The Geometry of Relevant Implication II

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Abstract

This note extends earlier results on geometrical interpretations of the logic **KR** to prove some additional results, including a simple undecidability proof for the four-variable fragment of **KR**.

1 The Logic KR

The logic **KR** results from logic **R** of relevant implication by adding the axiom *ex falso quodlibet*, that is to say, $(A \wedge \neg A) \rightarrow B$. The model theory for **KR** is elegantly simple. The usual ternary relational semantics for **R** includes an operation $*$ designed to deal with the truth condition for negation

$$x \models \neg A \Leftrightarrow x^* \not\models A.$$

The effect of adding *ex falso quodlibet* to **R** is to identify x and x^*; this in turn has a notable effect on the ternary accessibility relation. The postulates for an **R** model structure include the following implication:

$$Rxyz \Rightarrow (Ryxz \,\&\, Rxz^*y^*).$$

The result of the identification of x and x^* is that the ternary relation in a **KR** model structure (KRms) is *totally symmetric*. In detail, a KRms $\mathcal{K} = \langle S, R, 0\rangle$ is a 3-place relation R on a set containing a distinguished element 0, and satisfying the postulates:

1. $R0ab \Leftrightarrow a = b$;
2. $Raaa$;
3. $Rabc \Rightarrow (Rbac \,\&\, Racb)$ (total symmetry);
4. $(Rabc \,\&\, Rcde) \Rightarrow \exists f(Radf \,\&\, Rfbe)$ (Pasch's postulate).

2 Modular lattices and geometrical frames

In this section, we state the main results of the earlier paper [12] to which the current paper is a sequel. The proofs can be found in [12].

Given a **KR** model structure $\mathcal{K} = \langle S, R, 0\rangle$, we can define an algebra $\mathfrak{A}(\mathcal{K})$ as follows:

Definition 2.1 *The algebra $\mathfrak{A}(\mathcal{K}) = \langle \mathcal{P}(S), \cap, \cup, \neg, \top, \bot, t, \circ \rangle$ is defined on the Boolean algebra $\langle \mathcal{P}(S), \cap, \cup, \neg, \top, \bot \rangle$ of all subsets of S, where $\top = S, \bot = \emptyset, t = \{0\}$, and the operator $A \circ B$ is defined by*

$$A \circ B = \{c \mid \exists a \in A, b \in B(Rabc)\}.$$

The algebra $\mathfrak{A}(\mathcal{K})$ is a De Morgan monoid [1],[3] in which $a \wedge \overline{a} = \bot$, where $\bot$ is the least element of the monoid; we shall call any such algebra a **KR**-*algebra*. Hence the fusion operator $A \circ B$ is associative, commutative, and monotone. In addition, it satisfies the square-increasing property, and t is the monoid identity:

$$A \circ (B \circ C) = (A \circ B) \circ C,$$

$$A \circ B = B \circ A,$$

$$(A \subseteq B \wedge C \subseteq D) \Rightarrow A \circ C \subseteq B \circ D,$$

$$A \subseteq A \circ A,$$

$$A \circ t = A.$$

In what follows, we shall assume basic results from the theory of De Morgan monoids, referring the reader to the expositions in Anderson and Belnap [1] and Dunn and Restall [3] for more background. We have defined **KR**-algebras above as arising from De Morgan monoids by the addition of the axiom $a \wedge \neg a = \bot$. However, we could also have defined them using the construction of Definition 2.1, since any **KR**-algebra can be represented as a subalgebra of an algebra produced by that construction. This is not hard to prove by using the known representation theorems for De Morgan monoids – see for example [10]. **KR**-algebras are closely related to relation algebras. In fact, they can be defined as square-increasing symmetric relation algebras – for basic definitions the reader can consult the monograph [7] by Roger Maddux.

In a **KR**-algebra, we can single out a subset of the elements that form a lattice; this lattice plays a key role in the analysis of the logic **KR**.

Definition 2.2 *Let $\mathfrak{A}$ be a* **KR**-*algebra. The family $\mathcal{L}(\mathfrak{A})$ is defined to be the elements of $\mathfrak{A}$ that are $\geq t$ and idempotent, that is to say, $a \in \mathcal{L}(\mathfrak{A})$ if and only if $a \circ a = a$ and $t \leq a$. If $\mathcal{K}$ is a* **KR** *model structure, then we define $\mathcal{L}(\mathcal{K})$ to be $\mathcal{L}(\mathfrak{A}(\mathcal{K}))$.*

The following lemma provides a useful characterization of the elements of $\mathcal{L}(\mathfrak{A})$; it is based on some old observations of Bob Meyer.

Lemma 2.3 *Let $\mathfrak{A}$ be a* **KR**-*algebra. Then the following conditions are equivalent:*

1. $a \in \mathcal{L}(\mathfrak{A})$;
2. $a = (a \to a)$;
3. $\exists b[a = (b \to b)]$.

If $\mathcal{K} = \langle S, R, 0\rangle$ is a **KR** model structure, then a subset A of S is a *linear subspace* if it satisfies the condition

$$(a, b \in A \wedge Rabc) \Rightarrow c \in A.$$

A non-empty linear subspace must contain 0, since $Raa0$ holds in any **KR** model structure.

A lattice is *modular* if it satisfies the implication

$$x \geq z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z.$$

We include a zero element satisfying the equality $x \vee 0 = x$ in the modular lattices discussed below, and use the notation $\mathcal{M}_0$ for the variety of modular lattices with zero. For background on modular lattice theory, the reader can consult the texts of Birkhoff [2] or Grätzer [5].

A *chain* in a lattice L is a totally ordered subset of L; the length of a finite chain C is $|C| - 1$. A chain C in a lattice L is *maximal* if for any chain D in L, if $C \subseteq D$ then $C = D$. If L is a lattice, $a, b \in L$ and $a \leq b$, then the *interval* $[a, b]$ is defined to be the sublattice $\{c : a \leq c \leq b\}$.

Let L be a lattice with least element 0. We define the *height* function: for $a \in L$, let $h(a)$ denote the length of a longest maximal chain in $[0, a]$ if there is a finite longest maximal chain; otherwise put $h(a) = \infty$. If L has a largest element 1, and $h(1) < \infty$, then L has *finite height*.

Lemma 2.4 *If $\mathcal{K}$ is a **KR** model structure, then the elements of $\mathcal{L}(\mathcal{K})$ are exactly the non-empty linear subspaces of $\mathcal{K}$.*

Theorem 2.5 *If $\mathfrak{A}$ is a **KR**-algebra, then $\mathcal{L}(\mathfrak{A})$, ordered by containment, forms a modular lattice, with least element t, and the lattice operations of join and meet defined by $a \wedge b$ and $a \circ b$.*

Definition 2.6 *Let L be a lattice with least element 0. Define a ternary relation R on the elements of L by:*

$$Rabc \Leftrightarrow a \vee b = b \vee c = a \vee c,$$

and let $\mathcal{K}(L)$ be $\langle L, R, 0\rangle$.

Theorem 2.7 *$\mathcal{K}(L)$ is a **KR** model structure if and only if L is modular.*

In Definition 2.6, if a, b, c are distinct points in a projective space, then $Rabc$ holds if and only if the three points are collinear. Hence, the defined ternary relation can be considered as a generalized notion of collinearity that applies to any elements in a modular lattice.

Definition 2.8 *If L is a lattice, then an* ideal *of L is a non-empty subset I of L such that*

1. *If $a, b \in I$ then $a \vee b \in I$;*

2. If $b \in I$ and $a \leq b$, then $a \in I$.

The family of ideals of a lattice L, ordered by containment, forms a complete lattice $I(L)$. The original lattice L is embedded in $I(L)$ by mapping an element $a \in L$ into the *principal ideal* containing a, $(a] = \{b \mid b \leq a\}$. It is easy to verify that the mapping $a \longmapsto (a]$ is a lattice isomorphism between L and a sublattice of $I(L)$.

Theorem 2.9 *Let L be a modular lattice with least element 0, and $\mathcal{K}(L) = \langle L, R, 0\rangle$ the* **KR** *model structure constructed from L. Then $\mathcal{L}(\mathcal{K}(L))$ is identical with the lattice of ideals of L.*

Corollary 2.10 *Any modular lattice of finite height (hence any finite modular lattice) is representable as $\mathcal{L}(\mathcal{K})$ for some* **KR** *model structure $\mathcal{K}$. In addition, any modular lattice is representable as a sublattice of $\mathcal{L}(\mathcal{K})$ for some* **KR** *model structure $\mathcal{K}$.*

3 Applications of the main construction

Theorem 3.1 *Let $\mathfrak{A}$ be a* **KR***-algebra, and G a subset that freely generates $\mathfrak{A}$. If $G^* = \{a \to a : a \in G\}$, then G^* freely generates $\mathcal{L}(\mathfrak{A})$.*

Proof. Let L be the sublattice of $\mathcal{L}(\mathfrak{A})$ generated by G^*. If M is a modular lattice with least element 0, and $f : G^* \longmapsto M$ a function from G^* to M, then we need to show that f can be extended to a lattice homomorphism from L to M.

Using Definition 2.6, we can define the **KR** model structure $\mathcal{K}(M)$, and hence by Definition 2.1, the **KR**-algebra $\mathfrak{B} = \mathfrak{A}(\mathcal{K}(M))$. For $a \in G$, define $g(a) = f(a \to a)$. Since G freely generates $\mathfrak{A}$, g can be extended to a homomorphism h from $\mathfrak{A}$ to $\mathfrak{B}$. By Theorem 2.9, $\mathcal{L}(\mathfrak{B})$ is identical with the lattice of ideals of M, so that we can identify M with a sublattice of $\mathcal{L}(\mathfrak{B})$ by the embedding $a \rightleftharpoons (a]$ that maps an element $a \in M$ into the principal ideal generated by a.

For $a \in G^*$, let $a = b \to b$, for $b \in G$. Then

$$h(a) = h(b \to b) = h(b) \to h(b) = g(b) \to g(b)$$

$$= f(b \to b) \to f(b \to b) = f(a) \to f(a) = f(a).$$

Thus, h restricted to L is a lattice homomorphism from L to M extending f, showing that G^* freely generates L.

To complete the proof of the theorem, we show that $L = \mathcal{L}(\mathfrak{A})$. Define

$$X = \{a \in \mathfrak{A} : a \to a \in L\}.$$

Since $G^* \subseteq L$, $G \subseteq X$, and in addition, X is closed under $\wedge$ and $\circ$, since L is closed under these operations. If $a \in X$, then $(\neg a \to \neg a) = (a \to a) \in L$, so that $\neg a \in X$. Since G generates $\mathfrak{A}$, it follows that $X = \mathfrak{A}$, so that $L = \mathcal{L}(\mathfrak{A})$. $\square$

Corollary 3.2 *In the logic* **KR***, there are infinitely many distinct formulas built from the formulas* $p \to p, q \to q, r \to r$ *and* $s \to s$ *using only the connectives* $\wedge$ *and* $\circ$.

Proof. Theorem 3.1 shows that the formulas $p \to p, q \to q, r \to r$ and $s \to s$ generate an algebra of formulas isomorphic to the free modular lattice on four generators. This algebra is infinite [2, p. 64]. □

Corollary 3.2 was the only consequence deduced in [12] from Theorem 3.1. However, there are quite a few added conclusions that we can draw, as we explain in what follows.

Theorem 3.3 *The free* **KR***-algebra on three generators is finite.*

Proof. Let $\mathfrak{A}_3$ be the free **KR**-algebra on three generators. By Theorem 3.1, $\mathcal{L}(\mathfrak{A}_3)$ is an $\mathcal{M}_0$ lattice freely generated by three elements. This lattice is finite [2, pp. 63-64], so the **KR**-model structure $\mathcal{K}_3$ constructed from this lattice is finite. Since by Corollary 2.10, $\mathfrak{A}_3$ is isomorphic to $\mathcal{L}(\mathcal{K}_3)$, it follows that it is finite. □

Corollary 3.4 *The decision problem for* **KR** *for formulas in three variables is solvable.*

The main problem discussed in the paper [11], written in honour of my old friend Bob Meyer, was this: given a logic **L** intermediate between **T-W +A15** and **KR**, what is the smallest number of variables for which the decision problem for **L** is unsolvable? (**A15** is the axiom scheme $[(A \to B) \wedge A \wedge t] \to B$.) The main theorem of that paper shows that this number is at most four. For **KR**, Corollary 3.4 shows that this number is *exactly* four.

The construction proving the main theorem of [11] is a fairly intricate computation in coordinate frames. This seems to be necessary for logics weaker than **KR**; however, for **KR** itself, we can give a very quick and easy undecidability proof by exploiting some deep results from the theory of modular lattices.

Theorem 3.5 *The decision problem for* **KR** *in four variables is unsolvable.*

Proof. Define a *lattice formula* to be one built from $p \to p, q \to q, r \to r$ and $s \to s$ using only the connectives $\wedge$ and $\circ$, and a *lattice implication* to be a formula of the form $A \to B$, where A and B are lattice formulas.

If φ is a term in the theory of modular lattices in the variables x_1, x_2, x_3, x_4, we define its translation φ^τ as the lattice formula constructed from φ by replacing x_1, x_2, x_3, x_4 by $p \to p, q \to q, r \to r, s \to s$ and the lattice join $\vee$ by the fusion connective $\circ$. Theorem 3.1 shows that an inequality $\varphi \leq \psi$ in four variables holds in the free modular lattice if and only if $\varphi^\tau \to \psi^\tau$ is a theorem of **KR**.

Ralph Freese showed [4] that the word problem for the free modular lattice in five free generators is unsolvable. This was improved by Christian Herrmann [6] to four free generators. The results of Freese and Herrmann complete the proof of the theorem. □

4 Free associative connectives

Bob Meyer and his collaborators had an ingenious plan for proving **R** undecidable. The idea was to search for a free associative connective in **R** that could then be used to encode an undecidable problem in the theory of semigroups. As part of this search, they developed a suite of programs to help in the investigation. Their research program is described in the monograph by Thistlewaite, McRobbie and Meyer [9].

This monograph does not give a precise formal definition of the notion of free associative connective. However, the discussions in [9] and the examples given there suggest the following definition. Let **L** be a logic in the language of **R**, and $p \odot q$ a formula of **L** with two free variables. If σ is a term in the theory of semigroups with the free variables $x_1, \ldots, x_n$, then we define the translation of σ into **L** as the formula σ^t of **L** that results by replacing $x_1, \ldots, x_n$ by the propositional variables $p_1, \ldots, p_n$ and the semigroup operation by the defined connective $\odot$. Then we define $\odot$ to be a *free associative connective in* **L** if an equation $\sigma = \tau$ holds in the free semigroup if and only if $\sigma^t \leftrightarrow \tau^t$ is provable in the logic **L**.

The paper [11] contains a discussion of the question of the existence of such a connective in **R**, as well as the problem of proving or disproving its existence. However, for **KR**, we can settle the question easily.

Theorem 4.1 *There is no free associative connective in* **KR**.

Proof. Let $p \odot q$ be a formula of **L** with two free variables. Define an infinite sequence of semigroup terms in one variable x by setting $s_1 = x$ and $s_{k+1} = x \cdot s_k$, and let σ_i be the translation s_i^τ of the term s_i in **KR**. Since by Theorem 3.3, the free **KR**-algebra $\mathfrak{A}_1$ in a single generator is finite, it follows that in the infinite sequence $\sigma_1, \sigma_2, \ldots, \sigma_k \ldots$ there are distinct formulas σ_i and σ_j that take the same value in $\mathfrak{A}_1$. Hence, $\sigma_i \leftrightarrow \sigma_j$ is a theorem of **KR**. However, the terms s_i and s_j correspond to distinct elements in the free semigroup in one generator, so that the connective $\odot$ is not freely associative. □

At first sight, Theorem 4.1 appears to conflict with the undecidability of **KR**, since that result is proved by encoding a finitely presented semigroup in the logic. However, this problem is easily resolved by observing that the undecidability proof does not proceed by employing a free associative connective as defined above. The encoding of the semigroup operation $x \cdot y$ uses two auxiliary variables c_{23} and c_{31}, so that the key definition is:

$$x \cdot y = (x \otimes c_{23}) \otimes (c_{31} \otimes y),$$

where $b \otimes d$ is the operation defined on a modular n-frame as in §2 of [11]. Furthermore, the proof of associativity for the defined operation uses a number of added assumptions that formalize the concept of a three-dimensional coordinate frame.

The method used in the proof of Theorem 4.1 does not extend to **R**. This is because Meyer proved in [8] that the formulas in one free variable in the infinite sequence $p, p \circ p, (p \circ p) \circ p, \ldots$ are all mutually non-equivalent. Consequently, the questions of the existence of a free associative connective in **R** and the decidability of the one-variable fragment of **R** remain open.

References

[1] A. R. Anderson and N. D. Belnap. *Entailment*, volume 1. Princeton University Press, Princeton, NJ, 1975.

[2] G. Birkhoff. *Lattice Theory.* American Mathematical Society, 1967. A.M.S. Colloquium Publications Volume XXV.

[3] J. Michael Dunn and Greg Restall. Relevance logic. In D.M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, 2nd Edition*, volume 6, pages 1–128. Kluwer Academic Publishers, Dordrecht, Boston and London, 2002.

[4] Ralph Freese. Free modular lattices. *Trans. Amer. Math. Soc.*, 261:81–91, 1980.

[5] George Grätzer. *General Lattice Theory.* Academic Press, 1978.

[6] Christian Herrmann. On the word problem for the modular lattice with four free generators. *Mathematische Annalen*, 265:513–527, 1983.

[7] Roger D. Maddux. *Relation Algebras.* Elsevier, 2006. Studies in Logic and the Foundations of Mathematics Volume 150.

[8] Robert.K. Meyer. R_I – the bounds of finitude. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 16:385–387, 1970.

[9] P.B. Thistlewaite, M.A. McRobbie, and R.K. Meyer. *Automated theorem-proving in non-classical logics.* Pitman, London, 1988.

[10] Alasdair Urquhart. Duality for algebras of relevant logics. *Studia Logica*, 56:263–276, 1996.

[11] Alasdair Urquhart. Four variables suffice. *Australasian Journal of Logic*, 5:66–73, 2007.

[12] Alasdair Urquhart. The geometry of relevant implication. *IFCoLog Journal of Logics and their Applications*, 4:3:591–604, 2017.