Pure Refined Variable Inclusion Logics

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Abstract

In this article, we explore the semantic characterization of the (right) pure refined variable inclusion companion of all logics, which is a further refinement of the nowadays well-studied pure right variable inclusion logics. In particular, we will focus on giving a characterization of these fragments via a single logical matrix, when possible, and via a class of matrices, otherwise. In order to achieve this, we will rely on extending the semantics of the logics whose companions we will be discussing with infectious values in direct and in more subtle ways. This further establishes the connection between infectious logics and variable inclusion logics.

1 Background and aim

In recent years, there has been a renewed interest in certain logics where entailment can be understood in terms of *analytic implication*, content inclusion, or similar properties. These logics—belonging to a family of relevance systems that are sometimes referred to as Parry logics, in reference to the work of this logician in, e.g., [16, 17, 18, 19]—have been shown to be applicable to a plethora of different phenomena ranging from the aforementioned containment relations between hyperintensional contents in [11] to the more database-related cases of information retrieval from physically damaged processor memories in [9].

But perhaps their earliest application is to the *logics of nonsense* in [5, 13]. These are formal systems where reasoning is allowed to involve meaningless sentences in which a specific treatment is demanded for the overall inferential patterns that are deemed correct, as well as the semantic machinery that underlies the resulting framework. In terms of their semantic aspect, logics of nonsense have ties with the logics of analytic entailment in a very specific sense as shown by [8] and more recently by [15]. In fact, many systems of content inclusion can be characterized in terms of semantics counting with a truth value that behaves mathematically in an infectious or absorbent way—as the truth values that are usually associated with meaningless sentences in the context of the logics of nonsense do. This is why logics of nonsense belong in the more general collection of *infectious logics* as discussed in [22].

Our aim here is to go beyond the recent results on the characterization of these systems of analytic entailment, by broadening the class of logics that can be scrutinized and receive a semantic characterization with the help of logical matrices counting with infectious truth values. In doing so, we will actually focus on a subfamily of the logics of content inclusion comprehended in the set of Parry logics, discussing systems where *refined* inclusions or containments are required—in a sense to be made clear shortly.

Logics included in the family of Parry logics are usually informally understood as systems where the content of the conclusion is included somehow in the content of the premises. When dealing with propositional logics, this is usually formally represented by the fact that the set of propositional variables $Var(\varphi)$ appearing in the conclusion φ is a subset of the set of propositional variables $Var(\Gamma)$ appearing in the premises Γ. This naturally leads to consider, for any given logic L, what is called in [15] its pure right companion L^{pr}. Before moving on, let us mention that in what follows whenever we talk about a propositional language $\mathcal{L}$ we will be considering extensions of the set $\{\neg, \wedge, \vee\}$ with connectives intended to represent negation, conjunction, and disjunction, respectively. Also, if negation is singled out as $\neg$, it is assumed that the remaining connectives are not capable of defining it in any way.

Definition 1. For a logic $\mathrm{L} = \langle \mathbf{FOR}(\mathcal{L}), \vdash_{\mathrm{L}} \rangle$, its *pure right variable inclusion companion* L^{pr} is defined as follows:

$$\Gamma \vdash_{\mathrm{L}^{pr}} \varphi \quad \Longleftrightarrow \quad \begin{cases} \Gamma \vdash_{\mathrm{L}} \varphi, \text{ and} \\ Var(\varphi) \subseteq Var(\Gamma) \end{cases}$$

In this article, we are going to analyze a certain family of what we will call *refined* subsystems of the right companions of a given logical system. As a matter of informal motivation to these systems, which are related to the work of Angell in [1, 2], let us mention the following: logics of analytic entailment are usually motivated by the fact that one wouldn't like some content to appear vacuously in the conclusion if it didn't previously appear in the premises. This famously leads to the rejection of the rule of Disjunction Introduction or Addition (i.e., from φ infer $\varphi \vee \psi$) as well as to the rejection of the rule of Ex Falso Sequitur Quodlibet (i.e., from $\varphi \wedge \neg\varphi$ infer ψ), and similarly to the rejection of (a rule form of) Tertium Non Datur (i.e., from ψ infer $\varphi \vee \neg\varphi$). However, if the idea of content inclusion is only instantiated by the requirement that the propositional variables of the conclusion are a subset of the population of variables of the premises, this still allows for some undesirable cases where the content of the conclusion couldn't be intuitively said to be included in the content of the premises. To observe this, consider the following argument:

> For example, '(Jo died and Jo did not die and Flo wept)' does not mean the same as '(Jo died and Flo did not weep and Flo wept)'; for the first contains a false and inconsistent statement about Jo though the second does not, while the second contains a false and inconsistent statement about Flo though the first does not. How can two sentences mean the same thing if one contains a false and inconsistent statement about an individual while the other does not? A syntactical condition which will rule out such cases can be formulated using a distinction by Herbrand between 'positive' and 'negative' occurrences of a variable in a schema. [2, p. 121-122]

It could be said that the takeaway that Angell got from his reflection in the previous quote, and in the work where it is included, was that mere variable inclusion isn't a sufficient criterion for content inclusion.[1] That is why he developed his system of analytic containment, where this phenomenon is formally represented by a refined form of variable inclusion between premises and conclusions. Here the idea is that we don't only need to pay attention to the fact that the topics of the conclusion are a subset of the topics mentioned in the premises, but also that these are treated equally here and there—meaning by that that there is no content that is negatively treated in the conclusion but is not treated likewise in the premises, and similarly for contents being treated positively. Obviously, this will formally lead to a more refined conception of right variable inclusion companions, that can be defined for both the pure and

[1]Incidentally, one may take away from his position that under his understanding topics (and, therefore, contents) do not enjoy what Franz Berto calls the principle of negation transparency, i.e., that the topic of a sentence and its negation is the same—for more, see [4, p. 32]. Thanks to Thomas Ferguson for urging us to comment on this point.

the regular fragments. In fact, as observed in [9] and [11], (the first-degree entailment fragment of) Angell's own system AC can be understood as the pure refined right companion of Belnap and Dunn's four-valued logic $\mathsf{E}_{\mathtt{fde}}$ from [3, 6].

Definition 2. Let $Var^+(\varphi)$ be the set of all the positive atoms of φ and $Var^-(\varphi)$ the set of negative atoms of φ. These sets can be constructed recursively as follows:

a. If φ is atomic, $Var^+(\varphi) = \{\varphi\}$ and $Var^-(\varphi) = \emptyset$.

b. $Var^+(\neg\varphi) = Var^-(\varphi)$ and $Var^-(\neg\varphi) = Var^+(\varphi)$.

c. $Var^+(\varphi * \psi) = Var^+(\varphi) \cup Var^+(\psi)$ and $Var^-(\varphi * \psi) = Var^-(\varphi) \cup Var^-(\psi)$, for $* \in \{\wedge, \vee\}$.

With a slight abuse of notation, when Γ is a set of sentences and not a single sentence, we will also refer with $Var(\Gamma)$ to the set $\cup\{var(\gamma) \mid \gamma \in \Gamma\}$.

Definition 3. For a logic $\mathrm{L} = \langle \mathbf{FOR}(\mathcal{L}), \vdash_{\mathrm{L}} \rangle$, its *pure refined right variable inclusion companion* $\mathrm{L}^{pr\pm}$ is defined as follows:

$$\Gamma \vdash_{\mathrm{L}^{pr\pm}} \varphi \quad\Longleftrightarrow\quad \begin{cases} \Gamma \vdash_{\mathrm{L}} \varphi, \text{ and} \\ Var^+(\varphi) \subseteq Var^+(\Gamma), \text{ and} \\ Var^-(\varphi) \subseteq Var^-(\Gamma) \end{cases}$$

Thus, to state more clearly and concretely our goals for this article: we are going to explore the semantic characterizability of the right refined pure variable inclusion companions of any given logic. We will ponder the question of their characterizability in terms of a single matrix—trying to draw a parallel with what was done in [15] for the regular right and left pure variable inclusion companions of classical logic. For this investigation, we will draw inspiration on the semantic characterization of Angell's AC done by Ferguson in [9]. It's our hope, that we will be able to extract some consequences and observations that will point towards alternative perspectives on the semantics of refined variable inclusion logics, which we expect to explore in further work.

2 Pure companions

The aim of this section is to revisit the state of the art regarding how to characterize these logical fragments and to further generalize some important results for what will come later in handy.

2.1 Characterizability

The question in which we will inquire in this section is whether a logic can be characterized by a logical matrix or not. A logical matrix can be understood as a set of truth values, a set of truth tables, and a subset of designated values. The designated values will be those truth values that have to be preserved from premises to conclusion. A logic will be characterizable via a (possibly infinite) single logical matrix if and only if it has the *cancellation property.* Some informal although not necessarily intuitive readings can be given for this property. One may say that this property refers to the fact that whenever we have a valid inference where some of the premises do not share propositional variables with the other premises and with the conclusion, and additionally do not entail any formula whatsoever, then we can, as it were, cancel them out—and still have the inference with the remaining premises and the conclusion be valid. Another reading, a contrapositive one, pertains to the fact that whenever we have an invalid inference it can't be turned into a valid one by the addition of supplementary premises, if these new premises are built with propositional variables not appearing in the original premises and conclusion and if, furthermore, these supplementary formulas do not entail every formula whatsoever.

In what follows we will present the cancellation property in a formal manner and we will extract some important results for what will come next. Take $\mathcal{L}$ to be a propositional language, then:

Definition 4. A logical $\mathcal{L}$-matrix $\mathcal{M}$ is a pair $\langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is an algebra of the same similarity type as $\mathcal{L}$, and D is a subset of $\mathbf{A}$.

A logical $\mathcal{L}$-matrix $\mathcal{M}$ induces a consequence relation $\vDash_{\mathcal{M}}$ in the following way, where $\Gamma \cup \{\varphi\} \subseteq \mathbf{FOR}(\mathcal{L})$:

$$\Gamma \vDash_{\mathcal{M}} \varphi \text{ if and only if } \forall v \in Hom(\mathbf{FOR}(\mathcal{L}), \mathbf{A}) : \text{if } v(\Gamma) \subseteq D, \text{ then } v(\varphi) \in D.$$

Going forward, for any logical matrix $\mathcal{M} = \langle \mathbf{A}, D \rangle$ we will refer to functions v, v', v'', etc. in $Hom(\mathbf{FOR}(\mathcal{L}), \mathbf{A})$ as $\mathcal{M}$-valuations or valuations for short, as usual, when the context is clear enough.

When we have a (possibly infinite) class $\mathbb{M}$ of logical matrices for a language $\mathcal{L}$, the substitution-invariant Tarskian consequence $\vDash_{\mathbb{M}}$ is understood as $\bigcap\{\vDash_{\mathcal{M}} | \mathcal{M} \in \mathbb{M}\}$.

Definition 5. A logic $\mathrm{L} = \langle \mathbf{FOR}(\mathcal{L}), \vdash_{\mathrm{L}} \rangle$ has the *cancellation property* if and only if:

$$\Gamma \cup \{\Gamma_i | i \in I\} \vdash_{\mathrm{L}} \varphi \text{ implies } \Gamma \vdash_{\mathrm{L}} \varphi$$

for all $\varphi \in \mathbf{FOR}(\mathcal{L})$ and $\Gamma, \Gamma_i \subseteq \mathbf{FOR}(\mathcal{L})$, $i \in I$, such that:

a. $Var(\Gamma \cup \{\varphi\}) \cap Var(\bigcup\{\Gamma_i | i \in I\}) = \emptyset$,

b. $Var(\Gamma_i) \cap Var(\Gamma_j) = \emptyset$, for all $i \neq j$,

c. for any $i \in I$ there is a $\psi \in \mathbf{FOR}(\mathcal{L})$ such that $\Gamma_i \nvdash_{\mathrm{L}} \psi$.

Theorem 1 ([14, 21, 23])**.** *A logic* $\mathrm{L} = \langle \mathbf{FOR}(\mathcal{L}), \vdash_{\mathbf{L}} \rangle$ *has the cancellation property if and only if there is a single* $\mathcal{L}$*-matrix* $\mathcal{M}$ *such that* $\vdash_{\mathrm{L}} = \vDash_{\mathcal{M}}$.

In what follows, we will interchangeably refer to a logic as having the cancellation property and a logic having a single (possibly infinite) characteristic logical matrix. As it is shown in [15], CL^{pr} doesn't have the cancellation property, and thus it cannot be characterized by a single logical matrix. Now, as a matter of fact we can generalize this analysis for the pure right companion of any logic.

Theorem 2. *Let* $\mathcal{M} = \langle \mathbf{A}, D \rangle$ *be an* $\mathcal{L}$*-matrix and let* L *be a logic such that* $\vdash_{\mathrm{L}} = \vDash_{\mathcal{M}}$. L^{pr} *has the cancellation property if and only if* L *does not have anti-theorems.*

Proof. From left to right: by generalizing the proof of Lemma 3.1 in [15]. Take a logic L with anti-theorems, and consider its pure right companion L^{pr}. Let $\Gamma = p$, $\varphi = \neg p$, and let Γ_1 be one of the original system's anti-theorems such that $p \notin Var(\Gamma_1)$. Notice that, on the one hand, a. is satisfied because $Var(\Gamma \cup \{\varphi\}) \cap Var(\Gamma_1) = \emptyset$, also b. is vacuously satisfied, and more importantly c. is satisfied because $\Gamma_1 \nvdash_{\mathrm{L}^{pr}} \neg p$ since $Var(\neg p) \nsubseteq Var(\Gamma_1)$ by assumption. Now, observe that the cancellation property is not predicated of L^{pr} since the above are satisfied and nevertheless we have $p, \Gamma_1 \vdash_{\mathrm{L}^{pr}} \neg p$, and yet $p \nvdash_{\mathrm{L}^{pr}} \neg p$. Thus, L^{pr} does not have the cancellation property.

From right to left: by applying the ideas of Observation 1 in [8]. Now, by *reductio* assume that there are Γ, $\{\Gamma_i \mid i \in I\}$, and φ that satisfy items a. through c. of Definition 5, such that $\Gamma \cup \{\Gamma_i \mid i \in I\} \vdash_{\mathrm{L}^{pr}} \varphi$ though $\Gamma \nvdash_{\mathrm{L}^{pr}} \varphi$. Given this last bit, either $\Gamma \nvdash_{\mathrm{L}} \varphi$, or $Var(\varphi) \nsubseteq Var(\Gamma)$. If the latter, then since $\Gamma \cup \{\Gamma_i \mid i \in I\} \vdash_{\mathrm{L}^{pr}} \varphi$ we must conclude that $Var(\varphi) \cap Var(\bigcup\{\Gamma_i \mid i \in I\}) \neq \emptyset$. But this contradicts our assumption that Γ, $\{\Gamma_i \mid i \in I\}$, and φ satisfy items a. through c. of Definition 5. If the former, then by assumption there is a

$\mathcal{M}$-valuation v such that $v(\Gamma) \subseteq D$ while $v(\varphi) \notin D$. We also know that there is no $\mathcal{M}$-valuation v such that $v(\Gamma \cup \{\Gamma_i \mid i \in I\}) \subseteq D$ while $v(\varphi) \notin D$. But, since $Var(\Gamma \cup \{\varphi\}) \cap Var(\bigcup\{\Gamma_i \mid i \in I\}) = \emptyset$ by assumption, this guarantees that there is no $\mathcal{M}$-valuation v such that $v(\Gamma \cup \{\Gamma_i \mid i \in I\}) \subseteq D$. Thus, $\{\Gamma_i \mid i \in I\}$ is an anti-theorem of L. $\square$

These results notwithstanding, we know given the work of Wojcicki in [23] that whenever we have a Tarskian logic L, there is a (possibly infinite) class of (possibly infinite) logical matrices $\mathbb{M}$ such that $\vdash_{\mathrm{L}^{pr}} = \vDash_{\mathbb{M}}$. However, even if the previous results show that there is no single finite characteristic matrix for L^{pr} under some circumstances, one still could ask what would the class of logical matrices that characterize it look like. To answering this, whenever L itself is characterizable through a single matrix, we devote ourselves next.

2.2 Infectious extensions

For the purpose we set ourselves to achieve, we first provide the semantics in each case with the help of the notion of an infectious extension of an algebra and of the corresponding logical matrix.

Definition 6. Let $\mathbf{A}$ be an algebra of type $\mathcal{L}$ with universe A, its *infectious extension* is the algebra $\mathbf{A}[e]$ of type $\mathcal{L}$ with universe $A \cup \{e\}$ defined such that for all n-ary operations $\P^{\mathbf{A}[e]}$: $\P^{\mathbf{A}[e]}(a_1, \ldots, a_n) = \P^{\mathbf{A}}(a_1, \ldots, a_n)$ if $\{a_1, \ldots, a_n\} \subseteq A$, and $\P^{\mathbf{A}[e]}(a_1, \ldots, a_n) = e$ otherwise.

Definition 7. Let $\mathcal{M} = \langle \mathbf{A}, D \rangle$ be an $\mathcal{L}$-matrix, its *truth-preserving infectious extension* is the matrix $\mathcal{M}[e] = \langle \mathbf{A}[e], D \rangle$, and its *meaningfulness-preserving infectious* extension is the matrix $\mathcal{M}^*[e] = \langle \mathbf{A}[e], A \rangle$—where $\mathbf{A}[e]$ is the infectious extension of $\mathbf{A}$.

This nomenclature will be discussed afterward, but for now, and with these tools at hand, we can describe the semantics for L^{pr} in case it has, or hasn't a single characteristic logical matrix.

Theorem 3. *Let $\mathcal{M} = \langle \mathbf{A}, D \rangle$ be an $\mathcal{L}$-matrix and let* L *be a logic such that $\vdash_{\mathrm{L}} = \vDash_{\mathcal{M}}$. If L^{pr} has the cancellation property, then $\vdash_{\mathrm{L}^{pr}} = \vDash_{\mathcal{M}[e]}$.*

Proof. Suppose L^{pr} has the cancellation property. We show that $\vdash_{\mathrm{L}^{pr}} = \vDash_{\mathcal{M}[e]}$ by first proving the left-to-right inclusion of this equality, followed by the corresponding right-to-left inclusion.

From right to left: suppose that $\Gamma \nvdash_{\mathrm{L}^{pr}} \varphi$. Then, either, $\Gamma \nvdash_{\mathrm{L}} \varphi$ or $Var(\varphi) \nsubseteq Var(\Gamma)$. If $\Gamma \nvdash_{\mathrm{L}} \varphi$, then there is a $\mathcal{M}$-valuation v such that $v(\Gamma) \subseteq D$ and $v(\varphi) \notin D$. By assumption, this is also a $\mathcal{M}[e]$-valuation, whence $\Gamma \nvDash_{\mathcal{M}[e]} \varphi$. If $Var(\varphi) \nsubseteq Var(\Gamma)$, then there is a propositional variable $x \in Var(\varphi)$ and $x \notin Var(\Gamma)$ and a $\mathcal{M}[e]$-valuation v such that $v(x) = e$. We also know thanks to Theorem 2, that L doesn't have any anti-theorems. So, we can extend v to a $\mathcal{M}[e]$-valuation v' such that $v'(\Gamma) \subseteq D$ and $v'(\varphi) = e$. But then, $\Gamma \nvDash_{\mathcal{M}[e]} \varphi$. So if $\Gamma \nvdash_{\mathrm{L}^{pr}} \varphi$ then, $\Gamma \nvDash_{\mathcal{M}[e]} \varphi$.

From left to right: suppose that $\Gamma \nvDash_{\mathcal{M}[e]} \varphi$. Then, there is a $\mathcal{M}[e]$-valuation v such that, $v(\Gamma) \subseteq D$ and $v(\varphi) \notin D$. Given this, either $v(\varphi) \neq e$ or $v(\varphi) = e$. If $v(\varphi) \neq e$, then $\Gamma \nvdash_{\mathrm{L}} \varphi$ and, therefore, $\Gamma \nvdash_{\mathrm{L}^{pr}} \varphi$. If $v(\varphi) = e$, then there is a propositional variable x such that $x \in Var(\varphi) \setminus Var(\Gamma)$—for otherwise there would be some $\gamma \in \Gamma$ for which $v(\gamma) = e$, and this goes against our supposition. Therefore, $Var(\varphi) \nsubseteq Var(\Gamma)$, and this implies that $\Gamma \nvdash_{\mathrm{L}^{pr}} \varphi$. So, if $\Gamma \nvDash_{\mathcal{M}[e]} \varphi$, then $\Gamma \nvdash_{\mathrm{L}^{pr}} \varphi$. □

Theorem 4. *Let $\mathcal{M} = \langle \mathbf{A}, D \rangle$ be an $\mathcal{L}$-matrix and let* L *be a logic such that $\vdash_{\mathrm{L}} = \vDash_{\mathcal{M}}$. Then, regardless of* L^{pr} *having the cancellation property, we have that $\vdash_{\mathrm{L}^{pr}} = \vDash_{\mathbb{M}[e]}$, where $\mathbb{M}[e] = \{\mathcal{M}[e], \mathcal{M}^*[e]\}$.*

Proof. Suppose, for reductio, that $\vdash_{\mathrm{L}^{pr}} \neq \vDash_{\mathbb{M}[e]}$. Then there are Γ and φ such that $\Gamma \vdash_{\mathrm{L}^{pr}} \varphi$ and $\Gamma \nvDash_{\mathbb{M}[e]} \varphi$, or $\Gamma \nvdash_{\mathrm{L}^{pr}} \varphi$ and $\Gamma \vDash_{\mathbb{M}[e]} \varphi$. If the former, then either (a) $\Gamma \nvDash_{\mathcal{M}[e]} \varphi$, or (b) $\Gamma \nvDash_{\mathcal{M}^*[e]} \varphi$. In case (a), there is a $\mathcal{M}[e]$-valuation v, such that $v(\Gamma) \subseteq D$ and $v(\varphi) \notin D$. If $v(\varphi) \notin D$, then either $v(\varphi) \neq e$ or $v(\varphi) = e$. If the former, then $\Gamma \nvDash_{\mathcal{M}} \varphi$ whence $\Gamma \nvdash_{\mathrm{L}^{pr}} \varphi$. If the latter, then there is a propositional variable $x \in Var(\varphi)$, such that there is a $\mathcal{M}^*[e]$-valuation v for which $v(x) = e$. By this, we know that $x \notin Var(\Gamma)$, for otherwise then there would be a γ such that $v(\gamma) = e$ and that contradicts our assumption. So $Var(\varphi) \nsubseteq Var(\Gamma)$. Therefore, $\Gamma \nvdash_{\mathrm{L}^{pr}} \varphi$. In case (b) there is a $\mathcal{M}^*[e]$-valuation v such that, $v(\Gamma) \subseteq A$ and $v(\varphi) = e$. In which case, we know that there is a propositional variable $x \in Var(\varphi)$ and $x \notin Var(\Gamma)$, for otherwise there would be a γ such that $v(\gamma) = e$ and that contradicts our assumption. Then we know that $Var(\varphi) \nsubseteq Var(\Gamma)$. Therefore, $\Gamma \nvdash_{\mathrm{L}^{pr}} \varphi$.

Now, if there are Γ and φ such that $\Gamma \nvdash_{\mathrm{L}^{pr}} \varphi$ and $\Gamma \vDash_{\mathbb{M}[e]} \varphi$, then either $\Gamma \nvdash_{\mathrm{L}} \varphi$ or $Var(\varphi) \nsubseteq Var(\Gamma)$. If the former, then there is a $\mathcal{M}[e]$-valuation v, such that $v(\Gamma) \subseteq D$ and $v(\varphi) \notin D$, but then $\Gamma \nvDash_{\mathcal{M}[e]} \varphi$ and this implies $\Gamma \nvDash_{\mathbb{M}[e]} \varphi$. If the latter, then there is a propositional variable $x \in Var(\varphi)$ and

$x \notin Var(\Gamma)$ and, consequently, there is a $\mathcal{M}[e]$-valuation v such that $v(x) = e$ and $v(\Gamma) \subseteq A$. This implies $v(\varphi) = e$, for we know that e behaves in an infectious manner. But then, $v(\Gamma) \subseteq A$ and $v(\varphi) = e$, which implies $\Gamma \nvDash_{\mathcal{M}^*[e]} \varphi$ and, thus, $\Gamma \nvDash_{\mathbb{M}[e]} \varphi$. □

Still, these results do not necessarily illuminate what is going on and how to understand these notions of logical consequences and how to philosophically interpret these newly added truth values. To this task, we devote ourselves in the next section.

2.3 Interpreting the semantics

A few significant things can be said about how to read logical consequences in the case of infectious extensions for logical matrices. When L^{pr} can be characterized in terms of a single logical matrix, then logical consequence for it amounts to the preservation of the original designated values in the original logical matrix for L, but now in the context of $\mathcal{M}[e]$ where propositions are also able to receive the truth-value e. This situation amounts to propositions being allowed to be considered nonsensical or meaningless—which behaves infectiously, as documented in [5, 8, 12, 13] among others. In this respect, since we can reasonably consider the designated values of the original matrix to represent a generalized notion of truth, we can refer to $\mathcal{M}[e]$ as a truth-preserving infectious extension of the original semantics. In other words, this extended matrix represents the idea of truth-preservation when nonsensical propositions are allowed to be around.

In this vein, too, when L^{pr} cannot be characterized in terms of a single logical matrix, then logical consequence for it amounts to conjugating two things, as presented in the corresponding class of logical matrices $\mathbb{M}[e]$ of infectious extensions. On the one hand, is preservation of the original designated values in the original logical matrix for L, but now in the context of $\mathcal{M}[e]$ where propositions are also able to receive the truth-value e. We previously referred to this as truth-preservation in the context of allowing for nonsensical, and consequently neither true nor false propositions. On the other hand, preservation from premises to conclusions of the truth-values from the original matrix, but now in the context of $\mathcal{M}^*[e]$. This latter property can be understood as the preservation of meaningfulness from premises to conclusions, inasmuch as e represents meaninglessness or nonsense, and the values appearing in the original semantics are all meant to represent different ways in which a proposition can be meaningful. Therefore, logical consequence as dictated by $\mathbb{M}[e]$, where

$\mathbb{M}[e] = \{\mathcal{M}[e], \mathcal{M}^*[e]\}$, amounts to the joint preservation of truth and meaningfulness from premises to conclusions, when nonsensical propositions are allowed to come into the picture.

3 Pure Refined companions

Below, we present finite characterizability results for refined pure right companions and, when unfruitful, characterizability of such companions in terms of sets of matrices. In doing this, we link our results to and generalize those of [20], and show how they subsume the semantics for Angell's AC presented in [7].

3.1 Characterizability

We generalize the previous characterizability results for refined pure companions, as follows. To ease the understanding of the following material, we recall what the definition of $\mathrm{L}^{pr\pm}$ is, below.

$$\Gamma \vdash_{\mathrm{L}^{pr\pm}} \varphi \quad \Longleftrightarrow \quad \begin{cases} \Gamma \vdash_{\mathrm{L}} \varphi, \text{ and} \\ Var^{+}(\varphi) \subseteq Var^{+}(\Gamma), \text{ and} \\ Var^{-}(\varphi) \subseteq Var^{-}(\Gamma) \end{cases}$$

Theorem 5. *Let $\mathcal{M} = \langle \mathbf{A}, D \rangle$ be an $\mathcal{L}$-matrix and let* L *be a logic such that $\vdash_{\mathrm{L}} = \vDash_{\mathcal{M}}$.* $\mathrm{L}^{pr\pm}$ *has the cancellation property if and only if* L *does not have anti-theorems.*

Proof. From left to right: take a logic L with anti-theorems, and consider its pure refined right companion $\mathrm{L}^{pr\pm}$. Let $\Gamma = p \vee \neg p$, $\varphi = p \wedge \neg p$, and let Γ_1 be one of the original system's anti-theorems such that $p \notin Var(\Gamma_1)$. Notice that, on the one hand, a. is satisfied because $Var(\Gamma \cup \{\varphi\}) \cap Var(\Gamma_1) = \emptyset$, also b. is vacuously satisfied, and more importantly c. is satisfied because $\Gamma_1 \nvdash_{\mathrm{L}^{pr\pm}} p \wedge \neg p$ since $Var(p \wedge \neg p) \nsubseteq Var(\Gamma_1)$ by assumption and, thus, $Var^{+}(p \wedge \neg p) \nsubseteq Var^{+}(\Gamma_1)$ and $Var^{-}(p \wedge \neg p) \nsubseteq Var^{-}(\Gamma_1)$. Now, observe that the cancellation property is not predicated of $\mathrm{L}^{pr\pm}$ since the above are satisfied and nevertheless we have $p \vee \neg p, \Gamma_1 \vdash_{\mathrm{L}^{pr\pm}} p \wedge \neg p$, and yet $p \vee \neg p \nvdash_{\mathrm{L}^{pr\pm}} p \wedge \neg p$. Thus, $\mathrm{L}^{pr\pm}$ does not have the cancellation property.

From right to left: similar to the proof of Theorem 2, replacing the case where $Var(\varphi) \nsubseteq Var(\Gamma)$ for the case where either $Var^{+}(\varphi) \nsubseteq Var^{+}(\Gamma)$ or $Var^{-}(\varphi) \nsubseteq Var^{-}(\Gamma)$. □

However, notice that—just like before—these results are not informative by themselves, in terms of what the corresponding semantics look like. In the unfruitful case, what would the semantics in terms of a class of matrices look like? And in the fruitful case what would a single matrix semantics look like? We provide the answer to these questions together with the philosophical interpretation of the implemented formal instruments next.

3.2 Refined infectious extensions

Definition 8. Given a set A such that $e \notin A$, let $A[+]$ be the set $A \cup (A \times \{e\})$, let $A[-]$ be the set $\{e\} \times (A \cup \{e\})$, and let $A[\pm]$ be the set $A[+] \cup A[-]$.

In what follows, we require that the algebras $\mathbf{A}$ we work with, are such that for all a there is at least one b for which $\neg^{\mathbf{A}} b = a$. This, admittedly, sets a limitation on the generality of the logics and the semantics that we are capable of encompassing with the results below—something we embrace, with the hope of later generalizing this further in future work.

Definition 9. Let $\mathbf{A}$ be an algebra of type $\mathcal{L}$ with universe A, and let the set $A[\pm]$ be as described above. The operators π_0 and π_1 work as follows. For $a \in A$, we have $\pi_0(a) = a$ and $\pi_1(a) = \neg^{\mathbf{A}} a$. For $a = \langle x, y\rangle \in A[\pm] \setminus A$, we have $\pi_0(a) = x$ and $\pi_1(a) = y$. Additionally, for $a \in A$ we let π_2 be the operator such that $\pi_2(a) = b$ where $\neg^{\mathbf{A}} b = a$. In case there are $b_1 \neq b_2$ such that for some a, $\neg^{\mathbf{A}} b_1 = \neg^{\mathbf{A}} b_2 = a$, we allow $\pi_2(a)$ to choose indistinctly between them.

Definition 10. Let $\mathbf{A}$ be an algebra of type $\mathcal{L}$ with universe A, and let $\mathbf{A}[e]$ be its infectious extension. Then, its *refined infectious extension* $\mathbf{A}[\pm]$ is the algebra of type $\mathcal{L}$ with universe $A[\pm]$ defined such that:

$$\neg^{\mathbf{A}[\pm]} a = \begin{cases} \langle \pi_1(a), \pi_0(a)\rangle & \text{if } a \in A[\pm] \setminus A \\ \neg^{\mathbf{A}} a & \text{otherwise} \end{cases}$$

and for all n-ary operations $¶^{\mathbf{A}[\pm]}$:

$$¶^{\mathbf{A}[\pm]}(a_1, \dots, a_n) = \begin{cases} \langle ¶^{\mathbf{A}[e]}(\pi_0(a_1), \dots, \pi_0(a_n)), \neg ¶^{\mathbf{A}[e]}(\pi_1(a_1), \dots, \pi_1(a_n))\rangle & \text{if some } a_i \in A[\pm] \setminus A \\ ¶^{\mathbf{A}}(a_1, \dots, a_n) & \text{otherwise} \end{cases}$$

Definition 11. Let $\mathcal{M} = \langle \mathbf{A}, D\rangle$ be an $\mathcal{L}$-matrix, its *truth-preserving refined infectious extension* is the matrix $\mathcal{M}[\pm] = \langle \mathbf{A}[\pm], D[+]\rangle$, and its *meaningfulness-preserving refined infectious extension* is the matrix $\mathcal{M}^*[\pm] = \langle \mathbf{A}[\pm], A[+]\rangle$.

This nomenclature will be discussed afterward, but for now, and with these tools at hand, we can describe the semantics for $\mathrm{L}^{pr\pm}$ in case it has, or hasn't a single characteristic logical matrix.

Lemma 6. Given a $\mathcal{M}[\pm]$-valuation v, if we have that $e \notin \pi_0(v(Var^+(\varphi)))$ and also that $e \notin \pi_1(v(Var^-(\varphi)))$, then for any $\mathcal{M}[\pm]$-valuation v' such that $\pi_0(v(Var^+(\varphi))) = \pi_0(v'(Var^+(\varphi)))$ and $\pi_1(v(Var^-(\varphi))) = \pi_1(v'(Var^-(\varphi)))$, we have that $\pi_0(v(\varphi)) = \pi_0(v'(\varphi))$.

Proof. By induction on the complexity of φ. □

Theorem 7. *Let $\mathcal{M} = \langle \mathbf{A}, D \rangle$ be an $\mathcal{L}$-matrix and let* L *be a logic such that $\vdash_{\mathrm{L}} = \vDash_{\mathcal{M}}$. If $\mathrm{L}^{pr\pm}$ has the cancellation property, then $\vdash_{\mathrm{L}^{pr\pm}} = \vDash_{\mathcal{M}[\pm]}$.*

Proof. Suppose $\mathrm{L}^{pr\pm}$ has the cancellation property. We show that $\vdash_{\mathrm{L}^{pr\pm}} = \vDash_{\mathcal{M}[\pm]}$ by first proving the left-to-right inclusion of this equality, followed by the corresponding right-to-left inclusion.

From left to right: by contraposition, assume $\Gamma \nvDash_{\mathcal{M}[\pm]} \varphi$. Then, there is a $\mathcal{M}[\pm]$-valuation v such that $v(\Gamma) \subseteq D[+]$ but $v(\varphi) \notin D[+]$. Now, either (i) $v(\Gamma) \subseteq D$ or (ii) $v(\gamma) \in D[+] \setminus D$ for some $\gamma \in \Gamma$. In case (i), if $v(\varphi) \in A$, then v is a $\mathcal{M}$-valuation, whence $\Gamma \nvdash_{\mathrm{L}} \varphi$ and thus $\Gamma \nvdash_{\mathrm{L}^{pr\pm}} \varphi$. If, on the other hand, $v(\varphi) \notin A$, we know that either $\pi_0(v(\varphi)) = e$, or $\pi_1(v(\varphi)) = e$. Be that as it may, this witnesses the fact that $Var(\varphi) \nsubseteq Var(\Gamma)$, whence $\Gamma \nvdash_{\mathrm{L}^{pr\pm}} \varphi$. Now, in case (ii), we know that $\pi_0(v(\Gamma)) \subseteq D$, and that either $\pi_0(v(\varphi)) = e$, or $\pi_0(v(\varphi)) \neq e$. If the former, then this witnesses the fact that $Var^+(\varphi) \nsubseteq Var^+(\Gamma)$ or $Var^-(\varphi) \nsubseteq Var^-(\Gamma)$, whence $\Gamma \nvdash_{\mathrm{L}^{pr\pm}} \varphi$. If the latter, then consider a $\mathcal{M}[\pm]$-valuation v' such that:

$$v'(p) = \begin{cases} \pi_0(v(p)) & \text{if } v(p) \in A[\pm] \setminus A \text{ and } p \in Var^+(\Gamma) \cup Var^+(\varphi) \\ \pi_2(\pi_1(v(p))) & \text{if } v(p) \in A[\pm] \setminus A \text{ and } p \in Var^-(\Gamma) \cup Var^-(\varphi) \\ v(p) & \text{otherwise} \end{cases}$$

Appealing to Lemma 6, by induction on the complexity of the formula, it can be shown that $\pi_0(v(\gamma)) = \pi_0(v'(\gamma))$ for all $\gamma \in \Gamma$ and that $\pi_0(v(\varphi)) = \pi_0(v'(\varphi))$. But v' restricted to Γ and φ is a $\mathcal{M}$-valuation witnessing $\Gamma \nvDash_{\mathcal{M}} \varphi$, whence $\Gamma \nvdash_{\mathrm{L}} \varphi$ and thus $\Gamma \nvdash_{\mathrm{L}^{pr\pm}} \varphi$.

From right to left: by contraposition, assume that $\Gamma \nvdash_{\mathrm{L}^{pr\pm}} \varphi$. Then, either (i) $\Gamma \nvdash_{\mathrm{L}} \varphi$, or (ii) $Var^+(\varphi) \nsubseteq Var^+(\Gamma)$ or $Var^-(\varphi) \nsubseteq Var^-(\Gamma)$. In case (i), we know there is a $\mathcal{M}$-valuation v witnessing this fact such that $v(\Gamma) \subseteq D$ and $v(\varphi) \notin D$. But this is also a $\mathcal{M}[\pm]$-valuation, whence $\Gamma \nvDash_{\mathcal{M}[\pm]} \varphi$. In case (ii), then there is a propositional variable $p \in Var^+(\varphi) \setminus Var^+(\Gamma)$, or $p \in Var^-(\varphi) \setminus Var^-(\Gamma)$. Now, since L has no anti-theorems, consider a $\mathcal{M}$-valuation v such that $v(\Gamma) \subseteq D$ and then build the following $\mathcal{M}[\pm]$-valuation v' based on it:

$$v'(p) = \begin{cases} \langle e, e\rangle & \text{if } p \in Var^+(\varphi) \setminus Var^+(\Gamma) \text{ and } p \in Var^-(\varphi) \setminus Var^-(\Gamma) \\ \langle e, \pi_1(v(p))\rangle & \text{if } p \in Var^+(\varphi) \setminus Var^+(\Gamma) \text{ and } p \notin Var^-(\varphi) \setminus Var^-(\Gamma) \\ \langle \pi_0(v(p)), e\rangle & \text{if } p \notin Var^+(\varphi) \setminus Var^+(\Gamma) \text{ and } p \in Var^-(\varphi) \setminus Var^-(\Gamma) \\ v(p) & \text{otherwise} \end{cases}$$

Appealing to Lemma 6, by induction on the complexity of the formula, it can be shown that $\pi_0(v(\gamma)) = \pi_0(v'(\gamma))$ for all $\gamma \in \Gamma$, whence $v'(\Gamma) \subseteq D[+]$. On the other hand, the above allows to show that $\pi_0(v'(\varphi)) = e$, from which we can infer that $v'(\varphi) \notin D[+]$. Therefore, $\Gamma \nvDash_{\mathcal{M}[\pm]} \varphi$. □

Theorem 8. *Let $\mathcal{M} = \langle \mathbf{A}, D\rangle$ be an $\mathcal{L}$-matrix and let* L *be a logic such that $\vdash_{\mathrm{L}} = \vDash_{\mathcal{M}}$. Then $\vdash_{\mathrm{L}^{pr\pm}} = \vDash_{\mathbb{M}[\pm]}$, where $\mathbb{M}[\pm] = \{\mathcal{M}[\pm], \mathcal{M}^*[\pm]\}$.*

Proof. Suppose, for reductio, that $\vdash_{\mathrm{L}^{pr\pm}} \neq \vDash_{\mathbb{M}[\pm]}$. Then there are Γ and φ such that $\Gamma \vdash_{\mathrm{L}^{pr\pm}} \varphi$ and $\Gamma \nvDash_{\mathbb{M}[\pm]} \varphi$, or $\Gamma \nvdash_{\mathrm{L}^{pr\pm}} \varphi$ and $\Gamma \vDash_{\mathbb{M}[\pm]} \varphi$. If the former, then either (a) $\Gamma \nvDash_{\mathcal{M}[\pm]} \varphi$, or (b) $\Gamma \nvDash_{\mathcal{M}^*[\pm]} \varphi$. In case (a), there is a $\mathcal{M}[\pm]$-valuation v, such that $v(\Gamma) \subseteq D[\pm]$ and $v(\varphi) \notin D[\pm]$. If v is also a $\mathcal{M}$-valuation, then $\Gamma \nvDash_{\mathrm{L}} \varphi$, whence $\Gamma \nvdash_{\mathrm{L}^{pr\pm}} \varphi$. If v is not a $\mathcal{M}$-valuation, then either (i) $\pi_1(v(\gamma)) = e$ for some $\gamma \in \Gamma$, or (ii) $\pi_1(v(\varphi)) = e$, or (iii) or $\pi_0(v(\varphi)) = e$. In case (iii), we can guarantee to show that either $Var^+(\varphi) \nsubseteq Var^+(\Gamma)$ or $Var^-(\varphi) \nsubseteq Var^-(\Gamma)$, whence $\Gamma \nvdash_{\mathrm{L}^{pr\pm}} \varphi$. In cases (i) and (ii), without loss of generality, consider a $\mathcal{M}[\pm]$-valuation v' such that:

$$v'(p) = \begin{cases} \pi_0(v(p)) & \text{if } v(p) \in A[\pm] \setminus A \text{ and } p \in Var^+(\Gamma) \cup Var^+(\varphi) \\ \pi_2(\pi_1(v(p))) & \text{if } v(p) \in A[\pm] \setminus A \text{ and } p \in Var^-(\Gamma) \cup Var^-(\varphi) \\ v(p) & \text{otherwise} \end{cases}$$

Appealing to Lemma 6, by induction on the complexity of the formula, it can be shown that $\pi_0(v(\gamma)) = \pi_0(v'(\gamma))$ for all $\gamma \in \Gamma$ and that $\pi_0(v(\varphi)) = \pi_0(v'(\varphi))$.

But v' restricted to Γ and φ is a $\mathcal{M}$-valuation witnessing $\Gamma \nvDash_{\mathcal{M}} \varphi$, whence $\Gamma \nvdash_{\mathrm{L}} \varphi$ and thus $\Gamma \nvdash_{\mathrm{L}^{pr\pm}} \varphi$. Finally, in case (b) above, there is a $\mathcal{M}^*[\pm]$-valuation v for which $v(\Gamma) \subseteq A[+]$ while $v(\varphi) \notin A[+]$. By this, we know that either (i) there is a propositional variable $p \in Var(\varphi) \setminus Var(\Gamma)$ such that $\pi_0(v(p)) = e$, or (ii) there is a propositional variable $p \in Var(\varphi) \setminus Var(\Gamma)$ such that $\pi_1(v(p)) = e$. Be that as it may, we arrive at the fact that $\Gamma \nvdash_{\mathrm{L}^{pr\pm}} \varphi$.

Now, if there are Γ and φ such that $\Gamma \nvdash_{\mathrm{L}^{pr\pm}} \varphi$ and $\Gamma \vDash_{\mathbb{M}[\pm]} \varphi$, then either $\Gamma \nvdash_{\mathrm{L}} \varphi$ or either $Var^+(\varphi) \nsubseteq Var^+(\Gamma)$ or $Var^-(\varphi) \nsubseteq Var^-(\Gamma)$. If the former, then there is a $\mathcal{M}$-valuation v, such that $v(\Gamma) \subseteq D$ and $v(\varphi) \notin D$, and this implies $\Gamma \nvDash_{\mathbb{M}[\pm]} \varphi$. If the latter, then there is a propositional variable $p \in Var^+(\varphi) \setminus Var^+(\Gamma)$, or $p \in Var^-(\varphi) \setminus Var^-(\Gamma)$. Now, consider the following $\mathcal{M}^*[\pm]$-valuation v, where $a \in A$:

$$v(p) = \begin{cases} \langle e, e\rangle & \text{if } p \in Var^+(\varphi) \setminus Var^+(\Gamma) \text{ and } p \in Var^-(\varphi) \setminus Var^-(\Gamma) \\ \langle e, a\rangle & \text{if } p \in Var^+(\varphi) \setminus Var^+(\Gamma) \text{ and } p \notin Var^-(\varphi) \setminus Var^-(\Gamma) \\ \langle a, e\rangle & \text{if } p \notin Var^+(\varphi) \setminus Var^+(\Gamma) \text{ and } p \in Var^-(\varphi) \setminus Var^-(\Gamma) \\ a & \text{otherwise} \end{cases}$$

By an easy induction on the logical complexity of the formulas, we can guarantee that $v(\Gamma) \subseteq A[+]$ while $v(\varphi) \notin A[+]$. This implies $\Gamma \nvDash_{\mathcal{M}^*[\pm]} \varphi$ and, thus, $\Gamma \nvDash_{\mathbb{M}[\pm]} \varphi$. □

Once again, these results do not necessarily illuminate what is going on and how to understand these notions of logical consequences and how to philosophically interpret these newly added truth values. To this task, we devote ourselves in the next section.

3.3 Interpreting the semantics

For the task at hand, we borrow inspiration from the nine-valued semantics for the pure refined right companion of $\mathsf{E_{fde}}$ presented by Ferguson in [10], that is to say, the (first-degree fragment of the) logic AC by R. B. Angell. In this nine-valued semantics, we have two entries for the semantic information regarding a proposition: what the semantic status of this proposition is, and what the semantic status of its negation is—both regarded as independent matters. In this context, Ferguson allows for the semantic status of each item to be one of three between truth, false, and also meaningless or nonsensical. It's worth noting, then, that propositions can be either of these three semantic statuses and their negations can independently be this way, too. In the aforementioned semantics for AC, we, therefore, have nine combinations available, corresponding to the nine semantic values referred to previously. Motivations for embracing a semantic division of this kind can perhaps be borrowed from the fact that the literature sometimes regards meaningless propositions as having nevertheless meaningful negations—as in saying that "Colorless greens ideas sleep furiously" is meaningless whereas "It is not the case that colorless greens ideas sleep furiously" is a meaningful and true proposition. Although we do not push forward this narrative to its full extent, maybe a modification of this motivation calls to be implemented to grant the reasonableness of having the aforementioned picture.

In the case of our proposed semantics, however, we will only loosely understand them according to the model of Ferguson. If we look at the carrier set of the refined infectious extension of a given algebra, we will see that we are including all the original values alongside new values that have the form of pairs. Of these, some have original values as their first coordinate and an infectious value as their second coordinate, some have an infectious value as their first coordinate and an original value as their second coordinate, and one of them is the infectious value of the resulting algebra—represented by the pair counting with the element e as both its first and second coordinate. We may, in light of the reflections on Ferguson's semantics from before, understand all these elements as representing, on the one hand, propositions that have original values, and therefore the negations would have the corresponding value assigned to them in the original semantics, or on the other hand, as representing propositions that have original values but their negations are nevertheless nonsensical—or, alternatively, the other way around. Finally, we could also conceive the case of a completely nonsensical proposition, referring with this to a proposition that is nonsensical and also has a nonsensical negation.

In this context, if $\mathrm{L}^{pr\pm}$ has a single characteristic matrix, we can interpret $\mathcal{M}[\pm]$ as representing the notion of truth preservation while also allowing for nonsensical propositions to be around, although this time in a very different manner than what we allowed for in the case of $\mathcal{M}[e]$. Notably, since we allow for propositions to receive all sorts of truth values while their negations are nonsensical, and the other way around, we may also account for the preservation of truth in this refined case. This is what is represented by the membership in the set $D[+]$.

If, on the other hand, $\mathrm{L}^{pr\pm}$ does not have a single characteristic matrix, we needed to understand its notion of logical consequence as the conjugation of two different things, as dictated by the matrix class $\mathbb{M}[\pm]$. First, the previously discussed requirement of truth preservation while meaningless propositions are around, with all the subtleties of the present case. Secondly, a new and refined form of meaningfulness preservation from premises to conclusions. This new phenomenon, represented by the logical consequence in the matrix $\mathcal{M}^*[\pm]$, accounts for the fact that propositions can be meaningful in the context of refined infectious extensions even though their negations may not be so. This situation is depicted by membership in the set $A[+]$.

4 Epilogue: signed infectious extensions

In order to close this article, let us point to one interesting connection to a recent piece of literature. In his recent work [20], Randriamahazaka defines what he calls signed infectious extensions of a given algebra, with the help of which he shows it is possible to analyze certain variable inclusion systems related to pure companions. Thus, let us first provide his definitions in order to later highlight the connection to our previously discussed notion of refined infectious extension.

Definition 12. Let the signed infectious algebra be the structure counting with elements $\{\div, +, -, \times\}$ and the following two operations:

	ι
$\div$	$\div$
$+$	$-$
$-$	$+$
$\times$	$\times$

$\sqcup$	$\div$	$+$	$-$	$\times$
$\div$	$\div$	$+$	$-$	$\times$
$+$	$+$	$+$	$\times$	$\times$
$-$	$-$	$\times$	$-$	$\times$
$\times$	$\times$	$\times$	$\times$	$\times$

Furthermore, let $\bigsqcup\{a_1, \ldots, a_n\}$ stand in for $a_1 \sqcup \cdots \sqcup a_n$ and, when $\{a_1, \ldots, a_n\} = \{a\}$ let $\sqcup\{a\} = a$.

Definition 13. Let $\mathbf{A}$ be an algebra of type $\mathcal{L}$ with universe A, its *signed infectious extension* is the algebra $\mathbf{A}^s$ of type $\mathcal{L}$ with universe $A \cup \{+, -, \times\}$ defined such that for all n-ary operations $\P^{\mathbf{A}^s}$ except for negation: $\P^{\mathbf{A}^s}(a_1, \ldots, a_n) = \P^{\mathbf{A}}(a_1, \ldots, a_n)$ if $\{a_1, \ldots, a_n\} \subseteq A$, and $\P^{\mathbf{A}^s}(a_1, \ldots, a_n) = (\bigsqcup\{a_1, \ldots, a_n\}) \cap \{+, -, \times\}$ otherwise, where $\bigsqcup$ is the operation of the signed infectious algebra. Negation, in turn, behaves as in $\mathbf{A}$ for elements in A and as ι for $+, -, \times$.

In this respect, it is fairly simple to observe that the elements of the signed infectious algebra can be seen to represent certain collections of elements of the refined infectious extension $\mathbf{A}[\pm]$ of a given algebra $\mathbf{A}$. To wit, the element $\div$ serves as a surrogate for the elements in the original carrier set A, the element $+$ as a surrogate for the elements in the set $A[\pm]$ that are not in A which have an element of A as a first coordinate, whereas the element $-$ stands in for the elements in the set $A[\pm]$ that are not in A which have an element of A as a second coordinate and, finally, the element $\times$ represents the infectious value $\langle e, e\rangle$. All this can be verified by noting that there is a homomorphism from $\mathbf{A}[\pm]$ to $\mathbf{A}^s$ respecting these mappings—something we leave to the reader as an exercise.

These observations amount to establishing a different characterization of the pure refined right variable inclusion companion of a logic, now in terms of signed infectious extensions of logical matrices instead of refined infectious extensions of logical matrices, like we did in the previous sections. In light of the embedding mentioned in the paragraph before, it should come as no surprise that we can prove the results below, as they state in a different fashion that logical consequence for a pure refined companion can be equated to the concurrent phenomena of truth preservation and meaningfulness preservation from premises to conclusion.

Definition 14. Let $\mathcal{M} = \langle \mathbf{A}, D\rangle$ be an $\mathcal{L}$-matrix, its *signed infectious* extension is the matrix $\mathcal{M}^s = \langle \mathbf{A}^s, A \cup \{+\}\rangle$.

Theorem 9. *Let $\mathcal{M} = \langle \mathbf{A}, D\rangle$ be an $\mathcal{L}$-matrix and let* L *be a logic without anti-theorems such that $\vdash_{\mathrm{L}} = \vDash_{\mathcal{M}}$. Then, regardless of $\mathrm{L}^{pr\pm}$ having the cancellation property, we have that $\vdash_{\mathrm{L}^{pr\pm}} = \vDash_{\mathbb{M}^s}$—where $\mathbb{M}^s = \{\mathcal{M}, \mathcal{M}^s\}$.*

Proof. Suppose, for reductio, that $\vdash_{\mathrm{L}^{pr\pm}} \neq \vDash_{\mathbb{M}^s}$. Then there are Γ and φ such that $\Gamma \vdash_{\mathrm{L}^{pr\pm}} \varphi$ and $\Gamma \nvDash_{\mathbb{M}^s} \varphi$, or $\Gamma \nvdash_{\mathrm{L}^{pr\pm}} \varphi$ and $\Gamma \vDash_{\mathbb{M}^s} \varphi$. If the former, then either (a) $\Gamma \nvDash_{\mathcal{M}} \varphi$, or (b) $\Gamma \nvDash_{\mathcal{M}^s} \varphi$. In case (a), there is a $\mathcal{M}$-valuation v, such that $v(\Gamma) \subseteq D$ and $v(\varphi) \notin D$. Then $\Gamma \nvdash_{\mathrm{L}} \varphi$, whence $\Gamma \nvdash_{\mathrm{L}^{pr\pm}} \varphi$. In case (b), there is a $\mathcal{M}^s$-valuation v for which $v(\Gamma) \subseteq A \cup \{+\}$ while $v(\varphi) \in \{-, \times\}$. By this, we know that either (i) there is a propositional variable $x \in Var(\varphi) \setminus Var(\Gamma)$ such that $v(x) = \times$, or (ii) there is a propositional variable $x \in Var^+(\varphi) \setminus Var^+(\Gamma)$ such that $v(x) = -$, or (iii) there is a propositional variable $x \in Var^-(\varphi) \setminus Var^-(\Gamma)$ such that $v(x) = +$. Be that as it may, we arrive at the fact that $\Gamma \nvdash_{\mathrm{L}^{pr\pm}} \varphi$.

Now, if there are Γ and φ such that $\Gamma \nvdash_{\mathrm{L}^{pr\pm}} \varphi$ and $\Gamma \vDash_{\mathbb{M}^s} \varphi$, then either $\Gamma \nvdash_{\mathrm{L}} \varphi$ or either $Var^+(\varphi) \nsubseteq Var^+(\Gamma)$ or $Var^-(\varphi) \nsubseteq Var^-(\Gamma)$. If the former, then there is a $\mathcal{M}$-valuation v, such that $v(\Gamma) \subseteq D$ and $v(\varphi) \notin D$, and this implies $\Gamma \nvDash_{\mathbb{M}^s} \varphi$. If the latter, then there is a propositional variable $x \in Var^+(\varphi)$ and $x \notin Var^+(\Gamma)$, or $x \in Var^-(\varphi)$ and $x \notin Var^-(\Gamma)$. Now, consider the following $\mathcal{M}^s$-valuation v:

$$v(x) = \begin{cases} - & \text{if } x \in Var^+(\varphi) \setminus Var^+(\Gamma) \text{ and } x \notin Var^-(\varphi) \setminus Var^-(\Gamma) \\ + & \text{if } x \notin Var^+(\varphi) \setminus Var^+(\Gamma) \text{ and } x \in Var^-(\varphi) \setminus Var^-(\Gamma) \\ \times & \text{if } x \in Var^+(\varphi) \setminus Var^+(\Gamma) \text{ and } x \in Var^-(\varphi) \setminus Var^-(\Gamma) \\ a \in A & \text{otherwise} \end{cases}$$

By an easy induction on the logical complexity of the formulas, we can guarantee that $v(\Gamma) \subseteq A \cup \{+\}$ while $v(\varphi) \notin A \cup \{+\}$. This implies $\Gamma \nvDash_{\mathcal{M}^s} \varphi$ and, thus, $\Gamma \nvDash_{\mathbb{M}^s} \varphi$. □

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