Australasian Journal of Logic

Interpreting mereotopological connection

Nathaniel Gan

Department of Philosophy
National University of Singapore

Abstract. Classical mereotopology is sometimes thought to be represented by General Extensional Mereotopology with Closure Conditions (GEMTC). One reason typically given in favour of GEMTC is its relation to set-theoretic topology. However, the connection primitive in GEMTC lacks an obvious topological interpretation, and the alignment between GEMTC and topology varies across possible interpretations. This paper identifies, among several natural candidates, an interpretation that best aligns GEMTC with topology. Ten possible topological interpretations of mereotopological connection are examined, and for each, we identify (i) the conditions under which topological spaces can provide models of GEMTC, and (ii) the extent to which the definitions of GEMTC agree with their topological analogues in these models. It is observed that when connection is interpreted as the intersection of one set with the closure of another, the non-empty sets of any symmetric topology are a model of GEMTC, with agreement between several mereotopological notions and their topological analogues. This represents a stronger relation between GEMTC and topology than has thus far been observed in the literature. The results of the investigation also bear on issues like Peirce's puzzle, the possibility of external connection between regions, and our intuitive understanding of connection in terms of boundaries.

Mereology is the study of the parthood relation. With mereological theories, we may go some way toward modelling and reasoning about the spatial structure of regions. For instance, we may describe Australia as the mereological sum of its states and territories, and describe Sydney as part of New South Wales; we may also infer from these descriptions and the transitivity of parthood that Sydney is part of Australia. However, mereology on its own has limitations. A mereological theory cannot describe the breaking of a glass cup, because the shattered and whole cups have all the same parts, albeit differently arranged. To make sense of such cases, we turn to *mereotopology*, which extends mereology with topological notions such as connection, interiors, boundaries, and the like. Mereotopological theories were initially formulated with the goal of modelling relations between temporal or spatial entities like events (Whitehead, 1919), temporal intervals (Bentham, 1983), and regions (Casati & Varzi, 1999), and have also found applications to document classification (Fujihara & Mukerjee, 1991), geographic informational systems (Winter, 2000), natural language processing (Asher & Vieu, 1995), robot navigation (Kuipers & Levitt, 1988), and visual representation (Gooday & Cohn, 1995), among other things.

General Extensional Mereotopology with Closure Conditions (henceforth *GEMTC*) is a mereotopological theory sometimes held to represent classical mereotopology (Casati & Varzi, 1999; Varzi, 1996). One reason often given in favour of GEMTC is its close relation to set-theoretic topology. It is known that significant classes of topological spaces can provide models of GEMTC or sub-theories thereof. For instance, Rachavelpula (2017, pp. 9–12) showed that the non-empty sets of a Hausdorff topology are a model of GEMTC, and Grzegorczyk (1960) showed that the non-empty regular open sets of a Hausdorff topology are a model of General Extensional Mereotopology. Moreover, several mereotopological properties defined within GEMTC, such as self-connection, interior, closure, exterior, and boundary, have analogues in topology. The mereotopological and topological definitions of these notions are similar, and presumably are intended to align in topological models of GEMTC. Casati and Varzi take these relations between GEMTC and topology as indicative that the connection relation in GEMTC 'is germane to that of standard set-theoretic topology,' and hence that 'GEMTC may be considered as the archetype of a mereotopology theory' (Casati & Varzi, 1999, p. 59).

However, the topological aspect of GEMTC also has potentially unintuitive consequences, some of which are illustrated by *Peirce's puzzle* (Peirce, 1933). The Northern and Southern hemispheres of Earth seem to connect without overlapping. According to GEMTC, the Equator exists as the boundary of these hemispheres, which raises the question: to which hemisphere does the Equator belong? If it is taken to belong to both hemispheres, this entails that they overlap; if neither, then there is a region between them that is not part of either—both seem unintuitive. And if the Equator is taken to belong to just one hemisphere, the choice would seem arbitrary. Peirce took cases like this to suggest that boundaries do not exist, while others have chosen to adopt non-classical approaches to boundaries (e.g., Weber & Cotnoir, 2015).

Furthermore, the relation between GEMTC and topology is somewhat uncertain. Despite the many points of analogy between the two, the connection relation in GEMTC lacks an obvious topological interpretation. While connection is defined in topology as a property of topological spaces and as a property of individual sets, it is not defined in topology as a binary relation, as it is in GEMTC. Indeed, interpretations of connection in topological models of mereotopological theories have been somewhat varied in the literature. Rachavelpula interpreted connection in terms of separation by open neighbourhoods, while Grzegorczyk interpreted connection as the intersection of one set with the closure of the other, as do others (Casati & Varzi, 1999; Varzi, 2007). In some other mereotopological theories, connection is interpreted as intersection of sets (Clarke, 1981; Gotts, 1996) or of closures (Cotnoir, 2010; Pratt-Hartmann, 2007; Russell, 2008). These interpretations turn out not to be generally equivalent: the open intervals $(0,1)$ and $(1,2)$ are connected if connection is interpreted as intersection of closures, but not if connection is interpreted as intersection of sets. And, the choice of interpretation affects the alignment between GEMTC and topology. As will be seen, the conditions under which topological spaces can provide models of GEMTC varies across different interpretations of connection. And in these models, whether the mereotopological and topological definitions of self-connection, interior, closure, and the like agree also depends on how connection is interpreted.

This paper aims to resolve this uncertainty by identifying an interpretation of connection that best aligns GEMTC with topology. It will be observed that among several natural candidates for a topological interpretation of connection, one is superior to alternatives in the sense of admitting models of GEMTC in the broadest class of topological spaces and, in these models, countenancing the intended alignments between the definitions of GEMTC and their topological analogues to the greatest extent. Namely, when connection is interpreted as the intersection of one set with the closure of another, the non-empty sets of any symmetric topology are a model of GEMTC, with alignment between the mereotopological and topological definitions of (self-)connection, open and closed entities, interior, exterior, closure, and boundary, which is a stronger result than has thus far been shown in the literature. The results of the investigation will also bear on issues like Peirce's puzzle, the possibility of external connection between open or closed regions, and our intuitive understanding of connection in terms of boundaries.

§1 reviews the axioms and definitions of GEMTC. §2 reviews the relevant ideas from topology and identifies several intended points of analogy with mereotopology, which will provide desiderata for an interpretation of connection. §3 lays out ten possible interpretations, which §§4–5 assess against the desiderata. §6 summarises the results of the investigation and §7 discusses implications.

1. GEMTC

Casati and Varzi (1999) formulate the mereological aspect of GEMTC with the (improper) parthood relation Pxy as primitive. Parthood is stipulated to be reflexive, antisymmetric, and transitive:

(1)	Pxx	(Reflexivity of P)
(2)	$(Pxy \wedge Pyx) \rightarrow x = y$	(Antisymmetry of P)
(3)	$(Pxy \wedge Pyz) \rightarrow Pxz$	(Transitivity of P)

The overlap relation Oxy is defined in terms of parthood as the sharing of a part:

$$Oxy \coloneqq \exists z(Pzx \land Pzy) \qquad \text{(Overlap)}$$

The extensional aspect of GEMTC comes from the antisymmetry of P together with the following Strong Supplementation axiom, which says that whenever one thing is not part of another, some part of the first thing has no part in common with the second:

(4) $\neg Pyx \to \exists z(Pzy \land \neg Ozx)$ (Strong Supplementation)

GEMTC countenances unrestricted composition, as given by the Fusion axiom schema:

(5) $\exists x\phi x \to \exists z\forall y(Oyz \leftrightarrow \exists x(\phi x \land Oyx))$ (Fusion)

Intuitively, Fusion says that for any satisfied property, there is an object that comprises just all bearers of that property. The extensionality of GEMTC guarantees the uniqueness of witnesses to $\exists z$ in instances of Fusion, hence we may define (general) sums, products, and complements like so:

$$x + y \coloneqq \iota z\forall w\big(Owz \leftrightarrow (Owx \lor Owy)\big) \qquad \text{(Sum)}$$
$$\sigma x\phi x \coloneqq \iota z\forall y\big(Oyz \leftrightarrow \exists x(\phi x \land Oyx)\big) \qquad \text{(General sum)}$$
$$x \times y \coloneqq \iota z\forall w\big(Owz \leftrightarrow (Owx \land Owy)\big) \qquad \text{(Product)}$$
$$\sim x \coloneqq \sigma z(\neg Ozx) \qquad \text{(Complement)}$$

The topological aspect of GEMTC is formulated with the connection relation Cxy as an additional primitive. Connection is stipulated to be reflexive, symmetric, and monotonic with respect to parthood:

(6) Cxx (Reflexivity of C)
(7) $Cxy \to Cyx$ (Symmetry of C)
(8) $Pxy \to \forall z(Cxz \to Cyz)$ (Monotonicity)

Connection is a binary relation between entities, but the related property of *self-connection* may be defined for individuals, which says that an object cannot be partitioned into disconnected parts. In Casati and Varzi's formulation, self-connection is defined like so (the possibility of alternative definitions will be considered in §4):

$$SCx \coloneqq \forall y\forall z(x = y + z \to Cyz) \qquad \text{(Self-connection)}$$

The internal parthood relation, which roughly says that one thing is enclosed within another, is also defined in terms of connection:

$$IPxy \coloneqq Pxy \land \forall z(Czx \to Ozy) \qquad \text{(Internal parthood)}$$

In terms of internal parts, interiors, exteriors, closures, and boundaries may be defined:

$$ix \coloneqq \sigma z\, IPzx \qquad \text{(Interior)}$$
$$ex \coloneqq i({\sim}x) \qquad \text{(Exterior)}$$
$$cx \coloneqq {\sim}(ex) \qquad \text{(Closure)}$$
$$bx \coloneqq {\sim}(ix + ex) \qquad \text{(Boundary)}$$

Intuitively, the interior of an object comprises everything within its boundary, the exterior of an object comprises everything outside its boundary, and the closure of an object comprises it and its boundary. A region is mereotopologically *open* (*closed*) if it is equal to its own interior (closure), that is, if it excludes (includes) every part of its boundary.

The closure conditions of GEMTC govern the behaviour of interiors and closures and are given by the following axioms:[1]

(9)	$\exists z\, IPzx \rightarrow P(ix)x$	(Inclusion)
(10)	$\exists z\, IPzx \rightarrow i(ix) = ix$	(Idempotence)
(11)	$(\exists zIPzx \wedge \exists zIPzy) \rightarrow i(x \times y) = ix \times iy$	(Product)

GEMTC is the mereotopological theory comprising axioms (1)–(11) and the above definitions.

2. Set-theoretic topology

It is known that the mereological aspect of GEMTC has models in set-theory (Cotnoir & Varzi, 2021; Lewis, 1991, 1993; Pietruszczak, 2018; Pontow & Rainer, 2006; Tarski, 1983). When parthood is interpreted as the inclusion relation $\subseteq$, the non-empty subsets of a set X are a model of axioms (1)–(5).[2] Axioms (1)–(3) are satisfied because $\subseteq$ is a partial order, and axioms (4) and (5) follow from the standard axioms of set theory. In these models, the implied interpretations of the other mereological notions are as follows:[3]

$$Oxy \equiv x \cap y \neq \emptyset$$
$$x + y \equiv \mathrm{x} \cup \mathrm{y}$$
$$\sigma x \phi x \equiv \bigcup \{\mathrm{x} \subseteq \mathrm{X}: \phi \mathrm{x}\}$$
$$x \times y \equiv x \cap y$$
$$\sim x \equiv \mathrm{X} - \mathrm{x}$$

To model the topological aspect of GEMTC, we may give a collection of sets some topological structure.[4] A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X that includes $\emptyset$ and X, and that is closed under finite intersection and arbitrary union. A set with an associated topology $(X, \mathcal{T})$ is a *topological space*, in which the members of $\mathcal{T}$ are the *open sets* of $(X, \mathcal{T})$ and their set-theoretic complements are the *closed sets* of $(X, \mathcal{T})$. In terms of open and closed sets, topological interiors, closures, exteriors, and boundaries can be defined. The *interior* of a set is the union of all open subsets of that set, and the *closure* of a set is the intersection of all closed supersets of that set. The *exterior* of a set is the set-theoretic complement of its closure, and the *boundary* of a set is the complement of its interior relative to its closure. The topological interior, closure, exterior, and boundary of a set A will be denoted $Int(A)$, $\overline{A}$, $Ext(A)$, and ∂A respectively. Presumably, the topological and mereotopological definitions of these terms are intended to coincide in topological models of GEMTC.

In topology, connection is not typically defined as a binary relation between sets, hence the intended interpretation of mereotopological connection is left somewhat open. Nevertheless, we have connection defined as a property of a single set. A set x in a topological space is topologically *connected* if it cannot be partitioned into two non-empty subsets $y \cup z = x$ admitting open sets Y, Z such that $y \subseteq Y$, $z \subseteq Z$, and $y \cap Z = Y \cap z = \emptyset$. This property of topological connection will be denoted TCx. Presumably, mereotopological self-connection is intended to align with topological connection.

Ideally, the interpretation of mereotopological connection would be such that the axioms of (the topological aspect of) GEMTC can be satisfied while all the intended alignments just observed are countenanced. In fact, the desiderata can be simplified, because the relations between the topological notions of open and

[1] Axioms (9)–(11) parallel the Kuratowski closure axioms for topology. The significance of this parallel will be considered in §2.

[2] The restriction to non-empty sets is necessary because $\emptyset$ is a subset of every set, so models with $\emptyset$ would violate Strong Supplementation.

[3] We use $\coloneqq$ to denote definition, $\equiv$ to denote interpretation, and $=$ to denote identity.

[4] The definitions and theorems in this paper concerning familiar topological concepts can be found in standard texts like Munkres (2014).

closed sets, interiors, exteriors, closures, and boundaries parallel those between the corresponding mereotopological notions. Namely, the following are known theorems of topology:

Theorem 1. $Ext(A) = Int(\mathrm{X} - A)$

Theorem 2. $\overline{A} = \mathrm{X} - Ext(A)$

Theorem 3. $\partial A = \mathrm{X} - \big(Int(x) \cup Ext(x)\big)$

Theorem 4. A is open iff $A = Int(A)$.

Theorem 5. A is closed iff $A = \overline{A}$.

These theorems imply that if mereological complement is interpreted as set-theoretic complement and the mereotopological and topological definitions of 'interior' align, so will those for 'exterior', 'closure', 'boundary', 'open', and 'closed'. Moreover, topological interiors are known to satisfy the Kuratowski closure axioms, namely

Theorem 6. $Int(A) \subseteq A$

Theorem 7. $Int\big(Int(A)\big) = Int(\mathrm{A})$

Theorem 8. $Int(A \cap B) = Int(A) \cap Int(B)$

So an interpretation of mereotopological connection under which the mereotopological and topological definitions of 'interior' agree will also satisfy axioms (9)–(11) of GEMTC (regardless of how $\sim x$ is interpreted). Therefore, with parthood interpreted as $\subseteq$, for a topological model comprising all the non-empty subsets of a set to satisfy the axioms of GEMTC while yielding all the desired alignments, it is necessary and sufficient that connection be interpreted such that we have

(a) axioms (6)–(8) of GEMTC satisfied,
(b) $SCx \leftrightarrow TCx$ for all x, and
(c) $ix = Int(x)$ for all x.

Anticipating cases to be considered later, we might consider how these desiderata play out in topological models where mereotopological complement is not interpreted as set-theoretic complement. Call a set x in a topological space *regular open* if $x = Int(\overline{x})$. Maintaining the interpretation of parthood as $\subseteq$, it is known that the non-empty regular open (henceforth *NERO*) sets of a topological space are a model of axioms (1)–(5) with the implied interpretations of the other mereological notions like so:[5]

$$\begin{aligned}
&Oxy \equiv x \cap y \neq \emptyset \\
&x + y \equiv Int\big(\overline{\mathrm{x} \cup \mathrm{y}}\big) \\
&\sigma x \phi x \equiv Int\left(\overline{\bigcup\{x\colon \mathrm{x} \subseteq \mathrm{X}, \phi \mathrm{x}\}}\right) \\
&x \times y \equiv x \cap y \\
&{\sim} x \equiv Int(\mathrm{X} - \mathrm{x})
\end{aligned}$$

In such models, if $ix = Int(x)$, axioms (9)–(11) will be satisfied, and the mereotopological and topological definitions of 'interior', 'open', and 'exterior' will agree. However, under this interpretation of $\sim x$, if we have $ix = Int(x)$ generally, this also implies $cx = x$ generally; so the mereotopological closures of non-(topologically-)closed sets will not be their topological closures. Therefore, under a restriction to NERO

[5] A restriction to the non-empty open (or closed) sets of a topological space is insufficient, because it is not guaranteed that these sets satisfy Strong Supplementation.

sets, we can at best have a topological model of GEMTC in which the mereotopological definitions of 'connection', 'interior', 'open', and 'exterior' agree with their topological analogues. This optimal result would still be represented by desiderata (a)–(c).

Dually, call a set x *regular closed* if $x = \overline{Int(x)}$. The non-empty regular closed (henceforth *NERC*) sets of a topological space are a model of axioms (1)–(5) of GEMTC (when parthood is interpreted as ⊆) with the other mereological notions interpreted so:

$$
\begin{aligned}
&Oxy \equiv Int(x) \cap \mathrm{Int}(y) \neq \emptyset \\
&x + y \equiv \mathrm{x} \cup \mathrm{y} \\
&\sigma x \phi x \equiv \overline{\bigcup \{\mathrm{x} \subseteq \mathrm{X} : \phi \mathrm{x}\}} \\
&x \times y \equiv \overline{Int(x \cap y)} \\
&{\sim}x \equiv \overline{\mathrm{X} - \mathrm{x}}
\end{aligned}
$$

With only NERC sets in view, it would not be reasonable to have $ix = Int(x)$ as a desideratum because topological interiors are topologically open. Instead, it would make more sense to expect $cx = \overline{x}$ (this is equivalent to $ix = Int(x)$ in models comprising all non-empty subsets of a set). If this desideratum is satisfied, the mereotopological and topological definitions of 'closed' and 'closure' will agree. But under the present interpretation of ${\sim}x$, $cx = \overline{x}$ implies (and is equivalent to) $ix = x$, so the definitions of 'open', 'interior', and 'exterior' will not align for non-open sets. Nevertheless, $ix = x$ also implies that axioms (9)–(11) are satisfied. So under a restriction to NERC sets, an interpretation of connection that satisfies desiderata (a)–(b) and $cx = \overline{x}$ will admit topological models of GEMTC in which the definitions of 'connection', 'closed', and 'closure' align.

3. Interpretations of connection

Anthony Cohn and Achille Varzi (2003) considered three possible interpretations of connection, as follows (the predicate symbols have been changed to anticipate other candidate interpretations to come):

$$
\begin{aligned}
&C_4xy \equiv x \cap y \neq \emptyset \\
&C_6xy \equiv \overline{x} \cap y \neq \emptyset \text{ or } x \cap \overline{y} \neq \emptyset \\
&C_{7c}xy \equiv \overline{x} \cap \overline{y} \neq \emptyset
\end{aligned}
$$

These interpretations represent some of the more popular interpretations in the literature. Connection is interpreted as C_4 in the theories of Clarke (Clarke, 1981, 1985), the Region Connection Calculus (Cohn et al., 1995; Randell et al., 1992),[6] Roeper's region-based topology (Lando & Scott, 2019; Roeper, 1997), and others (Chen, 2020; White, 1974). Haemmerli and Varzi (2014), as well as Weber and Cotnoir (2015), formulate mereotopological theories in which connection is interpreted as C_6. Cotnoir (2010), Pratt-Hartmann (2007), and Russell (2008) adopt interpretation C_{7c}.

C_6 suggests that the following might also be a possible interpretation, though this interpretation is not typically considered in the literature:

$$C_5xy \equiv \overline{x} \cap y \neq \emptyset \text{ and } x \cap \overline{y} \neq \emptyset$$

Cohn and Varzi suggested in a note (2003, n.2) that interpretations in terms of interiors might be possible, so we shall also consider the duals of interpretations C_5 to C_{7c}

$$C_1xy \equiv Int(x) \cap \mathrm{Int}(y) \neq \emptyset$$

[6] Topological models of the Region Connection Calculus typically restrict consideration to just the NERC sets of a topological space. Under this restriction, all three interpretations here are equivalent.

$C_2xy \equiv \mathrm{Int}(x) \cap y \neq \emptyset$ and $x \cap \mathrm{Int}(y) \neq \emptyset$
$C_3xy \equiv \mathrm{Int}(x) \cap y \neq \emptyset$ or $x \cap \mathrm{Int}(y) \neq \emptyset$

In this order, C_1, C_2, C_3, C_4, C_5, C_6, and C_{7c} decrease in strength. It follows from known topological properties of interiors and closures that $C_i\mathrm{xy}$ implies C_jxy whenever $j > i$ or $j = 7c$, and there are counterexamples to the converses in the standard topology of the real numbers:

$(0,1)$ and $(1,2)$ are related by C_{7c} but not C_6,
$(0,1)$ and $[1,2]$ are related by C_6 but not C_5,
$(0,1) \cup [2,3]$ and $[1,2)$ are related by C_5 but not C_4,
$[0,1]$ and $[1,2]$ are related by C_4 but not C_3,
$(0,2) \cup \{3\}$ and $(2,4)$ are related by C_3 but not C_2,
$(0,2) \cup \{3\}$ and $(2,4) \cup \{1\}$ are related by C_2 but not C_1.

While enumerating possible topological interpretations of mereotopological connection, it might be helpful also to consider a few topological ideas relating to separation. In topological spaces, separation properties are sometimes considered, which describe various ways in which sets may be disconnected in some sense.

Definition 9. Sets a and b are *topologically distinguishable* if there is an open set A such that either $a \subseteq A$ and $b \cap A = \emptyset$, or $b \subseteq A$ and $a \cap A = \emptyset$.[7]

Definition 10. Sets a and b are *separated* if there are open sets A, B such that $a \subseteq A$, $b \subseteq B$, and $a \cap B = A \cap b = \emptyset$.

Definition 11. Sets a and b are *separated by open neighbourhoods* if there are open sets A, B such that $a \subseteq A$, $b \subseteq B$, and $A \cap B = \emptyset$.

Definition 12. Sets a and b are *separated by closed neighbourhoods* if there are open sets A, B such that $a \subseteq$ A, $b \subseteq B$, and $\overline{A} \cap \overline{B} = \emptyset$.

Definition 13. Sets a and b are *separated by a continuous function* if there is a continuous[8] $f: (X, \mathcal{T}) \to \mathbb{R}$ such that $f(a) \subseteq \{0\}$ and $f(b) \subseteq \{1\}$.[9]

The negations of separation properties might yield viable interpretations of mereotopological connection. Indeed, C_5 above is equivalent to topological indistinguishability, C_6 is equivalent to non-separation, and the interpretation of connection adopted by Rachavelpula (2017, p. 9) is non-separation by open neighbourhoods. After defining predicates corresponding to the negations of Definitions 11–13, we arrive at the following list of ten candidate interpretations:

$C_1xy \equiv \mathrm{Int}(x) \cap \mathrm{Int}(y) \neq \emptyset$
$C_2xy \equiv \mathrm{Int}(x) \cap y \neq \emptyset$ and $x \cap \mathrm{Int}(y) \neq \emptyset$
$C_3xy \equiv \mathrm{Int}(x) \cap y \neq \emptyset$ or $x \cap \mathrm{Int}(y) \neq \emptyset$
$C_4xy \equiv x \cap y \neq \emptyset$
$C_5xy \equiv \overline{x} \cap y \neq \emptyset$ and $x \cap \overline{y} \neq \emptyset$
$C_6xy \equiv \overline{x} \cap y \neq \emptyset$ or $x \cap \overline{y} \neq \emptyset$
$C_{7o}xy \equiv x$ and y are not separated by open neighbourhoods

[7] Topological distinguishability is typically defined only for points. Here, the definition has been extended to include sets in general. It may be verified that for pairs of singleton sets containing just a point each, the definition here coincides with the usual definition of topological distinguishability.
[8] A function is continuous if the preimage of every open set is open.
[9] The property of being *precisely* separated by a continuous function is also sometimes defined, in which f satisfies the additional condition that *only* points in a and b are mapped to 0 and 1. This property will not be considered here because the interpretation of connection that comes from it is not monotonic with respect to parthood—$[0,1]$ and $[2,3]$ are precisely separated by a continuous function but $(0,1)$ and $(2,3)$ are not.

$C_{7c}xy \equiv \overline{x} \cap \overline{y} \neq \emptyset$
$C_8xy \equiv x$ and y are not separated by closed neighbourhoods
$C_9xy \equiv x$ and y are not separated by a continuous function

It is known that the separation properties, in the order of the definitions above, strictly increase in strength. Hence, we have implications between the candidate interpretations as follows:

Theorem 14. $C_ixy \to C_jxy$ for all x and y iff one of the following holds

(i) $i < j$,
(ii) $i < 7$ and $j = 7c$ or $7o$, or
(iii) $i = 7c$ or $7o$ and $j > 7$.

$C_{7o}xy \to C_{7c}xy$, $C_{7c}xy \to C_{7o}xy$, and the converse implications do not generally hold. However, some converses hold in restricted classes of sets. With only open sets in view, all of C_1 to C_{7o} (excluding C_{7c}) are equivalent, and C_{7c} is equivalent to C_8. With only closed sets in view, interpretations from C_4 to C_{7c} (excluding C_{7o}) are equivalent.

The ten interpretations listed above by no means exhaust all the possibilities, but they perhaps represent some of the more natural options for interpreting connection in topological models, and include the popular interpretations in the literature. Furthermore, it may be verified that under any of these interpretations, connection is reflexive, symmetric, and monotonic with respect to parthood. So these interpretations all satisfy desideratum (a), which implies that any interpretation among these satisfying desideratum (c) will admit topological models of GEMTC.

4. Self-connection

We first consider the possibility of satisfying desideratum (b) in models comprising all the non-empty sets of a topological space. The definition of topological connection can be stated in terms of C_6: a set x is topologically connected if it cannot be partitioned into two separated subsets; that is, if for every partition $y \cup z = x$, we have C_6yz. This definition would coincide with that of mereotopological self-connection if connection is interpreted as C_6; therefore, C_6 satisfies desideratum (b).

If an interpretation other than C_6 is adopted, Theorem 14 implies that one direction of $SCx \leftrightarrow TCx$ will still generally hold, but the converse will not. For instance, if connection is interpreted as C_8, $TCx \to SCx$ generally holds. For, given any topologically connected x and every partition $x = y + z$, the definition of topological connection implies C_6yz, which further implies C_8yz by Theorem 14; hence x is self-connected. But it is not generally the case that $SCx \to TCx$ under this interpretation, with counterexamples given by pairs of sets that are related by C_8 but not by C_6. In the real numbers with the standard topology, $(0,1) \cup (1,2)$ is not topologically connected because $(0,1)$ and $(1,2)$ are separated, but $(0,1) \cup (1,2)$ is mereotopologically self-connected under interpretation C_8. In particular, $(0,1)$ and $(1,2)$ are related by C_8 because the closures of their smallest open neighbourhoods (namely themselves) intersect. Similarly, if connection is interpreted as C_i for $i < 6$, $SCx \to TCx$ will generally hold but $TCx \to SCx$ will not, with counterexamples given by pairs of sets that satisfy C_6 but not C_i, such as $[0,2]$ with the partition $[0,1) \cup [1,2]$.

Nevertheless, even under an interpretation other than C_6, there might still be ways of satisfying desideratum (b). Three possible strategies will be explored here. First, the mereotopological definition of self-connection could be revised, so that SC lines up with TC under the revised definition of SC and the adopted interpretation of connection. For instance, Clarke (1985), while formulating another mereotopological theory, defined self-connection equivalently to

$$SCx \coloneqq \forall y \forall z(x = y + z \to (C(cy)\mathrm{z} \vee Cy(cz))$$

Clarke interpreted C as C_4, which makes this definition equivalent to

$$SCx \coloneqq \forall y \forall z(x = y + z \to \mathrm{C}_6\mathrm{yz})$$

which coincides with the definition of topological connection.

A second way of satisfying desideratum (b) is to consider a restricted subclass of sets in the topological spaces under consideration. In models comprising just NERC sets, the definition of SCx is weaker than the original (because the quantifiers are restricted to NERC sets), while the definition of topological connection remains unchanged. So for any given interpretation of connection, if $TCx \to SCx$ generally holds in models comprising all the non-empty sets of a topological space, this will also be in the case in models comprising the NERC sets of a topological space. Moreover, it was observed in §3 that interpretations C_4 to C_{7c} (excluding C_{7o}) are equivalent for closed sets, so $TCx \to SCx$ also generally holds under these interpretations when only NERC sets are in view. Toward the converse, we observe that for a NERC set x, if $y \cup z = x$ and $\neg C_6 yz$, y and z are NERC.[10] So whenever x is NERC and not topologically connected, there are NERC y and z such that $y \cup z = x$ and $\neg C_6 yz$; hence x is not self-connected under interpretations C_1 to C_{7c} (excluding C_{7o}). So while $SCx \leftrightarrow TCx$ generally holds only under interpretation C_6 when all non-empty sets are in view, the biconditional holds under interpretations C_4 to C_{7c} when consideration is restricted to NERC sets.

Another natural option to consider is a restriction to NERO sets. With only NERO sets in view, $TCx \to SCx$ generally holds under interpretations C_{7c}, C_8 and C_9. For, if some NERO x is not self-connected under these interpretations, then $x = Int\left(\overline{y \cup z}\right)$ for some NERO y, z where $\neg C_{7c} yz$, from which it follows that $x = y \cup z$ and $\neg C_6 yz$,[11] and that x is not topologically connected. But $TCx \to SCx$ does not hold for the other interpretations, under which $(0,2) = Int\left(\overline{(0,1) \cup (1,2)}\right)$ is not self-connected despite being topologically connected. The converse $SCx \to TCx$ generally holds under interpretations C_1 to C_{7o} (excluding C_{7c}). For, if a NERO set x is not topologically connected, then $x = y \cup z$ where $\neg C_6 yz$. From this it follows that y and z are NERO,[12] and that $x = Int(\overline{x}) = Int\left(\overline{y \cup z}\right)$ with $\neg C_{7o} yz$ (because C_6 is equivalent to C_{7o} for NERO sets); hence x is not self-connected. But for the other interpretations, the union of two open balls of radius 1 centred at $(0,0)$ and $(2,0)$ is self-connected in the standard topology of the Euclidean plane despite being not topologically connected. Therefore, no interpretation among the candidates satisfies desideratum (b) generally.

A third possible strategy is to restrict consideration to a subclass of topological spaces. In topology, *separation axioms* are sometimes stipulated to hold of topological spaces, which entail that sets meeting certain conditions are separated in some sense. For instance, *completely normal* topological spaces are characterised by the property that separated sets are separated by open neighbourhoods. In these spaces, C_{7o} entails C_6;

---

[10] *Proof.* It suffices to show that $y = \overline{Int(y)}$. Since $\overline{Int(y)} \subseteq \overline{Int(x)} = x$, and $\overline{Int(y)} \subseteq \overline{y}$ which is disjoint from z, we have $\overline{Int(y)} \subseteq y$. Conversely, let $p \in y$. Since y and z are separated, there is an open set $U \supseteq y \ni p$ disjoint from z. Now suppose toward a contradiction that $p \notin \overline{Int(y)}$. Then there is an open set $V \ni p$ disjoint from $Int(y)$. Since $p \in x = \overline{Int(x)}$, $U \cap V$ intersects $Int(x)$, and $W = U \cap V \cap Int(x)$ is open. Since W is disjoint from z, W is an open subset of V and y, contradicting that V is disjoint from $Int(y)$. Hence, $y \subseteq \overline{Int(y)}$.

[11] *Proof.* To show $x = y \cup z$, it suffices to show that $y \cup z = Int\left(\overline{y \cup z}\right)$. Since y is an open set in $\overline{y} \subseteq \overline{y \cup z}$, we have $y \subseteq Int\left(\overline{y \cup z}\right)$; and likewise for z. Conversely, if $p \in Int\left(\overline{y \cup z}\right)$, there is an open set U with $p \in U \subseteq \overline{y \cup z}$. Since the closures of y and z do not intersect, p is not in both their closures—say $p \notin \overline{z}$. Then there is an open set V disjoint from z with $p \in V$. $U \cap V$ is an open set containing p in $\overline{y}$, hence $p \in Int(\overline{y}) = y$. $\neg C_6 yz$ follows from Theorem 14.

[12] *Proof.* It suffices to show that $y = Int(\overline{y})$. Since $Int(\overline{y}) \subseteq Int(\overline{x}) = x$ and $Int(\overline{y}) \subseteq \overline{y}$ which is disjoint from z, we have $Int(\overline{y}) \subseteq y$. Conversely, let $p \in y \subseteq x = Int\left(\overline{y \cup z}\right)$. Then there is an open set $U \ni p$ contained in $\overline{y \cup z}$. Since y and z are separated, there is an open set $V \supseteq y \ni p$ disjoint from $\overline{z}$. $U \cap V$ is now an open set containing p and contained in $\overline{y}$, hence $p \in Int(\overline{y})$.

so $SCx \rightarrow TCx$ holds under interpretation C_{7o} in models comprising all the non-empty sets of a completely normal topological space. This strategy can also be used in conjunction with the previous one. For instance, *normal* topological spaces are characterised by the property that disjoint closed sets are separated by a continuous function. In models comprising just the NERC sets of a normal topological space, C_9xy entails C_4xy, so all interpretations from C_4 to C_9 (including C_{7o} and C_{7c}) are equivalent and $SCx \leftrightarrow TCx$ holds under each.

5. Interior

We turn now to desideratum (c)—the alignment of ix with $Int(x)$. The following is a known property of topological interiors:

Theorem 15. A point $y \in Int(x)$ iff there is an open set Y such that $y \in Y \subseteq x$. Hence, a set $y \subseteq Int(x)$ iff there is an open set Y such that $y \subseteq Y \subseteq x$.

Intuitively, Theorem 15 says that to be in the topological interior of a region is to be enclosed within that region, in the sense of admitting a neighbourhood fully contained in the region. The internal parthood relation also says that one region is enclosed within another, but in the sense of being disconnected from its complement. Aligning the topological and mereotopological notions of interior, therefore, is a matter of interpreting connection such that these two senses of enclosure agree. However, it turns out that in models comprising all the non-empty sets of a topological space, none of the candidate interpretations countenance this alignment.

Under interpretations C_i with $i < 6$, all open subsets of x are internal parts of x. Theorems 1 and 2 imply that $X - Int(x) = \overline{X - x}$, so any open subset of x, because it does not intersect $X - Int(x)$, also does not intersect $\overline{X - x}$. These subsets are thus not connected to the complement of x under interpretation C_5 and, by Theorem 14, under any interpretation C_i with $i < 6$. So for these interpretations, regions within the topological interior of x also fall within an internal part of x, and we have $Int(x) \subseteq ix$. But $ix \subseteq Int(x)$ would not generally hold, with $[0,1]$ as a counterexample. $[0,1]$ is not connected to its complement under these interpretations, and is hence an internal part of itself; but $[0,1] \nsubseteq Int([0,1])$.

Under interpretations C_i with $i \geq 6$, every internal part of x admits an open neighbourhood fully contained within x. By Theorem 14, for y not to be connected to the complement of x under these interpretations is for it not to be so connected under interpretation C_6. And if y does not intersect the closure of $X - x$, then y admits an open neighbourhood not intersecting $X - x$ and hence fully contained within x. So for these interpretations, we have $ix \subseteq Int(x)$. But now $Int(x) \subseteq ix$ will not generally hold. As a counterexample, consider the topology on $\{0,1\}$ in which $\emptyset$, $\{1\}$, and $\{0,1\}$ are open. Since $\{1\}$ is open, 1 is in the interior of $\{1\}$; but since $\overline{\{1\}} = \{0,1\}$, $\{1\}$ is related to its complement by C_6 (and hence by C_i for any $i \geq 6$) and $\{1\}$ is not an internal part of itself.

Nevertheless, it might still be possible to satisfy desideratum (c) by employing some of the strategies suggested in §4. Alternative definitions of the relevant mereotopological ideas (such as interior or internal part) are not often considered in the literature, so we will focus on the possibility of restricting the sets or topological spaces under consideration.

With only NERO sets in view, it can be shown that $ix \subseteq Int(x)$ generally holds, and that we have $Int(x) \subseteq ix$ under interpretations C_1 to C_{7o} (excluding C_{7c}). $ix \subseteq Int(x)$ follows from the observations that all internal parts of x are subsets of x, and that $x = Int(x)$ when x is topologically open. For the converse, we note that for any $y \subseteq Int(x)$, $Int(\overline{x})$ is a NERO set containing y. At the same time, $Int(\overline{x})$ is a subset of x, so $Int(\overline{x}) \cap Int(X - x) = \emptyset$, and $Int(\overline{x})$ does not intersect the mereotopological complement of x. $Int(\overline{x})$ is thus an internal part of x under any interpretation equivalent to C_4, namely all interpretations from C_1 to C_{7o}. Under these interpretations, $y \subseteq ix$ and hence $Int(x) \subseteq ix$.

With only NERC sets in view, we seek to have $\overline{x} = cx$ generally instead of desideratum (c). It can be shown that $\overline{x} \subseteq cx$ generally holds, and that we have $cx \subseteq \overline{x}$ under interpretations C_1 to C_{7c} (excluding C_{7o}). To see this, we observe that the desideratum is equivalent to the condition that $ix = x$ generally, because $cx = {\sim}i({\sim}x)$ and $\overline{x} = x = {\sim}({\sim}x)$. Since all internal parts are subsets, we always have $ix \subseteq x$ and hence $\overline{x} \subseteq cx$. Conversely, for any NERC $y \subseteq Int(x)$, y and the mereotopological complement of x (namely $X - Int(x)$) are non-intersecting closed sets. So under any interpretation from C_1 to C_{7c} (excluding C_{7o}), y is an internal part of x. Under these interpretations, $x = \overline{Int(x)} = \overline{\cup\{y: y \subseteq Int(x)\}} \subseteq \sigma y\, IPyx = ix$ and hence $cx \subseteq \overline{x}$.

For interpretations C_i with $i > 5$, it is also possible to restrict the topological spaces under consideration such that $Int(x) \subseteq ix$ generally holds without restrictions on sets. *Symmetric* topological spaces are characterised by the property that for any open set A and point $y \in A$, $\overline{\{y\}} \subseteq A$. In such spaces, $y \in Int(x)$ implies that $\{y\}$ and $X - x$ are separated by the closed sets $\overline{\{y\}}$ and $X - Int(x)$. Hence $\{y\}$ is an internal part of x under interpretations C_6 and C_{7c}. If moreover the topologies are assumed to be normal, the disjoint closed sets $\overline{\{y\}}$ and $X - Int(x)$ are separated by a continuous function, so $\{y\}$ is also an internal part of x under interpretations C_8 and C_9.

For C_{7o}, we may consider *regular* topological spaces, whose characteristic property is that for any open set A and point $y \in A$, there is an open set Y such that $y \in Y \subseteq \overline{Y} \subseteq A$. In regular spaces, $y \in Int(x)$ implies that $y \in Y \subseteq \overline{Y} \subseteq Int(x)$ for some open Y. Now Y and $X - x$ are separated by the disjoint open sets Y and $X - \overline{Y}$, hence Y is an internal part of x.

6. Taking stock

Summarising the results of the investigation so far, each direction of $SCx \leftrightarrow TCx$ holds in the following topological models (keeping the definitions in the initial formulation of GEMTC):

	$SCx \to TCx$	$TCx \to SCx$
C_1	In all cases considered above	–
C_2	In all cases considered above	–
C_3	In all cases considered above	–
C_4	In all cases considered above	All NERC sets of any topological space
C_5	In all cases considered above	All NERC sets of any topological space
C_6	In all cases considered above	All non-empty sets or NERC sets of any topological space
C_{7o}	All non-empty sets of any completely normal space, or all NERC sets of any normal space, or NERO sets of any space	All non-empty sets or NERC sets of any topological space
C_{7c}	All NERC sets of any topological space	In all cases considered above
C_8	All NERC sets of any normal topological space	In all cases considered above
C_9	All NERC sets of any normal topological space	In all cases considered above

Table 1. Sufficient conditions for aligning mereotopological self-connection with topological connection.

And each direction of $ix = Int(x)$ or $\overline{x} = cx$ holds under the following conditions:

	$ix \subseteq Int(x)$ or $\overline{x} \subseteq cx$	$Int(x) \subseteq ix$ or $cx \subseteq \overline{x}$
C_1	All NERC sets or NERO sets of any topological space	In all cases considered above
C_2	All NERC sets or NERO sets of any topological space	In all cases considered above
C_3	All NERC sets or NERO sets of any topological space	In all cases considered above
C_4	All NERC sets or NERO sets of any topological space	In all cases considered above
C_5	All NERC sets or NERO sets of any topological space	In all cases considered above
C_6	In all cases considered above	All non-empty sets of any symmetric topological space, or all NERO sets of any space, or all NERC sets of any space

C_{7o}	In all cases considered above	All non-empty sets of any regular topological space, or all NERO sets of any topological space
C_{7c}	In all cases considered above	All non-empty sets of any symmetric topological space, or all NERC sets of any topological space
C_8	In all cases considered above	All non-empty sets of any normal symmetric topological space
C_9	In all cases considered above	All non-empty sets of any normal symmetric topological space

Table 2. Sufficient conditions for aligning the definitions of interior or closure.

It may be observed that all the interpretations considered admit sufficient conditions under which either $ix = Int(x)$ or $\overline{x} = cx$ holds. Therefore, under each of these interpretations, there are topological models of GEMTC in a significant class of topological spaces, in which some key mereotopological ideas align with their topological analogues. Moreover, interpretations C_6 and C_{7o} admit sufficient conditions under which desideratum (b) is additionally satisfied without any restriction on the sets under consideration (apart from the exclusion of $\emptyset$). Hence, these two interpretations admit topological models of GEMTC in which all the topological definitions in §2 align with their mereotopological analogues as intended. In summary:

	Model of GEMTC	**Definitions aligned**
C_1	NERO sets of any topological space	Interior, open, exterior
C_2	NERO sets of any topological space	Interior, open, exterior
C_3	NERO sets of any topological space	Interior, open, exterior
C_4	NERC sets of any topological space	Connection, closure, closed
C_5	NERC sets of any topological space	Connection, closure, closed
C_6	Non-empty sets of any symmetric space	Connection, interior, open, exterior, closure, closed, boundary
C_{7o}	Non-empty sets of any completely normal regular space	Connection, interior, open, exterior, closure, closed, boundary
C_{7c}	Non-empty sets of any symmetric space	Interior, open, exterior, closure, closed, boundary
C_8	Non-empty sets of any normal symmetric space	Interior, open, exterior, closure, closed, boundary
C_9	Non-empty sets of any normal symmetric space	Interior, open, exterior, closure, closed, boundary

Table 3. Models of GEMTC with mereotopological-topological alignment.

Of particular interest might be interpretation C_6, for which we have the closest observed relation between GEMTC and topology. The stipulation of the symmetric property is relatively weak compared to the other separation axioms mentioned thus far,[13] so if one is interested just in seeking stronger relations between GEMTC and topology, C_6 seems to be the interpretation that best relates the two, aligning the most mereotopological ideas in the broadest class of topological spaces with the weakest restriction on sets. In particular, all Hausdorff spaces are symmetric, but not conversely. Hence, this relation between GEMTC and topology is stronger than those observed thus far in the literature.

Moreover, the stipulation of the symmetric property is the *minimal* condition under which the given alignments hold under interpretation C_6 without further restrictions on the sets considered.

Theorem 16. When parthood is interpreted as $\subseteq$ and connection as C_6, in models comprising the non-empty sets of a non-symmetric topological space $(X, \mathcal{T})$, there is an $x \subseteq X$ such that:

(a) $ix \neq Int(x)$,
(b) x is topologically open but not mereotopologically open,
(c) $e(\sim x) \neq Ext(X - x)$,
(d) $c(\sim x) \neq \overline{X - x}$,
(e) $\sim x$ is topologically closed but not mereotopologically closed.

[13] Roughly, the separation axioms defining symmetric, Hausdorff, regular, normal, and completely normal topological spaces, in this order, strictly increase in strength. More precisely, Hausdorff topological spaces are always symmetric, and completely normal topological spaces are always normal. If the topological spaces in question are *Kolmogorov* (a property corresponding to another separation axiom), regular topological spaces are Hausdorff. If moreover the topological spaces in question are also *accessible* (a strictly stronger property than being Kolmogorov), normal topological spaces are regular. Most typical cases of topological spaces are accessible, so for the most part, regular topological spaces are a subclass of Hausdorff topological spaces, and so on.

Proof. Since $(X, \mathcal{T})$ is non-symmetric, there is a topologically open set x and a point $y \in x$ such that $\overline{\{y\}} \nsubseteq x$. Since x is topologically open, $y \in Int(x)$. We first show that $y \notin ix$. Suppose toward a contradiction that there is an internal part Y of x such that $y \in Y$. By the interpretation of connection, $\overline{Y} \cap {\sim}x = \emptyset$, hence $\overline{Y}$ is a topologically closed set in x containing y, contradicting $\overline{\{y\}} \nsubseteq x$. Hence $y \notin ix$ and $ix \neq Int(x)$, which yields (a). Since x is not equal to its own mereotopological interior, it is not mereotopologically open, which yields (b). We have (c) from $e({\sim}x) = i(x) \neq Int(x) = Ext(X - x)$ and (d) from $c({\sim}x) = {\sim}i(x) \neq {\sim}Int(x) = \overline{X - x}$. Now ${\sim}x = X - x$ is topologically closed and hence equal to its own topological closure, which is unequal to its mereotopological closure; hence (e).

The upshot is that insofar as the relation between GEMTC and set-theoretic topology is taken to be a reason in favour of GEMTC as a mereotopological theory, the case for GEMTC would best be served by interpretation C_6. And under this interpretation, the strongest possible relation between GEMTC and topology is the one identified above.

7. Discussion

The results above suggest that those who endorse GEMTC have reasons to interpret connection as the intersection of one set with the closure of the other. This choice of interpretation has several implications; here we mention three.

First, how connection is interpreted may provide some (though perhaps non-decisive) guidance when attempting to make sense of cases like those that arise in Peirce's puzzle. It initially appears that there are three ways in which the Northern and Southern hemispheres may be connected: either the Equator belongs to both hemispheres, or to just one, or to neither. But if connection is interpreted as the intersection of one set with the closure of another, then the latter is no longer an option. For, the closure of each hemisphere comprises just itself and the Equator, so if neither hemisphere contains the Equator, then neither intersects the closure of the other. The choice for those who adopt interpretation C_6, therefore, is between holding that the two hemispheres overlap, and holding that they are asymmetric in the sense of one being open and the other closed. Generalising over interpretations of connection: if connection is interpreted as C_i, and the Northern and Southern hemispheres are connected, then the Equator cannot belong to neither hemisphere iff i is between 1 and $7o$ (excluding $7c$), and the Equator cannot belong to just one hemisphere iff $i \leq 5$.

Relatedly, the possibility of external connection between open or closed regions is also affected by how connection is interpreted. When overlap is interpreted as intersection (as in many typical cases), the last sentence of the previous paragraph can be stated in terms of external connection: if connection is interpreted as C_i, two open regions cannot be externally connected iff i is between 1 and $7o$ (excluding $7c$), and a closed region cannot be externally connected with an open region iff $i \leq 5$. It may also be verified that two closed regions cannot be externally connected if i is between 1 and $7c$ (excluding $7o$),[14] and external connection is altogether ruled out iff $i \leq 4$. In particular, under interpretation C_6, even when a book is resting on a desk, the two objects would not be connected (assuming that the book and desk are closed, that is, that they contain their surfaces).

Third, under interpretation C_6, we have what is sometimes taken to be an intuitive understanding of connection in terms of boundaries. According to Casati and Varzi, it is part of 'standard' set-theoretic topology that two things are connected iff they share a boundary (Casati & Varzi, 1999, p. 59), that is:

$$Cxy \leftrightarrow (Oxy \vee Ox(cy) \vee O(cx)y)$$

If connection is interpreted as the intersection of one set with the closure of another and overlap is interpreted as intersection, this biconditional holds in all topological models where the mereotopological and topological definitions of closure align. Under alternative interpretations of connection, this might not

[14] Whether the converse is true depends on whether space is well-characterised by normal topological spaces.

be the case. For instance, if connection is interpreted as intersection, $(0,1)$ and $[1,2]$ are not connected despite sharing a boundary, and despite the latter overlapping the closure of the former. Insofar as it is intuitive to characterise connection by boundary-sharing, the fact that this characterisation is endorsed more naturally under interpretation C_6 than under alternatives might be further reason to think that classical mereotopology might incline toward this interpretation.

The investigation above has largely focused on relating mereotopology with topology, without considering other factors that might provide independent reasons to consider a restricted class of topological spaces, or a restricted class of sets within each topological space. The results of the investigation can also help in identifying the interpretations of connection that best relate mereotopology with topology subject to such considerations. We conclude this paper with an example.

GEMTC may be extended to give an *atomless* or *atomic* theory, where the former holds that everything has proper parts, and the latter holds that everything is ultimately composed of objects without proper parts. Topological models of GEMTC given by all the non-empty subsets of a topological space are models of atomic GEMTC, because a set containing just a point has no proper parts. However, it might be thought that atomless GEMTC comes closer to the way we intuitively think about space, because we do not perceive indivisible objects (at least not directly). To model the atomless property topologically, we require a restriction to regular topological spaces, and a restriction to either the NERO or the NERC sets of these spaces. Consider first models comprising the NERO sets of a regular topological space. The results above suggest that no further restrictions on sets or topological spaces are necessary to yield models of atomless GEMTC under interpretations C_1 to C_{7o} (which are equivalent). However, it was also observed above that when overlap is interpreted as intersection, as it is under a restriction to NERO sets, external connection is ruled out. We might thus think that these models, despite being formal models of atomless GEMTC, do not represent the theory as intended. Now consider models comprising the NERC sets of a regular topological space. The results above suggest that no further restrictions on sets or topological spaces are necessary to yield models of atomless GEMTC under interpretations C_1 to C_{7c} (excluding C_{7o}). Moreover, interpretations C_4 to C_{7c} (which are equivalent) allow for external connection between NERC regions. Therefore, it seems, those who endorse atomless GEMTC have reasons to interpret connection as intersection in these models.

Acknowledgments. Thanks to Ben Blumson, Lavinia Picollo, Joshua Thong, and an anonymous reviewer for helpful comments on earlier drafts.

Funding information. This work was supported by the National Research Foundation, Singapore, under its Medium Sized Centre Programme—Centre for Advanced Robotics Technology Innovation (CARTIN), Subaward A-0009428-08-00.

REFERENCES

Asher, N., & Vieu, L. (1995). Toward a geometry of common sense: A semantics and a complete axiomatization of mereotopology. *Proceedings of the International Joint Conference on Artificial Intelligence.*

Bentham, J. v. (1983). *The Logic of Time: A Model-Theoretic Investigation Into the Varieties of Temporal Ontology.* Springer Dordrecht.

Casati, R., & Varzi, A. C. (1999). *Parts and Places: The Structures of Spatial Representations.* MIT Press. https://doi.org/10.7551/mitpress/5253.001.0001

Chen, L. (2020). Infinitesimal gunk. *Journal of Philosophical Logic*, *49*, 981–1004. https://doi.org/10.1007/s10992-020-09544-x

Clarke, B. L. (1981). A calculus of individuals based on 'connection'. *Notre Dame Journal of Formal Logic*, *22*(3), 204–218. https://doi.org/10.1305/ndjfl/1093883455

Clarke, B. L. (1985). Individuals and points. *Notre Dame Journal of Formal Logic*, *26*(1), 61–75. https://doi.org/10.1305/ndjfl/1093870761

Cohn, A. G., Randell, D. A., & Cui, Z. (1995). Taxonomies of logically defined qualitative spatial regions. *International Journal of Human Computer Studies*, *43*, 831–846. https://doi.org/10.1006/ijhc.1995.1077

Cohn, A. G., & Varzi, A. C. (2003). Mereotopological connection. *Journal of Philosophical Logic*, *32*(4), 357–390. https://doi.org/10.1023/A:1024895012224

Cotnoir, A. J. (2010). *Non-Classical Mereologies and their Applications* [PhD dissertation, University of Connecticut].

Cotnoir, A. J., & Varzi, A. C. (2021). *Mereology*. Oxford University Press. https://doi.org/10.1093/oso/9780198749004.003.0001

Fujihara, H., & Mukerjee, A. (1991). *Qualitative reasoning about document design*.

Gooday, J. M., & Cohn, A. G. (1995). Using spatial logic to describe visual languages. *Artificial intelligence Review*, *10*, 1–2.

Gotts, N. M. (1996). *An axiomatic approach to topology for spatial information systems*. Research report 96.25. School of Computer Studies, University of Leeds.

Grzegorczyk, A. (1960). Axiomatizability of Geometry without Points. *Synthese*, *12*(2/3), 228–235. https://doi.org/10.1007/978-94-010-3667-2_10

Haemmerli, M., & Varzi, A. C. (2014). Adding convexity to mereotopology. In *Formal Ontology in Information Systems. Proceedings of the Eighth International Conference* (pp. 65–78). IOS Press. https://doi.org/10.1007/s10992-018-9496-8

Kuipers, B. J., & Levitt, T. S. (1988). Navigating and mapping in large-scale space. *AI Magazine*, *9*(2), 25–43.

Lando, T., & Scott, D. (2019). A calculus of regions respecting both measure and topology. *Journal of Philosophical Logic*, *48*, 825–850. https://doi.org/10.1007/s10992-018-9496-8

Lewis, D. (1991). *Parts of Classes*. Blackwell. https://doi.org/10.2307/2219902

Lewis, D. (1993). Mathematics is megethology. *Philosophia Mathematica*, *1*(3), 3–23. https://doi.org/10.1093/philmat/1.1.3

Munkres, J. (2014). *Topology*. Pearson Education Limited.

Peirce, C. (1933). The logic of quantity. In C. Hartshorne & P. Weiss (Eds.), *Collected papers of Charles Sanders Peirce*. Harvard University Press.

Pietruszczak, A. (2018). *Metamereology*. Nicolaus Copernicus University Scientific Publishing House. https://doi.org/10.12775/3961-4

Pontow, C., & Rainer, S. (2006). A mathematical analysis of theories of parthood. *Data and Knowledge Engineering*, *59*(1), 107–138. https://doi.org/10.1016/j.datak.2005.07.010

Pratt-Hartmann, I. (2007). First-order mereotopology. In M. Aiello, I. Pratt-Hartmann, & J. v. Benthem (Eds.), *Handbook of Spatial Logics* (pp. 13–97). Springer. https://doi.org/10.1007/978-1-4020-5587-4_2

Rachavelpula, S. (2017). The category of mereotopology and its ontological consequences. *University of Chicago Mathematics Research Program*.

Randell, D. A., Cui, Z., & Cohn, A. G. (1992). A spatial logic based on regions and connection. *3rd International Conference on Knowledge Representation and Reasoning*, 165–176.

Roeper, P. (1997). Region-based topology. *Journal of Philosophical Logic*, *26*(3), 251–309. https://doi.org/10.1023/A:1017904631349

Russell, J. S. (2008). The structure of gunk: Adventures in the ontology of space. In D. Zimmerman (Ed.), *Oxford Studies in Metaphysics* (Vol. 4). Oxford University Press.

Tarski, A. (1983). Foundations of the geometry of solids. In *Logic, Semantics, Metamathematics* (pp. 24–29). Hackett Publishing.

Varzi, A. C. (1996). Parts, wholes, and part-whole relations: The prospects of mereotopology. *Data and Knowledge Engineering*, *20*(3), 259–286. https://doi.org/10.1016/S0169-023X(96)00017-1

Varzi, A. C. (2007). Spatial reasoning and ontology: Parts, wholes and locations. In M. Aiello, I. Pratt-Hartmann, & J. Benthem (Eds.), *Handbook of Spatial Logic* (pp. 945–1038). Springer. https://doi.org/10.1007/978-1-4020-5587-4_15

Weber, Z., & Cotnoir, A. J. (2015). Inconsistent boundaries. *Synthese*, *192*, 1267–1294. https://doi.org/10.1007/s11229-014-0614-z

White, N. P. (1974). What numbers are. *Synthese*, *27*(1/2), 111–124.

Whitehead, A. N. (1919). *An Enquiry Conerning the Principles of Natural Knowledge*. Cambridge University Press.

Winter, S. (2000). Topology in raster and vector representation. *GeoInformatica*, *4*, 35–65.