Boole's indefinite symbols re-examined

David Makinson
School of Historical and Philosophical Inquiry
University of Queensland
d.makinson@uq.edu.au

Abstract

We show how one can give a clear formal account of Boole's notorious "indefinite" (or "auxiliary") symbols by treating them as variables that range over functions from classes to classes rather than just over classes while, at the same time, following Hailperin's proposal of binding them existentially.

1 Introduction

Boole's system of logic, set out in his 1847 booklet *The Mathematical Analysis of Logic*, the brief 1848 outline "The calculus of logic" and the extended 1854 volume *An Investigation of the Laws of Thought*, presents the reader with many tangles. Two of the most perplexing are the use of *indefinite* (also termed *auxiliary*) symbols to express certain kinds of proposition as equations, and the application of division as an inference step when solving problems.[1]

Both were already noticed by De Morgan in a draft letter to Boole of 27 November 1847, written immediately after the simultaneous publication, and exchange of copies, of *The Mathematical Analysis of Logic* and De Morgan's *Formal Logic*. There, De Morgan says (see Smith 1982 p. 26):

> With regard to the syllogistic process, there are unexplained difficulties about v and division by y. Here you have recourse to verbal monitions about the meaning of v. The process of division is not *per se* allowable.

This remark, along with illustration of problems arising from division, was omitted from the final letter dispatched the next day, presumably because, following friendly discussions on various topics in mathematics, De Morgan did not want to initiate their logical exchanges with a negative remark. But in the following decades, others did publish their reservations and both tangles were gradually cut away as Boole's insights were transformed into what we now know as the theory of Boolean algebras.

Nevertheless, it remains of historical interest, and perhaps more, to try to unravel them. The deepest discussion of "division by y" is perhaps that of Hailperin 1986 (see also

Burris & Sankappanavar 2014, 2014a). Our concern in this note is with the "difficulties about v" alone. Hailperin also makes a brief proposal about them, which goes some of the way towards unpicking the knot but does not suffice to make sense of Boole's notorious representation of "Some Xs are Ys" as $vx = vy$. We combine Hailperin's proposal with an interpretation of indefinite symbols as ranging over functions from classes to classes, rather than just over classes. This step, we suggest, is in accord with Boole's rather obscure "verbal monitions" as well as almost all of his symbolic practice and, with a suitable constraint on the domain and range of such functions, handles the recalcitrant equality as desired. There do remain occasional incoherencies in Boole's manipulation of indefinite symbols, arising when he commutes them with other variables as if application of a function were the same thing as intersection of classes. However those glitches can be removed from his text without damage, at least to his reconstruction of syllogistic.

The perspective of this article thus contrasts with a widespread attitude that was expressed emphatically by Dummett 1959 (p. 205) when he said of Boole: "His method of dealing with particular propositions, by means of his symbol 'v', is irretrievably confused" and by Van Evra 1977 (p. 370) who wrote in the same vein: "While Boole's intent [when using auxiliary symbols] is clear enough ... the device itself suffers from a fatal flaw, namely, the impossibility of a class whose only defining characteristic is that it have members". Our assessment is more nuanced: Boole's treatment of indefinite symbols is confused and flawed, yes, but neither fatally nor irretrievably so.

2 Ordinary variables

Before examining what Boole does with indefinite symbols, we need to look at his use of ordinary variables. They share features with the indefinite ones and, even for them, there are complications that can impede clear understanding.

One complication is that there is a significant difference between the formal level, that is, what Boole does with ordinary variables in his equations, and the informal perspective on them as revealed in his commentaries. In this section we focus on the formal level; section 7 discusses the informal one.

Associated with this are some delicate matters of terminology. Boole never uses the now standard term "variable" in *LT*, and hardly ever in *MAL* (Burris 2019 p. xxiii). The words actually employed change over time. In *MAL*, $x, y, ...$ are almost always called "elective symbols", for reasons connected with his informal perspective. In *LT*, on the other hand, $x, y, ...$ are referred to as "literal symbols", "symbols" and occasionally "logical" or "class" symbols. So our term "ordinary variables" is achronic but not, we believe, misleading.

Considering the formal level only, one sees that in each of *MAL*, *CL* and *LT* ordinary variables $x, y, ...$ range over classes, more precisely, over subclasses of a "universe". But within that framework there is a well-known twist. In the context of Boole's analysis of categorical propositions and traditional syllogisms (which looms large in the pages of *MAL* and *CL*), a certain constraint is imposed on the range of ordinary variables that is, however, largely ignored in his general theory of development for complex expressions (which plays a major role in *LT*). The constraint is that the values of $x, y, ...$ are required to be *non-empty* and *non-universal*. For brevity, we will refer to this as the *nenu* constraint.

Non-emptiness is required because Boole wishes to preserve the traditional principle of subalternation, that "All Xs are Ys" implies "Some Xs are Ys" (see e.g. *MAL* pp. 26-30); in other words, he wants to guarantee existential import for the universal. His preferred ways of representing "All Xs are Ys" are $x = xy$ and $x(1 - y) = 0$, where concatenation is for meet (intersection), 1 is the "universe", $1 - y$ is the complement of y with respect to 1, and 0 is the empty class. But when there are no Xs, i.e. the set x is empty, both these equalities are trivially true. Since Frege, modern logic has been happy to accept that limiting case, but neither Aristotelian logic nor Boole were willing to do so. Non-universality is also required because Boole wants the complement operation always to take values in the range of the variables, and the complement of the universe would be the empty class, already excluded.

Of course, the *nenu* constraint gives rise to a number of inconveniences. In particular, the meet of two non-empty classes may be empty, and the join of two non-universal classes may be universal (even when they are disjoint). For that reason, even within the confines of syllogistic the constraint was gradually abandoned by logicians from Jevons 1864 onward.[2] It is perhaps for the same reason that when he leaves the confined context of syllogistic to articulate his general theory of development, Boole himself largely ignores the *nenu* constraint and allows 0, 1 to be substituted freely for ordinary variables. Moreover, as noted by Burris 2019 (pp. x, xvii-xviii), at one point Boole inadvertently counts as invalid a certain syllogism with negative terms although it comes out as valid when those terms in it are required to be non-empty (see *MAL* p. 35).

3 Indefinite Symbols

Boole's indefinite symbols, which he writes as $v, v', ...$, share several of the features of ordinary variables. In particular, there is a gap between what is done with them in equations and what is said about them in the verbal commentaries. There is also a shift in terminology: whereas in *MAL* and *CL* Boole writes of "auxiliary" or "separate" symbols, "indefinite" is the dominant term in *LT*. Again, just as for ordinary variables, the *nenu* constraint is applied to the indefinite ones when representing traditional Aristotelian syllogisms, but not in the general theory – where, indeed, he explicitly allows the empty class and the universe as values of v:

> ... the expression 0/0 here indicates that *all, some*, or *none* of the class to whose expression it is affixed must be taken ... We may properly term 0/0 an indefinite class symbol, and may, if convenience should require, replace it by

> an uncompounded symbol v (*LT* chapter VI section 10, p. 90; see also the replacements of this kind made in *LT* pp. 90-92 and *MAL* pp. 74-76).

Despite these commonalities, indefinite symbols are not treated in the same way as ordinary ones. They are given a distinctive kind of verbal explanation and, on the formal level, are handled differently. As far as the author is aware, in the resolution of deductive problems they are introduced at only two moments – at the beginning of the analysis when symbolizing premises, and at the end when eliminating uninterpretable numerical expressions like $0/0$ in favour of v, which is then "interpreted" in English as "some". In between those limits, indefinite symbols are manipulated but not introduced. Understanding what is going on with them is the main purpose of this paper.

3.1 Implicit existential quantification

For Boole, indefinite symbols are sometimes optional, sometimes indispensable. They are optional for representing traditional universal propositions "All Xs are Ys" and "No Xs are Ys", in the sense that while we may use them for that task, we can also do the job perfectly well without them. But they are indispensable for representing equationally the particular propositions "Some Xs are Ys" and "Some Xs are not Ys". It is in this context that they make their first appearance in *MAL*, where Boole represents the particular affirmative "Some Xs are Ys" as $v = xy$ (p. 21) and alternatively as $vx = vy$ (p. 22).[3]

A naïve but fruitful question arises for both representations. If one writes "Some Xs are Ys" as $v = xy$ and likewise "Some Zs are Ys" as $v = zy$, then we may immediately infer $xy = zy$, which is clearly not a valid consequence of the two verbally presented propositions. Similarly, if the symbolizations are $vx = vy$ and $vz = vy$ we get $vx = vz$, which reads back to the words "Some Xs are Zs", again not a valid consequence of the given items.

Boole is fully aware of this, and provides a procedural safeguard: whenever we are representing a proposition with the help of an indefinite symbol, then it must be distinct from any indefinite symbol already used in the symbolization of any other proposition in the same inferential context. But, today, one can go beyond the procedural to the declarative level, and say that the problem arises because an existential quantifier is implicitly associated with representations using indefinite symbols. Notoriously, Boole did not have a clear conception of an existential quantifier as an operation binding a variable, even less an adequate notation for it; they came later, with Frege, Peirce, and the latter's student Mitchell. But he did have a vague feeling for the situation, which he expressed informally in his conditions of interpretation (the "monitions" of De Morgan's draft letter), with the clumsy Aristotelian terminology available to him. For example, in *MAL* he comments (p. 21):

> If Some Xs are Ys, there are some terms common to the classes X and Y. Let those terms constitute a separate class V, to which there shall correspond a separate elective symbol v, then $v = xy$. And as v includes all terms common

> to the classes X and Y, we can indifferently interpret it, as Some Xs, or Some Ys.

Accordingly, Hailperin 1986 proposed that when an equality $v = xy$ symbolizes a premise or the conclusion of an inference, we should write it as $\exists v(v = xy)$, where $\exists$ is an existential quantifier ranging over subclasses v of the universe which, in the context of syllogistic, should be required to satisfy the *nenu* constraint mentioned in section 2. The formula $\exists v(v = xy)$ belongs to predicate logic – of second or first degree according as we take our domain to be Boole's universe or its power-set. With the *nenu* constraint, it says that the meet of x with y is a non-empty class, which is just what we need for a representation of "Some Xs are Ys".[4]

As Hailperin remarks, to make inferential use of a premise of the form $\exists v(v = xy)$, one would immediately apply $\exists-$ (existential instantiation) to get unquantified $v = xy$ or, if instantiation to v has already been carried out from another premise, $v' = xy$ where v' is a fresh variable. One would continue from that point on as does Boole. If the conclusion obtained contains a free occurrence of an indefinite symbol, one would use $\exists+$ to bind it existentially again. To be sure, the $\exists-$ rule itself carries a "fresh variable" constraint so, at first glance, one may not seem to have emerged from the procedural woods. But the declarative justification of the $\exists-$ rule in the context of twentieth-century natural deduction, as a device for compressing longer indirect inferences into shorter direct sequences, is well known.[5]

One can therefore say that Hailperin's suggestion suffices to make good sense of Boole's representation of particular propositions as $v = xy$ (likewise for their negative counterparts), and thus also of Venn's limited use of indefinite symbols (described in note 2). But that is not the end of the story, for Boole has another symbolization of "Some Xs are Ys", for which Hailperin's proposal, as it stands, is not adequate.

3.2 Implicit Functionality

For an equational rendering of "Some Xs are Ys", Boole offers not only $v = xy$ but also $vx = vy$. While it plays only a secondary role in *MAL*, the latter takes on more importance in *CL* (in the variant form $vx = v'y$), while in *LT* it becomes the dominant way of expressing the particular affirmative.

As has long been recognized, under a class reading of v, this equation departs markedly from any natural understanding of "Some Xs are Ys" since it is true when x is disjoint from y so long as the (non-empty) class v has an empty intersection with each of the (non-empty) classes x, y. The same discrepancy arises for the variant representation $vx = v'y$ that is favoured in *CL*. It is preserved, even amplified, by Hailperin's introduction of an existential quantifier, since $\exists v(vx = vy)$ is true whenever there is at least one element of the universe that is in neither of x, y, for one can then choose v to be the singleton of such an element. Thus, while Hailperin's attempt to make declarative sense of indefinite symbols suffices for

Boole's initial representation of particular affirmatives, it does not suffice for another one that became dominant in Boole's later exposition.

To deal with this situation, it is helpful to return to Boole's verbal explanations. For example in *CL* (p. 3) he remarks:

> The term Some Xs will be expressed by vx, in which v is an elective symbol appropriate to a class V, some members of which are Xs, but which is in other respects arbitrary.

Again, in *LT* chapter IV section 13 (p. 63) he tells us that v is:

> ... the symbol of a class indefinite in all respects but this, that it contains some individuals of the class to whose expression it is prefixed ....

Lamentably vague, and reminiscent of the "arbitrary triangles" ridiculed more than a century earlier by Berkeley, these explanations have given rise to scornful dismissals by later logicians. But if we follow Boole's line of thought with the resources currently at our disposal, instead of simply dismissing it, we can give it a perfectly clear formal expression. The thesis of this paper is that we may understand an indefinite symbol v as *a variable ranging over functions $v(\cdot)$ that take a non-empty proper subclass x of the universe as argument and return a non-empty subclass $v(x)$ of x as value.*

These are, indeed, selection functions, which are widely used in current mathematics and logical theory, notably in the study of non-monotonic reasoning and the logic of belief change. They may be called *proper* selection functions, for the only difference is that they do not take as argument the class serving as local universe. Selection functions, of course, also generalize in another way choice functions, the difference being that for a choice function, $v(x)$ is an element, rather than a non-empty subset, of x.

It must be emphasized, however, that the above notion of a proper selection function differs significantly from what Boole had in mind when, in the verbal commentaries of *MAL* and *CL* (but not *LT*), he described ordinary variables as representing mental operations that make selections (see e.g. *CL* p. 2). That terminology will be discussed in section 7.

4 Representing categoricals functionally

In this section we show how such a reading of indefinite symbols plays out neatly in Boole's representations of the four traditional forms of categorical proposition. In the representations, we rewrite v as $v(\cdot)$ to make explicit its functional status and allowing us to keep track, as a calculation progresses, of the argument to which v is applied. We retain Hailperin's existential quantifiers, but now ranging over functions rather than over classes.

The universal affirmative. Boole has several ways of representing "All Xs are Ys" without v but also one with it, namely $x = vy$. We rewrite this equality as $x = v(y)$ and render the existential quantification visible by expanding that to $\exists v(x = v(y))$. This asserts that there is a function v (taking non-empty proper subclasses of the universe to non-empty subclasses of themselves) such that $x = v(y)$. It is a higher-order formula (of third or second order, according to the choice of domain), but it is clearly equivalent to one saying that x is a non-empty subset of y, thus expressing "All Xs are Ys" with the traditional existential import desired by Boole.

The particular affirmative. We recall from section 3.1 that in *MAL*, Boole's initial representation of "Some Xs are Ys" is $v = xy$. When we treat v as a function rather than a class, this becomes $v(x) = xy$ or alternatively $v(y) = xy$, corresponding to Boole's hint, mentioned above, that here "$v =$ Some Xs, or Some Ys". With quantification brought in, but this time over functions, the two equalities become respectively $\exists v(v(x) = xy)$ and $\exists v(v(y) = xy)$. Of course, for these formulations to have the intended meaning of "Some Xs are Ys", the function v must always return a non-empty subset of its argument, as we have required it to do.[6]

As we saw in section 3.2, Boole's alternative expression of "Some Xs are Ys" as $vx = vy$ faces a difficulty under a class reading of v, even when provided with Hailperin's existential quantifier. But when we treat v as ranging over functions, writing $v(x) = v(y)$ and completing with quantifier to $\exists v(v(x) = v(y))$, the difficulty vanishes. The last formula tells us that there is a function v returning non-empty subclasses $v(x)$, $v(y)$ of x, y respectively, such that $v(x) = v(y)$. That is the case just when there is an item that is in both x and y. The same analysis carries over to the variant representation $vx = v'y$.[7]

Negative categoricals. Boole handles these by treating negativity as acting on the predicate and reapplying to that context his symbolization of their positive counterparts. In more detail, his representation $v = x(1 - y)$ of the particular negative "Some Xs are not Ys" may be written functionally, with quantifier also explicit, as $\exists v(v(x) = x(1 - y))$ or $\exists v(v(1 - y) = x(1 - y)$, which says just what it should. For the universal negative "No Xs are Ys", he has several representations without an indefinite symbol, in parallel with those for the universal affirmative, but one that does make such use is $x = v(1 - y)$. For Boole, the parentheses on the right are just for grouping to form the meet of v with $(1-y)$. We re-read the same equality with the parentheses expressing application of the function v to the class $1 - y$, and make the implicit existential quantification visible by expanding to $\exists v(x = v(1 - y))$. Again, this holds just if no Xs are Ys.

In summary, while Boole's representation of "Some Xs are Ys" as $vx = vy$ or $\exists v(vx = vy)$ is aberrant under a class reading of v, it behaves correctly under a functional reading, as also do his representations of the other categoricals.

5 Commuting an indefinite symbol

Thus far, it has been straightforward to read indefinite symbols as variables ranging over functions. But a residual difficulty arises when, in his calculations, Boole commutes a function variable with another variable, in effect treating the non-commutative operation of *application* (of a function to a class) as if it were the commutative one of *meet* (of two classes). We illustrate this with two examples from *MAL*. One commutes v with a class variable, the other commutes it with another function variable.

5.1 First example: commuting zvx to vzx

On page 35 of *MAL*, Boole considers what conclusion may be drawn from the premises "All Ys are Xs", expressed as $y = vx$, and "No Zs are Ys", expressed without an indefinite symbol as $0 = zy$. From those two equalities, Boole infers $0 = vzx$, evidently in two steps: substituting vx for y in the second equality to get $0 = zvx$ and then commuting the indefinite symbol v with the ordinary variable z. He raises the question of how this conclusion $0 = vzx$ should be interpreted – whether as "Some Zs are not Xs" or as "Some Xs are not Zs". Despite the ordering of the variables in vzx where v is contiguous with z, but in accord with intuitive reasoning, he opts for the latter interpretation, with the following clumsy but evocative explanation:

> The reason why we cannot interpret $vzx = 0$ into Some Zs are not Xs, is that by the very terms of the first equation [i.e. $y = vx$] the interpretation of vx is fixed, as Some Xs; v is regarded as the representative of Some, only with reference to the class X.

As we see it, Boole is struggling to say that v is really a function, not a class. Suppose we rewrite the example in that way, at the same time making the existential quantifier explicit. The first premise becomes $\exists v(y = v(x))$, where v is applied to x, not intersected with it, while the second premise remains unchanged at $0 = zy$. After applying $\exists-$ to the first premise, we can follow Boole's calculation by substituting $v(x)$ for y in $0 = zy$ to get $zv(x) = 0$, which is understood as $z \cap (v(x)) = 0$. But the functional reading does not allow us to commute z with v to get $vz(x) = 0$ which is, moreover, meaningless since z is a class, not a function. And while $v(z)x = 0$, understood as $(v(z)) \cap x = 0$, is meaningful, it does not follow from $zv(x) = z \cap (v(x)) = 0$ because it gratuitously permutes variables x and z. In brief, Boole's confusion of application with intersection gets him into a symbolic muddle, commuting unjustifiably, although the intuition expressed in his verbal commentary is basically sound.

5.2 Second example: commuting $v'vx$ to $vv'x$

This example appears in a footnote of *MAL* (p. 42) and is repeated in *CL* (p. 9). Boole considers the *Barbara* syllogism from "All Ys are Xs" and "All Zs are Ys", represented by the respective equalities $y = vx$ and $z = v'y$, to the conclusion "All Zs are Xs", which he writes as $z = vv'y$. Now, when indefinite symbols are read as variables ranging over functions, one can indeed follow Boole from $y = v(x)$ and $z = v'(y)$ to $z = v'(v(x))$ by substituting $v(x)$ for y in $v'(y)$. But Boole writes the indefinite symbols in reverse order

vv' (presumably out of a sense of notational tidiness), and one cannot pass by substitution from $y = v(x)$ and $z = v'(y)$ to $z = v(v'(x))$.

However, this illegitimate commutation does no harm, because Boole uses the equality $z = v(v'(x))$ only to infer, in effect, that $z = w(x)$ where w is a composition of v with v', yielding $\exists w(z = w(x))$ symbolizing the desired conclusion "All Zs are Xs". This is legitimate since, although we cannot choose w as $(v \circ v')$, we can put it to be $(v' \circ v)$.

6 Sources of the function/class conflation

Why was Boole willing to treat functions taking classes to classes as if they themselves were classes? Part of the answer seems to be that he and others had been doing something similar for differential equations without running into serious difficulties. It was common practice to play rather loosely between reals, functions taking reals to reals, and operations taking such functions into others, as was signalled by Boole in chapter XVI, article 3 of his *Treatise on Differential Equations*, a textbook bringing together much of a lifetime's work in the area.

> If a convention, it is at least a very natural one that we should express an operation performed on a subject by attaching, in some way, the symbol denoting the operation to the symbol denoting the subject. The order of writing ... has doubtless determined the mode of connection ... which is the same as if the symbol of operation were a symbol of quantity employed as a co-efficient or multiplier. It comes to pass, moreover, that the formal laws of combination, in the direct cases investigated in Article 2, prove to be the same for the symbol d/dx as for a co-efficient or multiplier.

If such play is successful in the one sphere, why not in the other? To be sure, classes are not real numbers but meet is sufficiently analogous to multiplication to inspire confidence in applying the same gambit to logic.

Another source of Boole's assimilation of class-to-class functions to classes themselves may lie in his non-extensional notion of a function. For him, as for others of his time, a function was not a set of ordered pairs. The only available conceptualizations were as a mental act or a syntactic expression. Boole vacillated between the two, but in his formal work on logic opted for the latter, as is illustrated by the following definition from *LT* chapter V section 8 (p. 71), with a similar passage in *MAL* (p. 16):

> Any algebraic expression involving a symbol x is termed a function of x, and may be represented under the abbreviated general form $f(x)$.

With such an understanding, it can be difficult to make distinctions that are routine under an extensional account of a function as a set of ordered pairs satisfying appropriate existence and uniqueness conditions.

We remark in passing that such factors may also have encouraged a related ambiguity in expositions of probability in many texts of the nineteenth and early twentieth centuries. Until Kolmogorov and others cleaned up the conceptual and notational landscape, it was common practice to use the same variable to range indifferently over arbitrary probability functions on a given domain, and the reals that may appear as values of a given such function. The line between productive and confusing "abuse of notation" can be fuzzy.

7 The evolution of Boole's intuitive perspective on ordinary variables

The preceding sections have focussed on Boole's formal treatment of variables, both ordinary and indefinite, as well as his intuitive conception of the indefinite ones. In this section we return to his intuitive outlook on ordinary variables.

As noticed in section 2, between *MAL* and *LT* there was a shift in terminology for ordinary variables: the "elective symbols" $x, y, ...$ of *MAL* and *CL* became "literal symbols" or just "symbols" in *LT*. This modification of terminology reflected a significant conceptual evolution. Its endpoint, in *LT*, can be described quite briefly: there, Boole speaks of $x, y, ...$ simply as ranging over classes, thus putting the shifting verbal perspective in line with an unchanging formal treatment. But in *MAL* and *CL*, the vision offered for ordinary variables is much more complex and deserves careful attention.

Roughly speaking, in the two early publications, Boole saw $x, y, ...$ as representing "mental operations" that effect some kind of "selection" from classes while the classes themselves are signified informally by corresponding upper-case letters X,Y,... that do not appear in the formal equations. In *CL*, this is put quite briefly:

> Suppose that we have the conception of any group of objects consisting of Xs, Ys, and others, and that x, which we shall call an elective symbol, represents the mental operation of selecting from that group all the Xs which it contains ... (*CL* p. 2)

MAL explains his thought at greater length, from which we select the central parts:

> The symbol x operating upon any subject comprehending individuals as classes, shall be supposed to select from that subject all the Xs which it contains .... When no subject is expressed, we shall suppose 1 (the Universe) to be the subject understood, so that we shall have $x = x(1)$, the meaning of either term being the selection from the Universe of all the Xs which it contains ... From these premises it will follow, that the product xy will represent, in succession, the selection of the class Y, and the selection from the class Y of such individuals of the class X as are contained in it, the result being the class whose members are both Xs and Ys. (*MAL* p. 15)

Close reading of these passages suggests that the mental operations in question are context-dependent. When x is standing alone, a selection is made from the universe, but when x is immediately followed by another variable y then it is effected on the result of the selection for y. From this point of view, the selections might more accurately have been described as restrictions. Concatenation of variables is not so much a convention for expressing intersections as one for effecting such restrictions, and meet is seen not so much as an independently given operation, as a by-product of restriction.

Boole does not attempt to express this understanding formally – it precedes all his formal work. But if we were to take on the task with present-day resources and notation, we could do so as follows. The variable x stands for any one-place function $f_a(\cdot)$ that is defined by putting $f_a(b) = a \cap b$ for any subclass b of the universe (in syllogistic, these classes being required to satisfy the *nenu* condition of section 2). In brief, an elective symbol stands for any left projection of the two-place operation of intersection, and we will write it more suggestively as $\cap_a(\cdot)$ where a is any subclass of the universe.

Whether such projection functions $\cap_a(\cdot)$ are described as effecting restrictions or selections, they are quite different from selection functions as the latter are understood in current mathematics and used in section 3.2 to make sense of Boole's indefinite symbols. The differences can be brought out in several ways.

On the one hand, not every projection function $\cap_a(\cdot)$ is a selection function, since $\cap_a(b) = a \cap b$ may be empty even when b is non-empty. Conversely, not every proper selection function is such a projection function: there are proper selection functions v such that there is no class a such that $v(b) = \cap_a(b) = a \cap b$ for all classes b. For example, consider a three-element universe $\{1, 2, 3\}$ and a proper selection function v that puts, *inter alia*, $v(\{1\}) = \{1\}$ and $v(\{1, 2\}) = \{2\}$. Then for any $a \subseteq \{1, 2, 3\}$, if both $a \cap \{1\} = \{1\}$ and $a \cap \{1, 2\} = \{2\}$ we have $1 \in a$ and $1 \notin a$.

Moreover, the two kinds of function behave quite differently. Functions $\cap_a(b) = a \cap b$ trivially satisfy the equalities $\cap_a(a) = a$, $\cap_a(b \cap c) = (\cap_a(b)) \cap (\cap_a(c)) = (\cap_a(b)) \cap c) = b \cap (\cap_a(c))$, $\cap_a(b + c) = (\cap_a(b)) + (\cap_a(c))$, where $+$ may be read as either union, symmetric difference or Boole's partial operation defined only when its arguments are disjoint (over which range union and symmetric difference coincide). None of the counterparts $v(a) = a$, $v(b \cap c) = v(b) \cap v(c) = v(b) \cap c = b \cap v(c)$, $v(b + c) = v(b) + v(c)$ hold in general for selection functions. There are also equalities for $\cap_a(\cdot)$ that are difficult even to translate into the language of selection functions because they involve more than one index, for example $\cap_a(b) = \cap_b(a)$ and $\cap_a(\cap_b(c)) = \cap_{a \cap b}(c)$ expressing commutation and association respectively. One could entertain more or less analogous principles such as $v(v'(a)) = (v'(v(a))$ and $v(v'(a)) = (v'(v))(a)$ but, for selection functions, the first of these fails while the second is meaningless.

As mentioned at the beginning of this section, in *LT* the verbal perspective of ordinary variables is radically simplified by being brought in line with the unchanging formal treatment. In the commentary there, Boole speaks of $x, y, ...$ as ranging over classes, without reference to mental operations of selection. For example, in chapter II section 5 (p. 28) he says simply: "Let us then agree to represent the class of individuals to which a particular name or description is applicable, by a single letter x". This was, indeed, a significant step towards clarity that met with general acquiescence – and occasional regret, as in Bryant 1888.

On the other hand, Boole's intuitive view of $v, v', ...$ as indicating "indefinite classes" remained unchanged from *MAL* through *LT*. Accompanying this, however, was a secondary conceptual evolution paralleling that for ordinary variables. In *MAL* the term "separate elective symbol" (e.g. *MAL* p. 21) suggests that he also envisaged $v, v', ...$ as standing for some kind of class-selection (or, as we have described it, class-restriction) operation; while in *LT*, that term is replaced by ones like "indefinite class symbol" (e.g. chapter IV section 14, p. 64). But the most salient feature of $v, v', ...$ did not change over time: the class concerned, whether envisaged via a restriction or directly, was not a specific one, but "indefinite".

8 Conclusions

Table 1 displays in graphic form our account of Boole's treatment of variables in *MAL, CL* and *LT*. Its upper four rows record Boole's conceptual evolution in the matter, decomposed along three dimensions: his formal work *vs* his verbal explanations; his distinction between ordinary variables $x, y, ...$ and indefinite symbols $v, v', ...$; his perspective in *MAL* and *CL* vs that of *LT*. The bottom row recapitulates our formal reconstruction of Boole's troublesome verbal explanations of both kinds of symbol.

Table 1: Boole's treatment of variables with our reconstructions

<table>
<tr><td>formal work</td><td colspan="4">verbal explanations</td></tr>
<tr><td>$x, y, ..., v, v', ...$</td><td colspan="2">$x, y, ...$</td><td colspan="2">$v, v', ...$</td></tr>
<tr><td>MAL, CT, LT</td><td>MAL, CL</td><td>LT</td><td>MAL, CL</td><td>LT</td></tr>
<tr><td rowspan="2">classes</td><td>mental operations restricting a class to a subclass</td><td rowspan="2">classes</td><td>mental operations restricting a class to an "indefinite" subclass</td><td>indefinite classes</td></tr>
<tr><td>left projections $\cap_a(\cdot)$ of the meet operation</td><td colspan="2">existentially quantified class-to-class selection functions</td></tr>
</table>

The main thesis of the paper is signalled in the bottom right cell of the table. It suggests that a functional reading of Boole's indefinite symbols, combined with their existential quantification as suggested by Hailperin, makes good declarative sense of his equational representations of categorical propositions, including his much maligned rendering of the particular affirmative as $vx = vy$. The reconstructed equations can be expressed in the notation of predicate logic of second or third order (depending on whether we take our basic domain to be made up of classes or their elements). Some incoherencies do remain in Boole's analysis of syllogistic inference, arising when in the course of a calculation he commutes a variable for functions with another variable in a way that conflates application with intersection. Nevertheless, those muddles are marginal, and may be edited away manually without harming the line of reasoning. In brief, Boole's clumsily expressed intuitions on what he is doing with indefinite symbols may be made precise.

On the other hand, this does not imply that other tangles in Boole's logical work can always be unpicked. And if, like Cayley 1871, one is happy to express propositions by inequalities like $xy \neq 0$, then the need for indefinite symbols does not even arise.

Notes

[1]For brevity, we use the customary acronym *MAL* for *The Mathematical Analysis of Logic*, *CL* for "The calculus of logic", and *LT* for *An Investigation of the Laws of Thought.* Most of the comments on the two kinds of symbol are located in pp. 15-47 of *MAL*, pp. 1-4 of *CL*, and chapters II, IV (sections 11-14) and XV of *LT*. Page-calls for *MAL* follow the original 1847 edition as reproduced in the annotated version of Burris 2019; this can differ by a page or two from that of other editions. Page-calls for *CL* follow the web version mentioned in the bibliography while for *LT* we cite by the Dover edition, which is a facsimile of the original with unaltered pagination. In quotations from Boole, we preserve his distribution of roman and italic fonts and expressions such as "&c". In the list of references, whenever easily accessible electronic and print-on-demand versions are known to exist, they are added to the often hard-to-obtain originals.

[2]Jevons 1864 chapter XV criticised Boole's system in several respects, notably its restriction of join (union) to disjoint classes, its operation of division, and those applications of arithmetic multiplication, sum and subtraction that produce uninterpretable expressions. On the other hand, he accepted several other features that later became subject to criticism. As far as indefinite symbols are concerned, he preferred to avoid them in the representation of universal propositions, but continued to use them to represent particular ones, without further comment or clarification (see chapter XII paragraphs 144-145). Ladd 1883 (more widely known by her married name Ladd-Franklin) reports that while MacColl and Peirce dispensed with Boole's indefinite symbols, they continued to be used in one way or another, not only by Jevons but also by Schröder and Grassmann. To that list we would add Venn 1881 where, following an extended and rather tortuous discussion in chapter VII, he settles for representing "Some Xs are Ys" as $xy = v$, avoiding Boole's other symbolisation of it as $vx = vy$.

[3]Readers unfamiliar with Boole's work will naturally ask why he didnt represent "Some Xs are Ys" simply as $xy \neq 0$, thus eliminating any need for indefinite variables. Although he does not answer this question, or even pose it, the answer is clearly that he very much wanted all his symbolic expressions to be equalities, never inequalities, to facilitate his algebraic manipulations. The first to propose using the inequality $xy \neq 0$ to represent the particular affirmative appears to have been Cayley 1871.

[4]Hailperin's 1986 account of the role of ordinary variables $x, y, ...$ in Boole's system is given in his sections 1.4 and 1.5, while the indefinite symbols $v, v', ...$ are discussed in section 2.5. On the formal level, both are treated as ranging over proper, non-empty subclasses of the universe U. In particular, in section 2.5 Hailperin does not suggest resolving the difficulty of understanding indefinite symbols by treating them functionally; they are explicitly called "class variables" (p. 153) and range only over classes. The casual reader should not be misled by the fact that, beginning with section 3.1, Hailperin goes on to reconstruct

Boole's system "in accordance with modern standards" with a quite different treatment of variables. Boole's $v, v', ...$ disappear from the apparatus, while ordinary variables are divided into two kinds. Those written in plain italics range over classes while those written in bold italics range over functions ("operators", in Hailperin's terminology) from classes to classes – but in contexts quite different from those in which Boole would have used an indefinite symbol. For example, the distribution principle that Boole writes as $x(y+z) = xy + xz$, is written as $\mathbf{x}(y+z) = \mathbf{x}y + \mathbf{x}z$ with $\mathbf{x}$ representing a function; evidently, this is quite distinct from Boole's use of indefinite symbols to get "Some Xs are Ys" into equational form.

[5]See e.g. Makinson 2020 section 10.4.3, where such compression is called "flattening".

[6]It is tempting to allow the universe of discourse to be an argument of v and rewrite Boole's $v = xy$ functionally as $v(1) = xy$ to avoid the asymmetry of choosing arbitrarily between $v(x)$ and $v(y)$. But the gain in symmetry is offset by a loss. In so far as, in his treatment of syllogistic, Boole's ordinary variables range only over non-empty non-universal (*nenu*) subsets of the universe, it would be awkward to allow the universe to serve as argument for a class-to-to-class function. However, in the general theory of elimination and development, where this constraint is not applied, the symmetric representation would be the most elegant.

[7]Hailperin, who (as we saw in note 4 above) follows a class reading of indefinite variables, makes a quite different proposal for handling $vx = vy$. He proposes that we conjoin it with a clause $xy \neq 0$ to obtain the representation $vx = vy \wedge xy \neq 0$, becoming $\exists v(vx = vy \wedge xy \neq 0)$ when the quantifier is made explicit (Hailperin 1986, section 2.5, page 154). However, this is rather odd since Boole, from beginning to end, seeks to represent everything as an equality. If we are going to introduce an inequality, then we might as well represent "Some Xs are Ys" directly by the simplest one possible, namely $xy \neq 0$.

Acknowledgements

Thanks to Alexei Muravitsky and Werner Stelzner for helpful comments on a draft. Special thanks are due to an anonymous reviewer, whose detailed and penetrating comments led to important revisions throughout the paper.

References

[1] Boole, G. 1847. *The Mathematical Analysis of Logic.* Cambridge: MacMillan, Barclay & MacMillan. Reprinted several times, including by CreateSpace Independent Publishing Platform, California, 2016. Also available in html from Google Books.

[2] Boole, G. 1848. The calculus of logic. *Cambridge and Dublin Mathematical Journal* 3: 183-198. Also available in pdf at https://www.maths.tcd.ie/pub/HistMath/People/Boole/CalcLogic/CalcLogic.pdf

[3] Boole, G. 1854. *An Investigation of the Laws of Thought.* Reprinted by Dover Books, New York, 1951 with later reprints. Also available in html from Google Books.

[4] Boole, G. 1859/1877. *A Treatise on Differential Equations.* London: MacMillan. First edition 1859, fourth edition 1877. Also available in html from Google Books.

[5] Bryant, S. 1888. On the nature and function of a complete symbolic language. *Mind* 13: 188-207. Also available via https://ia600708.us.archive.org/

[6] Burris, S.N. 2019. *Annotated Version of Boole's Algebra of Logic 1847*. Version of 21 January 2019 available at https://www.math.uwaterloo.ca/ snburris/htdocs/MAL.pdf

[7] Burris, S.N. & H.P. Sankappanavar 2014. *Boole's principles of symbolical reasoning.* ArXiv: 1412.2953v1 [math.LO], 9 December 2014.

[8] Burris, S.N. & H.P. Sankappanavar 2014a. Boole's method II. The original version. Available via https://www.researchgate.net/

[9] Cayley, A. 1871. A note on the calculus of logic. *Quarterly Journal of Pure and Applied Mathematics* 7: 282-3.

[10] De Morgan, A. 1847. *Formal Logic.* Taylor & Walton.

[11] Dummett, M. 1959. Review of G. Boole, *Studies in Logic and Probability*, ed. R. Rhees. *The Journal of Symbolic Logic* 24: 203-209. Reprinted in J. Gasser ed. 2000, A Boole Anthology, Dordrecht: Kluwer and Springer, pp. 79-86.

[12] Hailperin, T. 1986. *Boole's Logic and Probability*, 2nd edition. Amsterdam: North-Holland.

[13] Jevons, S. 1864. *Pure Logic, or the Logic of Quality apart from Quantity; with Remarks on Boole's System and on the Relation of Logic and Mathematics.* London: Stanford. Reprinted several times, including by Franklin Classics 2018. Also available in html from Google Books.

[14] Ladd, C. 1883. On the algebra of logic, pp. 17-71 of C.S. Peirce, *Studies in Logic, by Members of the John Hopkins University*, Boston: Little, Brown & Co. Reprinted by Forgotten Books 2012. Also available in html from Google Books.

[15] Makinson, D. 2020. *Sets, Logic and Maths for Computing*, third edition. Switzerland: Springer.

[16] Smith, G.C. 1982. *The Boole – De Morgan Correspondence.* Oxford: Clarendon Press.

[17] Van Evra, J.W. 1977. A reassessment of George Boole's theory of logic. *Notre Dame Journal of Formal Logic* 18: 363-377. Reprinted in J. Gasser ed. 2000, *A Boole Anthology*, Dordrecht: Kluwer and Springer, pp. 87-100.

[18] Venn, J. 1881. *Symbolic Logic.* London: MacMillan. Reprinted by Forgotten Books 2018. Also available in pdf from Google Books.