A note on negation inconsistent variants of **FDE**-negation

Satoru Niki
Department of Philosophy I
Ruhr University Bochum, Germany
Satoru.Niki@rub.de

Abstract

H. Omori and H. Wansing introduced in a recent paper possible alternatives for the negation of the logic of first-degree entailment. One of their observations with regard to these alternative negations is that some of them turn out to induce negation inconsistency, meaning that some contradictions become provable (under an arbitrary premise) when used in place of the original negation. Omori and Wansing also considered a non-deterministic generalisation of such operators, but it was left open whether the generalised negation similarly induces negation inconsistency. In this paper, we provide an answer to this question in the positive, and moreover look into further generalisation and characterisation of non-deterministic operators which satisfy the formal criteria of negation inconsistency and its pair notion of negation incompleteness in the setting of Omori and Wansing.

Keywords— Contradictory logics; First-degree entailment; Negation inconsistency; Negation incompleteness; Non-deterministic semantics.

1 Introduction

The negation in *the logic of first-degree entailment* **FDE**, also known as *Belnap-Dunn logic* [9, 10, 12, 13], can be characterised by a four-valued semantics with values **t** (true), **f** (false), **b** (both true and false), and **n** (neither true nor false).

There are, on the other hand, other unary operators that are proposed as representing a type of negation (see e.g. [18]). In particular, H. Omori and H. Wansing [17] considered possible variants of **FDE**-negation under the criterion that $\sim A$ is true if and only if A is false. In other words, the variants are obtained by changing the falsity condition of the **FDE**-negation, an approach sometimes called the

Bochum plan [14]. Furthermore, Omori and Wansing also introduced a further generalisation by considering *non-deterministic matrices (Nmatrices)* introduced by A. Avron and I. Lev [6, 7].

One observation made in [17] is that there are four notions of negation which makes the system negation inconsistent, in the sense that there is a formula A such that both A and its negation $\sim A$ are consequences of any arbitrary formula B. Furthermore, the matrices of the four negations were generalised into a non-deterministic matrix, and corresponding natural deduction systems was given. However, it remained open whether this generalised system is negation inconsistent as well. Similar observations are also made with respect to the negation incomplete variants, and in this case as well it remained open whether the non-deterministic generalisation of the negation incomplete variants is itself negation incomplete.

In the above cases, the problems are equivalent semantically to the definability of a constant (**b**/**n** constant) which always gives the value **b** and **n**, respectively. The definability of connectives in **FDE** and its neighbouring systems (with different sets of connectives) has been well-studied. In [1, 3], O. Arieli and A. Avron characterised sets of definable connectives in various languages in the context of the bilattices $\mathcal{FOUR}$ and $\mathcal{THREE}$, in terms of the properties of their corresponding operations. The papers also address the issue of functional completeness, as is also done by e.g. A.P. Pynko [20], H. Omori and K. Sano [15] as well as L.Yu. Devyatkin [11]. Omori and Sano [15] moreover give a classification of certain expansions of **FDE**. A. Přenosil [19] offers a description of expansions that is independent of the choice of primitive connectives. Also related to our enquiry is a work by Arieli, Avron and A. Zamansky [2] which investigates notions of maximality in paraconsistent three-valued logics given by non-deterministic semantics. Devyatkin [11] makes analogous consideration into the deterministic expansions of **FDE**.

In this note, we shall first show that the non-deterministic system is indeed negation inconsistent. Then we shall attempt to generalise this result, by considering operators obtainable by introducing further non-determinacy, and in the end we shall present a characterisation of unary operators which formally satisfy the criterion of negation inconsistency. Moreover, we shall make similar observations with respect to the negation incomplete variants of **FDE**-negation. Finally, we shall look at negation inconsistency obtainable by adding non-deterministic unary operators to the full **FDE** with the **FDE**-negation.

2 Preliminaries

Let us first specify the propositional language to be used in our enquiry.

$$(\mathcal{L}_{\mathbf{FDE}})\ A ::= p \mid A \wedge A \mid A \vee A \mid {\sim} A.$$
$$(\mathcal{L}_{\mathbf{FDE}_\circ})\ A ::= p \mid A \wedge A \mid A \vee A \mid {\sim} A \mid\ \circ A.$$

The set of all formulas in $\mathcal{L}_{\mathbf{FDE}}$ ($\mathcal{L}_{\mathbf{FDE}_\circ}$) will be denoted by Form ($\mathsf{Form}_\circ$).

We will occasionally attach subscripts to $\sim$ in order to differentiate negations in multiple systems, but they do not constitute a proper part of the languages.

We next introduce the definition of non-deterministic matrices (for more details, see e.g. [8]).

Definition 2.1. A *non-deterministic matrix* (or *Nmatrix*) for a language $\mathcal{L}$ is a triple $(\mathcal{V}, \mathcal{D}, \mathcal{O})$, where $\mathcal{V} \neq \emptyset$ stands for the set of *truth values*, $\emptyset \subsetneq \mathcal{D} \subsetneq \mathcal{V}$ stands for the set of *designated* values, and $\mathcal{O}$ contains $\tilde{\diamond} : \mathcal{V}^n \longrightarrow 2^{\mathcal{V}} \setminus \emptyset$ for each n-ary connective $\diamond$ in $\mathcal{L}$. A *valuation*[1] is then a function $v : \mathsf{Form} \to \mathcal{V}$ satisfying the following condition for each connective $\diamond$ in $\mathcal{L}$:

$$v(\diamond(A_1, \ldots, A_n)) \in \tilde{\diamond}(v(A_1), \ldots, v(A_n)).$$

Then the *semantical consequence relation* $\Gamma \models A$ is defined by: if for any valuation v, $v(B) \in \mathcal{D}$ for all $B \in \Gamma$ implies $v(A) \in \mathcal{D}$.

For instance, if an Nmatrix is such that $\mathcal{V} = \{\mathbf{t}, \mathbf{b}, \mathbf{f}\}$ and $\tilde{\diamond} \in \mathcal{O}$ is such that $\tilde{\diamond}(\mathbf{t}) = \{\mathbf{t}, \mathbf{b}\}$, then when $v(A) = \mathbf{t}$, $v(\diamond A)$ must be one of $\mathbf{t}$ or $\mathbf{b}$. An n-ary connective $\diamond$ will be called *deterministic* with respect to an Nmatrix if $\tilde{\diamond}(x_1, \ldots, x_n)$ is a singleton set for all $(x_1, \ldots, x_n) \in \mathcal{V}^n$; otherwise it will be called *non-deterministic.*

Occasionally, we add a subscript for an element of $\mathcal{O}$ to indicate the Nmatrix it is from. Also, given a set Γ of formulas we will use the notation $v(\Gamma)$ for $\{v(A) : A \in \Gamma\}$. Then the specific Nmatrix of our (initial) concern is given as follows [17, Definition 69].

Definition 2.2 (four-valued negation inconsistent **FDE**-Nmatrix)**.** The *four-valued negation inconsistent* **FDE**-*Nmatrix* for $\mathcal{L}_{\mathbf{FDE}}$ is a triple $(\mathcal{V}, \mathcal{D}, \mathcal{O})$, where $\mathcal{V} = \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$, $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$ and $\mathcal{O} = \{\tilde{\wedge}, \tilde{\vee}, \tilde{\sim}\}$. The next tables for $\tilde{\wedge}, \tilde{\vee}$ are identical to those of **FDE**. (We abbreviate braces, e.g. $\mathbf{n}, \mathbf{f}$ instead of $\{\mathbf{n}, \mathbf{f}\}$.):

$\tilde{\wedge}$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**b**	**b**	**f**	**f**
n	**n**	**f**	**n**	**f**
f	**f**	**f**	**f**	**f**

$\tilde{\vee}$	**t**	**b**	**n**	**f**
t	**t**	**t**	**t**	**t**
b	**t**	**b**	**t**	**b**
n	**t**	**t**	**n**	**n**
f	**t**	**b**	**n**	**f**

	$\tilde{\sim}$
t	**n**,**f**
b	**t**,**b**
n	**f**
f	**b**

A valuation and the semantical consequence relation of the *four-valued negation inconsistent* **FDE**-*Nmatrix* will be called a *four-valued negation inconsistent* **FDE**-*valuation* and the *four-valued negation inconsistent* **FDE**-*consequence relation.* The latter will be denoted $\models_{4b}$ and is identified with the logic **b-FDE**.

[1] We will focus on the so-called *dynamic valuations*, which are more general than another notion called *static valuations*; see [5] for the details. At the same time, we note that one of the reviewers expressed an expectation in favour of static valuations in the present context.

Remark 2.3. In order to compactly represent possible values a formula can take under a valuation, we adopt an informal notational convention. Suppose, as an example, that we have a four-valued negation inconsistent **FDE**-valuation v and a formula A such that $v(A) = \mathbf{b}$. Consider then a case of representing which of the four values $\sim A \wedge \sim\sim A$ might take under v.

There are two steps where the value of formula is not deterministically given. Firstly, $v(\sim A)$ may take either the value $\mathbf{t}$ or $\mathbf{b}$ according to Definition 2.2. Let us denote these possibilities by an ordered pair $(\mathbf{t}, \mathbf{b})_1$ (the comma will be abbreviated) with a label 1 representing a step at which the non-determinism occurs.

Secondly, there are also multiple possibilities for $v(\sim\sim A)$, in both of the cases when $v(\sim A) = \mathbf{t}$ and when $v(\sim A) = \mathbf{b}$. In the former case, $v(\sim\sim A)$ either takes the value $\mathbf{n}$ or $\mathbf{f}$, and this will be denoted by $(\mathbf{nf})_2$ with a new label 2 for the new instance of non-determinism. In the latter case, $v(\sim\sim A)$ either takes the value $\mathbf{t}$ or $\mathbf{b}$, and this will be denoted by $(\mathbf{tb})_2$: we use the same label because they concern the same formula $\sim\sim A$. To denote all possibilities, then, we substitute these pairs for $\mathbf{t}$ and $\mathbf{b}$ in $(\mathbf{tb})_1$, obtaining an expression $((\mathbf{nf})_2(\mathbf{tb})_2)_1$.

This simply represents that two possibilities are introduced at the step labelled by 1, and in each of the possibilities a further pair of possibilities is introduced at the step labelled by 2. The situation is also representable in a table.

A	$1 : \sim A$	$2 : \sim\sim A$	$\sim A \wedge \sim\sim A$
$\mathbf{b}$	$(\mathbf{tb})_1$	$((\mathbf{nf})_2(\mathbf{tb})_2)_1$	$((\mathbf{nf})_2(\mathbf{bb})_2)_1$

Now this presentation makes it easy to calculate the possible values for $v(\sim A \wedge \sim\sim A)$. There are four possible values for $v(\sim\sim A)$, and if it has one of the value $\mathbf{n}$ or $\mathbf{f}$ from $(\mathbf{nf})_2$, then it is a result of $v(\sim A)$ being $\mathbf{t}$, because of the convention we adopted. So $v(\sim A \wedge \sim\sim A)$ is either $\mathbf{n}$ or $\mathbf{f}$ in this case, from $\tilde{\wedge}$ in Definition 2.2. Otherwise, it has one of the values $\mathbf{t}$ or $\mathbf{b}$ from $(\mathbf{tb})_2$, and it is a result of $v(\sim A)$ being $\mathbf{b}$; so $v(\sim A \wedge \sim\sim A)$ is $\mathbf{b}$ in both cases. Hence the possibilities for $v(\sim A \wedge \sim\sim A)$ can be expressed as $((\mathbf{nf})_2(\mathbf{bb})_2)_1$ or more simply as $((\mathbf{nf})_2\mathbf{b})_1$ since the second non-determinism does not affect the value when $v(\sim A) = \mathbf{b}$.

One important notion in non-deterministic matrices is that of *refinement*, of which we give a specific version for our specific purpose; for the full definition, see e.g. [8, Definition 30].

Definition 2.4 (refinement)**.** Let $\mathcal{M}_1 = (\mathcal{V}, \mathcal{D}, \mathcal{O}_1)$ and $\mathcal{M}_2 = (\mathcal{V}, \mathcal{D}, \mathcal{O}_2)$ be Nmatrices in a language $\mathcal{L}$. We say $\mathcal{M}_1$ is a *refinement* of $\mathcal{M}_2$ if for each n-ary connective $\diamond$ in $\mathcal{L}$ and for all $x_1, \ldots, x_n \in \mathcal{V}$, $\tilde{\diamond}_{\mathcal{M}_1}(x_1, \ldots, x_n) \subseteq \tilde{\diamond}_{\mathcal{M}_2}(x_1, \ldots, x_n)$.

Essentially, this just means that a refinement of an Nmatrix is obtained by eliminating some entries from its tables.

Proposition 2.5. Let $\mathcal{M}_1$ be a refinement of $\mathcal{M}_2$, and $\models_{\mathcal{M}_1}$, $\models_{\mathcal{M}_2}$ be their semantical consequence relations. Then $\Gamma \models_{\mathcal{M}_2} A$ implies $\Gamma \models_{\mathcal{M}_1} A$.

Proof. See e.g. [4, Proposition 3.3]. □

3 Negation-inconsistent variants of FDE-negation

According to [17], there are four types of negation which produce a negation inconsistent system (**FDE**^{i} for $i \in \{4, 8, 12, 16\}$) in their framework. Moreover, the negations can be generalised into a non-deterministic type of negation, whose table is identical to the one we saw above.

A	$\sim_4$ A	$\sim_8 A$	$\sim_{12}$ A	$\sim_{16} A$	$\sim A$
t	**f**	**f**	**n**	**n**	**n**,**f**
b	**b**	**t**	**b**	**t**	**t**,**b**
n	**f**	**f**	**f**	**f**	**f**
f	**b**	**b**	**b**	**b**	**b**

The question [17, Remark 73] that remained unanswered is whether $\sim A$ gives rise to negation inconsistency. Here, a logic **L** is said to be *negation inconsistent* if for some A, both $B \models_{\mathbf{L}} A$ and $B \models_{\mathbf{L}} \sim A$ for any B. (cf. [17, Definition 28]). The formula A will sometimes be referred to as a *provable contradiction*.

3.1 Negation inconsistency of b-FDE

We will show that the answer to the open question is positive. This is more easily seen from a proof-theoretic perspective. The following natural deduction system is proved in [17] to be sound and complete with respect to the four-valued negation inconsistent **FDE**-Nmatrix.

Definition 3.1 (Nb-FDE). The following rules define a natural deduction **Nb-FDE** in $\mathcal{L}_{\mathbf{FDE}}$.

$$\frac{A \qquad B}{A \wedge B}\ (\wedge\text{I}) \qquad\qquad \frac{A_1 \wedge A_2}{A_i}\ (\wedge\text{E})$$

$$\frac{A_i}{A_1 \vee A_2}\ (\vee\text{I}) \qquad\qquad \frac{A \vee B \qquad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\ (\vee\text{E})$$

$$\frac{\sim A_i}{\sim(A_1 \wedge A_2)}\ (\sim\wedge\text{I}) \qquad\qquad \frac{\sim(A \wedge B) \qquad \begin{matrix}[\sim A]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[\sim B]\\ \vdots \\ C\end{matrix}}{C}\ (\sim\wedge\text{E})$$

$$\frac{\sim A \qquad \sim B}{\sim(A \vee B)}\ (\sim\vee\text{I}) \qquad\qquad \frac{\sim(A_1 \vee A_2)}{\sim A_i}\ (\sim\vee\text{E})$$

$$\frac{}{A \vee \sim\sim A}\ (\sim\sim) \qquad\qquad \text{(where } i \in \{1, 2\}.\text{)}$$

The derivability in **Nb-FDE** is denoted by $\vdash^b_{\mathbf{FDE}}$.

The following formulas witness the negation inconsistency of **b-FDE**.

Proposition 3.2. The following formulas are derivable in **Nb-FDE**.

(i) $\sim(A\wedge\sim\sim A) \wedge \sim((A\wedge\sim\sim A)\wedge\sim\sim(A\wedge\sim\sim A))$.

(ii) $\sim(\sim(A\wedge\sim\sim A) \wedge \sim((A\wedge\sim\sim A)\wedge\sim\sim(A\wedge\sim\sim A)))$.

Proof. For (i), the left conjunct can be derived from an instance of $(\sim\sim)$, namely $\sim A \vee \sim\sim\sim A$. Similarly, the right conjunct can be obtained from another instance $\sim(A \wedge \sim\sim A) \vee \sim\sim\sim(A \wedge \sim\sim A)$.

For (ii), we use the instance of $(\sim\sim)$:

$$((A\wedge\sim\sim A)\wedge\sim\sim(A\wedge\sim\sim A)) \vee \sim\sim((A\wedge\sim\sim A)\wedge\sim\sim(A\wedge\sim\sim A))),$$

from which we infer $\sim\sim(A\wedge\sim\sim A) \vee \sim\sim((A\wedge\sim\sim A)\wedge\sim\sim(A\wedge\sim\sim A)))$. Then apply $(\sim\wedge\mathrm{I})$. □

The contradictory formulas may be seen as an instance of a general schema $\sim A \wedge \sim(A \wedge \sim\sim A)$. What does it signify? To answer this question, we go back to the semantical perspective. We introduce first what we mean in this paper when a deterministic connective is said to be *definable* in a logic.

Definition 3.3. Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an Nmatrix, and let $\diamond$ be a deterministic n-ary connective given by $\tilde{\diamond} : \mathcal{V}^n \longrightarrow 2^{\mathcal{V}} \setminus \emptyset$. We say $\tilde{\diamond}$ is *definable* in $\mathcal{M}$, if there is a formula $A(p_1, \ldots, p_n)$ such that for each $(x_1, \ldots, x_n) \in \mathcal{V}^n$, if $\tilde{\diamond}(x_1, \ldots, x_n) = \{y\}$ then $v(B_i) = x_i$ for $1 \leq i \leq n$ implies $v(A(B_1, \ldots, B_n)) = y$. Then $\diamond$ is *definable* in a logic **L** when $\tilde{\diamond}$ is definable in an Nmatrix whose associated semantical consequence relation gives rise to **L**.

Now we can see what $\sim A \wedge \sim(A \wedge \sim\sim A)$ amounts to in **b-FDE**.

Proposition 3.4. $\sim_4$ is definable in **b-FDE**.

Proof. The following table shows that for any four-valued negation inconsistent **FDE**-valuation v, the valuation of $\sim A \wedge \sim(A \wedge \sim\sim A)$ matches that of $\sim_4$.

A	$1 : \sim A$	$2 : \sim\sim A$	$A\wedge\sim\sim A$	$3 : \sim(A\wedge\sim\sim A)$	$\sim A\wedge\sim(A\wedge\sim\sim A)$
t	$(\mathbf{nf})_1$	$(\mathbf{fb})_1$	$(\mathbf{fb})_1$	$(\mathbf{b}(\mathbf{tb})_3)_1$	**f**
b	$(\mathbf{tb})_1$	$((\mathbf{nf})_2(\mathbf{tb})_2)_1$	$(\mathbf{fb})_1$	$(\mathbf{b}(\mathbf{tb})_3)_1$	**b**
n	**f**	**b**	**f**	**b**	**f**
f	**b**	$(\mathbf{tb})_2$	**f**	**b**	**b**

□

Proposition 3.2 can now be double-checked semantically.

Corollary 3.5. The following statements hold.

(i) $v({\sim}(A{\wedge}{\sim}{\sim}A){\wedge}{\sim}((A{\wedge}{\sim}{\sim}A){\wedge}{\sim}{\sim}(A{\wedge}{\sim}{\sim}A))) = \mathbf{b}$ for any four-valued negation inconsistent **FDE**-valuation v.

(ii) **b-FDE** is negation inconsistent.

Proof. For (i), note that the formula in Proposition 3.2 can be read as ${\sim}_4(A \wedge {\sim}{\sim}A)$. Moreover, the table in Proposition 3.4 shows that $A \wedge {\sim}{\sim}A$ must attain either the value **b** or **f**. Hence the value of ${\sim}_4(A \wedge {\sim}{\sim}A)$ must be **b**. For (ii), by (i), ${\sim}_4(A \wedge {\sim}{\sim}A)$ attains the value **b** for any v, and so $v({\sim}{\sim}_4(A \wedge {\sim}{\sim}A)) \in \{\mathbf{t}, \mathbf{b}\}$. Therefore $B \models_{4b} {\sim}_4(A \wedge {\sim}{\sim}A)$ and $B \models_{4b} {\sim}{\sim}_4(A \wedge {\sim}{\sim}A)$ for any B, i.e. **b-FDE** is negation inconsistent. □

Remark 3.6. It is also possible to reuse the example of a provable contradiction for $\mathbf{FDE}_4$ given in [17, Theorem 29]. The formula ${\sim}_4{\sim}_4A$ becomes in **b-FDE** an abbreviation of:

$${\sim}({\sim}A{\wedge}{\sim}(A{\wedge}{\sim}{\sim}A)) \wedge {\sim}(({\sim}A{\wedge}{\sim}(A{\wedge}{\sim}{\sim}A)) \wedge {\sim}{\sim}({\sim}A{\wedge}{\sim}(A{\wedge}{\sim}{\sim}A))).$$

Yet another example is the formula $(A{\vee}{\sim}{\sim}A){\wedge}{\sim}({\sim}(A{\vee}{\sim}{\sim}A){\wedge}{\sim}{\sim}{\sim}(A{\vee}{\sim}{\sim}A))$. In this case, the schema $A \wedge {\sim}({\sim}A \wedge {\sim}{\sim}{\sim}A)$ defines a connective giving the value **b** when $v(A) \in \{\mathbf{t}, \mathbf{b}\}$, and **f** when $v(A) \in \{\mathbf{n}, \mathbf{f}\}$. An obvious consequence is that A is a theorem of **b-FDE** iff $A \wedge {\sim}({\sim}A \wedge {\sim}{\sim}{\sim}A)$ is a provable contradiction.

At the same time, it is worth noting that the negation inconsistency of **b-FDE** is restricted in an important way.

Proposition 3.7. For no formula A, $B \vdash^b_{\mathbf{FDE}} {\sim}A$ and $B \vdash^b_{\mathbf{FDE}} {\sim}{\sim}A$ for all B.

Proof. We shall confirm the statement semantically. It suffices to show that for any A there is a four-valued negation inconsistent **FDE**-valuation v which gives the value $v(A) = \mathbf{b}$. Choose v such that $v(p) = \mathbf{b}$ for all p, and $v(C) = \mathbf{b}$ implies $v({\sim}C) = \mathbf{b}$, except when $C \equiv {\sim}A$. Then $v(A) = \mathbf{b}$ and $v({\sim}A) = \mathbf{t}$. Thus $v({\sim}{\sim}A) \notin \mathcal{D}$ and so $A \not\models_{4b} {\sim}{\sim}A$. □

3.2 Interdefinability of inconsistent negations

In the last subsection, we saw that ${\sim}_4$ is definable from the non-deterministic negation ${\sim}$. This raises the question of whether other inconsistent negations, namely ${\sim}_8, {\sim}_{12}$ and ${\sim}_{16}$, are definable as well. First we recall the definitions of the sixteen variants of negation in [17].

A	$\sim_1 A$	$\sim_2 A$	$\sim_3 A$	$\sim_4 A$	$\sim_5 A$	$\sim_6 A$	$\sim_7 A$	$\sim_8 A$
t	**f**	**f**	**f**	**f**	**f**	**f**	**f**	**f**
b	**b**	**b**	**b**	**b**	**t**	**t**	**t**	**t**
n	**n**	**n**	**f**	**f**	**n**	**n**	**f**	**f**
f	**t**	**b**	**t**	**b**	**t**	**b**	**t**	**b**

$\sim_9 A$	$\sim_{10} A$	$\sim_{11} A$	$\sim_{12} A$	$\sim_{13} A$	$\sim_{14} A$	$\sim_{15} A$	$\sim_{16} A$
n	**n**	**n**	**n**	**n**	**n**	**n**	**n**
b	**b**	**b**	**b**	**t**	**t**	**t**	**t**
n	**n**	**f**	**f**	**n**	**n**	**f**	**f**
t	**b**	**t**	**b**	**t**	**b**	**t**	**b**

The consequence relations $\models^i_{\mathbf{FDE}}$ for $i \in \{1, \ldots, 16\}$ are defined from that of $\models^b_{\mathbf{FDE}}$ by using an Nmatrix whose $\tilde{\sim}$ is defined according to the i-th negation above. v_i will be used to denote a valuation with respect to $\models^i_{\mathbf{FDE}}$.

We can now answer the question we posed earlier.

Proposition 3.8. b-FDE does not define any of $\sim_8, \sim_{12}, \sim_{16}$.

Proof. Since any valuation for $\models^4_{\mathbf{FDE}}$ can be mimicked by a four-valued negation inconsistent **FDE**-valuation, it suffices to show the existence of a valuation v_4 for $\models^4_{\mathbf{FDE}}$ that contradicts the definability of the respective negation. Firstly, if $\sim_8$ is definable, then $v_4(A) = \mathbf{b}$ implies $v_4(\sim_8 A) = \mathbf{t}$. But as the set $\{\mathbf{b}\}$ is closed under the matrix operations for $\models^4_{\mathbf{FDE}}$, taking $v_4(p) = \mathbf{b}$ forces $v_4(\sim_8 A) = \mathbf{b}$, a contradiction. Thus $\sim_8$ and similarly $\sim_{16}$ are not definable in $\models^4_{\mathbf{FDE}}$. For $\sim_{12}$ to be definable, $v_4(A) = \mathbf{t}$ must imply $v_4(\sim_{12} A) = \mathbf{n}$. But since the set $\{\mathbf{t}, \mathbf{b}, \mathbf{f}\}$ is closed under the matrix operations, we obtain a counterexample to the definability by again taking the valuation assigning $\mathbf{b}$ to all propositional variables. □

We can extend the result to observe that none of the consistent negations are definable in **b-FDE**.

Corollary 3.9. b-FDE does not define any of the sixteen variants except for $\sim_4$.

Proof. We again argue via the undefinability of the negations from $\sim_4$. The consideration in Proposition 3.8 rules out $\sim_5$–$\sim_{16}$. For $\sim_1$ and $\sim_3$, when $v_4(A) = \mathbf{f}$ we have to have $v_4(\sim_1 A) = (\sim_3 A) = \mathbf{t}$ if they are definable. But this is not possible if we take v_4 such that $v_4(p) = \mathbf{b}$ for all p, because the set $\{\mathbf{b}, \mathbf{f}\}$ is closed under the matrix operations.

Finally, for $\sim_2$, consider a formula A and another formula B in which A occurs as a subformula. Let S be the set of pairs of possible values B can take given a valuation v, where the first coordinate is the value when $v_4(A) = \mathbf{t}$, and the second when $v_4(A) = \mathbf{n}$. We claim $S = \{(\mathbf{t}, \mathbf{n}), (\mathbf{b}, \mathbf{f}), (\mathbf{t}, \mathbf{t}), (\mathbf{b}, \mathbf{b}), (\mathbf{n}, \mathbf{n}), (\mathbf{f}, \mathbf{f})\}$. Then it follows that $\sim_2$ is undefinable, for that would require the combination $(\mathbf{f}, \mathbf{n})$.

The claim can be checked by induction on the complexity of B. If $B \equiv A$, then we have the pair $(\mathbf{t}, \mathbf{n})$. Otherwise, if $B \equiv C \wedge D$ or $B \equiv C \vee D$, then there are two cases depending on whether A occurs in both C and D as a subformula. It is however straightforward to check that in each case, the possible combination of values is contained in S. When $B \equiv \sim_4 C$, then the possible combinations are either $(\mathbf{b}, \mathbf{b})$ or $(\mathbf{f}, \mathbf{f})$. Thus the claim follows. □

It is also possible to consider non-deterministic semantics that are intermediate between the four-valued negation inconsistent **FDE**-Nmatrix and the deterministic ones. They are of interest to see if less non-determinacy enables us to define more types of negation.

	$\tilde{\sim}_{b1}$	$\tilde{\sim}_{b2}$	$\tilde{\sim}_{b3}$	$\tilde{\sim}_{b4}$
t	**f**	**n**	**n**,**f**	**n**,**f**
b	**t**,**b**	**t**,**b**	**b**	**t**
n	**f**	**f**	**f**	**f**
f	**b**	**b**	**b**	**b**

Let the **b***i*-**FDE**-*Nmatrix* be the semantics obtained from the four-valued negation inconsistent **FDE**-Nmatrix by replacing $\tilde{\sim}$ with $\tilde{\sim}_{bi}$ for $i \in \{1, 2, 3, 4\}$. (The subscripts will be omitted when there is no fear of confusion.) The logics **bi-FDE** for $i \in \{1, 2, 3, 4\}$ are then defined by setting their consequences relations as in Definition 2.1.

Proposition 3.10. The following statements hold.

(i) $\sim_{12}$ is definable in **b2-FDE**.

(ii) $\sim_8$ is definable in **b4-FDE**.

Proof. (i) Recall that $\sim_4$ is definable in **b-FDE**; the connective is therefore also definable in the **b2-FDE**. Then we claim that $\sim_{12} A$ is definable as $(A \wedge \sim_{b2} A) \vee \sim_4 A$.

A	$1 : \sim_{b2} A$	$A \wedge \sim_{b2} A$	$\sim_4 A$	$(A \wedge \sim_{b2} A) \vee \sim_4 A$
t	**n**	**n**	**f**	**n**
b	$(\mathbf{tb})_1$	**b**	**b**	**b**
n	**f**	**f**	**f**	**f**
f	**b**	**f**	**b**	**b**

(ii) We claim $\sim_8 A$ is definable as $\sim_{b4} A \wedge \sim_{b4}(A \wedge \sim_{b4} A)$.

A	$1 : \sim_{b4} A$	$A \wedge \sim_{b4} A$	$2 : \sim_{b4}(A \wedge \sim_{b4} A)$	$\sim_{b4} A \wedge \sim_{b4}(A \wedge \sim_{b4} A)$
t	$(\mathbf{nf})_1$	$(\mathbf{nf})_1$	$(\mathbf{fb})_1$	**f**
b	**t**	**b**	**t**	**t**
n	**f**	**f**	**b**	**f**
f	**b**	**f**	**b**	**b**

□

In other cases, the deterministic negations are not definable.

Proposition 3.11. The following statements hold.

(i) $\sim_8$ is not definable in **bi-FDE** for $i \in \{1, 2, 3\}$.

(ii) $\sim_{12}$ is not definable in **bi-FDE** for $i \in \{1, 3, 4\}$.

(iii) $\sim_{16}$ is not definable in **bi-FDE** for $i \in \{1, 2, 3, 4\}$.

Proof. The arguments are analogous to the one in Proposition 3.8. □

4 Introducing more non-determinacy

Having noted the negation inconsistency of **b-FDE**, one may question if the same holds for a weaker system obtained by introducing more non-determinacy for $\sim$. For such generalisations, we must consider operators outside the framework for what counts as a negation in [17]. Hence $\sim$ in this section should be seen as a kind of 'sub-negation' operator, rather than a full-fledged negation: nonetheless, for the sake of simplicity, we will keep using the term 'negation inconsistency' to refer to the property of a logic that satisfies the condition explained at the beginning of the last section.

What interests us at present are Nmatrices which can be refined to the four-valued negation inconsistent **FDE**-Nmatrix by changing $\tilde{\sim}$. Aside from deepening our understanding of the phenomenon of negation inconsistency, such Nmatrices may turn out to be useful for ones with a more liberal view on the definition of negation than in [17]. There are many options to introduce additional values to the table of negation in the four-valued negation inconsistent **FDE**-Nmatrix. We can however check that most of them remove negation inconsistency, either by allowing a valuation which either excludes the target designated value (**b** in this case), or damages the closure of the target value under matrix operations. Stating the conclusion first, we claim that the following three operators are the only candidates for negation inconsistency that can be obtained in this manner.

	$\tilde{\sim}_{w1}$	$\tilde{\sim}_{w2}$	$\tilde{\sim}_{w3}$
t	**b**,**n**,**f**	**n**,**f**	**b**,**n**,**f**
b	**t**,**b**	**t**,**b**	**t**,**b**
n	**f**	**b**,**f**	**b**,**f**
f	**b**	**b**	**b**

Proposition 4.1. The other ways of introducing a new value to the four-valued negation inconsistent **FDE**-Nmatrix produce negation consistency.

Proof. It suffices to check all the cases where one value is added to one of $\tilde{\sim}(\mathbf{t})$, $\tilde{\sim}(\mathbf{b})$, $\tilde{\sim}(\mathbf{n})$, $\tilde{\sim}(\mathbf{f})$: if the addition of one value breaks negation inconsistency, the addition of more values does not restore it.

To $\tilde{\sim}(\mathbf{t})$ we can add $\mathbf{t}$, i.e. $\tilde{\sim}(\mathbf{t}) = \{\mathbf{t}, \mathbf{n}, \mathbf{f}\}$. For negation inconsistency, we would need a formula attaining $v(A) = \mathbf{b}$ for any v. However, if we set $v(p) = \mathbf{t}$ and $v(\sim B) = \mathbf{t}$ for all p and B occurring in A, we get $v(A) = \mathbf{t}$.

If we add further values to $\tilde{\sim}(\mathbf{b})$ then even when $v(A) = \mathbf{b}$ we can set $v(\sim A)$ to be undesignated; so we no longer have negation inconsistency.

To $\tilde{\sim}(\mathbf{n})$ we can add $\mathbf{t}$ or $\mathbf{n}$, but then we can choose a valuation which assigns values only from $\{\mathbf{t}, \mathbf{n}\}$. So negation inconsistency is lost in both cases.

To $\tilde{\sim}(\mathbf{f})$ we can add $\mathbf{t}$, $\mathbf{n}$ or $\mathbf{f}$, but then we can choose a valuation which assigns values only from $\{\mathbf{t}, \mathbf{n}, \mathbf{f}\}$. □

Now it remains to show that $\sim_{w1}$–$\sim_{w3}$ above keep the negation inconsistency. Let us denote by $\models_{w1} - \models_{w3}$ the consequence relations defined by the Nmatrices with the operators (to be called the **w1-FDE** – **w3-FDE**-Nmatrices), which give rise to systems **wi-FDE** for $i \in \{1, \ldots, 3\}$. (Similar remarks apply for the systems introduced afterwards.)

Theorem 4.2. w1-FDE – **w3-FDE** are negation inconsistent.

Proof. It suffices to show that **w3-FDE** is negation inconsistent, as the other two are based on refinements of **w3-FDE**-Nmatrix. We will show the formula

$$(B\vee{\sim}{\sim}B)\wedge({\sim}B\vee{\sim}{\sim}{\sim}B)\wedge{\sim}(B\wedge{\sim}B\wedge{\sim}{\sim}B\wedge{\sim}{\sim}{\sim}B)$$

where $B \equiv (A \wedge \sim A)$ gives an instance of negation inconsistency. This can be checked through the following tables.

B	$1 : \sim B$	$2 : \sim\sim B$	$B\vee\sim\sim B$
$\mathbf{t}$	$(\mathbf{bnf})_1$	$((\mathbf{tb})_2(\mathbf{bf})_2\mathbf{b})_1$	$\mathbf{t}$
$\mathbf{b}$	$(\mathbf{tb})_1$	$((\mathbf{bnf})_2(\mathbf{tb})_2)_1$	$((\mathbf{btb})_2(\mathbf{tb})_2)_1$
$\mathbf{n}$	$(\mathbf{bf})_1$	$((\mathbf{tb})_2\mathbf{b})_1$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{b}$	$(\mathbf{tb})_2$	$(\mathbf{tb})_2$

$3 : \sim\sim\sim B$	$\sim B\vee\sim\sim\sim B$
$(((\mathbf{bnf})_3(\mathbf{tb})_3)_2((\mathbf{tb})_3\mathbf{b})_2(\mathbf{tb})_3)_1$	$(((\mathbf{btb})_3(\mathbf{tb})_3)_2\mathbf{t}(\mathbf{tb})_3)_1$
$(((\mathbf{tb})_3(\mathbf{bf})_3\mathbf{b})_2((\mathbf{bnf})_3(\mathbf{tb})_3)_2)_1$	$(\mathbf{t}((\mathbf{btb})_3(\mathbf{tb})_3)_2)_1$
$(((\mathbf{bnf})_3(\mathbf{tb})_3)_2(\mathbf{tb})_3)_1$	$(((\mathbf{btb})_3(\mathbf{tb})_3)_2(\mathbf{tb})_3)_1$
$((\mathbf{bnf})_3(\mathbf{tb})_3)_2$	$((\mathbf{btb})_3(\mathbf{tb})_3)_2$

$(B\vee\sim\sim B)\wedge(\sim B\vee\sim\sim\sim B)$	$B\wedge\sim B$	$B\wedge\sim B\wedge\sim\sim B$
$(((\mathbf{btb})_3(\mathbf{tb})_3)_2\mathbf{t}(\mathbf{tb})_3)_1$	$(\mathbf{bnf})_1$	$(\mathbf{bff})_1$
$((\mathbf{btb})_2((\mathbf{btb})_3\mathbf{b})_2)_1$	$\mathbf{b}$	$((\mathbf{bff})_2\mathbf{b})_1$
$(((\mathbf{btb})_3(\mathbf{tb})_3)_2(\mathbf{tb})_3)_1$	$\mathbf{f}$	$\mathbf{f}$
$((\mathbf{btb})_3\mathbf{b})_2$	$\mathbf{f}$	$\mathbf{f}$

$B\wedge{\sim}B\wedge{\sim\sim}B\wedge{\sim\sim\sim}B$	$4: {\sim}(B\wedge{\sim}B\wedge{\sim\sim}B\wedge{\sim\sim\sim}B)$
$(((\mathbf{bff})_3\mathbf{b})_2\mathbf{ff})_1$	$((((\mathbf{tb})_4\mathbf{bb})_3(\mathbf{tb})_4)_2\mathbf{bb})_1$
$((\mathbf{bff})_2((\mathbf{bff})_3\mathbf{b})_2)_1$	$(((\mathbf{tb})_4\mathbf{bb})_2(((\mathbf{tb})_4\mathbf{bb})_3(\mathbf{tb})_4)_2)_1$
$\mathbf{f}$	$\mathbf{b}$
$\mathbf{f}$	$\mathbf{b}$

$(B\vee{\sim\sim}B)\wedge({\sim}B\vee{\sim\sim\sim}B)\wedge{\sim}(B\wedge{\sim}B\wedge{\sim\sim}B\wedge{\sim\sim\sim}B)$
$((\mathbf{b}(\mathbf{t}\text{ or }\mathbf{b}))_2\mathbf{bb})_1$
$\mathbf{b}$
$\mathbf{b}$
$\mathbf{b}$

(In the last table, '(**t** or **b**)' represents that the possible values depend on more than one non-determinacy, in this case the ones introduced at 3 and 4.)

Now since $A \wedge {\sim}A$ never obtains the value **t**, then taking the formula as B gives the desired formula which attains the value **b** under any valuation. □

Does $\tilde{\sim}_{w3}$ give the most general negation inconsistent operator in the present setting? While this is true if we start from the four-valued negation inconsistent **FDE**-Nmatrix, it is possible to obtain another generalisation by starting instead from the **b1-FDE**-Nmatrix, namely the one given by the following table.

	$\tilde{\sim}_{w4}$
t	**b**,**f**
b	**t**,**b**
n	**t**,**b**,**f**
f	**b**

Theorem 4.3. w4-FDE is negation inconsistent.

Proof. First note that if $v(B) \in \{\mathbf{b}, \mathbf{f}\}$, then the formula we used in Theorem 4.2:

$$(B\vee{\sim\sim}B)\wedge({\sim}B\vee{\sim\sim\sim}B)\wedge{\sim}(B\wedge{\sim}B\wedge{\sim\sim}B\wedge{\sim\sim\sim}B)$$

attains the value **b** in v as well, because v assigns one of the values from $\{\mathbf{t}, \mathbf{b}, \mathbf{f}\}$ for formulas constructed from B, and in particular there are less choices in $\tilde{\sim}_{w4}$ than in $\tilde{\sim}_{w3}$ when it comes to these values. Now, taking $B \equiv {\sim}A \wedge {\sim\sim}A$ gives the desired instance of a formula attaining only **b** or **f**.

A	$1: {\sim}A$	$2: {\sim\sim}A$	${\sim}A \wedge {\sim\sim}A$
t	$(\mathbf{bf})_1$	$((\mathbf{tb})_2\mathbf{b})_1$	$(\mathbf{bf})_1$
b	$(\mathbf{tb})_1$	$((\mathbf{bf})_2(\mathbf{tb})_2)_1$	$((\mathbf{bf})_2\mathbf{b})_1$
n	$(\mathbf{tbf})_1$	$((\mathbf{bf})_2(\mathbf{tb})_2\mathbf{b})_1$	$((\mathbf{bf})_2\mathbf{bf})_1$
f	$\mathbf{b}$	$(\mathbf{tb})_2$	$\mathbf{b}$

□

The incomparability of **w3-FDE** and **w4-FDE** is witnessed by the following formulas.

Proposition 4.4. The following statements hold.

(i) $\models_{w3} A \vee {\sim}{\sim}A$ but $\not\models_{w4} A \vee {\sim}{\sim}A$.

(ii) $\models_{w4} {\sim}A \vee {\sim}{\sim}A$ but $\not\models_{w3} {\sim}A \vee {\sim}{\sim}A$.

Proof. It is not difficult to observe these from the tables given in Theorem 4.2 and Theorem 4.3. □

The proposition tells that contrary to the impression **b-FDE** might give, $A \vee {\sim}{\sim}A$ is not essential for negation inconsistency[2]. It also means the negation inconsistency w.r.t. ${\sim}_4$ and ${\sim}_8$ can be interpreted to come from two different sources.

5 Characterising negation inconsistency

So far, we have considered generalisations of negation inconsistent alternatives to the **FDE**-negation. In this section, we shall take a broader view in order to capture all non-deterministic unary operators which give rise to 'negation inconsistency' when combined with the deterministic conjunction and disjunction. This will characterise the limit of negation inconsistency in the setting. It will also provide someone with a liberal account of negation a larger pool of unary connectives to choose from, when negation inconsistency is desired.

The methodology for obtaining the operators remains the same as the one in the previous sections. The operators will be presented in some groups, to be followed by the observation that they exhaust all possibilities.

5.1 First group

The operators in the first group are such that negation inconsistency is obtained when and only when there is a formula which attains the value **b** under any valuation: recall that ${\sim}_{w3}$ and ${\sim}_{w4}$ are also of this character.

	$\tilde{\sim}_{w5}$	$\tilde{\sim}_{w6}$
t	**b**,**n**	**b**,**n**,**f**
b	**t**,**b**	**t**,**b**
n	**b**	**b**
f	**t**,**b**,**n**	**b**,**n**

Proposition 5.1. **w5-FDE** and **w6-FDE** are negation inconsistent.

[2]However, note also that $\models_{w4} {\sim}A \vee {\sim}{\sim}{\sim}A$ because ${\sim}A$ never attains the value **n**, which is the source of invalidity of $A \vee {\sim}{\sim}A$.

Proof. For $\models_{w5}$, let $C[p]$ be the formula:

$$((\sim p \wedge \sim\sim p) \vee (\sim\sim p \wedge \sim\sim\sim p)) \wedge \sim(\sim\sim p \wedge \sim\sim\sim p).$$

We claim the formula $\sim B \wedge (B \vee (C[B] \wedge \sim C[B]))$, where $B \equiv (\sim A \wedge \sim\sim A) \vee C[A]$ is a witness of the negation inconsistency.

B	$1 : \sim B$	$2 : \sim\sim B$	$3 : \sim\sim\sim B$
$\mathbf{t}$	$(\mathbf{bn})_1$	$((\mathbf{tb})_2\mathbf{b})_1$	$(((\mathbf{bn})_3(\mathbf{tb})_3)_2(\mathbf{tb})_3)_1$
$\mathbf{b}$	$(\mathbf{tb})_1$	$((\mathbf{bn})_2(\mathbf{tb})_2)_1$	$(((\mathbf{tb})_3\mathbf{b})_2((\mathbf{bn})_3(\mathbf{tb})_3)_2)_1$
$\mathbf{n}$	$\mathbf{b}$	$(\mathbf{tb})_2$	$((\mathbf{bn})_3(\mathbf{tb})_3)_2$
$\mathbf{f}$	$(\mathbf{tbn})_1$	$((\mathbf{bn})_2(\mathbf{tb})_2\mathbf{b})_1$	$(((\mathbf{tb})_3\mathbf{b})_2((\mathbf{bn})_3(\mathbf{tb})_3)_2(\mathbf{tb})_3)_1$

$\sim B \wedge \sim\sim B$	$\sim\sim B \wedge \sim\sim\sim B$	$(\sim B \wedge \sim\sim B) \vee (\sim\sim B \wedge \sim\sim\sim B)$
$(\mathbf{bf})_1$	$(((\mathbf{bn})_3\mathbf{b})_2\mathbf{b})_1$	$(((\mathbf{bt})_3\mathbf{b})_2\mathbf{b})_1$
$((\mathbf{bn})_2\mathbf{b})_1$	$((\mathbf{bf})_2((\mathbf{bn})_3\mathbf{b})_2)_1$	$((\mathbf{bn})_2((\mathbf{bt})_3\mathbf{b})_2)_1$
$\mathbf{b}$	$((\mathbf{bn})_3\mathbf{b})_2$	$((\mathbf{bt})_3\mathbf{b})_2$
$((\mathbf{bn})_2\mathbf{bf})_1$	$((\mathbf{bf})_2((\mathbf{bn})_3\mathbf{b})_2\mathbf{b})_1$	$((\mathbf{bn})_2((\mathbf{bt})_3\mathbf{b})_2\mathbf{b})_1$

$4 : \sim(\sim\sim B \wedge \sim\sim\sim B)$	$C[B]$
$((((\mathbf{tb})_4\mathbf{b})_3(\mathbf{tb})_4)_2(\mathbf{tb})_4)_1$	$\mathbf{b}$
$(((\mathbf{tb})_4(\mathbf{tbn})_4)_2(((\mathbf{tb})_4\mathbf{b})_3(\mathbf{tb})_4)_2)_1$	$((\mathbf{b}(\mathbf{nfn})_4)_2\mathbf{b})_1$
$(((\mathbf{tb})_4\mathbf{b})_3(\mathbf{tb})_4)_2$	$\mathbf{b}$
$(((\mathbf{tb})_4(\mathbf{tbn})_4)_2(((\mathbf{tb})_4\mathbf{b})_3(\mathbf{tb})_4)_2(\mathbf{tb})_4)_1$	$((\mathbf{b}(\mathbf{nfn})_4)_2\mathbf{bb})_1$

$5 : \sim C[B]$	$C[B] \wedge \sim C[B]$
$(\mathbf{tb})_5$	$\mathbf{b}$
$(((\mathbf{tb})_5(\mathbf{b}(\mathbf{tbn})_5\mathbf{b})_4)_2(\mathbf{tb})_5)_1$	$((\mathbf{bf})_2\mathbf{b})_1$
$(\mathbf{tb})_5$	$\mathbf{b}$
$(((\mathbf{tb})_5(\mathbf{b}(\mathbf{tbn})_5\mathbf{b})_4)_2(\mathbf{tb})_5(\mathbf{tb})_5)_1$	$((\mathbf{bf})_2\mathbf{bb})_1$

Now it is straightforward to see that if $v(B) \in \{\mathbf{b}, \mathbf{n}\}$ then $v(\sim B \wedge (B \vee (C[B] \wedge \sim C[B]))) = \mathbf{b}$. It is also not difficult to check using the above tables that $(\sim A \wedge \sim\sim A) \vee C[A]$ always attains one of the values $\mathbf{b}$ and $\mathbf{n}$; hence we obtain the desired negation inconsistency.

For $\models_{w6}$, we claim the following formula:

$$(B \wedge \sim B) \vee ((\sim B \wedge \sim\sim B) \wedge \sim(\sim B \wedge \sim\sim B)) \vee ((\sim\sim B \wedge \sim\sim\sim B) \wedge \sim(\sim\sim B \wedge \sim\sim\sim B))$$

where $B \equiv (A \wedge \sim A)$, witnesses the negation inconsistency.

B	$1 : \sim B$	$2 : \sim\sim B$	$3 : \sim\sim\sim B$
$\mathbf{t}$	$(\mathbf{bnf})_1$	$((\mathbf{tb})_2\mathbf{b}(\mathbf{bn})_2)_1$	$(((\mathbf{bnf})_3(\mathbf{tb})_3)_2(\mathbf{tb})_3((\mathbf{tb})_3\mathbf{b})_2)_1$
$\mathbf{b}$	$(\mathbf{tb})_1$	$((\mathbf{bnf})_2(\mathbf{tb})_2)_1$	$(((\mathbf{tb})_3\mathbf{b}(\mathbf{bn})_3)_2((\mathbf{bnf})_3(\mathbf{tb})_3)_2)_1$
$\mathbf{n}$	$\mathbf{b}$	$(\mathbf{tb})_2$	$((\mathbf{bnf})_3(\mathbf{tb})_3)_2$
$\mathbf{f}$	$(\mathbf{bn})_1$	$((\mathbf{tb})_2\mathbf{b})_1$	$(((\mathbf{bnf})_3(\mathbf{tb})_3)_2(\mathbf{tb})_3)_1$

$B \wedge {\sim}B$	${\sim}B \wedge {\sim}{\sim}B$	${\sim}{\sim}B \wedge {\sim}{\sim}{\sim}B$
$(\mathbf{bnf})_1$	$(\mathbf{bff})_1$	$(((\mathbf{bnf})_3\mathbf{b})_2\mathbf{b}(\mathbf{bf})_2)_1$
$\mathbf{b}$	$((\mathbf{bnf})_2\mathbf{b})_1$	$((\mathbf{bff})_2((\mathbf{bnf})_3\mathbf{b})_2)_1$
$\mathbf{f}$	$\mathbf{b}$	$((\mathbf{bnf})_3\mathbf{b})_2$
$\mathbf{f}$	$(\mathbf{bf})_1$	$(((\mathbf{bnf})_3\mathbf{b})_2\mathbf{b})_1$

The table for $B \wedge {\sim}B$ indicates that when a conjunction is formed by a formula and its 'negation', it does not affect the values **b**,**f** and the value **n** is transformed into **f**. We therefore obtain the following tables.

$({\sim}B \wedge {\sim}{\sim}B) \wedge {\sim}({\sim}B \wedge {\sim}{\sim}B)$	$({\sim}{\sim}B \wedge {\sim}{\sim}{\sim}B) \wedge {\sim}({\sim}{\sim}B \wedge {\sim}{\sim}{\sim}B)$
$(\mathbf{bff})_1$	$(((\mathbf{bff})_3\mathbf{b})_2\mathbf{b}(\mathbf{bf})_2)_1$
$((\mathbf{bff})_2\mathbf{b})_1$	$((\mathbf{bff})_2((\mathbf{bff})_3\mathbf{b})_2)_1$
$\mathbf{b}$	$((\mathbf{bff})_3\mathbf{b})_2$
$(\mathbf{bf})_1$	$(((\mathbf{bff})_3\mathbf{b})_2\mathbf{b})_1$

It is now easy to check that when $v(B) \in \{\mathbf{b}, \mathbf{n}, \mathbf{f}\}$, then the claimed formula attains the value **b**. Then it is clear from the above table that $A \wedge {\sim}A$ always attains one of the three values. □

5.2 Second group

The operators in the second group are such that a formula witnesses negation inconsistency if and only if it always attains the value **t**. We will consider four such operators.

	$\tilde{\sim}_{w7}$	$\tilde{\sim}_{w8}$	$\tilde{\sim}_{w9}$	$\tilde{\sim}_{w10}$
t	**t**,**b**	**t**,**b**	**t**,**b**	**t**,**b**
b	**t**,**n**	**t**,**n**	**t**,**n**,**f**	**t**,**n**,**f**
n	**t**,**b**	**t**,**b**,**f**	**t**,**b**	**t**,**b**,**f**
f	**t**,**b**,**n**	**t**,**b**	**t**,**n**	**t**

Proposition 5.2. w7-FDE – w10-FDE are negation inconsistent.

Proof. In each case we need to construct a formula such that $v(A) = \mathbf{t}$ for all respective valuations. For $\models_{w7}$, the formula ${\sim}_{w7}A \vee {\sim}_{w7}{\sim}_{w7}A$ witnesses the negation inconsistency. For $\models_{w8}$, the formula $A \vee {\sim}_{w8}A \vee {\sim}_{w8}{\sim}_{w8}A$ witnesses the negation inconsistency. These can be checked readily from the tables below.

A	$1 : {\sim}_{w7}A$	$2 : {\sim}_{w7}{\sim}_{w7}A$
t	$(\mathbf{tb})_1$	$((\mathbf{tb})_2(\mathbf{tn})_2)_1$
b	$(\mathbf{tn})_1$	$(\mathbf{tb})_2$
n	$(\mathbf{tb})_1$	$((\mathbf{tb})_2(\mathbf{tn})_2)_1$
f	$(\mathbf{tbn})_1$	$((\mathbf{tb})_2(\mathbf{tn})_2(\mathbf{tb})_2)_1$

A	$1 : {\sim}_{w8}A$	$2 : {\sim}_{w8}{\sim}_{w8}A$
t	$(\mathbf{tb})_1$	$((\mathbf{tb})_2(\mathbf{tn})_2)_1$
b	$(\mathbf{tn})_1$	$((\mathbf{tb})_2(\mathbf{tbf})_2)_1$
n	$(\mathbf{tbf})_1$	$((\mathbf{tb})_2(\mathbf{tn})_2(\mathbf{tb})_2)_1$
f	$(\mathbf{tb})_1$	$((\mathbf{tb})_2(\mathbf{tn})_2)_1$

For $\models_{w9}$, the formula $A \vee \sim_{w9} A \vee \sim_{w9}\sim_{w9} A$ witnesses the negation inconsistency. Finally, $A \vee \sim_{w10} A \vee \sim_{w10}\sim_{w10} A$ witnesses the negation inconsistency for $\models_{w10}$.

A	$1 : \sim_{w9} A$	$2 : \sim_{w9}\sim_{w9} A$
t	$(\mathbf{tb})_1$	$((\mathbf{tb})_2(\mathbf{tnf})_2)_1$
b	$(\mathbf{tnf})_1$	$((\mathbf{tb})_2(\mathbf{tb})_2(\mathbf{tn})_2)_1$
n	$(\mathbf{tb})_1$	$((\mathbf{tb})_2(\mathbf{tnf})_2)_1$
f	$(\mathbf{tn})_1$	$(\mathbf{tb})_2$

A	$1 : \sim_{w10} A$	$2 : \sim_{w10}\sim_{w10} A$
t	$(\mathbf{tb})_1$	$((\mathbf{tb})_2(\mathbf{tnf})_2)_1$
b	$(\mathbf{tnf})_1$	$((\mathbf{tb})_2(\mathbf{tbf})_2\mathbf{t})_1$
n	$(\mathbf{tbf})_1$	$((\mathbf{tb})_2(\mathbf{tnf})_2\mathbf{t})_1$
f	**t**	$(\mathbf{tb})_2$

□

5.3 Third group

The third group concerns operators which are closed under the designated values **t** and **b**. We will consider the next two tables.

	$\tilde{\sim}_{w11}$	$\tilde{\sim}_{w12}$
t	**t**,**b**	**t**,**b**
b	**t**,**b**	**t**,**b**
n	**t**,**b**	**t**,**b**,**f**
f	**t**,**b**,**n**	**t**,**b**

If we define $\models_{w11}$ and $\models_{w12}$ with these operators, then it is straightforward to observe that $\sim_{wi} A \vee \sim_{wi}\sim_{wi} A$ witnesses the negation inconsistency of **wi-FDE** for $i \in \{11, 12\}$.

5.4 A characterisation

Having observed the negation inconsistency of **w1-FDE** – **w12-FDE**, we are now ready to obtain a characterisation of operators which give rise to negation inconsistency. For a precise statement, we introduce another definition.

Definition 5.3. We define the *four-valued general* **FDE**-*Nmatrix* to be like four-valued negation inconsistent **FDE**-Nmatrix, except that $\tilde{\sim}$ is given by:

	$\tilde{\sim}$
t	**t**,**b**,**n**,**f**
b	**t**,**b**,**n**,**f**
n	**t**,**b**,**n**,**f**
f	**t**,**b**,**n**,**f**

We claim that $\tilde{\sim}_{w3}$–$\tilde{\sim}_{w12}$, listed again below, cover all the possibilities for negation inconsistent unary operators in $\mathcal{L}_{\mathbf{FDE}}$.

	$\tilde{\sim}_{w3}$	$\tilde{\sim}_{w4}$	$\tilde{\sim}_{w5}$	$\tilde{\sim}_{w6}$	$\tilde{\sim}_{w7}$
t	**b**,**n**,**f**	**b**,**f**	**b**,**n**	**b**,**n**,**f**	**t**,**b**
b	**t**,**b**	**t**,**b**	**t**,**b**	**t**,**b**	**t**,**n**
n	**b**,**f**	**t**,**b**,**f**	**b**	**b**	**t**,**b**
f	**b**	**b**	**t**,**b**,**n**	**b**,**n**	**t**,**b**,**n**

$\tilde{\sim}_{w8}$	$\tilde{\sim}_{w9}$	$\tilde{\sim}_{w10}$	$\tilde{\sim}_{w11}$	$\tilde{\sim}_{w12}$
t,**b**	**t**,**b**	**t**,**b**	**t**,**b**	**t**,**b**
t,**n**	**t**,**n**,**f**	**t**,**n**,**f**	**t**,**b**	**t**,**b**
t,**b**,**f**	**t**,**b**	**t**,**b**,**f**	**t**,**b**	**t**,**b**,**f**
t,**b**	**t**,**n**	**t**	**t**,**b**,**n**	**t**,**b**

Theorem 5.4. Let $\mathcal{M}$ be a refinement of the four-valued general **FDE**-Nmatrix. Then the logic defined by $\mathcal{M}$ is negation inconsistent if and only if $\mathcal{M}$ is also a refinement of one of the **w**i-**FDE**-Nmatrices for $i \in \{3, \ldots, 12\}$.

Proof. We first note that $\tilde{\diamond}_{\mathcal{M}} = \tilde{\diamond}$ for $\diamond \in \{\wedge, \vee\}$. Then the 'if' direction of the statement immediately follows from the fact that **w3-FDE** – **w12-FDE** are negation inconsistent. For the 'only if' direction, suppose the logic defined by $\mathcal{M}$ is negation inconsistent. Then one of $\tilde{\sim}_{\mathcal{M}}(\mathbf{t})$ or $\tilde{\sim}_{\mathcal{M}}(\mathbf{b})$ has to be contained in the set of designated values $\{\mathbf{t}, \mathbf{b}\}$. Hence we have the next three cases.

1. the containment holds for both $\tilde{\sim}_{\mathcal{M}}(\mathbf{t})$ and $\tilde{\sim}_{\mathcal{M}}(\mathbf{b})$

If the containment holds for both of them, then we claim that $\mathcal{M}$ has to be a refinement of the **w11-FDE**-Nmatrix or the **w12-FDE**-Nmatrix. For suppose it is not a refinement of the **w11-FDE**-Nmatrix. Then there must be values x and y such that $y \in \tilde{\sim}_{\mathcal{M}}(x)$ but $y \notin \tilde{\sim}_{w11}(x)$. x cannot be **t** or **b** because $\tilde{\sim}_{w11}$ contains both of the designated values. If $x = \mathbf{n}$, then first $y \neq \mathbf{n}$ since that would allow a valuation assigning all formulas the value **n**, which results in negation consistency. So y must be **f**. But then $\mathbf{n}, \mathbf{f} \notin \tilde{\sim}_{\mathcal{M}}(\mathbf{f})$ because otherwise a valuation assigning all formulas either a value from $\{\mathbf{f}\}$ or from $\{\mathbf{n}, \mathbf{f}\}$ becomes possible. Hence the Nmatrix must be a refinement of the **w12-FDE**-Nmatrix.

2. the containment holds only for $\tilde{\sim}_{\mathcal{M}}(\mathbf{t})$

If the containment holds only for $\tilde{\sim}_{\mathcal{M}}(\mathbf{t})$, then we claim that $\mathcal{M}$ is a refinement of one of the **w**i-**FDE**-Nmatrices for $i \in \{7, 8, 9, 10\}$. For if it is not a refinement of the **w7-FDE**-Nmatrix, then again there must be values x and y such that $y \in \tilde{\sim}_{\mathcal{M}}(x)$ but $y \notin \tilde{\sim}_{w7}(x)$. For the same reason as the last time, x cannot be **t**. It cannot be **f** either, so as to avoid a valuation assigning all formulas the value **f**.

If $x = \mathbf{b}$, then $y \neq \mathbf{b}$ because that would allow a valuation assigning formulas the value **b** up until a certain point, which results in negation consistency by the argument in Proposition 3.7. (Note by assumption, $\tilde{\sim}_{\mathcal{M}}(\mathbf{b})$ must contain an undesignated value.) So y must be **f**. In this case, we shall see that $\mathcal{M}$ must be a refinement of either the **w9-FDE**-Nmatrix or the **w10-FDE**-Nmatrix. If it is not a refinement of the **w9-FDE**-Nmatrix, then there are x' and y' such that $y' \in \tilde{\sim}_{\mathcal{M}}(x')$ but $y' \notin \tilde{\sim}_{w9}(x')$. Again $x' \neq \mathbf{t}$ and also $x' = \mathbf{b}$ implies $y' = \mathbf{b}$, which is impossible for the reason we saw earlier. Moreover, if $x' = \mathbf{f}$ then y' must be either $\mathbf{b}$ or $\mathbf{f}$, but this is impossible as they each enable a valuation assigning all formulas a value from $\{\mathbf{b}, \mathbf{f}\}$ or $\{\mathbf{f}\}$. Thus x' must be $\mathbf{n}$. Then y' must be $\mathbf{f}$ to avoid a valuation assigning all formulas the value $\mathbf{n}$. This necessitates $\tilde{\sim}_{\mathcal{M}}(\mathbf{f}) = \{\mathbf{t}\}$ since if other values are in the set then a valuation assigning all formulas a value from one of $\{\mathbf{b}, \mathbf{f}\}$, $\{\mathbf{n}, \mathbf{f}\}$ or $\{\mathbf{f}\}$ becomes possible. Therefore $\mathcal{M}$ is a refinement of the **w10-FDE**-Nmatrix.

If $x = \mathbf{n}$, then y must be $\mathbf{f}$. In this case, we shall see that $\mathcal{M}$ must be a refinement of either the **w8-FDE**-Nmatrix or the **w10-FDE**-Nmatrix. If it is not a refinement of the **w8-FDE**-Nmatrix, then there are x' and y' such that $y' \in \tilde{\sim}_{\mathcal{M}}(x')$ but $y' \notin \tilde{\sim}_{w8}(x')$. We can check that y' must be $\mathbf{b}$ and x' must be $\mathbf{f}$. This necessitates $\mathcal{M}$ being a refinement of the **w10-FDE**-Nmatrix.

3. the containment holds only for $\tilde{\sim}_{\mathcal{M}}(\mathbf{b})$

If the containment holds only for $\tilde{\sim}_{\mathcal{M}}(\mathbf{b})$, then $\mathcal{M}$ has to be a refinement of one of the **w**i-**FDE**-Nmatrices for some $i \in \{3, 4, 5, 6\}$. To establish this, assume that it is not a refinement of the **w3-FDE**-Nmatrix. Then there must be values x and y such that $y \in \tilde{\sim}_{\mathcal{M}}(x)$ but $y \notin \tilde{\sim}_{w3}(x)$. x cannot be $\mathbf{t}$ or $\mathbf{b}$, so it must be that $x = \mathbf{n}$ or $x = \mathbf{f}$.

If $x = \mathbf{n}$, then y must be $\mathbf{t}$; but then $\mathbf{n}$ must not be an element of $\tilde{\sim}_{\mathcal{M}}(\mathbf{t})$ nor $\tilde{\sim}_{\mathcal{M}}(\mathbf{f})$. From this we can infer that $\mathcal{M}$ is a refinement of the **w4-FDE**-Nmatrix.

If $x = \mathbf{f}$, then either $y = \mathbf{t}$ or $y = \mathbf{n}$. In the former case, it must be the case that $\mathbf{f} \notin \tilde{\sim}_{\mathcal{M}}(\mathbf{t})$ and consequently we must have $\mathbf{n} \in \tilde{\sim}_{\mathcal{M}}(\mathbf{t})$, as $\tilde{\sim}_{\mathcal{M}}(\mathbf{t})$ is not contained in the set of designated values. This necessitates $\tilde{\sim}_{\mathcal{M}}(\mathbf{n}) = \{\mathbf{b}\}$ for otherwise a valuation assigning all formulas a value from $\{\mathbf{t}, \mathbf{n}, \mathbf{f}\}$ becomes possible. Hence $\mathcal{M}$ is a refinement of the **w5-FDE**-Nmatrix. In the latter case, we claim $\mathcal{M}$ is a refinement of either the **w5-FDE**-Nmatrix or the **w6-FDE**-Nmatrix. If it is not a refinement of the **w5-FDE**-Nmatrix, then there are x' and y' such that $y' \in \tilde{\sim}_{\mathcal{M}}(x')$ but $y' \notin \tilde{\sim}_{w5}(x')$. One can again straightforwardly see that x' is neither $\mathbf{b}$ nor $\mathbf{f}$. Furthermore, if x' is $\mathbf{n}$ then y' is either $\mathbf{t}$ or $\mathbf{f}$. This however enables a valuation assigning all formulas a value from $\{\mathbf{t}, \mathbf{n}, \mathbf{f}\}$, since one of $\mathbf{n}$ or $\mathbf{f}$ has to be in $\tilde{\sim}_{\mathcal{M}}(\mathbf{t})$. Thus x' is $\mathbf{t}$, and y' must be $\mathbf{f}$. This necessitates $\tilde{\sim}_{\mathcal{M}}(\mathbf{n}) = \{\mathbf{b}\}$ and so $\mathcal{M}$ is a refinement of the **w6-FDE**-Nmatrix. □

6 Negation-incomplete variants

Since it is possible to argue in an analogous manner, we shall briefly study negation-incomplete variants of the **FDE**-negation, whose known instances are $\sim_{13}$–$\sim_{16}$. We may motivate these operators in a mirroring way, namely that it provides possible options for somebody interested in negation incompleteness. First we look at a non-deterministic generalisation of these, given by the next table:

	$\tilde{\sim}$
t	**n**
b	**t**
n	**n**, **f**
f	**t**, **b**

A valuation/the semantical consequence relation with respect to this negation is called a *four-valued negation incomplete* **FDE**-*valuation*/the *four-valued negation incomplete* **FDE**-*consequence relation*, and defined analogously to the negation inconsistent case. The consequence relation is denoted by $\models_{4n}$.

It remained open in [17] whether the resulting system **n-FDE** is *negation incomplete*, that is to say whether there there is a formula A such that $A \models_{4n} B$ and $\sim A \models_{4n} B$ for all B.

Definition 6.1. For a formula A, we define its dual formula A^* by setting $p^* = p$, $(B \wedge C)^* = B^* \vee C^*$, $(B \vee C)^* = B^* \wedge C^*$ and $(\sim A)^* = \sim A^*$

Note that $(A^*)^* \equiv A$. We can then show the negation incompleteness of the four-valued negation incomplete **FDE**-consequence relation.

Theorem 6.2. **n-FDE** is negation incomplete.

Proof. Let v be a four-valued negation incomplete **FDE**-valuation. Let $d : \mathcal{V} \to \mathcal{V}$ be a mapping s.t. $d(\mathbf{t}) = \mathbf{f}$, $d(\mathbf{b}) = \mathbf{n}$, $d(\mathbf{n}) = \mathbf{b}$ and $d(\mathbf{f}) = \mathbf{t}$. Define $v' : \mathsf{Form} \to \mathcal{V}$ by setting $v'(A) = d(v(A^*))$. We claim v' is a four-valued negation inconsistent **FDE**-valuation.

We show this by induction on the complexity of A. When $A \equiv p$, there is nothing to prove. When $A \equiv (B \wedge C)$, then $A^* = B^* \vee C^*$. By I.H., $v'(B) = d(v(B^*))$ and $v'(C) = d(v(C^*))$ are in accordance with the condition for a four-valued negation inconsistent **FDE**-valuation. So for instance, if $v'(B) = \mathbf{b}$ and $v'(C) = \mathbf{n}$, then $v(B^*) = \mathbf{n}$ and $v(C^*) = \mathbf{b}$. Thus $v(B^* \vee C^*) = \mathbf{t}$ and as a result $v'(B \wedge C) = \mathbf{f} \in \tilde{\wedge}(\mathbf{b}, \mathbf{n})$, as required. Other cases can be checked similarly. The case $A \equiv B \vee C$ is analogous. Finally, when $A \equiv \sim B$, then $v'(B) = \mathbf{t}$ implies $v(B^*) = \mathbf{f}$, and so $v(\sim B^*)$ is either $\mathbf{t}$ or $\mathbf{b}$. Therefore $v'(\sim B)$ is either $\mathbf{f}$ or $\mathbf{n}$, which are included in $\tilde{\sim}(\mathbf{t}) = \{\mathbf{n}, \mathbf{f}\}$. The other cases are analogous.

Now, as we saw in Corollary 3.5, there is a formula A which attains the value $\mathbf{b}$ in any four-valued negation inconsistent **FDE**-valuation. In particular, $v'(A) = \mathbf{b}$ for any four-valued negation incomplete **FDE**-valuation v. This implies $v(A^*) = \mathbf{n}$ for all v; whence it follows that $A^* \models_{4n} B$ and $\sim A^* \models_{4n} B$ for all B. □

We can thus obtain instances of negation incompleteness from instances of negation inconsistency in **b-FDE**. For example,

$$\begin{aligned} &((A \vee \sim\sim A) \wedge \sim(\sim(A \vee \sim\sim A) \wedge \sim\sim\sim(A \vee \sim\sim A)))^* \\ =&(A \wedge \sim\sim A) \vee \sim(\sim(A \wedge \sim\sim A) \vee \sim\sim\sim(A \wedge \sim\sim A)) \end{aligned}$$

can be checked to be such an instance.

6.1 Characterising negation incomplete variants

We shall generalise the preceding observation by introducing more non-determinacy, as we did in the negation inconsistent case. We introduce $\tilde{\sim}_{w3'}$–$\tilde{\sim}_{w12'}$, with which the corresponding Nmatrices and semantical consequence relations are defined.

	$\tilde{\sim}_{w3'}$	$\tilde{\sim}_{w4'}$	$\tilde{\sim}_{w5'}$	$\tilde{\sim}_{w6'}$	$\tilde{\sim}_{w7'}$
t	**n**	**n**	**b,n,f**	**b,n**	**b,n,f**
b	**t,n**	**t,n,f**	**n**	**n**	**n,f**
n	**n,f**	**n,f**	**n,f**	**n,f**	**b,f**
f	**t,b,n**	**t,n**	**b,n**	**t,b,n**	**n,f**

$\tilde{\sim}_{w8'}$	$\tilde{\sim}_{w9'}$	$\tilde{\sim}_{w10'}$	$\tilde{\sim}_{w11'}$	$\tilde{\sim}_{w12'}$
n,f	**b,f**	**f**	**b,n,f**	**n,f**
t,n,f	**n,f**	**t,n,f**	**n,f**	**t,n,f**
b,f	**t,b,f**	**t,b,f**	**n,f**	**n,f**
n,f	**n,f**	**n,f**	**n,f**	**n,f**

We shall use **wi′-FDE** to refer to the logics defined by the Nmatrix with $\tilde{\sim}_{wi'}$ (to be called the **wi′-FDE**-Nmatrix) for $i \in \{3, \ldots, 12\}$.

Theorem 6.3. **w3′-FDE–w12′-FDE** are negation incomplete.

Proof. The argument is analogous to Theorem 6.2. We take v' to be a **wi-FDE**-valuation for each of $i \in \{3, \ldots, 12\}$. In checking that v' is well-defined, the cases for conjunction and disjunction are identical, so the only difference is in the case of negation. But it is routine to observe that $\tilde{\sim}_{w3'}$–$\tilde{\sim}_{w12'}$ are defined in such a way that this works. □

Hence we obtain a characterisation for negation incompleteness.

Theorem 6.4. Let $\mathcal{M}$ be a refinement of the four-valued general **FDE**-Nmatrix. Then the logic defined by $\mathcal{M}$ is negation incomplete if and only if $\mathcal{M}$ is also a refinement of one of the **wi′-FDE**-Nmatrices for $i \in \{3, \ldots, 12\}$.

Proof. Analogous to Theorem 5.4, but we appeal to Theorem 6.3 instead. □

7 Negation inconsistency induced by another operator

We have so far looked at unary operators whose combination with the conjunction and disjunction of **FDE** bring about ‘negation inconsistency’. In the same vein, one may also add another non-deterministic operator $\circ$ to the proper language of **FDE** i.e. $\mathcal{L}_{\mathbf{FDE}}$, and consider which operators cause negation inconsistency (in terms of $\sim$). As none of the other negations (except for $\sim_{16}$ in [16]) seems to have an independent motivation, this might be a more plausible approach to some. The *material connexive logic* **MC** [23], which adds an implication to **FDE**, is one example of negation inconsistent systems obtained in such a manner. Here we shall first take up the case for unary operators, by applying the same methodology as before. After characterising unary operators which create negation inconsistency, we shall also observe its consequence for the case of binary operators.

Let us begin by introducing another type of Nmatrix for this purpose.

Definition 7.1. We define the $\circ_i$-*Nmatrix* ($i \in \{1, \ldots, 6\}$) in $\mathcal{L}_{\mathbf{FDE}_\circ}$ as $(\mathcal{V}, \mathcal{D}, \mathcal{O})$, where $\mathcal{V} = \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$, $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$ and $\mathcal{O} = \{\tilde{\wedge}, \tilde{\vee}, \tilde{\sim}, \tilde{\circ}_i\}$. $\tilde{\wedge}$ and $\tilde{\vee}$ are as in the four-valued negation inconsistent **FDE**-Nmatrix, and $\tilde{\sim} = \tilde{\sim}_1$ in Section 3.2. Each $\tilde{\circ}_i$ is given by the table below.

	$\tilde{\circ}_1$	$\tilde{\circ}_2$	$\tilde{\circ}_3$	$\tilde{\circ}_4$	$\tilde{\circ}_5$	$\tilde{\circ}_6$
t	**t**,**b**,**n**,**f**	**t**,**b**,**n**,**f**	**t**,**b**,**n**,**f**	**b**	**n**	**b**,**n**
b	**t**,**b**,**n**,**f**	**t**,**b**,**n**,**f**	**t**,**b**,**f**	**t**,**b**,**n**,**f**	**t**,**b**,**n**,**f**	**t**,**b**,**f**
n	**t**,**b**,**f**	**b**	**b**	**t**,**b**,**f**	**b**	**b**
f	**b**	**n**	**b**,**n**	**t**,**b**,**n**,**f**	**t**,**b**,**n**,**f**	**t**,**b**,**n**,**f**

We shall use $\models_{\circ i}$ for the semantical consequence relations, and call the resulting systems $\circ i$-**FDE**.

As already mentioned, in this section the negation inconsistency of a system is evaluated only with respect to $\sim$, and not with respect to $\circ$. Hence the definability of the **b** constant is what we are after.

Lemma 7.2. $\circ$1-**FDE** – $\circ$6-**FDE** are negation inconsistent.

Proof. Since $\circ_1$–$\circ_3$ are replicable in terms of $\circ_4$–$\circ_6$ respectively as $\sim\circ_i$, it suffices to show the negation inconsistency of $\circ$1-**FDE**–$\circ$3-**FDE**.

For $\circ$1-**FDE**, we claim $B \vee \sim(B \vee \circ_1 B)$, where $B \equiv (A \wedge \circ_1(A \wedge \sim A)) \wedge \sim \circ_1 (A \wedge \sim A)$ witnesses the negation inconsistency.

A	$\sim A$	$A\wedge\sim A$	$1 : \circ_1(A\wedge\sim A)$	$\sim\circ_1(A\wedge\sim A)$	$A\wedge\circ_1(A\wedge\sim A)$
t	**f**	**f**	**b**	**b**	**b**
b	**b**	**b**	$(\mathbf{tbnf})_1$	$(\mathbf{fbnt})_1$	$(\mathbf{bbff})_1$
n	**n**	**n**	$(\mathbf{tbf})_1$	$(\mathbf{fbt})_1$	$(\mathbf{nff})_1$
f	**t**	**f**	**b**	**b**	**f**

B	$2 : \circ_1 B$	$B\vee\circ_1 B$	$\sim(B\vee\circ_1 B)$	$B \vee \sim(B\vee\circ_1 B)$
b	$(\mathbf{tbnf})_2$	$(\mathbf{tbtb})_2$	$(\mathbf{fbfb})_2$	**b**
$(\mathbf{fbff})_1$	$(\mathbf{b}(\mathbf{tbnf})_2\mathbf{bb})_1$	$(\mathbf{b}(\mathbf{tbtb})_2\mathbf{bb})_1$	$(\mathbf{b}(\mathbf{fbfb})_2\mathbf{bb})_1$	**b**
f	**b**	**b**	**b**	**b**
f	**b**	**b**	**b**	**b**

For $\circ 2$-**FDE**, we claim $\sim B \wedge (B \vee \sim \circ_2 \circ_2 B)$, where $B \equiv A \wedge \sim A \wedge \circ_2 A$ witnesses the negation inconsistency.

B	$\sim B$	$1 : \circ_2 B$	$2 : \circ_2 \circ_2 B$	$\sim \circ_2 \circ_2 B$
t	**f**	$(\mathbf{tbnf})_1$	$((\mathbf{tbnf})_2(\mathbf{tbnf})_2\mathbf{bn})_1$	$((\mathbf{fbnt})_2(\mathbf{fbnt})_2\mathbf{bn})_1$
b	**b**	$(\mathbf{tbnf})_1$	$((\mathbf{tbnf})_2(\mathbf{tbnf})_2\mathbf{bn})_1$	$((\mathbf{fbnt})_2(\mathbf{fbnt})_2\mathbf{bn})_1$
n	**n**	**b**	$(\mathbf{tbnf})_2$	$(\mathbf{fbnt})_2$
f	**t**	**n**	**b**	**b**

$B \vee \sim \circ_2 \circ_2 B$	$\sim B \wedge (B \vee \sim \circ_2 \circ_2 B)$
t	**f**
$((\mathbf{bbtt})_2(\mathbf{bbtt})_2\mathbf{bt})_1$	**b**
$(\mathbf{ntnt})_2$	**n**
b	**b**

It is now straightforward to deduce from the table that $A \wedge \sim A \wedge \circ_2 A$ always attains either the value **b** or **f**.

For $\circ 3$-**FDE**, we claim $--A$, where

$$-B := (\circ_3(A \wedge \sim A) \vee \sim \circ_3(A \wedge \sim A)) \wedge (\sim A \vee \sim (\circ_3(A \wedge \sim A) \vee \sim \circ_3(A \wedge \sim A)))$$

witnesses the negation inconsistency.

A	$\sim A$	$A \wedge \sim A$	$1 : \circ_3(A \wedge \sim A)$	$\sim \circ_3(A \wedge \sim A)$	$\circ_3(A \wedge \sim A) \vee \sim \circ_3(A \wedge \sim A)$
t	**f**	**f**	$(\mathbf{bn})_1$	$(\mathbf{bn})_1$	$(\mathbf{bn})_1$
b	**b**	**b**	$(\mathbf{tbf})_1$	$(\mathbf{fbt})_1$	$(\mathbf{tbt})_1$
n	**n**	**n**	**b**	**b**	**b**
f	**t**	**f**	$(\mathbf{bn})_1$	$(\mathbf{bn})_1$	$(\mathbf{bn})_1$

$\sim (\circ_3(A \wedge \sim A) \vee \sim \circ_3(A \wedge \sim A))$	$\sim A \vee \sim (\circ_3(A \wedge \sim A) \vee \sim \circ_3(A \wedge \sim A))$	$-A$
$(\mathbf{bn})_1$	$(\mathbf{bn})_1$	$(\mathbf{bn})_1$
$(\mathbf{fbf})_1$	**b**	**b**
b	**t**	**b**
$(\mathbf{bn})_1$	**t**	$(\mathbf{bn})_1$

Now, since $-A$ attains either the value **b** or **n**, it is immediate that $--A$ always attains the value **b**. □

The next unary operator is also related to our interest.

Definition 7.3. We define the $\circ_7$-Nmatrix as the same as the $\circ_i$-Nmatrix except for the following $\tilde{\circ}$.

	$\tilde{\circ}_7$
t	**b**,**n**
b	**n**
n	**b**
f	**b**,**n**

$\circ_7$-**FDE** is then defined analogously to $\circ_1$-**FDE**–$\circ_6$-**FDE**.

Lemma 7.4. $\circ_7$-**FDE** is negation consistent.

Proof. Let v be a valuation for the $\circ$7-Nmatrix. We define another valuation v' by:

$$v'(A) = \begin{cases} \mathbf{t} & \text{if } v(A) = \mathbf{t}. \\ \mathbf{b} & \text{if } v(A) = \mathbf{n}. \\ \mathbf{n} & \text{if } v(A) = \mathbf{b}. \\ \mathbf{f} & \text{if } v(A) = \mathbf{f}. \end{cases}$$

We check that v' is well-defined by induction on the complexity of formulas. For $A \equiv B \wedge C$, if e.g. $v(A) = \mathbf{b}$, then the pair of $(v(B), v(C))$ is either $(\mathbf{t}, \mathbf{b})$, $(\mathbf{b}, \mathbf{t})$ or $(\mathbf{b}, \mathbf{b})$. So $(v'(B), v'(C))$ is either $(\mathbf{t}, \mathbf{n})$, $(\mathbf{n}, \mathbf{t})$ or $(\mathbf{n}, \mathbf{n})$. In each case, $v'(A) = \mathbf{n}$, as required. Other cases for conjunction and the cases for disjunction and negation are analogously checked. For $A \equiv \circ B$, if $v(A) = \mathbf{b}$ then either $v(B) = \mathbf{t}, \mathbf{n}$ or $\mathbf{f}$. Hence $v'(B) = \mathbf{t}, \mathbf{b}$ or $\mathbf{f}$. Hence it is allowed to take $v'(B) = \mathbf{n}$, as required.

Now, by the stipulation either $v(A) \neq \mathbf{b}$ or $v'(A) \neq \mathbf{b}$. Therefore no formula attains the value $\mathbf{b}$ under all valuations. □

We similarly obtain negation consistency when $\tilde{\circ}(\mathbf{t}) = \{\mathbf{t}\}$ or $\{\mathbf{f}\}$ instead.

Remark 7.5. Note that the negation consistency is obtained even though the $\circ_7$-Nmatrix does not have a valuation assigning all formulas a value from the set $\{\mathbf{t}, \mathbf{n}, \mathbf{f}\}$. On the other hand, the refinements of the $\circ_7$-Nmatrix where $\tilde{\circ}(\mathbf{t}) = \{\mathbf{b}\}$ and $\tilde{\circ}(\mathbf{t}) = \{\mathbf{n}\}$ both create negation inconsistency, as witnessed by the formulas $\circ(\circ A \vee \circ \circ A)$ and $\circ\circ(\circ A \vee \circ \circ A)$, respectively. This is in contrast to the case for **b-FDE**, in which the negation inconsistency is preserved when generalised from $\sim_4, \sim_8, \sim_{12}$ and $\sim_{16}$.

Let the *four-valued general* $\circ$-**FDE**-*Nmatrix* be defined from the $\circ_i$-Nmatrices in such a way that $\tilde{\circ}$ is identical to $\tilde{\sim}$ in the four-valued general **FDE**-Nmatrix. Then we obtain a characterisation of unary operators whose addition to **FDE** induce negation inconsistency.

Theorem 7.6. Let $\mathcal{M}$ be a refinement of the four-valued general $\circ$-**FDE**-Nmatrix. Then the logic defined by $\mathcal{M}$ is negation inconsistent if and only if $\mathcal{M}$ is also a refinement of one of the $\circ i$-**FDE**-Nmatrices for $i \in \{1, \dots, 6\}$.

Proof. The 'if' direction follows from Lemma 7.2. For the 'only if' direction, since $\{\mathbf{n}\}$ is closed under $\tilde{\wedge}, \tilde{\vee}, \tilde{\sim}$, it must be that $\tilde{\circ}(\mathbf{n}) \subseteq \{\mathbf{t}, \mathbf{b}, \mathbf{f}\}$. First consider the case $\tilde{\circ}(\mathbf{n}) = \{\mathbf{b}\}$. In this case, to avoid a valuation assigning all formulas a value from $\{\mathbf{t}, \mathbf{f}\}$, we must have either $\mathbf{t}, \mathbf{f} \notin \tilde{\circ}(\mathbf{t})$ or $\mathbf{t}, \mathbf{f} \notin \tilde{\circ}(\mathbf{f})$. We shall consider the latter case: the former case is similarly argued. In this case, if $\tilde{\circ}(\mathbf{f}) = \{\mathbf{b}, \mathbf{n}\}$ then $\mathbf{n} \in \tilde{\circ}(\mathbf{b})$ and $(\{\mathbf{b}, \mathbf{n}\} \subseteq \tilde{\circ}(\mathbf{t})$ or $\mathbf{t} \in \tilde{\circ}(\mathbf{t})$ or $\mathbf{f} \in \tilde{\circ}(\mathbf{t}))$ results in negation consistency, by Lemma 7.4. Hence either $\mathbf{n} \notin \tilde{\circ}(\mathbf{b})$ or ($\{\mathbf{b}, \mathbf{n}\} \nsubseteq \tilde{\circ}(\mathbf{t})$ and $\mathbf{t} \notin \tilde{\circ}(\mathbf{t})$ and $\mathbf{f} \notin \tilde{\circ}(\mathbf{t})$). But then $\mathcal{M}$ is a refinement of the $\circ_3$-**FDE**-Nmatrix or one of the $\circ_4$-**FDE**-Nmatrix or the $\circ_5$-**FDE**-Nmatrix, respectively. Otherwise, $\tilde{\circ}(\mathbf{f})$ is either $\{\mathbf{b}\}$ or $\{\mathbf{n}\}$, in which case $\mathcal{M}$ is a refinement of the $\circ_1$-**FDE**-Nmatrix or the $\circ_2$-**FDE**-Nmatrix.

When $\tilde{\circ}(\mathbf{n}) \neq \{\mathbf{b}\}$, we must have either $\tilde{\circ}(\mathbf{t}) = \{\mathbf{b}\}$ or $\tilde{\circ}(\mathbf{f}) = \{\mathbf{b}\}$ in order to avoid a valuation assigning all formulas a value from $\{\mathbf{t}, \mathbf{n}, \mathbf{f}\}$. But then $\mathcal{M}$ is a refinement of the $\circ_1$-**FDE**-Nmatrix or the $\circ_4$-**FDE**-Nmatrix. □

Remark 7.7. The above unary operators are also useful for defining some binary operators whose addition to **FDE** creates negation inconsistency. In the tables below (where commas are abbreviated), $A \star_{ia} A$, $\sim A \star_{ib} \sim A$, $A \star_{ic} \sim A$ and $\sim A \star_{id} A$ each replicates $\circ_i$ for $i \in \{1, \ldots, 3\}$. Hence the **b** constant is definable when a connective defined by one of the tables below is added to **FDE**, instead of $\circ_1, \ldots, \circ_6$.

$\tilde{\star}_{1a}$	**t**	**b**	**n**	**f**
t	**tbnf**	**tbnf**	**tbnf**	**tbnf**
b	**tbnf**	**tbnf**	**tbnf**	**tbnf**
n	**tbnf**	**tbnf**	**tbf**	**tbnf**
f	**tbnf**	**tbnf**	**tbnf**	**b**

$\tilde{\star}_{1b}$	**t**	**b**	**n**	**f**
t	**b**	**tbnf**	**tbnf**	**tbnf**
b	**tbnf**	**tbnf**	**tbnf**	**tbnf**
n	**tbnf**	**tbnf**	**tbf**	**tbnf**
f	**tbnf**	**tbnf**	**tbnf**	**tbnf**

$\tilde{\star}_{1c}$	**t**	**b**	**n**	**f**
t	**tbnf**	**tbnf**	**tbnf**	**tbnf**
b	**tbnf**	**tbnf**	**tbnf**	**tbnf**
n	**tbnf**	**tbnf**	**tbf**	**tbnf**
f	**b**	**tbnf**	**tbnf**	**tbnf**

$\tilde{\star}_{1d}$	**t**	**b**	**n**	**f**
t	**tbnf**	**tbnf**	**tbnf**	**b**
b	**tbnf**	**tbnf**	**tbnf**	**tbnf**
n	**tbnf**	**tbnf**	**tbf**	**tbnf**
f	**tbnf**	**tbnf**	**tbnf**	**tbnf**

$\tilde{\star}_{2a}$	**t**	**b**	**n**	**f**
t	**tbnf**	**tbnf**	**tbnf**	**tbnf**
b	**tbnf**	**tbnf**	**tbnf**	**tbnf**
n	**tbnf**	**tbnf**	**b**	**tbnf**
f	**tbnf**	**tbnf**	**tbnf**	**n**

$\tilde{\star}_{2b}$	**t**	**b**	**n**	**f**
t	**n**	**tbnf**	**tbnf**	**tbnf**
b	**tbnf**	**tbnf**	**tbnf**	**tbnf**
n	**tbnf**	**tbnf**	**b**	**tbnf**
f	**tbnf**	**tbnf**	**tbnf**	**tbnf**

$\tilde{\star}_{2c}$	**t**	**b**	**n**	**f**
t	**tbnf**	**tbnf**	**tbnf**	**tbnf**
b	**tbnf**	**tbnf**	**tbnf**	**tbnf**
n	**tbnf**	**tbnf**	**b**	**tbnf**
f	**n**	**tbnf**	**tbnf**	**tbnf**

$\tilde{\star}_{2d}$	**t**	**b**	**n**	**f**
t	**tbnf**	**tbnf**	**tbnf**	**n**
b	**tbnf**	**tbnf**	**tbnf**	**tbnf**
n	**tbnf**	**tbnf**	**b**	**tbnf**
f	**tbnf**	**tbnf**	**tbnf**	**tbnf**

$\tilde{\star}_{3a}$	**t**	**b**	**n**	**f**
t	**tbnf**	**tbnf**	**tbnf**	**tbnf**
b	**tbnf**	**tbf**	**tbnf**	**tbnf**
n	**tbnf**	**tbnf**	**b**	**tbnf**
f	**tbnf**	**tbnf**	**tbnf**	**bn**

$\tilde{\star}_{3b}$	**t**	**b**	**n**	**f**
t	**bn**	**tbnf**	**tbnf**	**tbnf**
b	**tbnf**	**tbf**	**tbnf**	**tbnf**
n	**tbnf**	**tbnf**	**b**	**tbnf**
f	**tbnf**	**tbnf**	**tbnf**	**tbnf**

$\tilde{\star}_{3c}$	**t**	**b**	**n**	**f**
t	**tbnf**	**tbnf**	**tbnf**	**tbnf**
b	**tbnf**	**tbf**	**tbnf**	**tbnf**
n	**tbnf**	**tbnf**	**b**	**tbnf**
f	**bn**	**tbnf**	**tbnf**	**tbnf**

$\tilde{\star}_{3d}$	**t**	**b**	**n**	**f**
t	**tbnf**	**tbnf**	**tbnf**	**bn**
b	**tbnf**	**tbf**	**tbnf**	**tbnf**
n	**tbnf**	**tbnf**	**b**	**tbnf**
f	**tbnf**	**tbnf**	**tbnf**	**tbnf**

Then e.g. the connectives $\rightarrow_{13}$, $\rightarrow_{16}$ (the implication of **MC**) and $\leftarrow_{16}$ in [18] fall under the scope of possible refinements for $\tilde{\star}_{1a}$. Note however that the above tables do not exhaust all binary connectives which induce negation inconsistency. For instance,

$\tilde{\star}$	**t**	**b**	**n**	**f**
t	**n**	**n**	**b**	**n**
b	**n**	**n**	**n**	**n**
n	**b**	**n**	**t**	**n**
f	**n**	**n**	**n**	**n**

cannot be obtained from any of the above connectives via a refinement, but $(A \star A) \star (A \star A) \star (A \star A)$ defines the **b** constant.

8 Concluding remarks

The starting point of our enquiry was to answer the question raised by Omori and Wansing [17] concerning the negation inconsistency and incompleteness of systems with a non-deterministic negation. We observed that the two non-deterministic operators given by them do give rise, respectively, to negation inconsistency and incompleteness. We then attempted to generalise the result by defining and characterising non-deterministic unary operators which give rise to negation inconsistency and incompleteness in the same setting. Moreover, we studied the case when a non-deterministic unary connective is added to the full language of **FDE** as well.

The value of taking (non-trivial) negation inconsistent logics seriously is vigorously defended by Wansing [21], who also endorses [22] an informational view of logical values (e.g. **t** interpreted as signifying the *support of truth (only)* rather than *truth (only)*) inspired by the reading of the values in **FDE** by N. Belnap [9, 10], and through which (non-trivial) negation inconsistent systems can be seen to support the logico-informational view which he calls *dimathematism*. It would then be an interesting next step to consider how and what kind of systems with non-deterministic negation (or other connectives) can be taken as sensible logics in this type of framework. A possible hint for this may be the 'partial' tables Belnap [9] presents in his three-step strategy to motivate the deterministic tables for conjunction and disjunction.

Acknowledgements This research has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme, grant agreement ERC-2020-ADG, 101018280, ConLog. The author thanks Hitoshi Omori, Heinrich Wansing and two anonymous referees for their valuable comments and suggestions.

References

[1] Ofer Arieli and Arnon Avron. The value of the four values. *Artificial Intelligence*, 102(1):97–141, 1998.

[2] Ofer Arieli, Arnon Avron, and Anna Zamansky. Maximal and premaximal paraconsistency in the framework of three-valued semantics. *Studia Logica*, 97(1):31–60, 2011.

[3] Arnon Avron. On the expressive power of three-valued and four-valued languages. *Journal of Logic and Computation*, 9(6):977–994, 1999.

[4] Arnon Avron. A non-deterministic view on non-classical negations. *Studia Logica*, 80(2):159–194, 2005.

[5] Arnon Avron and Beata Konikowska. Multi-valued calculi for logics based on non-determinism. *Logic Journal of IGPL*, 13(4):365–387, 2005.

[6] Arnon Avron and Iddo Lev. Canonical propositional Gentzen-type systems. In *International Joint Conference on Automated Reasoning*, pages 529–544. Springer, 2001.

[7] Arnon Avron and Iddo Lev. Non-deterministic multiple-valued structures. *Journal of Logic and Computation*, 15(3):241–261, 2005.

[8] Arnon Avron and Anna Zamansky. Non-deterministic semantics for logical systems. In Dov M. Gabbay and Franz Guenthner, editors, *Handbook of philosophical logic*, volume 16, pages 227–304. Springer, 2011.

[9] Nuel D. Belnap. How a computer should think. In G. Ryle, editor, *Contemporary aspects of philosophy*, pages 30–55. Oriel Press, 1977.

[10] Nuel D. Belnap. A useful four-valued logic. In *Modern uses of multiple-valued logic*, pages 5–37. Springer, 1977.

[11] Leonid Yu. Devyatkin. On the expressive power of maximally paraconsistent and maximally paracomplete expansions of FDE. *Logical Investigations*, 27(2):66–92, 2021.

[12] J. Michael Dunn. *The algebra of intensional logics.* PhD thesis, University of Pittsburgh, 1966.

[13] J. Michael Dunn. Intuitive semantics for first-degree entailments and 'coupled trees'. *Philosophical Studies*, 29(3):149–168, 1976.

[14] Luis Estrada-González. The Bochum plan and the foundations of contra-classical logics. *CLE e-Prints*, 19(1):1–21, 2020.

[15] Hitoshi Omori and Katsuhiko Sano. Generalizing functional completeness in Belnap-Dunn logic. *Studia Logica*, 103(5):883–917, 2015.

[16] Hitoshi Omori and Heinrich Wansing. On contra-classical variants of Nelson logic N4 and its classical extension. *The Review of Symbolic Logic*, 11(4):805–820, 2018.

[17] Hitoshi Omori and Heinrich Wansing. Varieties of negation and contra-classicality in view of Dunn semantics. In Katalin Bimbó, editor, *Relevance Logics and other Tools for Reasoning. Essays in Honor of J. Michael Dunn*, pages 309–337. College Publications, London, 2022.

[18] Yaroslav Petrukhin and Vasily Shangin. Correspondence analysis and automated proof-searching for first degree entailment. *European Journal of Mathematics*, 6(4):1452–1495, 2020.

[19] Adam Přenosil. De Morgan clones and four-valued logics. *Algebra universalis*, 82(2):1–42, 2021.

[20] Alexej P Pynko. Functional completeness and axiomatizability within Belnap's four-valued logic and its expansions. *Journal of Applied Non-Classical Logics*, 9(1):61–105, 1999.

[21] Heinrich Wansing. Beyond paraconsistency. A plea for a radical breach with the Aristotelean orthodoxy in logic. To appear in: Abílio Rodrigues, Henrique Antunes and Alfredo Freire (eds), *Walter Carnielli on Reasoning, Paraconsistency, and Probability.* Springer.

[22] Heinrich Wansing. One heresy and one orthodoxy: On dialetheism, dimathematism, and the non-normativity of logic. *Erkenntnis*, pages 1–25, 2022.

[23] Heinrich Wansing. Connexive Logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy.* Metaphysics Research Lab, Stanford University, Summer 2023 edition, 2023.