Orlov Ninety-Six Years On: A Guide to "The calculus of the compatibility of propositions"

David Makinson and Werner Stelzner

Ivan[1] Efimovich Orlov's paper "The calculus of the compatibility of propositions" ["Ischislenie sovmestnosti predlozhenij"], published in Russian in 1928, is fascinating for anyone interested in the early history of relevance, modal or intuitionistic logic.

In this paper Orlov formulated the first axiomatic system of relevant logic, deductively equivalent to the implication-negation fragment of the system R, later developed by Anderson/Belnap independently of Orlov.

In order to express and to discuss Brouwer's basic ideas of intuitionism, he also formulated for the first time (within the framework of his relevant logic) the modal basis of the system S4, on which Gödel based his interpretation of intuitionistic logic in 1933.

The following remarks are intended to help the reader get to grips with the English translation.

1. Background

1.1 The author

Not much is known of Orlov's life. At present, the main sources are Povarov & Biryukov 2000 (in Russian), Bazhanov 2003 (in English) and Bazhanov 2007 (in Russian). They tell us that Orlov was born in 1886, and Povarov & Biryukov 2000 reports that he died on 13 October 1936.

Orlov studied at the Natural Sciences Department of the Physical-Mathematical Faculty of Moscow University and graduated in 1912.

Regarding his professional career, it is known that after the Russian revolution of 1917, he was in the 1920s an employee of the "Natural and Exact Sciences Section of the Methodology Department of the Communist Academy". The "Communist Academy" of the 1920s (from 1918 to 1924 "Socialist Academy") was the predecessor of the Institute of Philosophy of the Academy of Sciences of the USSR, now the Institute of Philosophy of the Russian

[1] Transliteration of Cyrillic characters is performed according to the GOST 7.79-2000 (B) standard.

Academy of Sciences. In addition, in the 1920s Orlov was a staff member of the “Timiryazov State Research Institute for the Study and Propaganda of the Natural Scientific Foundations of Dialectical Materialism.” After leaving these two institutes, Orlov, in accordance with his scientific training, worked in a chemical-pharmaceutical research institute until his death.

Beyond what may be gleaned from the above sources, little is known about his circumstances, personal beliefs and intentions in any of these periods. Presumably they were influenced by general political and philosophical developments after 1917, but for specifics one remains in the realm of conjecture. In particular, we do not know the reasons for the absence of any work of a philosophical nature after 1928 and the shift to chemistry. Thus, one can only speculate whether this change occurred as a result of unavoidable administrative coercion or that Orlov himself decided, because of the circumstances, to turn completely to the ideologically safe field of chemistry, both professionally and in his publishing activities.

One conjecture would see enthusiasm followed by disillusion leading to refuge in technology. Circumstantial support for this conjecture may be found in the politicization of theoretical controversies within Soviet philosophy in the period. Already in the early 1920s, one of the central disputes was between adherents of a mechanistic world view (the “mechanicists”) and representatives of a dialectical view (the “dialecticians”). At the end of the 1920s this dispute became extreme hostile and politically charged. Bazhanov 2007 (p. 265) describes Orlov’s role as follows: “Apparently, Orlov tried not to join either side of this polemic. Nevertheless, his scientific attitudes were close to mechanisticism, while in philosophy he considered himself a consistent dialectician. Therefore, in his 1928 article ‘On Dialectical Tactics in Natural Science,’ Orlov declares that dialectical criticism of ‘mechanisticists’ should not be excessive.”

However, such an attempt to mediate could not succeed because of the political instrumentalization of the issues that had already taken place. Bazhanov 2007, p. 269 again: “Events were unfolding in the direction of an intensification of polemics, their translation into the political plane, and the scale of repressions and, accordingly, the repressive apparatus was expanding. It seems very likely that Orlov anticipated the possibility of repressions against him as a scientist who had published in the pre-revolutionary journal *Questions of Philosophy and Psychology* [*Voprosy filosofii i psixologii*], where ideological enemies of the Bolsheviks had collaborated, so he tried to change his field of activity to one that was ideologically neutral and, moreover, extremely necessary for the development of industry–chemical research and production of iodine and bromine.”

This move to a purely scientific-industrial institute was also connected with a radical change in Orlov’s publication activity. Whereas before 1930 he had published mainly works on the philosophy of science, methodology and logic, in the 1930s he published exclusively in the field of chemistry.[2]

Orlov’s first publication “Basic formulas of the principle of relativity from the point of view of classical mechanics” [“Osnovnye formuly principa otnosi-

[2]Bazhanov 2003 and 2007 contain lists of Orlov’s publications, from which one can see the broad outlines of his intellectual evolution.

tel'nosti s tochki zrenija klassicheskoj mexaniki"] appeared in 1914. In 1916, one year before the Bolshevik Revolution, he published two articles (Orlov 1916, 1916a) on the philosophy of natural sciences in the Russian journal *Questions of Philosophy and Psychology*. Their main topic was the role of deduction and induction in the natural sciences. In this context, he paid special attention to the importance and treatment of implicit or hidden presuppositions.

After a gap of several years, he began publishing again, beginning in 1923 with short reviews and texts on the philosophy of mathematics and, from the following year, on the philosophy of science more generally. These articles were all published in party-controlled magazines such as *Under the Banner of Marxism* [*Pod znamenem marksizma*] and, later, *The Militant Materialist* [*Voinstvuyushhij materialist*]. Consequently, these articles by Orlov had to fit into the official political-ideological framework with its emphasis on Marxist materialism and dialectics. In addition to factually based positions, the articles always contain ideologically influenced passages and are characterized by a polemical tone.

Among them was a very short text of 1924 on "Formal logic, natural-scientific logic and dialectics" ["Logika formal'naya, estestvenno-nauchnaya i dialektika"], scorning formal logic of any kind and especially modern developments in the area. In 1925 he published the article "Logical Calculus and Traditional Logic" ["Logicheskoe ischislenie i tradicionnaya logika"], in which he tried to show the superiority of traditional logic over modern symbolic logic. However, as Bazhanov 2007, p. 268 remarks: "To be fair, unlike other authors of *Under the Banner of Marxism*, *The Militant Materialist*, etc. Orlov resorts to ideological stamps and terminology much less frequently. He does not provoke others into polemics, throwing accusations of departures from Marxism, etc. He only responds to attacks on him (Z. Cejtlin, È. Kol'man, Gr. Bammel)."

In 1925 Orlov's book *The Logic of Natural Science* [*Logika estestvoznaniya*] (Orlov 1925b) appeared, which is completely free from the language of party philosophy found in the articles after 1923. In this book, on the basis of traditional logic and mathematics, a realistic philosophy and methodology of natural science is presented, which ties in with the positions of the 1916 articles. However, despite its non-ideological character, the 1925 book does not contain any elements of modern symbolic logic ("logistics" or "logical calculus"), but remains entirely within the scope of traditional logic.

In the following three years Orlov seems to have studied in detail the very ideas of modern symbolic logic which he had castigated in his articles as useless and limited, and to which he had devoted no attention in his book *The Logic of Natural Science*. Abandoning his previous framework of traditional logic and now using the technical means of modern symbolic logic, he succeeded in producing his famous "The calculus of the compatibility of propositions", a paper on relevance logic and its combination with modality that was, in important respects, some years ahead of its time.

At some time in the years 1926–1928 there was a turning point. His last article in *Under the Banner of Marxism* appeared in 1928, as did his last article in an ideologically oriented anthology. On the other hand, Orlov's first two publications after the revolution in serious scientific collections appeared in 1926;

both were on musical acoustics. The paper "The calculus of the compatibility of propositions" appeared in 1928 in *Mathematical Collection* [*Matematicheskij sbornik*], a prestigious mathematics journal. After 1928 Orlov published only papers on chemistry and chemical engineering, and exclusively in scientific journals and collections. His last publication was a book on chemical analysis and control methods that appeared posthumously in 1939.

1.2 Logical context: influences, limitations, goals

The footnotes of the 1928 publication reveal the main influences in Orlov's study of mathematical logic. Foremost among them appears to have been Hilbert's "Über das Unendliche" of 1926, but he also consulted the same author's 1923 paper "Die logischen Grundlagen der Mathematik", Brouwer 1925, 1925a, as well as Ackermann 1925 (all in German). Two books are mentioned: Peano's 1894 *Notations de logique mathématique*, and Whitehead & Russell's monumental *Principia Mathematica, Volume I* of 1910. Although Kolmogorov had published an article in Russian on the principle of excluded middle in 1925, he is not mentioned in the footnotes. Kolmogorov was very young (only 22 years old) at the time of that publication, and relatively unknown. Orlov obviously preferred to cite the already internationally recognized authorities Brouwer and Hilbert, to both of whom Kolmogorov also referred extensively.

As one would expect, Orlov's text shows limitations of its time, lacking features that nowadays are in the repertoire of any well-educated logic student. For example, his work is severely hampered by not having a notion of consequence relation/operation, whose conceptualization began to emerge with Herz in the 1920s and most clearly with Tarski in the 1930s. Again, while Orlov was aware of a distinction between object-language and metalanguage, he was not always very clear in its deployment. For example, when he uses the Russian word that translates literally as 'inference', he sometimes seems to have that operation in mind, but sometimes implication as a relation between propositions, and on other occasions the implication connective building compound formulae out of simpler ones. Some difficulties that may have their origin in vagueness about the object/metalanguage distinction are discussed in sections 2.2 and 3.2 below.

Arguably, he was about as clear in such matters as were some of the authors to whom he refers, for example Whitehead and Russell, but he was less so than some others he mentions, in particular Hilbert and Ackermann. From the historical investigations of Zach 1999, it appears that the first author to be really clear about the distinction between object-language and metalanguage was Bernays, in his 1918 *Habilitationschrift*, submitted in Göttingen under the direction of Hilbert. Unfortunately, this was not published until 1926, and then only in part, and seems not to have been known to Orlov.

As a point of departure, Orlov took from Whitehead and Russell, Ackermann and especially Hilbert the idea of axiomatizing propositional logic, while from Brouwer he took on board the idea of understanding mathematical statements as assertions of provability or absurdity. His own goals were: (i) to develop an alternative to Hilbert's axioms in which the implication connective could be understood as expressing a "connection of senses" between antecedent and consequent (this takes up sections 1 through 5), (ii) to introduce an explicit provability operator into the object-language (section 6), (iii) to use the resulting propositional language to propose a formal articulation of Brouwer's idea (section 7).

1.3 The reception of Orlov's paper

Although Orlov's paper was was included by Alonzo Church in "A Bibliography of Symbolic Logic" (Church 1936, p. 201, № 379), as far as the present authors have been able to ascertain it was not discussed inside Russia in any publication before the 1960s, nor outside Russia until the 1990s. This state of affairs calls for some explanation since the journal *Mathematical Collection* [*Matematicheskij sbornik*], in which the paper appeared, had a high reputation in both its country of publication and the wider scientific world. It published papers on mathematics, including mathematical logic, mainly in Russian but also, from 1922 to 1947, in other languages such as German, English and French, attracting the attention of readers of those languages.

The reasons for the neglect seem to have been rather different for Russia and for the West.

In the case of the West, two factors seem fairly obvious. One was the language barrier, exacerbated by the Cyrillic rather than Latin script, which most westerners working in logic would have found insurmountable. Another was that a total silence on Orlov's work in Russia itself before the 1960s meant that no further stimuli came from that quarter to jog western attention. A third circumstance is that one of the topics of Orlov's paper, namely relevance logic, was not a recognizable field of mathematical investigation in the West until later. Apart from the isolated note of Parry 1933 taking a rather different direction on the matter, and the papers of Moh 1950 on the deduction theorem and Church 1951 on weak implication, the subject gained momentum only after Ackermann 1956, which served as inspiration for the work of Anderson and Belnap in the 1960s and 70s. Incidentally, although Church must have been aware of the existence of Orlov's paper, having mentioned it in the 1936 bibliography, he did not cite it in his 1951 paper, leaving readers to presume that the slate had hitherto been blank.

However, in the present authors' opinion, the role of the third factor above should not be exaggerated, since two of the topics of Orlov's paper — Brouwer's intuitionistic conception of logic and its interpretation in terms of modal logic — were already receiving lively attention from western logicians in the 1930s. They included Gödel 1933 with his modal interpretation of Heyting's system of intuitionistic logic, calling upon the same modal principles as Orlov had done five years earlier. Thus, as far as the world outside Russia is concerned, the main reasons for ignoring Orlov's paper seem not to have been uncharted subject matter but the language barrier and lack of further development from within the country of publication. Incidentally, an untitled notebook that was maintained by Gödel in the 1930s contains an annotated list of papers that he consulted, and the list does include Orlov's 1928 paper, with three lines mentioning its modal content but not the aspect of "connection of sense" (von Plato 2022, p. 132). But neither Gödel nor any of his his contemporaries are known to have mentioned Orlov's paper in print.

In the case of Russia, linguistic matters evidently had nothing to do with the long delay, over several decades, to recognition of Orlov's paper. In his "Reflections on Orlov", Priest 2001 p. 10 suggests that it may have been due simply to the fact that Orlov had "an idea before its time." A reviewer whom Priest quotes in a footnote offers a more substantive reason: "Mathematical logic in the USSR as a whole (including, of course, its non-classical subtrees) was considered as alien to dialectical thinking, as an example of a bourgeois metaphysics. Therefore, the development of logic was to a large extent mainly inhibited until mid and late 1950's" (Priest 2021, p. 10).

Certainly, there was open hostility towards formal logic in discourse on philosophy and the humanities in universities, academies, party institutions and the press, where ideology reigned supreme. But it was far more attenuated in purely mathematical contexts. The works of A. N. Kolmogorov, I. I. Zhegalkin, V. Glivenko, A. I. Uzkov, A. I. Mal'cev, D. A. Bochvar and P. S. Novikov published in *Mathematical Collection* show the development of logic within mathematics in the USSR in the 1920s to 1940s, reaching contemporaneous western standards. In particular, as from the middle of the 1920s there emerged a strong tradition of intuitionistic/constructivistic logic, connected with Kolmogorov and his school. Clearly, Orlov wished to anchor his 1928 article within mathematics, as is apparent from its content, style and chosen journal, contrasting with his earlier publications.

So it seems that there may have been further reasons why mathematicians in Russia ignored Orlov's endeavour. One could have been that although well trained in mathematics, he was not a professional mathematician, but a chemical engineer. Another may have been that while Orlov's article was technically competent, it did not reach the level of formal sophistication that was already standard in Russian mathematical logic by the late 1920s. Again, while Orlov considered intuitionistic logic, he did not do so from the perspective of the constructive or algorithically identifiable existence of mathematical objects, which was its prime source of interest for the Kolmogorov school.

There was presumably a further ground for wariness, indeed suspicion, on the part of mathematical logicians in Russia. For most of the 1920s, Orlov had been employed by institutes of Marxist philosophy and ideology, where he had played an active role in philosophical/ideological discussions of science and mathematics, publishing in journals such as *Under the Banner of Marxism* with a negative attitude towards formal logic in general and modern symbolic logic in particular. This may have established a reputation which, while no longer corresponding to Orlov's intellectual stance on logic in 1928, could well have been an obstacle to taking his article at face value.

In summary, it would appear that professional credentials, level of sophistication, and content merged with an acquired reputation to prevent recognition of Orlov's 1928 paper among mathematical logicians in its country of publication.

After the end of the long-lasting ideological blockade of modern logic in philosophical institutions of the USSR, the first printed reference to Orlov's "calculus of the compatibility of propositions" may be found in 1962, in A. A. Zinov'ev's book *Propositional Logic and Theory of Inference* [*Logika vyskazyvanij i teoriya vyvoda*], where he describes Orlov's system as "claiming to be a general theory of deduction and different from the classical calculus of propositions" (Zinov'ev 1962, p. 68f.). He goes on to present Orlov's axioms and discuss its motivation, saying: "At the heart of classical mathematical logic, Orlov claims, lies the notion of material inference, which can connect two propositions in one formula, which have no connection in sense. Orlov intends to consider the sense connections of propositions" (Zinov'ev 1962, p. 69).

The exposition is accompanied by some reasoned criticism. Concerning Orlov's idea of "connection by sense", he says: "The expression 'connection by sense' does not distinguish itself by its clarity. In any case, the reference to the terms 'presupposes', 'impossible', 'necessary condition', etc. does not produce any logical clarity." (Zinov'ev 1962, p. 69f.).

Zinov'ev also criticises Orlov's requirement of compatibility of premises in valid deductive inference: "After all, the cases $x \,\&\, {\sim}x \rightarrow x \,\&\, {\sim}x$ and $x \,\&\, {\sim}x \rightarrow {\sim}x \,\&\, x$, which satisfy Orlov's system, contain incompatible premises. For the possibility of deductive inference, the requirement of compatible premises is not necessary." (Zinov'ev 1962, p. 70). On the other hand, Zinov'ev does not discuss Orlov's modal interpretation of intuitionistic principles.

In 1978 and 1986, V. M. Popov published two papers on Orlov's system. They were significant for drawing attention to connections between Orlov's system and Anderson & Belnap's logic R, whose formulation became available only after Zinov'ev's publication. However, Popov's papers were written in Russian and were at the time, as still today, difficult to obtain outside the country of publication. For that reason, they too were unknown in the west until much later. As Belnap & Dunn later declared, referring to the earlier volume Anderson & Belnap 1975: "[. . .] we passed on our belief that the earliest versions of relevance logic were those of Moh 1950 and Church 1951. [. . .] we certainly missed the truth by over two decades: relevance logic was already treated with insight and rigor by Orlov 1928! This we first learned from an engaging report of Došen 1990. We subsequently learned to our increased chagrin that the work of Orlov had already been brought to light by V. M. Popov in 1978." (Anderson, Belnap & Dunn 1992 p. xvii).

In the first of his two papers, Popov refers to a system of Smirnov 1974 called RAO, recalling Smirnov's observation that if one adds to it an axiom of distributivity then one obtains a system equivalent to Anderson and Belnap's system R. Popov claims, without proof, that "the implication-negation fragment of RAO is equivalent in essence to I. E. Orlov's calculus of the compatibility of propositions, which is the first relevant system in the history of logic" (Popov 1978 p. 118). The qualification "in essence" reflects the fact that while Orlov's calculus had been defined as a Hilbertian system, both Smirnov 1974 and Popov 1978 worked with calculi of sequents. From this equivalence, together with the equivalence of the implication-negation fragments of RAO and R (also established by Smirnov 1974, but not explicitly mentioned by Popov), it follows that Orlov's system and the implication-negation fragment of R are equivalent. Thus Popov 1978 implicitly claimed the equivalence of those two systems.

In his second paper, Popov offers a proof of his claim (Popov 1986 pp. 95–96). However the formulation given there of what the proof achieves, is misleading. For Popov no longer refers to the implication-negation fragment of R, but rather to the earlier weak implicational calculus with negation of Church 1951 pp. 22–25, which Popov mistakenly presents with the scheme $\sim\sim a \rightarrow a$ among the axioms. In fact, while that scheme is an axiom of R (and of an axiomatization of the implication-negation fragment of R that was given by Anderson & Belnap 1975 section 14.1.3), it is not an axiom, nor a theorem, of Church's system, where negation is defined by the rule $\sim$A := A→f from implication and a primitive zero-ary connective f that is devoid of axiomatic constraints. On the basis of this misindentification of the implication-negation fragment of R with Church's weak implicational calculus with negation, Popov inaccurately described himself as proving the equivalent of Orlov's system with the latter.

The equivalence of Orlov's calculus of the compatibility of propositions with the implication-negation fragment of R was finally proven in an internationally accessible journal, in English, by Došen 1992 pp. 342–344, based on a semi-publication of 1990. Došen's paper also discusses Orlov's axiomatization of the S4 modal principles (in the context of his relevant logic) and Orlov's modal interpretation of the basic principles of intuitionistic logic.

2. Relevant implication

For relevance logic, Orlov set out to construct an axiom system paralleling that of Hilbert for classical propositional logic but whose implication connective conveys a connection of sense. This was to be accompanied by an account of inference in which premises are required to be, in some sense, *compatibl*e with each other.

On the one hand, his formal system develops the requirement of relevance with uncanny insight and skill. On the other hand, as we shall see, it is rather equivocal in its treatment of compatibility, and struggles to connect inferences between sentences to the implication connective that occurs within sentences.

2.1 Orlov's system

Whitehead and Russell had chosen negation and disjunction as primitive connectives. Hilbert made different choices in different publications: in Hilbert 1923 it was negation and implication while in Hilbert 1926, which was Orlov's main reference point, it was negation, implication and (redundantly) conjunction. Ackermann chose those plus (more redundantly still) disjunction. As Orlov's focus of attention was on implication with a non-classical reading, it was natural for him to include implication among his primitives, and that was accompanied by just negation, as in Hilbert 1923. But that option presented him with a dilemma: what should he do about the other connectives, disjunction and conjunction?

Since the implication connective in Orlov's system is intended to require a "connection of sense" that lacks some of the regularities of material implication, Hilbert's definitions of those connectives no longer support all their classical properties, thus creating a quandary. Should he provide definitions of conjunction and disjunction and investigate their resulting not-quite-classical behavior, or should he introduce additional primitives and provide them with full classical axioms?

Orlov chose the former option while, decades later, Anderson & Belnap 1975 set the pattern for most subsequent investigations by choosing the latter. From their perspective, one would say that Orlov considered the implication/negation fragment of the full language of relevance logic, whose primitive connectives also include classical conjunction and disjunction, undefinable from relevance-sensitive implication and negation alone.

In Orlov's paper, implication is written as an arrow, $\rightarrow$, while negation is expressed by a bar over what is negated. In the present guide, we retain the arrow but use the standard modern sign $\neg$ for negation. Given those two connectives, Orlov introduces others by a cumulative series of definitions already available in the literature but now read differently as a result of the relevance-sensitive understanding of the arrow.

Specifically, sense-related conjunction $a{\cdot}b$ is defined as $\neg(a \rightarrow \neg b)$. What Orlov calls "incompatibility" $a|b$ is then defined as $\neg(a{\cdot}b)$ thus, when unwound to primitive notation, as $\neg\neg(a \rightarrow \neg b)$ which, given his axioms, simplifies to $a \rightarrow \neg b$. Sense-related disjunction, which he writes $a{\vee}b$, is defined as $\neg a|\neg b$, unwinding to $\neg(\neg a{\cdot}\neg b)$ and hence, in primitive notation, $\neg\neg(\neg a \rightarrow \neg\neg b)$, simplifying to $\neg a \rightarrow b$. Finally, sense-related co-implication $a \rightleftarrows b$ is defined as $(a \rightarrow b){\cdot}(b \rightarrow a)$ and thus, in primitive notation, $\neg((a \rightarrow b) \rightarrow \neg(b \rightarrow a))$. On the level of notation, he has a mixed policy: for sense-connected conjunction and equivalence, he adopts signs different from those for their truth-functional counterparts in Hilbert and Whitehead & Russell, but for sense-connected disjunction he keeps their sign $\vee$.

Orlov works with six axioms (or axiom schemes, as we would say, since they admit substitution). Only two of them, his axioms (2) and (6), are among the four axioms for propositional logic of Hilbert 1926, and only two, namely his axioms (5) and (6), are among the six axioms of Hilbert 1923. Orlov remarks in a footnote that of the twelve axioms of Ackermann 1925, only five are even derivable in his system.

To generate theorems from the axioms, Orlov has one "axiom of a non-formal character" (or derivation rule, as we would now describe it), namely detachment. This terminological separation indicates that he had at least some grasp of the object-language/metalanguage distinction, for the axioms are formulas of the object-language while the "axiom of a non-formal character" is a closure condition, expressed in the metalanguage.

Using these, he systematically derives formulae expressing properties of the defined connectives, and it is surprising how much of their classical behavior continues into Orlov's relevance-sensitive framework. The dexterity of the derivations reveals that even though Orlov was not a professional mathematician, he was well trained in algebraic manipulations.

In later years, the Anderson-Belnap tradition reinvented the above-mentioned non-classical versions of conjunction and disjunction (sometimes as defined in Orlov's manner, sometimes as primitive) under names such as 'relevant' or 'intensional' conjunction and disjunction or, more briefly, respectively 'fusion' and 'fission'. The term 'co-tenability' has also been used for intensional conjunction, and yet other names have been used for analogous connectives in the linear logic literature.

For today's reader, a striking feature is that Orlov had no rigorous semantics for his notion of a "connection of senses". However, it should be remembered that even for classical logic, rigorous semantics had tended to lag somewhat behind axiomatics; for non-classical systems, it was not until after the Second World War that one began to see serious semantic constructions. This was done first for modal logics, then in cascade for counterfactuals, relevance logics and others. Orlov's semantic poverty was thus on par for the epoch.

Nevertheless, he evidently worked with some kind of intuition of what "connection of senses" might involve, and this intuition served not only as an initial motivation and, presumably, heuristic guide to the formulation of his axioms, but also as a rough-and-ready technique for obtaining negative results. For instance, in the last paragraphs of section 2, he 'demonstrates' with examples the possible falsehood of various formulas under his reading of their connectives, regarding them as undesirable and, as he hints correctly but without proof, underivable in his system.

We remark in passing that, since the publication of Orlov's paper, alternative syntactic approaches to classical and non-classical logic have been devised, complementing the Frege-style axiomatizations that he learned from Hilbert. Most notably, Gentzen-style sequent calculi and Gentzen/Jaśkowski systems of natural deduction emerged for classical and intuitionistic logic in the 1930s, then for modal and other non-classical logics as from the 1960s. Hybrid semantic/syntactic accounts in terms of truth-trees also came well after Orlov's paper. They were developed for classical and intuitionistic logic in the post-war period, again followed by adaptation to modal and other logics. But it would be quite anachronistic to expect any of these in Orlov's paper.

Although Orlov's implication connective is meant to convey a 'connection of sense', it does not seem that he intended it to express any necessity in the connection, and his formal system includes a principle that would not be in accord with such a requirement, namely axiom 5, $(a \to (b \to c)) \to (b \to (a \to c))$, often called 'permutation of antecedents'. We recall that this formula is absent from the system E of Anderson & Belnap 1975, whose arrow connective is meant to express relevance-with-necessity, but is a theorem of the Anderson-Belnap system R, intended for relevance-with-or-without-necessity, and is accepted in the truth-tree system of Makinson 2021.

2.2 Inferences with incompatible premises

Orlov's paper contains some assertions that may startle a modern reader. One of them concerns the possibility of inference with incompatible premises. Section 1 contains the following rather obscure passage (all our quotations from Orlov use Stelzner's translation).

> For the possibility of a deductive inference, the requirement of truth of the premises, generally speaking, is not necessary; it is sufficient that the weaker requirement of compatibility of the premises be satisfied. From false propositions true consequences can be inferred, but from premises that are incompatible with each other inferences are not possible at all. From this it follows that the requirement of compatibility of propositions is all that we need, and the requirement of their joint truth is exaggerated.

In section 2, after presenting his axioms, Orlov reiterates the message rather more clearly.

> If, however, the inference is made not from one but from two or more premises, then the derived proposition is no longer a condition of the truth of the premises, but only of the compatibility of the premises ... Thus, the conclusions that can be derived from any premises, generally speaking, can only be regarded as necessary conditions for the compatibility of the premises, and nothing more.

Understanding of these passages is assisted if we notice the equivocal way in which Orlov uses the Russian term ≪совместно≫ [sovmestno] that is translated as 'compatible'.

- On the one hand, there is the account in his Definition (1). This tells us: "The relation of compatibility of two propositions (logical product) can be defined formally as follows: $\neg(a \to \neg b) = a{\cdot}b$". Orlov goes on to define a formula for incompatibility, written $a|b$ and defined as $\neg(a{\cdot}b)$, which unwinds to $\neg\neg(a \to \neg b)$, equivalent to $a \to \neg b$. To read the definitions, note first that Orlov writes the *definiens* on the left and the *definiendum* on the right, contrary to present convention. More importantly, note that $a{\cdot}b$ and $a|b$ are *formulae* of the object-language or, rather, abbreviations of formulae. *Pace* the phrasing, they are not *relations* between formulae.
- On the other hand, there is the informal discussion in section 1 and elsewhere. There, Orlov treats a as incompatible with b iff the formula $a|b$, that is, $a \to \neg b$, is a *theorem of his axiom system*. In this sense, incompatibility is indeed a relation between formulae.

In the present authors' view, the best way of making sense of the declarations quoted above is to see them as using both senses of incompatibility. They equate the validity of an inference from premises a, b to conclusion c with the logical truth or theoremhood of the corresponding formula $(a{\cdot}b)|\neg c$, which reads intuitively as saying that the compatibility of a with b is incompatible with $\neg c$. The requirement of the *theoremhood of the entire formula* uses the strong sense of incompatibility from section 1 of his paper, while the *fusion connective buried inside* creates a compatibility sub-formula in the sense of his Definition (1).

So understood, Orlov's declaration becomes rather more familiar. The formula $(a{\cdot}b)|\neg c$ abbreviates $\neg((a{\cdot}b){\cdot}\neg c)$, which unwinds to $\neg\neg((a{\cdot}b) \to \neg\neg c)$, simplifying to $(a{\cdot}b) \to c$, unwinding again to $\neg(a \to \neg b) \to c$. Thus, for the validity of an inference scheme from premises a,b to conclusion c, Orlov is requiring that $\neg(a \to \neg b) \to c$ is a theorem of his axiom system. But it can be shown that, for that system (as for the later relevance logic R of Anderson & Belnap 1975 and for the system of Makinson 2021), $\neg(a \to \neg b) \to c$ is a theorem iff $a \to (b \to c)$ is a theorem (indeed, the formula stating the equivalence of the two is itself a theorem of those systems).

Putting all this together, Orlov is in effect defining the validity of an inference scheme $a,b/c$ as the logical truth or theoremhood of the corresponding formula $a \to (b \to c)$. Given the commutative and associative properties of the 'fusion' connective, the definition easily generalizes from two to n premises: $a_1,\ldots,a_n/c$ is valid iff $(a_1{\cdot}\ldots{\cdot}a_n)|\neg c$ is a theorem of his system, i.e. iff $a_1 \to (a_2 \to \ldots (a_n \to c)\ldots)$ is a theorem. That coincides with a well-known proposal, made independently and much later in the Anderson/Belnap tradition, for defining relevance-sensitive consequence relations (see the discussion in Makinson 2021 section 10).

2.3 Leibniz' praeclarum theorem

Another puzzling declaration appears in sections 5 and 6. The following citation brings together three separate passages from those sections, as signaled by suspension points.

> Let us prove the well-known "*praeclarum* theorem" of Leibniz:
>
> $$(a \to b){\cdot}(c \to d) \to (a{\cdot}c \to b{\cdot}d) \qquad (15a)$$
>
> [...] Leibniz' theorem is needed to justify deduction. It has a very general character; all possible forms of deductive inference can be obtained as special cases of (15a) by simple transformations [...] With the derivation of Leibniz' theorem, the problem of the general justification of deduction without applying the principle of simplification can be considered to be solved.

But why should formula (15a), or the intuitive principle that it expresses, be thought to generate all possible forms of deduction by simple transformations? Orlov gives no reasons, writing as if the point is both established and well known. Although he refers in a footnote to *Principia Mathematica Vol I* pp. 115–116, Whitehead & Russell make no claim of that kind there — they merely give the formula as an *example* of one that can be derived in their system.

The name 'praeclarum', which Leibniz gave to the principle, means 'splendid'. When formulating it, he also described it as "a fine theorem" (see Leibniz

1679–1686), but he does not seem to have claimed that all possible forms of deductive inference can be obtained as special cases of it. The present authors have not been able to determine the origins of that apparently fanciful view. However, a glance at the internet reveals that in recent decades, the formula has become a popular example for displaying the powers of both theorem-proving software and logical graphs in the tradition of C. S. Peirce.

3. Modality

In section 6 of his paper, Orlov shows how to introduce modal operators into his relevance-sensitive propositional logic. We outline briefly the resulting system, assuming that the reader has at least a little knowledge of the basics of modal logic as we know it today, and then discuss a puzzling question about Orlov's methodology.

3.1 Orlov's modal system

An operator for provability, written Φ, is added to the primitive connectives $\rightarrow$, $\neg$ of Orlov's language for relevance-sensitive logic. Intuitively, Φa is understood as "a is provable" and is provided with (what we now know as) the extra-classical axioms for the modal logic S4. This anticipates work of Oskar Becker in 1930, followed by that of Gödel 1933. Notations have differed over time; currently the box sign $\Box$ is standard.

The statement of the modal derivation rule, now often known as necessitation, is quite vague — more so than was detachment earlier in the paper. It bears no name, no number, no categorization, and is stated as follows: "Since we consider all axioms and the sentences derived from them to be assertible and consequently trustworthy, every expression admitted as an axiom, or derived in the preceding paragraphs, can be written as a function $\Phi(a)$." Nevertheless, the rule is employed without hitch in deriving some basic theorems of S4, including ones expressed using a defined operator that Orlov writes as X, where Xa abbreviates $\Phi\neg a$ and is read as stating the absurdity of a. In contrast to most later developments of modal logic, no possibility operator is introduced, whether by definition or as primitive; this reflects his tight focus on expressing Brouwer's basic intuitionistic principles, where provability and absurdity, but not possibility, take centre stage.

3.2. Modality with/without relevance

For readers today it is natural to ask why Orlov introduced modal operators into his relevance-sensitive logic rather than into plain classical propositional logic as axiomatized by Hilbert. He needed them to formalize ideas of Brouwer, but what do those ideas have to do with "connection of sense"? The acceptance of principles such as $(a\wedge\neg a)\rightarrow b$, $\neg a\rightarrow(a\rightarrow b)$, $a\rightarrow(b\rightarrow a)$, by both Brouwer and later intuitionists such as Heyting, suggests that they are not bothered by an absence of relevance between antecedent and consequent of an implication.

Given what we know today, Orlov's answer to that question is surprising: he believed that it is not possible to bring a modal operator into classical logic without stripping the endeavor of all meaning. The claim is anticipated in the introduction and made explicit in the very last paragraph of the paper; it is not just a tentative thought but a considered position. Nevertheless, Orlov's reasoning is difficult to understand. We recall the paragraph in section 7 that gives his argument.

> The introduction of the above functions in classical mathematical logic is impossible, since the interpretation of the notion of "follow" as a material inference deprives all the expressions proved for the functions we have introduced of sense. In addition, in classical theory, propositions are accepted and derived which, from our point of view, cannot be evaluated otherwise than as false. For example, from the proposition proved in classical theory, "all true propositions are equivalent," the following consequence emerges:
>
> $$a \rightleftarrows \Phi a \rightleftarrows X\Phi a \rightleftarrows XX\Phi a$$
>
> Such a consequence renders the introduction of the above kind of functions devoid of any meaning; in this case, in order to construct schemes of transfinite conclusions there would be no other way but to deny the law "tertium non datur".

The point seems to be that if we introduce into classical logic a modal operator governed by the S4 postulates, then we can prove that it collapses into plain assertion; in other words, we can then derive in the system the formula $\Phi a \rightleftarrows a$ (written $\Box a \leftrightarrow a$ in current notation).

Došen 1992, p. 341 comments laconically: "Orlov thinks that such propositional operations could not be introduced consistently into classical logic, but can be so introduced into his new logic. With hindsight, we are able to say that they can be introduced consistently into both". Of course, Došen is right, but it is intriguing to ask *why* Orlov made the mistake. Was there anything interesting, though wrong, going on in his mind?

Before attempting to answer this question, it is worth drawing attention to an apparent misprint in the display $a \rightleftarrows \Phi a \rightleftarrows X\Phi a \rightleftarrows XX\Phi a$ that does not appear to have been noticed in the literature. Assuming, with Orlov, that Φ collapses into plain assertion, so that X collapse into mere denial, this display reduces to $a \rightleftarrows a \rightleftarrows \neg a \rightleftarrows \neg\neg a$, where the third term is clearly anomalous. It should also reduce to a or its double negation. As mentioned earlier, in the Russian original negation was written as a bar over what is negated. Such bars are easily overlooked in typesetting, so one possibility is that one has been lost over the X (or perhaps even over the a) in the third term.

Leaving aside the anomalous third term in the display and returning to Orlov's basic claim of impossibility, several conjectures may be entertained. We mention just two.

- Priest 2021 has suggested that it may have arisen from "a misunderstanding of reasoning under assumption" in modal contexts. Perhaps Orlov had in mind a proof by cases from the tautology $a \vee \neg a$, applying the necessitation rule of modal logic within the scopes of one or both of the respective assumptions a, $\neg a$ without being aware of the constraints on legitimately doing so. That diagnosis connects with Orlov's text via his reference, in both the quoted paragraph and the Introduction, to saving the law of excluded middle.

- An alternative diagnosis hooks instead onto Orlov's reference to the classical principle that "all true propositions are equivalent". Recall that one way of expressing that principle is as the tautology $a \to (b \to (a \leftrightarrow b))$, which is not derivable from Orlov's axioms under the definition of $a \leftrightarrow b$, written $a \rightleftarrows b$ in his notation, as $\neg((a \to b) \to \neg(b \to a))$. Using that tautology, one can see that whenever a formula a is a theorem of S4 (or even of the weaker system K), then so too is $\Box a \leftrightarrow a$. For, if a is a theorem then the necessitation rule tells us that $\Box a$ is a theorem, allowing us to apply detachment twice to $a \to (\Box a \to (a \leftrightarrow \Box a))$ to get $a \leftrightarrow \Box a$. Now, if one is unclear about the distinction between the status of a formula as a theorem of the system and its status as true (under some specific but unarticulated valuation), then one might take this fact as telling us that whenever formula a is *true* then so is $a \leftrightarrow \Box a$, so that the box collapses into plain assertion.

4. Expressing Brouwer's ideas modally

Although Orlov had seen some papers by Brouwer, the more formal account of intuitionistic logic by Heyting 1930 was yet to appear. Unlike Gödel 1933, who took Heyting's paper as a starting point and mapped intuitionistic propositional logic into the modal logic S4, Orlov does not formally define a translation function from one language to another. Indeed, in section 7 he is not very clear about what the language of intuitionistic logic should be, and its connectives for conjunction and disjunction are left unmentioned — presumably, they are taken to be the relevance-sensitive ones defined from $\neg$, $\to$ earlier in the paper.

Apart from its very last paragraph (where he argues that a modal connective cannot coherently be added to classical logic, as discussed above) the focus in section 7 is on making formal sense of Brouwer's dicta about the logic of iterated absurdity assertions. Orlov represents such an assertion in his formalism by the sign X which, we recall, abbreviates the expression $\varPhi\neg$ or, as we would now write it, $\Box\neg$. His main point is that a formula of the kind $XXX\varPhi a$ is equivalent, in his system, to $X\varPhi a$ that is, we may strike out the initial pair XX. On the other hand, in opposition to a claim that he attributes to Brouwer 1925a, he remarks that we cannot do this without $\varPhi$, since $XXXa \to Xa$ is not a theorem of his system.

The positive point is shown by constructing a suitable derivation in the system. However, while the negative remark is correct, its proof is not at all rigorous. In the absence of a semantics to provide 'separating interpretations', or a Gentzen-style syntax to support an inductive argument, Orlov offers an informal example about Martians where the formula intuitively appears to fail, and passes from that to the conclusion that it is not provable in his system.

Translations

Passages originally in Russian that are here quoted in English have been translated by Werner Stelzner.

Acknowledgements

Thanks to three anonymous reviewers and the editor for helpful comments on the text.

References

Ackermann, W. 1925. Begründung des "tertium non datur" mittels der Hilbertschen Theorie der Widerspruchsfreiheit. *Mathematische Annalen* 93: 1–36.

Ackermann, W. 1956. Begründung einer strengen Implikation. *The Journal of Symbolic Logic* 21: 113–128.

Anderson, A. R. & N. D. Belnap Jr. 1975. *Entailment: The Logic of Relevance and Necessity Vol I.* Princeton University Press.

Anderson, A. R., N. D. Belnap Jr. & J. M. Dunn 1992. *Entailment: The Logic of Relevance and Necessity Vol II.* Princeton University Press.

Bazhanov, V. A. 2003. The scholar and the "wolfhound era": the fate of Ivan E. Orlov's ideas in logic, philosophy and science. *Science in Context* 16: 535–550.

Bazhanov, V. A. 2007. *History of Logic in Russia and the USSR.* Moscow, Kanon. [= В. А. Бажанов 2007. *История Логики в России и СССР.* Москва, Канон].

Becker, O. 1930. Zur Logik der Modalitäten. *Jahrbuch für Philosophie und phänomenologische Forschung* 11: 497–548.

Bernays, P. 1918. *Beiträge zur axiomatischen Behandlung des Logik-Kalküls.* Habilitationsschrift, Universität Göttingen.

Bernays, P. 1926. Axiomatische Untersuchungen des Aussagen-Kalküls der "Principia Mathematica". *Mathematische Zeitschrift* 25: 305–20.

Brouwer, L. E. J. 1925. Zur Begründung der intuitionistischen Mathematik. *Mathematische Annalen* 93: 244–257.

Brouwer, L. E. J. 1925a. Intuitionistische Zerlegung mathematischer Grundbegriffe. *Jahresbericht der Deutschen Mathematiker-Vereinigung* 33: 251–256.

Church, A. 1936. A bibliography of symbolic logic. *The Journal of Symbolic Logic* 1: 121–216.

Church, A. 1951. The weak theory of implication. In: A. Menne, A. Wilhelmi & H. Angstl (eds.), *Kontrolliertes Denken. Untersuchungen zum Logikkalkül und der Logik der Einzelwissenschaften*, Freiburg Brsg. & Munich, Kommissions-Verlag Karl Alber: 22–37.

Došen, K. 1992. The first axiomatization of relevant logic. *Journal of Philosophical Logic* 21: 339–356.

Gödel, K. 1933. Eine Interpretation des intuitionischen AussagenkalkÃ¼ls. *Ergebnisse eines mathematischen Kolloquiums* 4: 39–40. Repinted with English translation in Gödel, K. 1986. *Collected Works, Vol 1: Publications 1929–1936.* Oxford University Press.

Heyting, A. 1930. Die formalen Regeln der intuitionischen Logik. *Sitzungberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klasse* II: 42–56.

Hilbert, D. 1923. Die logischen Grundlagen der Mathematik. *Mathematische Annalen* 88: 151–165.

Hilbert, D. 1926. Über das Unendliche. *Mathematische Annalen* 95: 161–190.

Kolmogorov, A. N. 1925. On the principle of tertium non datur. *Mathematical Collection* 32 (4): 646–667. [= А. Н. Колмогоров 1925. О принципе tertium non datur. *Математический сборник*, том 32 (4): 646–667]. English translation "On the principle of excluded middle" in J. van Heijenoort ed 1967: 416–437.

Leibniz, G. W. 1679–1686. Adenda to the specimen of the universal calculus. In G. H. R. Parkinson (editor and translator) *Leibniz: Logical Papers*, 1986 Oxford University Press.

Makinson, D. 2021. Relevance-sensitive truth-trees. In Duntsch & Mares eds, *Alasdair Urquhart on Nonclassical and Algebraic Logic and Complexity of Proofs*, Springer. A more recent version under the title "Crashing with parity" is accessible at https://sites.google.com/site/davidcmakinson/

Moh, S.-K. 1950. The deduction theorems and two new logical systems. *Methodos* 2: 56–75.

Orlov, I. E. 1914. Basic formulas of the principle of relativity from the point of view of classical mechanics. *Journal of the Russian Physico-Chemical Society. Part Physics*. Vol. 46. Issue 4: 163–175 [= И. Е. Орлов 1914. Основные формулы принципа относительности с точки зрения классической механики. *Журнал русского Физико-химического общества. Часть физическая.* Т. 46. Выпуск 4: 163–175].

Orlov, I. E. 1916. Realism in natural science and the inductive method. *Questions of Philosophy and Psychology* 131 (1): 1–35 [= И. Е. Орлов 1916. Реализм в естествознании и индуктивный метод. *Вопросы философии и психологии* 131 (1): 1–35].

Orlov, I. E. 1916a. On inductive proof. *Questions of Philosophy and Psychology* 135 (1): 356–388 [= И. Е. Орлов 1916. Об индуктивном доказательстве. *Вопросы философии и психологии* 135 (1): 356–388].

Orlov, I. E. 1923. "Pure geometry" and factual reality. *Under the Banner of Marxism* 11–12: 213–219 [= И. Е. Орлов 1923. «Чистая геометрия» и реальная действительность. *Под знаменем марксизма* 11–12: 213–219].

Orlov, I. E. 1924. Formal logic, natural-scientific logic and dialectics. *Under the Banner of Marxism* 6–7: 69–90 [= И. Е. Орлов 1924. Логика формальная, естественно-научная и диалектика. *Под знаменем марксизма* 6–7: 69–90].

Orlov, I. E. 1925. Logical Calculus and Traditional Logic. *Under the Banner of Marxism* 4: 69–73 [= И. Е. Орлов 1925. Логическое исчисление и традиционная логика. *Под знаменем марксизма* 4: 69–73].

Orlov, I. E. 1925a. On the principles of scientific explanation of phenomena. *The Militant Materialist* 3: 277–293 [= И. Е. Орлов 1925. О принципах научного объяснения явлений. *Воинствующий материалист* 3: 277–293].

Orlov, I. E. 1925b. *Logic of natural science*. Moscow, Leningrad, State Publishing House [= И. Е. Орлов 1916. *Логика естествознания*. Москва, Ленинград, Государственное издательство.]

Orlov, I. E. 1928. The calculus of the compatibility of propositions. *Mathematical Collection* 35 (3–4): 263–286 [= И. Е. Орлов 1928. Исчисление совместности предложений. *Математический сборник* 35 (3–4): 263–286]. English translation by W. Stelzner to be published alongside the present text.

Parry, W. T. 1933. Ein Axiomensystem für eine neue Art von Implikation (Analytische Implikation). *Ergebnisse eines mathematischen Kolloquiums* 4: 5–6.

Peano, G. 1894. *Notations de logique mathematique.* Publié par la *Rivista di Matematica.* Turin, Imprimerie Charles Guadagnini.

Popov, V. M. 1978. On the decidability of the relevance logic RAO. In *Modal and Intensional Logics* (*Theses of the coordination meeting, Moscow, June 5–7, 1978*), Moscow, Institute of Philosophy, USSR Academy of Sciences Moscow: 115–119 [= В. М Попов 1978. О разрешимости релевантной системы RAO. В сборнике *Модальные и интенсиональные логики* (*Тезисы координационного совещания, Москва, июнь 5–7, 1978 г.*), Москва, Институт философии АН СССР Москва: 115–119].

Popov, V. M. 1986. I. S. Orlov's System and Relevant Logic.[3] *Philosophical problems of the history of logic and methodology of science*. Part 1. Moscow: Institute of Philosophy, Academy of Sciences of the USSR: 93-98 [= В. М. Попов

[3]There is a typographical error in the title of this paper: Orlov's intitials are printed there as "I.S.", instead of the correct "I. E." (Cyrillic "И. С." instead of "И. Е."). In all other places of his paper Popov uses the correct initials "И. Е."

1986. Система И. С. Орлова и релевантная логика. *Философские проблемы истории логики и методологии науки*. Часть 1. Москва: Институт Филочофии АН СССР: 93–98].

Povarov G. N. & B. V. Biryukov 2010. ORLOV Ivan Efimovich. *The New Encyclopedia of Philosophy. In four volumes*. Institute of Philosophy of the Russian Academy of Sciences. Editorial Board: V. S. Stepin, A. A. Guseinov, G. Y. Semigin, A. P. Ogurcov. Volume III, Moscow, Publishing House Thought: 165. [= Г. Н. Поваров & Б. В. Бирюков 2010. *ОРЛОВ Иван Ефимович*. *Новая философская энциклопедия. В четырех томах*. Институт философии РАН. Научно-редакционный совет: В. С. Степин, А. А. Гусейнов, Г. Ю. Семигин, А. П. Огурцов. Том 3. Москва, Издательство Мысль: 165].

Priest, G. 2021. Reflections on Orlov. *History and Philosophy of Logic* 42: 118-128

van Heijenoort, J. (ed.) 1967. *From Frege to Gödel: A Source Book in Mathematical Logic 1879–1931.* Harvard University Press.

von Plato, J. 2022. *Chapters from Gödel's Unfinished Book on Foundational Research in Mathematics.* Vienna Circle Institute Library 6. Cham, Springer.

Whitehead A. N. & B. Russell 1910. *Principia Mathematica, Vol I.* Cambridge University Press.

Zach, R. 1999. Completeness before Post: Bernays, Hilbert, and the development of propositional logic. *Bulletin of Symbolic Logic* 5: 331–364.

Zinov'ev, A. A. 1962. *Propositional logic and theory of inference.* Moscow, Publishing house of the USSR Academy of Sciences. [= А. А. Зиновьев 1962, *Логика высказываний и теория вывода*. Москва, Издательство Академии Наук СССР.]