The Evolution of Chihara's Anti-Nominalistic Nominalism Regarding the Existence of Mathematical Objects

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Abstract

Charles Chihara argues that many scholars in the philosophy of mathematics have adopted a false assumption, the Fregean Assumption, that mathematical theorems are propositions about the actual world. Chihara, who does not believe in the existence of abstract objects but does believe in abstract notions such as satisfiability in a structure, contends that mathematical theorems are rather only true in the sense that they are satisfied by appropriate kinds of structures. In this paper, Chihara's thesis is presented and analyzed particularly in the context of Set Theory.

1 Introduction

Charles Chihara sadly passed away in April 2021, and he left behind a rich legacy of philosophical work. Inherently pragmatic, eminently logical, Chihara's philosophy on the existence of mathematical objects is both meticulous and profound. I trust his work will stand the test of time and will be a beacon to other philosophers. His elevation of a modal primitive to a central role in his philosophy articulated within his work *Constructibility and Mathematical Existence* and developed further in *The Worlds of Possibility* and *A Structural Account of Mathematics* is incisive and a wonderful contribution into the problem of existence in mathematics, although it is not without problematic areas. Chihara's family donated his last unpublished work '*The Fregean Assumption or Does the Philosophy of Mathematics Rest* *on a Mistake?*'[1] to the Tarski Group in Logic at The University of California at Berkeley. In the *Fregean Assumption* Chihara synthesizes all of his previous works and diagnoses what he views as the central problem with Indispensability Arguments, namely an implicit and hidden assumption he claims is false. In his boldest assertion in all his works Chihara contends that many great philosophers have made an false assumption in the philosophy of mathematics, the Fregean assumption, which has several adverse consequences. The work also extends his previous work on his theory of Constructibility. Chihara advocates for a Hilbertian structural account of mathematical truth within the confines that he does not believe there are abstract objects. Before delving into the *Fregean Assumption*, Chihara's new thesis, and an analysis of his arguments, I shall provide a brief summary of Chihara's previous work so that one can see how his thinking evolved over time.

1.1 *Ontology and the Vicious Circle Principle 1973*

Chihara's first major work was *Ontology and the Vicious Circle Principle* in 1973 where he put forth a nominalistic version of mathematics within a predicative version of set theory. His motivation was to respond to Quine's Indispensability Argument and Quine's challenge to the nominalist to produce a version of mathematics adequate for the needs of empirical scientists without requiring its quantifiers to range over mathematical objects. In the work Chihara argues that the Continuum Hypothesis has no definitive truth value by relating the axioms of set theory to a story about laws in a mythical kingdom story. True facts about the mythical kingdom of Myo would be relative to the axioms, and statements which could not be formally derived from the laws of the mythical kingdom would lack any genuine truth value.[2] Thus, the Continuum Hypothesis would lack any genuine truth value in virtue of it being independent of the axioms of Zermelo-Fraenkel Set theory. A central idea for Chihara was the development of a nominalistic version of mathematics within a predicative version of set theory. Chihara wanted to use open sentences instead of sets within Σ_ω, a nominalistic version of Wang's predicative set theory.

[1] Chihara (2021)
[2] Chihara (1973: 67-70)

1.2 *Constructibility and Mathematical Existence 1990*

Chihara's anti-nominalistic nominalism objects to abstract entities but not abstract notions such as satisfiability in a structure.[3] Chihara refers to his brand of nominalism as "not nominalistic" from "the ideological point of view".[4] In response to the limitations of the predicative version for his nominalistic approach and in response other versions of the Indispensability Argument put forth by Putnam, Resnik and Shapiro, Maddy and Burgess, Chihara expanded his previous work to a nominalistic reconstruction of impredicative mathematics in *Constructibility and Mathematical Existence* 1990: The impredicative version would have greater mathematical power than the previous predicative version. Chihara discusses the problem of how we are to understand existence assertions of mathematics,

> The classical mathematician is regarded as asserting the existence of 'abstract' entities not found in the physical world. The mathematician who believes the theorems of classical mathematics to be true, according to this position, ought to believe that there really are such entities as natural numbers, functions, sets, ordered pairs, and the like. The view that emerges then is that of the mathematician investigating a realm of entities that cannot be seen, felt, heard, smelled, or tasted, even with the most sophisticated instruments. But if this is so, how can the mathematician know that such things exist?[5]

Chihara implemented his theory of Constructibility in the work as a nominalistic version of simple type theory; Constructibility would not assert the existence of sets, but would instead assert that it is possible to construct various kinds of open-sentence tokens.[6] Chihara describes the construction of open-sentence tokens: "to say it is possible to construct an open sentence of a certain sort, is to say that in some possible world there is a token of the type of open constructed," where we "can imagine a possible world in which some people who have an appropriate language, do something that can be described as the production of a token."[7]

Chihara maintains throughout Constructibility and all his subsequent works that his Constructibility theory is "ontologically" 'nominalistic' but

[3]Chihara (2004: 47)
[4]Chihara (2004: 47)
[5]Chihara (1998: 5)
[6]Chihara (2021: 122)
[7]Chihara (1990: 40)

not "ideologically" 'nominalistic'.[8] This distinction is central to Chihara's defense of his work from various arguments put forth against Constructibility, but it may expose him to other objections. While Chihara does not believe in the existence of abstract objects, he does believe in certain abstract notions such as satisfiability in a structure. Chihara is adamant that he is not providing an interpretation of statements of real mathematics. In his final work, the *Fregean Assumption*, Chihara responds to criticism by Balaguer that his constructibility is certainly a non-standard reconstruction by stating, "It should have been clear to Balaguer, from this passage, that my aim was not to supply an interpretive analysis of real classical mathematics, but rather was to provide us philosophers with a 'another kind of mathematics' which would give us a 'new perspective from which to view' classical mathematics."[9] Chihara contends that his theory is a kind of nominalistic model (a kind of "theoretical instrument") to facilitate carrying out and assessing logical, mathematical and philosophical reasoning.

> Thus, if an adequate model of mathematical reasoning can be constructed which does not 'quantify over' or presuppose the existence of mathematical objects, then one would have reasons for being skeptical of the claim that mathematical objects must be presupposed in order to explain why mathematics has proved to be so useful for scientific reasoning. Thus, to argue that such a model is not an accurate 'interpretation' of actual mathematics would be to miss completely the point of the Constructibility Theory.[10]

John Steel argues that Chihara's theory of Constructibility looks very much like Simple Type Theory (STT) and questions that "if you have a genuinely new interpretation of the language of STT, that novelty has to show up somewhere. In the case of mathematical constructivists, it shows up in their rejection of various set-existence assertions. But you do not seem willing to reject any specific axiom or theorem of STT."[11] Chihara responds to Steel's objection in his 1998 work *Worlds of Possibility*, but his response may not be entirely satisfactory in that it avoids the heart of the question.

[8] Chihara (1990: 47)
[9] Chihara (2021: 126)
[10] Chihara (2021: 127)
[11] Chihara (1998: 316)

1.3 *Worlds of Possibility 1998*

In *Worlds of Possibility* Chihara argues that the success of possible world semantics has given rise to modal realism; however Chihara rejects modal realism. Nonetheless, Chihara advocates for the usefulness of modality in science and he quotes Putnam, "we may argue that the notion of possibility is intuitively evident and necessary for science."[12] Chihara certainly relies on such notions of mathematics as satisfiability, while continuing to eschew the existence of abstract objects, and Chihara uses modal reasoning within Constructibility theory, but he is not a realist when it comes to possible worlds. Thus, the main purpose of his 1998 work was to demonstrate that modal principles and modal arguments may be analyzed with systems of modal quantificational logic such as S5 without making a commitment to the existence of possible worlds. Chihara views the semantics of modal logic useful for investigating validity of modal sentences, but a connecting theorem is needed to bridge the divide between possible worlds semantics and absolute truth for a non modal realist.[13] Chihara is essentially responding to Davidson's problem: How is the notion of truth relativized to an interpretation related to the absolute notion of truth.[14] Chihara casts his connecting theorem in terms of NL proto-interpretation conforming to some S5 interpretation. For Chihara's definition of conformance of proto-interpretation to an S5 interpretation, modality is the underlying primitive in how "the world could have been such".[15] The centrality of the modal intuition in terms of the Connecting theorem is a rather stellar insight of Chihara to accomplish his stated goal. However, the commitment to the intuition of modality in the broadly logical sense may open up problems in other areas.

Chihara cites four conditions, for an NL proto-interpretation of M*, Chihara's version of S5 quantificational modal logic, to conform to an S5 interpretation. The first is that the conforming NL proto-interpretation ϑ be such that every possibility represented by the S5 interpretation is indeed a possibility. The second, requires that the NL proto-interpretation be such that there be no possibility that the S5 interpretation fails to represent. The third is a consistency condition and the fourth is a serious actualism condition.[16]

[12] Chihara (1990: 52)
[13] Chihara (1998: 203)
[14] Chihara (1998: 190)
[15] Chihara (1998: 232)
[16] Chihara (1998: 233)

Definition 1.1. The world could have been such that there was a function f_w via which $< \langle d(w), r(\alpha), e(\xi, w)\rangle_\vartheta >$ represented the world means: the world could have been such that there was a bijection f_w from $d(w)$ to the extension of $[\theta/\vartheta]$ such that

1) for every predicate θ of any degree n, and for every n-tuple of members of d(w), $< \delta_1, \delta_2, ..., \delta_n >$: $< \delta_1, \delta_2, ..., \delta_n > \ \in e(\theta, w)$ iff
$< f_w(\delta_1), f_w(\delta_2), ..., f_w(\delta_n) > \ \in ext[\theta/\vartheta]$,

2) for every even-subscripted individual constant $\alpha \in d(w)$:
$f_w(r(\alpha)) = den[\alpha/\vartheta]$,

given S5 interpretation $\langle W, D, w*, d, r, e\rangle_\vartheta$ and proto-interpretation ϑ.

Definition 1.2. C1 (the NL proto-interpretation is such that every possibility the S5 interpretation represents is indeed a possibility)

For every $w \in W$ the world could have been such that there was a function via which $\langle d(w), r(\alpha), e(\xi, w)\rangle_\vartheta$ represented the world.

Definition 1.3. C2 (the NL proto-interpretation is such that that there be no possibility that the S5 interpretation fails to represent)

The world could not have been such that, for no $w \in W$, was there a function via which $\langle d(w), r(\alpha), e(\xi, w)\rangle_\vartheta$ represented the world.

Definition 1.4. C3 (Consistency Condition)

For all $\delta_1, \delta_2, ..., \delta_n \in D$ and for every $w, w_1, w_2, ..., w_n \in W$ such that $\delta_1 \in d(w_1), \delta_2 \in d(w_2), ..., \delta_n \in d(w_n)$

if every $\delta_1, \delta_2, ..., \delta_n \in d(w)$, then it is not the case that the world could have been such that there was a function f_1 via which $\langle d(w_1), r(\alpha), e(\xi, w_1)\rangle_\vartheta$ represented the world; the world could have been such that there was a function f_2 via which $\langle d(w_2), r(\alpha), e(\xi, w_2)\rangle_\vartheta$ represented the world;...; and the world could have been such that there was a function f_n via which $\langle d(w_n), r(\alpha), e(\xi, w_n)\rangle_\vartheta$ represented the world, such that had the world been such that there was a function via which $\langle d(w), r(\alpha), e(\xi, w)\rangle_\vartheta$ represented the world, it is not the case that there would have also been a function g via which $\langle d(w), r(\alpha), e(\xi, w)\rangle_\vartheta$ represented the world such that $g(\delta_1) = f(\delta_1), g(\delta_2) = f(\delta_2), ..., g(\delta_n) = f(\delta_n)$;

and if, for some k, $\delta_k \notin d(w)$, then it is not the case that the world could have been such there was a function f_k via which $\langle d(w_k), r(\alpha), e(\xi, w_k)\rangle_\vartheta$ represented the world such that had the world been such that there was a function via which $\langle d(w), r(\alpha), e(\xi, w)\rangle_\vartheta$ represented the world, it is not the case that there would have been a function g via which $\langle d(w), r(\alpha), e(\xi, w)\rangle_\vartheta$ represented the world, such that, for some $x \in d(w), g(x) = f_k(\delta_k)$.

Definition 1.5. C4 (Serious Actualism Condition)

For every $w, w_1, w_2, ..., w_n \in W$, for any atomic sentence θ consisting of a predicate of degree m immediately followed by m occurrences of individual constants, among which are occurrences of the odd-subscripted individual constants $\beta_1, \beta_2, ..., \beta_n$, and for all $\delta_1, \delta_2, ..., \delta_n \in D$, **if there is a u** $\in$ **W** such that, had the world been such that there was a function via which $\langle d(w), r(\alpha), e(\xi, w)\rangle_\vartheta$ represented the world, there would have been a function g via which $\langle d(u), r(\alpha), e(\xi, u)\rangle_\vartheta$ represented the world such that some $\vartheta(d)$-$[\beta_1/g(\delta_1), \beta_2/g(\delta_2), ..., \beta_n/g(\delta_n)]$-sequence satisfied the condition:

$[\theta/\vartheta]$ truly describes what would have been the case, had the world had been w,

then had the world been such that there was a function h via which $\langle d(w), r(\alpha), e(\xi, w)\rangle_\vartheta$ represented the world, for each $j \in 1, 2, ..., n$, there exists an $x \in d(w)$ such that $h(x) = f_j(\delta_j)$.

Chihara refers to an NL proto-interpretation which conforms to a S5 interpretation, as an NL-interpretation.

The Barcan formula is not provable within the system M*. This is desirable for Chihara because he does not want a constructibility assertion of the form (CF)(...F...) to assert the actual existence of an open-sentence token with certain properties, and the Barcan formula would entail that $\Diamond(\exists F)(KF \ \&...F...)$ implies $(\exists F)\Diamond(KF \ \&...F...)$. Chihara later in the *Fregean Assumption* cites complications over the Barcan formula and his unpreparedness in his prior 1990 work to fully develop a system of modal logic as a reason for not developing Constructibility in S5 within this earlier work.[17]

Chihara proves a Fundamental Connecting Theorem:

Theorem 1.6. *For every $w \in W$, and for every sentence θ for every $\delta_1, \delta_2, ..., \delta_n \in D$, if $\beta_1, \beta_2, ..., \beta_n$ are the n odd-subscripted individual constants that occur in θ then the following holds: θ is satisfied at w under I by some D-[$\beta_1/\delta 1, \beta_2/\delta_2, ...\beta_n/\delta_n$]-sequence iff if for some $z \in W$, had the world been z, there would have been a representing function g for z such that some $\vartheta(d)$-[$\beta_1/g(\delta 1), \beta_2/g(\delta_2), ...(\beta_n/g(\delta_n)$]-sequence satisfied the condition: [θ/ϑ] truly describes what would have been the case had the world been w.*[18]

The significance of the theorem is that Chihara was able to construct via a Connecting Theorem an applied semantics for modal logic without

[17] Chihara (2021: 144)
[18] Chihara (1998: 252)

any commitment to possible worlds and thus was able to creatively utilize S5 structures while retaining his philosophical opposition to modal realism.

1.4 *A Structural Account of Mathematics 2004*

Chihara explains how this theory of Constructibility can be used by the ontological nominalist to provide a "structural account of mathematics" without assuming the existence of mathematical objects. Chihara is sympathetic to certain aspects put forth by structuralists, but Chihara distinguishes his theory from structuralism of Shapiro and Resnik and refers to his view as a "structural account of mathematics". Chihara describes the distinction, "my own view of mathematics will not attempt to provide an account of the content of mathematical assertions in the way their accounts [structuralism] do."[19] However, Chihara believes that theorems of mathematical theories may provide us with "information that is structural in content," but his account does not tell us what the theorems assert in the literal sense.[20]

When Chihara speaks of what is possible in Constructibility he is using 'possible' in the broadly logical sense. He goes on to develop finite cardinality theory within a formalized axiomatic version of Constructibility to follow Frege's own development of finite cardinality theory. Chihara shows how a version of arithmetic can be developed without quantification over numbers, sets, extensions of concepts, or any other abstract objects:

An attribute N is a cardinal number attribute iff it is possible to construct some property F such that N is satisfiable by all and only those properties equinumerous with F.[21]

Chihara explains how the impredicative system of Constructibility differed from the predicative system in *Ontology and the Vicious Circle Principle* and provided greater mathematical power, but he stressed that he invented Constructibility not as an interpretive analysis of real classical mathematics but rather to provide a new perspective from which to view classical mathematics, to respond to Quine's challenge, and to explain the role mathematics plays in everyday life and science.[22]

Chihara argues that within Constructibility he can provide for sentences in natural language involving mathematical operations a corresponding sentence of Constructibility, the "c-version of the sentence" where the sentence

[19]Chihara (2004: 65)
[20]Chihara (2004: 65-66)
[21]Chihara (2004: 178)
[22]Chihara (2004: 163-168)

is true if and only if the c-version is true.[23] Chihara does not elaborate on limitations as to the scope of Constructibility and which sentences have c-versions.

Chihara has a discussion of Gödel's Platonism.[24] In Gödel's ontological platonism, mathematical intuition gives rise to mathematical knowledge. Gödel placed the same confidence in mathematical intuition as sense perception. Gödel states, "set-theoretic concepts and theorems describe some well-determined reality, in which Cantor's conjecture [the continuum hypothesis] must either be true or false."[25] (quoting Benacerraf and Putnam '*What is Cantor's Continuum Hypothesis*' p.263-264) Gödel believed that the Continuum Hypothesis had a definitive truth value irrespective of its formally undecidable nature from the axioms, and that we possess mathematical intuition which allows us to perceive sets.[26] Gödel also believed that we should have no less confidence in mathematical intuition than sense perception.[27] Given Chihara's use of broadly logical possibility in connecting results, why would he not consider it relevant as to whether such a perception was possible instead of only dismissing Gödel's mathematical intuition.

Chihara's use of possibility in the broadly logical sense might give rise to interesting consequences when he creates constructibility versions of mathematical statements. Chihara would be opposed to the Continuum Hypothesis having a truth value according to *Ontology and the Vicious Circle Problem.* He also is opposed to Gödel's ontological platonism:

> Gödel maintains that we have a kind of perception of the objects of transfinite set theory, and this position is supposed to follow from the fact that the axioms of set theory force themselves upon us as being true, but how do the axioms force themselves upon as being true? ' ...,' crudely put, the suggestion is that the only plausible explanation of this knowledge is that we have a kind of perception of sets, '...,' Gödel's justification for his claim that we must have a kind of perception of mathematical objects is simply not convincing.[28]

Chihara compares Gödel's belief in sets to the Platonic doctrine of forms as both are independent of the physical world. As such Gödel needed mathe-

[23]Chihara (2004: 231)
[24]Chihara (2004: 99-103)
[25]Chihara (2004: 101)
[26]Chihara (2004: 101)
[27]Chihara (2004: 102)
[28]Chihara (1990: 20)

matical intuition to explain how we can have access to such abstract objects and why the axioms of set theory force themselves upon us.[29] Chihara also discusses Maddy's views briefly and states, "Maddy adopted the view that her doctrines regarding our ability to see sets should not be taken to constitute an interpretation of Gödel's views but rather her form of set-theoretical realism- a view she has come most recently to reject."[30] Chihara does not provide a constructibility version for the Continuum Hypothesis. It may be that the translation would give CH an independent truth value given our modal intuition that assertions about what is possible in an absolute sense should be true or false. This would clash with his stated position.

Chihara rejects Gödel's mathematical intuition and ontological platonism yet accepts the abstract notion of modal intuition and accepts modality as a primitive. Chihara also accepts abstract notions such as satisfiability, but he does not believe in abstract objects. This forces him to use modal notions in connecting results; it is a difficult and narrow path to walk, as he defends his focus on structures while maintaining that by using structures he is not committing himself to any metaphysical entity. Chihara claims he can accomplish through constructible realizations,[31] where he will also implement his Constructibility Theory and use open sentences which are constructible.[32]

2 *The Fregean Assumption or Does the Philosophy of Mathematics Rest on a Mistake? 2021*

In his recent unpublished draft manuscript he was working on up until his death, Chihara returns full circle to Quine and the Indispensability Argument. The Fregean Assumption is the contention that mathematical theorems are propositions about the actual world. Chihara uses the word 'Fregean' to denote any scholar who accepts or assumes that mathematical theorems are propositions about the actual world.[33] Chihara includes in the Fregean camp a long list of scholars including Gottlob Frege, Bertrand Russell, Alfred North Whitehead, Kurt Gödel, Willard Quine, Hilary Putnam, David Lewis, Hartry Field, Mark Balaguer, John Burgess, Gideon Rosen, Penelope Maddy, Alvin Plantinga, Michael Resnik, Stewart Shapiro, Bob

[29] Chihara (2004: 101-102)
[30] Chihara (1990: 104)
[31] Chihara (2004: 221)
[32] Chihara (2004: 224)
[33] Chihara (2021: iv)

Hale, Crispin Wright, Mark Steiner, Philip Kitcher, Mark Colyvan, and James Brown.[34] Chihara writes,

> I claim that each of the philosophers in the above list has, at one time or other, expressed or assumed the belief that the truths of mathematics are all propositions about the actual world—a belief, in other words, in the Fregean Assumption. Thus, I claim that there are significant grounds for concluding: Much of the philosophy of mathematics that was put forward, developed, and widely accepted, during the Contemporary Period does indeed rest on a mistake—the mistake of accepting or simply assuming what I have called the Fregean Assumption.[35]

Chihara concludes that the Fregean Assumption in most cases leads to Platonism,[36] as existential statements in mathematics are truths about the actual world carrying ontological commitments.[37]

Chihara argues the Fregean Assumption has generally been something that philosophers often tacitly assume rather than something they directly express.[38] Chihara contends that because Frege accepted the a priori nature of Euclidean Geometry, Frege accepted three sources of all our knowledge: sense perception, a geometrical and temporal source of knowledge that he called "intuition" and "the logical source of knowledge."[39] Chihara contrasts the Fregean approach with that of Hilbert who saw theorems of geometry as not true in the straight forward sense by rather true in that they are satisfied by structures and are true in a "structural way."[40] Chihara saw Russell, Quine and Gödel as three of the most influential Fregeans.[41] Chihara seeks in the work to argue against the Indispensability Argument of Quine. Chihara contends the Fregean Assumption arose because those who adopted it were more comfortable with axiomatic systems than the model theoretic versions of mathematical theories developed by Hilbert and Tarski.[42]

To refute the Indispensability Argument for the existence of mathematical objects by Quine and Colyvan,[43] Chihara discusses the epistemological

[34] Chihara (2021: 21, 314)
[35] Chihara (2021: 315)
[36] Chihara (2021: xviii, 324)
[37] Chihara (2021: 8)
[38] Chihara (2021: xx)
[39] Chihara (2021: 8)
[40] Chihara (2021: 10)
[41] Chihara (2021: 26)
[42] Chihara (2021: 320)
[43] Chihara (2021: 30)

problem for Platonism: How are we able to obtain knowledge about these abstract objects to which theorems of mathematics refer? Although Chihara concludes that the Fregean Assumption must be rejected, he must explain why mathematical can be regarded as "true" in an appropriate sense, and thus he argues that mathematical theorems are satisfied by appropriate kinds of structures, where the structure is regarded as "abstract model of that part of the mathematical universe in which the sentence is satisfied."[44] Chihara states,

> [M]athematical theorems —theorems that we classify as 'expressing mathematical truths'—are not straightforward expressions of propositions about the actual world. The truths of mathematics are not propositions that are directly about the actual world at all, but instead are Hilbertian sentences that are true or false of (or satisfied by) appropriate kinds of structures. Obviously, I have taken the kind of view of geometry that Hilbert defended in his dispute with Frege (which I described in the first chapter) and applied the resulting Hilbertian view to number theory and analysis. This is why, in what follows, I shall call this account of mathematical truth my Hilbertian structural account of mathematical truth.[45]

I think this represents a potential problem for Chihara, given the significant role structures play in his philosophy while he rejects the existence of abstract objects, one that I will discuss later below.

In a provocative and probably the boldest conclusion of his career Chihara refers to the Fregean Assumption as adopted first by Frege, and subsequently by Russell and Whitehead, Quine and Gödel, as an "original sin that has led astray countless philosophers of mathematics of the Twentieth Century (and beyond)."[46] Chihara asks,

> Does philosophy of mathematics rest upon a mistake? One can now say 'yes' a great deal of the philosophy of mathematics of the Contemporary Period is, indeed, based upon a mistake: the mistake of accepting the Fregean Assumption and of thus thinking of mathematical theorems as propositions about the actual world.[47]

[44] Chihara (2021: 288)
[45] Chihara (2021: 290)
[46] Chihara (2021: 344)
[47] Chihara (2021: 343)

Gödel defended Russell's realism evident in Russell's remark "Logic is concerned with the real world, just as truly as zoology, though with its more abstract and general features".[48] Gödel believed that the objects of set theory exist independently of our constructions and our perception of individual sets enables us to recognize the truth of the axioms of set theory with regard to these abstract objects.[49] Chihara characterizes Gödel's Ontological Platonism within the context of Set Theory as "unintelligible."[50]

Chihara repeats his argument that Constructibility was never meant to "give a hermeneutic interpretation of standard mathematics, but only to give an alternative interpretation that would provide a new perspective from which to view classical mathematics."[51] Chihara writes, "The kind of possibility that concerns the Constructibility Theory (and hence the constructibility quantifiers of the interpretation in question) is what is called 'conceptual' or 'broadly logical' possibility"[52] where Possibility in the broadly logical sense is formalized by S5 versions of modal logic.[53]

Chihara introduces a Constructibility formal version for interpreting standard set theory, Simple Type Theory (STT)–and then formulates a constructibility interpretation of the formalism consistent with his anti-nominalistic nominalism.[54] Chihara explains that wherever in STT a set of a certain sort is asserted to exist, an open-sentence of a corresponding sort is asserted to be constructible. The STT existential and universal quantifiers in Constructibility become $(\mathrm{C}\phi)$ and $(\mathrm{A}\phi)$ where $(\mathrm{C}\phi)\Psi\phi$ asserts that open-sentence ϕ can be constructed such that $\Psi\phi$ and $(\mathrm{A}\phi)\Psi\phi$ asserts that every open-sentence ϕ for which it is possible to construct is such that $\Psi\phi$.[55] Chihara writes,

> Now in setting out to show that the above constructibility interpretation of the formalism of STT is useful and intuitively acceptable, I claimed, on the basis of modal reasoning, that the following two conditions are true: (i) The axioms of the theory come out intuitively true when the formal sentences are interpreted in the above way; (ii) the inference rules of the deriva-

[48] Feferman (1995: 104)
[49] Gödel (1964: 262)
[50] Chihara (2021: 44)
[51] Chihara (2021: 126)
[52] Chihara (2021: 139)
[53] Chihara (2021: 139-140)
[54] Chihara (2021: 129-130)
[55] Chihara (2021: 130)

tional system used in the theory preserves intuitive truth.[56]

Chihara uses a formal version of STT he calls Q-STT (after Quine) in which the superscripts on the individual symbols give the level of the symbol. The superscripts on the binary predicates will consist of ordered triples of Arabic numerals, where the first member of the ordered triple gives the arity of the predicate, the second gives the level of the first member in the relation, and the third gives the level of the second member in the relation. Chihara uses the binary S satisfaction symbol to play the role of set membership. To say that an open-sentence of a certain sort is constructible is not to imply that any such open-sentence actually exists—it only asserts what could exist. Constructibility quantifiers do not carry ontological commitments in the way the quantifiers of standard extensional logic do.[57]

The vocabulary of Q-STT will contain just one kind of nonlogical constant, the binary predicate $E^{2,n,n+1}$ for the membership relations of the various levels; in addition to the standard logical constants of Mates language of first order logic, there is one additional logical constant, $I^{2,n,n}$ for identity predicates.[58]

The axiom schema of comprehension and axiom of extensionality are the principle axioms of Q-STT. There are also an annex of two principles, referred to as choice and infinity.[59] Perhaps the principle of infinity is related to the Hypothesis of Infinity from his earlier 1990 work.[60]

Axiom 2.1. *Comprehension:* $\exists y^{n+1}(x^n)(x^n \in y^{n+1} \longleftrightarrow ---x^n---)$ *where* $---x^n---$ *is any formula of the theory that expresses a condition on* x^n*.*

Axiom 2.2. *Extensionality:* $(z^{n+1})(y^{n+1})((x^n)(x^n \in y^{n+1} \longleftrightarrow x^n \in z^{n+1}) \longrightarrow y^{n+1} = z^{n+1})$*.*

Objections were made to Chihara's 1990 Constructibility work concerning the cases in which one or more variables occur free in $---x^n---$ so Chihara now in the 2021 draft provides a reformulation using an S5 characterization. Chihara claims he originally refrained from using S5 in his original work on Constructibility over concerns about the Barcan formula, $\Diamond(\exists x)\neg Fx \rightarrow (\exists x)\Diamond\neg Fx$, concerns that were only later overcome with the development of M*, a version of S5 quantificational modal logic where the

[56] Chihara (2021: 132)
[57] Chihara (2021: 131)
[58] Chihara (2021: 135)
[59] Chihara (2021: 137-138)
[60] Chihara (1990: 68)

Barcan formula was not provable within his 1998 work on *Worlds of Possibility*.[61]

Chihara now employs the S5 characterization of the constructibility quantifier whereby he translates constructibility statements into assertions of possibility in S5 modal logic: $(CF)(...F...)$ iff $\Diamond(\exists F)(KF\ \&...F...)$, and Chihara uses this reformulation in the disputed case while otherwise retaining the proof from the 1990 work.[62]

3 Summary of Chihara's Arguments against Fregeans

3.1 Historical Argument

The historical argument is one of Chihara's strongest arguments. Frege believed that theorems of geometry were synthetic a priori truths.[63] With the adoption of Non Euclidean geometry following Einstein's special theory of Relativity Hilbert won the dispute with Frege over model theoretic consistency proofs. Chihara writes,

> From the contemporary point of view, Hilbert's view—especially in its use of mathematical interpretations of the axioms of his geometry to generate model theoretic consistency and independence proofs—is generally regarded by most mathematicians who have studied the dispute, as the one that is correct.[64]

However, mathematicians whom Chihara might classify as Fregean can be both comfortable with non-isomorphic models for a mathematical theory and the belief there is an intended interpretation.

3.2 Mathematical Practice

The Hilbertian structural account fits mathematical practice of Model theory. Mathematical theorems can be said to be "true" when they are true of, or satisfied by, an appropriate kind of mathematical structure.[65] Chihara explains the attractiveness of the Hilbertian Structural Account by stating,

> Acceptance of the account allows one to abandon the Platonist's view according to which the set theorist is seen to be a discoverer

[61]Chihara (2021: 144)
[62]Chihara (2021: 149)
[63]Chihara (2021: 7)
[64]Chihara (2021: 15)
[65]Chihara (2021: 289)

> of a vast realm of abstract entities that are interrelated by such mathematical relations as membership and less than. No longer will there be a need to explain how a set theorist, sitting alone in an office, can discover the existence and properties of things that are supposedly in another realm of existence that is beyond the reach of our most sophisticated scientific instruments. No longer will there be a need to postulate some sort of mysterious faculty of perception, of the sort that Gödel postulated, by means of which we are able to obtain knowledge of sets that are thought to exist in the actual world. Instead the set theorist proves theorems that are true of certain kinds of structures.[66]

However, Chihara has not explained why mathematicians would feel so compelled to study set theory and certain kinds of structures if they are so unrelated to reality, nor explained why a Hilbertian Structural Account would be incompatible with realism in the form of pluralism.

3.3 Scientific Practice

The Hilbertian account fits Scientific Practice: Chihara contends our best scientific theories do not require the existence in the actual world of any mathematical entities. "I see no reason at all for thinking that the existence in the actual world of mathematical entities is indispensable for our scientific theories."[67] Chihara responds to Maddy's demand: that "if mathematics isn't true [of the actual world], we need an explanation of why it is all right to treat it as true when we use it in physical science"[68] by stating,

> [T]he theorems of mathematical analysis are typically true of the kinds of structures that are to be found in the very basic framework of the theories utilized by physical scientists in describing, analyzing, and explaining physical phenomena. This is why such theorems hold in just the kinds of structures that provide the scientist with an excellent framework for theorizing about the physical world.[69]

Thus, Chihara would contend that mathematical theorems were only 'structurally true' of scientific models of physical reality, but not true about ab-

[66] Chihara (2021: 294)
[67] Chihara (2021: 340)
[68] Chihara (2021: 306)
[69] Chihara (2021: 308)

stract objects.[70] To explain how a theorem ϕ can be applied in science, it is not necessary to assume that ϕ is true. Rather the structural content of ϕ being true would be sufficient.[71]

3.4 Rejection of the Indispensability Argument by Quine and Colyvan

Chihara focuses on two key quotes by Quine about the Indispensability Argument:

"[The nominalist] is going to have to accommodate his natural sciences unaided by mathematics; for mathematics, except for some trivial portions such as very elementary arithmetic, is irredeemably committed to quantification over abstract objects," and "Classical mathematics . . . is up to its neck in commitments to an ontology of abstract entities. . . . The issue is clearer now than of old, because we now have a more explicit standard whereby to decide what ontology a given theory or form of discourse is committed to . . . ."[72]

Chihara argues that Quine adopted the Fregean Assumption and that the Fregean Assumption is responsible for Quine's belief that the truth of certain existential statements of mathematics commits him to Platonism. "Fundamentally then, underlying Quine's Indispensability Argument is the Fregean Assumption: it is argued that the truth of the existential theorems of mathematics provides us with the premise needed to yield the existence (in the actual world) of mathematical objects"[73] and "the Fregean Assumption plus the thesis of the truth of mathematics yields Platonism."[74] Chihara concludes, "Once one rejects Quine's assumption that mathematical theorems are propositions about the actual world (that is, once one rejects Quine's adoption of the Fregean Assumption), then his case for Platonism collapses."[75]

Chihara characterizes Colyvan's version of the Indispensability Argument:

(CV-1) We ought to have ontological commitment to all and only those entities that are indispensable to our best scientific theories;

[70] Chihara (2021: 310)
[71] Chihara (2021: 222)
[72] Chihara (2021: 323)
[73] Chihara (2021: 324)
[74] Chihara (2021: 324)
[75] Chihara (2021: 325)

(CV-2) Mathematical entities are indispensable to our best scientific theories;

From these premises, we are supposed to conclude:

(CV-3) We ought to have ontological commitment to mathematical entities.[76]

Chihara rejects Colyvan second premise, 'Mathematical entities are indispensable to our best scientific theories.'[77] Chihara rejects this on the grounds that his Constructibility theory has shown how Simple Type Theory can be developed without quantification over abstract mathematical objects and on the grounds that the structural account of mathematics does not require the existence of objects in the actual world.[78]

3.5 The Problem of Reference with regard to Mathematical Objects

Chihara states that the difficulty of determining which abstract object is the referent of typical mathematical terms has

> led mathematical Realists, such as Resnik, to infer that the mathematical objects denoted by these terms are 'incomplete' entities. However, from the perspective of the position described in the previous section, this 'partially interpreted' feature of mathematical terms is exactly what one would expect if one believed, as I do, that mathematical truths yield information about kinds of structures: this is because one would expect the mathematical terms occurring in the theorem to be functioning, not as names denoting particular entities, but rather as what might be called 'structural parameters' that stand for positions in structures—hence, the 'partially interpreted' nature of mathematical terms.[79]

By contrast Chihara argues that the Hilbertian Account is immune to such problems. "This is not a puzzle for the Hilbertian Account since it does not require the existence in the actual world of any mathematical objects to be referred to. That mathematicians are not concerned with reference is, according to the Hilbertian Account given here, a perfectly reasonable attitude to have."[80] However, Chihara's acceptance of structures could also

[76] Chihara (2021: 340)
[77] Chihara (2021: 340)
[78] Chihara (2021: 340)
[79] Chihara (2021: 294)
[80] Chihara (2021: 302)

lead to the existence of abstract objects, and it is curious that he both recognizes Platonism as a possible consequence of structuralism while at the same time rejecting that this is a problem for him.[81]

3.6 Russell's Axiom of Infinity

Chihara argues that the problems Russell encountered with his Axiom of Infinity in *Principia Mathematica* (PM) only arose because Russell had adopted the Fregean Assumption. The existence of infinitely many things in the actual world could neither be considered to a logical truth nor a necessary truth, and thus Russell's Axiom of Infinity differed from the other axioms of PM. The Hilbertian Account avoids this problem. Chihara contends the problem with the Russell's Axiom of Infinity can be directly traced to his acceptance of the Fregean Assumption. Chihara states,

> if mathematical theorems are thought to be propositions about the actual world, then to include the Axiom of Infinity among the axioms of PM is to regard it as a true proposition about the actual world. Who knows if there are, in fact, infinitely many individuals in the actual world? And how could one know?[82]

Chihara argues in the *Fregean Assumption* that the Hilbertian Structural account of mathematical truth avoids the problem Russell faced because,

> [I]t is not developed from the acceptance of the Fregean Assumption. Mathematical theorems are not, from my Hilbertian position, propositions about the actual world. Instead mathematical theorems are satisfied by structures of a certain sort-structures that have to be infinite if one is to have the structure of the natural numbers.[83]

3.7 Counter to Burgess and Rosen

Chihara contends that his non-Fregean constructibility version of simple type theory demonstrates that there is a model which allows for the development of classical analysis without requiring a commitment to mathematical objects, thus undercutting the Indispensability Argument.[84] Constructibility is in my opinion one of the most controversial aspect of Chihara's theory, and Chihara rightly so devotes a great deal of time to defend it against

[81]Chihara (2004: 218)
[82]Chihara (2021: 305)
[83]Chihara (2021: 305)
[84]Chihara (2021: 327)

criticism by Burgess and Rosen. Chihara contends that the question of whether a nominalistic reconstruction of mathematics is useful as a model for philosophical analysis is conflated by Burgess and Rosen with the different question of a nominalistic reconstruction of mathematics serving as a replacement.[85] Chihara emphatically reiterates,

> [M]y Constructibility Theory was devised to be an instrument or tool for philosophical and logical analysis and assessment and showed, by way of various examples, how the theory could function as a useful instrument for analyzing philosophical arguments. It was abundantly clear that my theory was not intended to be a 'rational reconstruction' of anything, in the sense in which Burgess uses the expression.[86]

Chihara contends that Burgess and Rosen adhere to the Fregean Assumption, and he refers to it as "an unstated premise of practically all of the Burgess and the Burgess-Rosen anti-nominalism arguments!"[87] "So I am left with the picture of a philosopher spending many years attacking the nominalist's conception of mathematics, but never taking the time and energy required to attempt to develop anything like an adequate alternative Platonic view of mathematics."[88] However, a Platonist would see no need to develop an alternative view, and Chihara shies away himself from a nominalistic reconstruction of mathematics which would serve as a replacement.

Rejection of Burgess's fork: Chihara refers to what he calls "Burgess's fork": Each nominalist targeted must be either a hermeneutic nominalist or else is a revolutionary nominalist.[89] Chihara rejects Burgess's fork outright. Chihara examines a quote by Burgess:

> Before we come to philosophy, we have a fairly uncritical attitude towards, for instance, standard results of mathematics, or such of them as we have learned about. Having studied Euclid's Theorem, we are prepared to say that there exist infinitely many prime numbers. Moreover, when we say so, we say so without consciously mental reservations or purpose of evasion. . . Why not just acquiesce in the minimal non- or un-nominalism many of us find ourselves coming to philosophy with? . . . [W]hile

[85]Chihara (2021: 228)
[86]Chihara (2021: 189-190)
[87]Chihara (2021: 194)
[88]Chihara (2021: 248)
[89]Chihara (2021: 178)

> our positive conception of the nature of the numbers in whose existence we thus acquiesce may be of the haziest, we at least understand that numbers are not supposed to be like ordinary concrete things like rocks or trees or people. . . . In this sense, we acquiesce not only in their existence, but also in their abstractness.[90]

Chihara links the 1997 Burgess quote [91] to the Fregean Assumption and rejects it.[92]

Chihara defends the 'conceptual' or 'broadly logical' possibility used for the constructibility quantifiers, 'it is possible to construct,' in the Constructibility Theory [93] and applies S5 to Constructibility: (CF)(... F...) can be characterized by there exists a possible world in which an open sentence token of the required sort existed, precisely $\Diamond(\exists F)$ (KF $\wedge$ ... F ...).[94]

Chihara examines a Burgess quote from 2005 that 'one would like to be told something more about the nature of the non-human intelligent being Chihara has in mind', and that Chihara must have in mind 'extraterrestrials with super powers' when Chihara responded to Resnik by arguing that the constructibility of open sentences does not have to be done by a human. Chihara responds, "[S]uggesting that (CF)(. . . F . . .) should be understood to be the assertion that an extraterrestrial with super powers can construct an open-sentence such that . . . . involves an even more egregious misrepresentation of my views than the one Resnik made."[95] Chihara argues that Burgess's mistake is analogous to interpreting 'For every set S, it is possible to construct a well-ordering of S' as 'for every set S, it is possible for an extraterrestrial with super powers to construct a well-ordering of S', and states that this is "not the sort of interpretation that any reasonable person, without an irresistible desire to liken mathematics to science fiction, would be tempted to make."[96] Chihara argues that the S5 characterization 'There exists a possible world in which an open-sentence token of the required sort has been constructed' "does not imply or presuppose that, in some possible world, an open-sentence token of the required sort has been constructed by a being of that type."[97]

[90] Chihara (2021: 193)
[91] Burgess and Rosen (1997: 10-11)
[92] Chihara (2021: 193-194)
[93] Chihara (2021: 132)
[94] Chihara (2021: 140)
[95] Chihara (2021: 142)
[96] Chihara (2021: 142)
[97] Chihara (2021: 142)

The remark by Burgess may be unnecessarily caustic, although the confusion might be traced to the awkward way in which Chihara originally described the constructibliity of open-sentences. Chihara argued that the ontological status of tokens is not something to be concerned about because "to say that someone has constructed an open-sentence is not to say that an entity of a certain sort has been constructed but only that the person has done something-he has performed a series of actions,"[98] where the actions might include utterances, writings, or even hand signals.[99] Chihara did not at that time clarify that the ontological status of the person constructing the token is equally unconcerning.

Chihara argues that Burgess and Rosen fail to list in their premises they feel are undeniable in advocacy for Realism, the essential assumption they have made, the Fregean Assumption.[100] Chihara illustrates his objection to Burgess's fork with two examples:

i. Chihara claims Principia Mathematica would fail the Burgess-Rosen test for a reconstruction of mathematics to have merit, namely, that a reconstruction of mathematics can only have scientific or mathematical merit by providing an analysis of the meanings of the terms and sentences of a mathematical system or by demonstrating its superiority on scientific grounds to the current mathematical system:[101] Chihara states,

> PM is not satisfactory as a hermeneutic account of mathematics, nor is it satisfactory as a revolutionary account of mathematics—but even so, one can argue that it has proven to be an enlightening model of real mathematics—especially in such areas as proof theory, foundations of mathematics, and mathematical logic.[102]

Chihara argues that Burgess and Rosen would be forced to conclude that "the reconstructed mathematics of PM is distinct from and inferior to standard real mathematics and is therefore of no significant value to science or mathematics. Such a conclusion would be shocking–would it not?"[103]

ii. Non standard analysis fails Burgess's fork: Chihara argues that nonstandard analysis should not be restricted to being either something wherein science can be better expressed or something inferior to standard analysis

[98] Chihara (1990: 40)
[99] Chihara (1990: 40)
[100] Chihara (2021: 200)
[101] Chihara (2021: 197)
[102] Chihara (2021: 226)
[103] Chihara (2021: 199)

and thereby not capable of making a significant contribution to science on its own. Chihara states that it is "neither practical nor reasonable to replace current versions of science by versions using nonstandard analysis, but that should not lead us to deny that nonstandard analysis can be fruitfully employed to advance mathematics and science."[104] I am not sure that non-standard analysis would fail Burgess's fork in that the Compactness of First Order Logic can be used to obtain a non-standard model of the Reals, and Burgess could argue that non standard analysis provides meaning to the term infinitesimal which had been used as an intuition in the development of Calculus and thus in that sense would be hermeneutic.

Lastly, Chihara claims the Burgess and Rosen argument that Euclid's proof that there exist infinitely many primes is a reason to reject nominalism is faulty as Euclid's proof of his theorem does not involve proving that prime numbers exist in the actual world. Chihara states, "the proof does not first show that there are prime numbers in the actual world, and then show that there are infinitely many of them in the actual world. It only proves that there are (in some appropriate mathematical sense) infinitely many prime numbers, given that there are (in that sense) prime numbers."[105] He continues, "[T]he supposition that begins the reductio ad absurdum proof must be the supposition that there exists in the actual world at most finitely many prime numbers—a supposition that is compatible with there being no prime numbers at all in the actual world."[106] However, I do not read the Burgess and Rosen quote on Euclid's proof to be a claim that infinitely many prime numbers are proven to exist in the actual world. Rather, it can be interpreted as expressing a comfort level with abstract objects prior to philosophical investigation.

Chihara would assign the burden to the Platonist to explain the existence of abstract objects, a task which Burgess and Rosen would at least admit appears initially difficult.[107] Burgess and Rosen would argue that the burden should be on Chihara to demonstrate a reconstruction that can effectively replace abstract objects. They argue that a general view of philosophers would be that "nominalism pretty clearly must be judged to be untenable unless an appropriate reconstruction or reconstrual of science in conformity with its tenets can be developed."[108] Nonetheless, Burgess and Rosen would still find value in nominalistic reconstructions for advancing the philosophical un-

[104]Chihara (2021: 209)
[105]Chihara (2021: 259-260)
[106]Chihara (2021: 261)
[107]Burgess and Rosen (1997: 12)
[108]Burgess and Rosen (1997: 12)

derstanding of science.[109] Thus, the dispute centers over which side should carry the burden and which among parsimony and fruitfulness is more important.[110] Burgess and Rosen argue Chihara's philosophical position with respect to set theory following his 1990 work on Constructibility remained "elusive."[111] Chihara's ontological and anti-ideological nominalism appears highly connected to the Hilbertian structural account for mathematics he puts forth in his final work. As such, is his position tenable without a more robust explanation of structures?

4 Appeal of the Hilbertian Account to CH

Chihara has argued that mathematical theorems are explainable by a Hilbertian structural account and that the Fregean Assumption is intrinsically linked to Platonism and is at the heart of a serious error within the philosophy of mathematics. He also does not seem to rule out that the Continuum Hypothesis can be translated somehow within Constructibility even if he makes no such claim.[112] Thus, it would seem appropriate to examine Chihara's broad advocacy for the Hilbertian structural account of mathematics and his ontological and non-ideological nominalism in the context of Set Theory in addition to the way he has advocated using it for finite arithmetic and classical analysis.

Gödel showed under $V = L$ the continuum hypothesis holds. The reals and the power set of the reals are relativized to a structure. Powerset of M for transitive models of ZF is generally not absolute. Thus the satisfiability of the Continuum Hypothesis in a model and whether a bijection exists depends on what functions and subsets of reals the model contains. As an application of the Downward Löwenheim Skolem Theorem, working in ZFC one can produce countable transitive models M for any finite list of axioms of ZFC. Cohen was able to use an outer model for countable transitive models via forcing to prove the relative consistency result for $ZFC + \neg CH$. Cohen understood that with any axiom system extending ZF consistent with $V = L$ proving the existence of an uncountable standard model in which AC and $\neg CH$ held was impossible and thus only through outer models could he achieve AC and $\neg CH$.[113] Forcing utilizes an outer model approach using

[109] Burgess and Rosen (1997: 243)
[110] Burgess and Rosen (1997: 12)
[111] Burgess and Rosen (1997: 199)
[112] Chihara (1998: 316)
[113] Cohen (1966: 108-110)

filters. We can describe the forcing extension using a model that contains an M-generic filter for a countable transitive model.

Forcing constructs a generic extension M[G] for a *c.t.m.* M of ZFC whereby given p.o. $\mathbb{P} \in M$

p $\Vdash \phi$ ("p forces ϕ") if and only if for all filters G which are $\mathbb{P}$-Generic over M (G contains members from any dense subset of $\mathbb{P}$ in M) if $p \in G$ then $M[G] \models \phi$.

Two significant aspects of Forcing are that it is definable in the ground model and that it equals truth: let $\tau_1, \tau_2, ..., \tau_n \in M^{\mathbb{P}}$.

Theorem 4.1. *Forcing is definable in the ground model M*

For all $p \in P$ $p \Vdash \phi(\tau_1, \tau_2, ..., \tau_n) \leftrightarrow p \Vdash^* \phi(\tau_1, \tau_2, ..., \tau_n)^M$.

Theorem 4.2. *Forcing equals Truth*
For all G which are $\mathbb{P}$*-Generic over M,*
$\phi(\tau_{1G}, \tau_{2G}, ..., \tau_{nG})^{M[G]} \leftrightarrow \exists p \in G$ $p \Vdash \phi(\tau_1, \tau_2, ..., \tau_n)$.[114]

Another approach to forcing is via Boolean-valued models.

Theorem 4.3. *Boolean-valued Model Approach: Let M be a generic transitive model of ZFC, B a complete Boolean algebra in M*

If G is an M-Generic Ultrafilter on B, then for all $x_1, \ldots, x_n \in M^B$, $M[G] \models \phi(x_1^G, \ldots, x_n^G) \iff ||\phi(x_1, \ldots, x_n)|| \in G$.[115]

Chihara's Hilbertian structural account of mathematical truth and his opposition to the Fregean Assumption is similar in some respects with the multiverse view expounded by Joel Hamkins in set theory. Hamkins uses a Naturalist account of Forcing, which allows for forcing over any model of ZFC without requiring consideration of countable transitive models.[116]

Theorem 4.4. *If V is a (the) universe of set theory and* $\mathbb{P}$ *is a notion of forcing, then there is in V a class model of the theory expressing what it means to be forcing extension of V. In the language with* $\in$*, constant symbols for every element of V, a predicate for V, and constant symbol G, the theory asserts:*

The full elementary diagram of V, relativized to the predicate for V.
The assertion that V is a transitive proper class in the (new) universe.
The assertion that G is a V-Generic ultrafilter on $\mathbb{P}$*.*
The assertion that the (new) universe is V[G], and ZFC holds there.[117]

[114]Kunen (1980: 200) For a more complete discussion of $\Vdash^*$, see p. 194-200.
[115]Jech (2002: 216) For a discussion of Boolean-valued models and M^B, see p. 206-216.
[116]Hamkins (2012: 423)
[117]Hamkins (2012: 423)

Hamkins points out that this is a theorem scheme and provides another description via an elementary embedding into a class model:

Theorem 4.5. *For any forcing notion* $\mathbb{P}$ *there is an elementary embedding*

$$V \precsim \overline{V} \subseteq \overline{V}[G]$$

of the universe V *into a class model* $\overline{V}$ *for which there is a* $\overline{V}$*-generic filter* $G \subseteq \overline{\mathbb{P}}$.[118]

Hamkins argues that the multiverse view "holds that there are diverse distinct concepts of set, each instantiated in a corresponding set theoretic universe, which exhibit diverse set-theoretic truths."[119] Hamkins appeals as does Chihara to the history of geometry.

> Meanwhile, set theorists continued, like the geometers a century ago, to gain experience living in the alternative set-theoretic worlds, and the multiverse view now makes the same step in set theory that geometers ultimately made long ago, namely, to accept the alternative worlds as fully real.[120]

Hamkins view of realism differs from that of Gödel in that Gödel believed in mathematical intuition that could lead one to apprehend truths of the axioms of set theory for a single absolute universe of sets in which all set-theoretic truths resided. From such a perspective forcing extensions of V would necessarily be illusory.[121] Hamkins would reject the view that there is an intended model for set theory. By contrast V under the multiverse view refers to whatever universe is under consideration and not the absolute universe of sets, and the forcing extensions V[G] are fully real.[122] This is also given Chihara's ontological nominalism where the similarity with Chihara's Hilbertian structural account ends.

5 Some Objections Considered

5.1 Scope

In *Worlds of Possibility* Chihara responds to Steel's objection over the Constructibility theory in that the novelty of a new interpretation of STT must

[118]Hamkins (2012: 423)
[119]Hamkins (2012: 416)
[120]Hamkins (2012: 426)
[121]Hamkins (2012: 419)
[122]Hamkins (2012: 419)

manifest itself somewhere and that Chihara should be forced to give up some set-existence assertions and some axiom or theorem of STT.

> I do not claim that the translation of anything assertable in any mathematical theory, say STT with such axioms as Choice and the Continuum Hypothesis, can be asserted in my Constructibility theory. On the other hand, if the translation of some theorem of such a set theory cannot be asserted in my constructibility theory, I would not consider that fact grounds for rejecting the theorem as false of meaningless. I again emphasize that I am not advocating a revisionist account of mathematics.[123]

Chihara's response to Steel is somewhat legalistic and evades the deeper question of whether his position of ontological and anti-ideological nominalism is ultimately sustainable.

Chihara has also not fully answered Steel's objection in that he has not addressed places where the novelty Steel mentioned would manifest nor has he addressed anything he is definitively willing to give up. He also has not explained why certain mathematical statements might not be explained by Constructibility and, if not, why the general methodology of using modal notions and connecting theorems could not be extended to translate such potentially untranslatable mathematical assertions as Choice or CH. He also has not explained why the use of modal primitives is restricted to "it is possible to construct an open sentence such that..."? His continual appeals to the Hilbertian structural account to explain mathematical truth together with his ontological nominalism and his use of modality as a primitive in the broadly logical sense seemingly undermine his contention that he is only offering a limited account to cast doubt on the Indispensability Argument. Chihara does exhibit a tendency throughout his works to make expansive revisionist sounding claims when critically analyzing the works of other philosophers, but retreats to the familiar refrain he is not a revisionist when his position is questioned, and in attempting to immunize his philosophical positions against criticism, he may detract from the profound and possibly more broadly applicable nature of his work.

5.2 Epistemology

A mysterious faculty of perception that Gödel postulated is downplayed by Chihara, but he appears to have no problem with intuitions about modality.

[123]Chihara (1998: 316)

Chihara uses modalized quantifiers in terms of connecting theorems in the effort to avoid the "ontologically weighty" commitment to mathematical objects. However, Chihara commits to a "epistemologically weighty" notion of modality. Chihara relates c-sentences to mathematical sentences. Resnik is skeptical that we have the epistemological means to know complicated modal facts. Chihara defends his use of broadly logical possibility within Constructibility by stating that we have plausible grounds for the existence of modal facts by theoretical reasoning.[124] However, a potential problem is lurking here with such a methodology. A broadly logical modal version of the Continuum hypothesis might have an independent truth value if modal intuition was used in an absolute sense to consider all possible reals, all possible subsets of reals, all possibles functions between a subset of the reals and the reals, but Chihara would reject any truth value for the Continuum hypothesis and consider its truth or falsity only relative to a model. The intuitive notion of all possible reals could be independent of a structure. In general given our modal intuitions, Chihara's use of modality as a primitive in connecting results could lead to truth values for formally undecidable mathematical statements, a feature of Platonism that he would certainly want to reject. This could be one of the areas in which the novelty Steel suggested might show up.

5.3 Circularity

Hans Sluga suggested to Chihara following a reading of his manuscript that Chihara's semantics for the language of Constructibility was given in terms of sets, but Chihara does not believe in the existence of sets as abstract objects.[125] Chihara recognizes that he uses "model theory to explain the logical significance of various elements of the constructibility theory, while at the same time developing this model theory within the constructibility theory itself," but he does not consider this unusual.[126] Chihara is very accepting of modal intuition and modality in the broadly logical sense. He claims his constructibility quantifiers are "primitives", and his appeal to possible worlds was made in order to aid the reader in achieving a proper grasp of these quantifiers and the reasoning.

> Talk of possible worlds in the explication of [his] mathematical system should be regarded as merely heuristic and didactic in nature, useful for aiding the uninitiated in getting the appropriate

[124] Chihara (2004: 217)
[125] Chihara (1990: 78)
[126] Chihara (1990: 79)

> understanding of the key concepts and in learning to apply and to develop these concepts in a correct manner. I see no vicious circularity in applying possible worlds semantics in this way, and then going on to explain apparent references to possible worlds in possible worlds semantical theory in terms of constructibility quantifiers.[127]

Thus, Chihara adroitly dances around objections of circularity. What is missing though is why possible world talk is so necessary to explicate the primitive quantifier.

Chihara also uses modality as a primitive in his connecting result, but when Chihara proves his connecting theorem he is undoubtedly not thinking in terms of constructibility quantifiers and reasoning within Constructibility; rather he is reasoning in terms of structures as sets. Furthermore, how would Chihara view his Fundamental Connecting theorem or his other various connecting theorems in terms of truth in the actual world? If he were to regard his connecting theorems as true in the actual world, then he may be adopting some version of the Fregean Assumption. Chihara asserts that the constructibility version of Q-STT is not Fregean, but neither is it Hilbertian,[128] as "its assertions are not assertions about the actual world, but rather are modal assertions about the constructibility of open-sentences (in possible world semantics, they are assertions about what holds in at least one possible world)."[129]

5.4 Structures

Structures and satisfiability in structures are the cornerstones to Chihara's thesis about the Hilbertian structural account regarding mathematical theorems and to his assertion that mathematical theorems are in some sense true but not true in way advocated by those who adopt the Fregean Assumption. Chihara claims he is not an ideological nominalist as he believes in the notion of satisfiability of a formula in a structure. However, as anti-realist Chihara does not believe in the existence of abstract objects such as sets. His ontological parsimony also extends to structures. Chihara does not regard structures to be abstract entities in the actual world.[130] Since structures do not exist as abstract objects according to Chihara, when he discusses "mathematical theories being true of structures", it is to be only

[127] Chihara (1998: 328)
[128] Chihara (2021: 173)
[129] Chihara (2021: 152)
[130] Chihara (2021: 291)

understood as "merely a manner of thinking."[131] Chihara turns to Constructibility and constructible realizations described in his 2004 work to explain structures.[132] Chihara contends he can give a Russellian "no-class" type of explanation of satisfiability in a structure via the constructibility of open-sentences. He states,

> [S]trictly speaking, there is no such abstract entity as a structure—when I assert that some sentence is satisfied by a structure, I am not presupposing the existence of some sort of abstract object, a structure, that somehow exists in the actual world. No, I am making use of Bertrand Russell's idea of 'incomplete symbol', which was the central logical device he used in developing the 'no-class' theory in PM.[133]

Despite structures being so fundamental to Chihara's advocacy for the Hilbertian structural account of truth, he gives structures less attention than he does to other aspects of his thesis. The examples provided in the 2004 work regarding constructible realizations are quite simplistic, and there is a brief discussion of Peano Arithmetic.[134] Given that structures are at the heart of his non-ideological nominalism and ontological nominalism, it seems curious why structures are not explored in more depth. According to the Upward Löwenheim-Skolem Theorem if X is an infinite L-structure and κ is a cardinal with $\kappa > |L|, |X|$ then there exists a structure Y with $|Y| = \kappa$ and $X \prec Y$. Structures can also be direct limits, ultraproducts, ultrapowers, or boolean valued models, which are also used in Forcing. It is questionable that Chihara can fully explain structures and satisfiability in a structure within Constructibility given his ontological constraints. He acknowledges that according to Parsons it is difficult to develop an explanation of structures without presupposing sets or becoming involved in a circularity, but claims this is only "a problem for the structuralist who is putting forward a far-reaching metaphysical overview of all of mathematics, but it does not arise for my view since I am not putting forward general account of what mathematical theories, literally construed, are about."[135] Thus, Chihara's defense is again legalistic in that he claims to circumvent any potential philosophical difficulties concerning his methodologies and his

[131] Chihara (2021: 292)
[132] Chihara (2004: 218-227)
[133] Chihara (2021: 343)
[134] Chihara (2004: 240-245)
[135] Chihara (2004: 218)

ontological and non-ideological nominalistic positions based solely upon his intent in putting them forward.

Given that Constructibility theory is about the possibility of constructing open-sentences, one might also wonder why Chihara would not advocate for $V = L$ as a philosophical position, in which case the Continuum Hypothesis would be true, a position he would also dispute. Chihara also refers to structures as "products of the mathematician's creative imagination".[136] How are these distinct from abstract objects? When he speaks of structures as "products of the mathematician's creative imagination" while eschewing the existence of abstract objects, he possibly touches on the fact that a fundamental activity of our consciousness is to form collections. While the role of consciousness in physics within quantum mechanics has been a subject of debate, it is not explicitly present within the axioms for set theory themselves, and the Formalist can work with ZFC freely without worrying about whether there is actually a set-theoretic universe. Chihara's explanation of the Fregean Assumption refers to the "actual world." What is the actual world? Consciousness is a part of the actual world, but Chihara seems to equate the actual world with the physical world alone. The question is difficult to answer given our understanding of the actual world is incomplete and constantly evolving.

6 Conclusion

Chihara writes with great insight as he attempts to adroitly walk a tightrope between non-ideological nominalism and ontological nominalism. Ultimately the success of his thesis will probably remain unsettled given the difficult nature of the issues. Two of the greats of Set Theory who are responsible for showing the consistency of CH under $V = L$ (L being the constructible universe) and consistency of $(\neg CH)$ via Forcing demonstrate Platonic leanings. Gödel's beliefs on Ontological Platonism are well known. Cohen, who developed Forcing to show the consistency of the negation of CH using outer models, believes the Continuum Hypothesis is false, and writes,

> [W]hether or not one believes that set theory refers to an existing reality, there is a beauty in its simplicity and in its scope. Someone who rejects that sets exist as 'completed wholes' swimming in an ethereal fluid beyond all direct human experience has the formidable task of explaining from whence this beauty derives.

[136] Chihara (2021: 294)

> On the other hand, how can one assert that the continuum exists when there is no way one could even in principle search it, or even worse, search the set of all subsets, to see if there was a set of intermediate cardinality? Faced with these two choices, I choose the first. ... We will debate, experiment, prove and conjecture until some picture emerges that satisfies this wonderful taskmaster that is our intuition. ... I think the consensus will be that CH is false.[137]

Chihara would adopt formalism in regard to CH. Hamkins, who believes CH can never be answered under the multiverse conception, nonetheless regards the universes of the multiverse as real:

> The multiverse view is one of higher-order realism—Platonism about universes—and I defend it as a realist position asserting actual existence of the alternative settheoretic universes into which our mathematical tools have allowed us to glimpse. The multiverse view, therefore, does not reduce via proof to a brand of formalism.[138]

While Hamkins and Gödel would both adopt realism with regard to abstract objects, Hamkins unlike Gödel would adopt a form of pluralism. If Chihara's non-ideological nominalism ultimately leads to some form of Platonism then a multiverse view might make some of the apparent distinctions between the Fregean Assumption and Chihara's position less dramatic in that if the multiverse is the ultimate reality then truth about the model could be construed as a truth within a part of that reality, as the model resides in the multiverse.

Overall, Chihara's work on the Fregean Assumption to explain his version of nominalism which rejects the existence of abstract objects while adopting satisfiability within a structure and to provide a critique of the philosophical debate surrounding the Indispensability Argument is the culmination of an extraordinary career in the philosophy of mathematics and should generate much debate, as should all great philosophical works.

[137]Cohen (2002: 1099)
[138]Hamkins (2012: 417)

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