as with two terms. . . . the test of inconsistency is that each of the letters A, B, C, a, b, c shall appear somewhere in the series of combinations; but I have not been able to discover any mode of calculating the number of cases in which inconsistency would happen . . . an exhaustive examination of the combinations in detail is the only method applicable (Jevons 1874, 157 – 158)

Now, there can be no doubt that Ladd-Franklin takes **Theorem II** to provide a full solution to the inverse logical problem involving three terms – she says as much, after all. But she also says that **Theorem II** "contains" such a solution, not that it "is" the solution. On the contrary, it seems clear that Ladd-Franklin *did* take her main purpose in (Ladd 1883), or, at the very least, one of her main purposes, to be the reduction of the valid syllogisms to a single formula. After all, once she has proven **Theorem II** she immediately spends two pages discussing its application to syllogisms, and her discussion of Jevons occurs a good bit later, in a section of the essay titled "Resolution of Problems".

That being said, Uckelman is absolutely right when she claims that Ladd-Franklin could not have taken herself to be solving a problem that had plagued the theory of syllogisms since Aristotle. But we do not need the subtle and detailed analysis Uckelman provides to see this – we need only note that Ladd-Franklin states that **Theorem II** provides a complete characterization of *ninety-six* valid syllogisms, not fifteen (or twenty-four). In short, Ladd-Franklin understood the notion of syllogism quite differently than those working strictly in the Aristotelian tradition, since she allowed syllogisms to contain propositions of any of the eight forms **A**, **∀**, **E**, **Ǝ**, **a**, **ɐ**, **e**, and **ə**. Thus, it is quite fair to attribute the complete characterization of the valid syllogisms to Ladd-Franklin – we just have to note that she meant something different by "syllogism" than what Aristotle meant, and thus a complete characterization of the valid syllogisms in his sense (which, again, as Uckelman notes, was already at least implicit in the work of Aristotle himself) falls far short of a complete characterization of the valid syllogisms in Ladd-Franklin's sense.

Finally, it seems likely that this problem – the complete characterization of the valid syllogisms in the extended, eight-proposition setting – was a problem inherited, not from Jevons, but from De Morgan. Ladd-Franklin is quite adamant, when she first introduces the eight forms of proposition with which she will be concerned when analyzing syllogisms, that the extended class of propositional forms is due to (De Morgan 1860), writing that:

> The eight propositions of De Morgan are then,– (Ladd 1883, 25)

immediately before producing the chart reproduced in §3 above.[22]

We now move from determining what problem Ladd-Franklin took herself to be solving to asking whether she actually solved any problems at all. In "The Syllogism's Final Solution", I. Susan Russinoff also provides a detailed overview of Ladd-Franklin's system, focusing on Ladd-Franklin's characterization of the valid syllogisms.[23] The centerpiece of (Russinoff 1999) is a modern, model-theoretic proof that any valid syllogism must have the form that Ladd-Franklin attributes to such syllogisms.[24] What is somewhat odd about Russinoff's discussion, however, is that she claims that Ladd-Franklin not only produced no such proof, but in principle could not have done so:

> Although she presents her *Rule of Syllogism* as a theorem, Ladd-Franklin does not give a rigorous proof of the correctness of her result. She is claiming that *all* inconsistent triads, or "antilogisms", as she called them, share a certain form yet the results necessary for a proof were unknown at the time she did this work. Although it is obvious that all triads with the form she describes are inconsistent, it is not obvious that every inconsistent triad has that form. It has been unrecognized by those who have written about Ladd-Franklin's work that not only did she give no proof of her theorem, but she could not have done so. Moreoever, those who have mentioned a proof seem unaware that it is not a trivial one. (Russinoff 1999, 463 – 4646)

[22](Russinoff 1999) also points out that the notion of a domain or universe of discourse also has its roots in the work of De Morgan. I leave a detailed examination of the influence of De Morgan's work on Ladd-Franklin to another time.

[23]Russinoff does not seem to recognize that Ladd-Franklin is working in an extended context with eight, rather than four, forms that propositions can take in a syllogism, which goes some ways towards explaining why she characterized Ladd-Franklin's work as directed at traditional Aristotelian logic.

[24]More carefully put, the proof given in (Russinoff 1999, 464 – 468) demonstrates this for traditional Aristotelian syllogisms, but can be easily modified to apply to all syllogisms that can be constructed using propositions of any of the forms **A**, **∀**, **E**, **∃**, **a**, **ɐ**, **e**, and **ə**.

Now, it is certainly true that Ladd-Franklin could not have given the contemporary-style, model theoretic proof that Russinoff provides, since that proof depends on formal tools (model theory in particular) that were not available to Ladd-Franklin. Nevertheless, Russinoff's criticism here seems too harsh.

First, it is not the case that Ladd-Franklin's argument for the claim that all syllogisms that are encoded by an instance of **Theorem II** are valid is merely "obvious". As we have seen, Ladd-Franklin gives a genuine proof of this fact (or, at the very least, provides enough of a sketch of the proof that we were able to reconstruct it in the previous section).

Second, Ladd-Franklin never presents anything titled *The Rule of Syllogism* as a theorem. On the contrary, the italicized occurrence of this phrase at (Ladd 1883, 41) is, strictly speaking, a section heading. The section in question then presents an algorithm for testing arguments for validity, not a theorem of any sort. Further, in the paragraph leading up to this occurrence, Ladd-Franklin set up this *Rule of Syllogism* as follows:

> It is then possible to give a perfectly general rule, easy to remember and easy of application, for testing the validity of any given syllogism, universal or particular, which is given in words. It is this:– (Ladd 1883, 41)

Simply put, Ladd-Franklin is explicit here that she is presenting an algorithm or rule, not stating a theorem. The only theorems she proves are theorems in her logic, such as **Theorem I** and **Theorem II**.

Further, although Ladd-Franklin never claims to have proven both the necessity and the sufficiency of the condition codified in the rule (which, to be fair, would amount to claiming not only that all syllogisms corresponding to instances of **Theorem II** are valid, but in addition that *only* syllogisms corresponding to instances of **Theorem II** are valid), it is not clear that, as Russinoff puts it:

> . . . not only did she give no proof of [*The Rule of Syllogism*], but she could not have done so. (Russinoff 1999, 463 – 464)

One simple (albeit tedious) way to do so would be to carry out a proof by exhaustion, testing all $8 \times 8 \times 8 \times 4 = 2048$ possible syllogisms (in the enriched, eight proposition setting). This is no doubt much more daunting than testing the two hundred and fifty-six possible Aristotelian syllogisms individually, but it is not impossible (and Ladd-Franklin would have been aware of techniques, such as Venn diagrams, that would to a certain extent mechanize this process).

More likely, however, Ladd-Franklin convinced herself of both the necessity and the sufficiency of conditions laid out in the *Rule of Syllogism* via carrying out an informal version of something like the technical argument provided in (Russinoff 1999).[25] At any rate, there is no reason to think that Ladd-Franklin's endorsement of the *Rule of Syllogism* was based merely on lucky guesswork rather than on what would have at the time been considered compelling evidence or even proof.

This is not to say that Russinoff's result is worthless. On the contrary, her model-theoretic reconstruction of Ladd-Franklin's results is extremely useful in helping us to understand why Ladd-Franklin's techniques work. But the fact that Ladd-Franklin did not give, and could not have given, *this* proof in no way implies that she did not have *some* proof, or some proof-like, mathematically compelling argument, for the correctness of the conditions laid out in the *Rule of Syllogism*.

This concludes our examination of Ladd-Franklin's logical system. There is, no doubt, much work left to be done in understanding the technical details of Ladd-Franklin's logic and the way that she understood it. Hopefully, this essay has progressed us a bit further in this important and enjoyable task.[26]

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[25] In fact, the careful reasoning through cases Ladd-Franklin carries out at (Ladd 1883, 39) when leading up to the *Rule of Syllogism* sounds exactly as if this is what Ladd-Franklin is doing.

[26] Thanks are owed to Cat Saint-Croix, Jasmin Özel, and an anonymous referee for helpful comments on, or conversations related to, earlier versions of this essay.

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