A Recipe for Paradox

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Abstract

In this paper, we provide a recipe that not only captures the common structure of semantic paradoxes but also captures our intuitions regarding the relations between these paradoxes. Before we unveil our recipe, we first discuss a well-known schema introduced by Graham Priest, namely, the Inclosure Schema. Without rehashing previous arguments against the Inclosure Schema, we contribute different arguments for the same concern that the Inclosure Schema bundles together the wrong paradoxes. That is, we will provide further arguments on why the Inclosure Schema is both too narrow and too broad.

We then spell out our recipe. The recipe shows that all of the following paradoxes share the same structure: The Liar, Curry's paradox, Validity Curry, Provability Liar, Provability Curry, Knower's paradox, Knower's Curry, Grelling-Nelson's paradox, Russell's paradox in terms of extensions, alternative Liar and alternative Curry, and hitherto unexplored paradoxes.

We conclude the paper by stating the lessons that we can learn from the recipe, and what kind of solutions the recipe suggests if we want to adhere to the Principle of Uniform Solution.

There has long been an interest in identifying the common structure of the semantic and logical paradoxes, going back to Russell's vicious circle principle. In ([23], p. 224), Russell proposes that the Liar, Richard's Paradox, Burali-Forti's Paradox, and other paradoxes all share a common feature: self-reference or reflexiveness. They all involve a total collection where the total collection is a member of itself ([23], p. 225). For example, the set R, where R is the set of *all* sets that are not members of themselves, exhibits this feature; it is a total collection that must be a member of itself since it applies to itself. However, Russell's proposal is too broad. Any set that is a member of itself would fulfill the aforementioned feature, even if it is not paradoxical. Additionally, any truth-teller-like sentence (e.g., a sentence that says of itself that it is true) would fulfill Russell's vicious circle principle, but such sentences are not paradoxical ([1], p. 185).

An influential recent attempt is Graham Priest's Inclosure Schema [15]. Priest claims that Russell was on the right track, so Priest introduces his Inclosure Schema as a slight modification to Russell's attempt in [22]. However, Priest's Inclosure Schema has received many objections (see, [1–4, 7–11, 14, 28, 31]). A glut of these objections were due to the failure to include Curry's paradox—making the Inclosure Schema too narrow, and a heap of these objections were due to the inclusion of the Sorites—rendering the Inclosure Schema too broad.

The aim of this paper is to present a better method that highlights the common structure of semantic paradoxes and thereby captures our intuitions regarding the relations between these paradoxes. However, before introducing our recipe for semantic paradoxes, we will provide new arguments against the Inclosure Schema, supporting the claim that the Inclosure Schema is both too narrow and too broad (§1). We will first argue that the Liar and the Curry are patently related. We will then show that there is a way of forcing the Curry into the Inclosure Schema, but, as it will be clear, this comes at too high a cost. Finally, we show that the Sorites and even more distant paradoxes fit the Schema, in ways that threaten to trivialize it.

We will then put forward our own method of capturing the commonality between different paradoxes: The Recipe (§2). The Recipe reveals how various paradoxes share the same structure, including: The Liar, Curry's paradox, Validity Curry, Provability Liar, Provability Curry, Knower's Curry, Grelling-Nelson's paradox, Russell's paradox in terms of extensions, and other semantic paradoxes..

We finally present some lessons that we can learn from our recipe (§3): how to generate hitherto unexplored paradoxes, why the Sorites Paradox does not fall into the same family, what the Recipe has to say with regard to Truth-tellers and loops, and most importantly, what possible uniform solutions the Recipe suggests.

1 Problems with the Inclosure Schema

1.1 The Inclosure Schema

In an attempt to capture the underlying common structure of paradoxes, Graham Priest introduces the Inclosure Schema [15]. The Inclosure Schema involves a set Ω, two unary predicates φ and ψ, and a one place function δ that satisfy the following conditions:

1. There is a set Ω such that $\Omega = \{x : \varphi(x)\}$, and $\psi(\Omega)$
2. If $X \subseteq \Omega$ and $\psi(X)$, then
   (a) $\delta(X) \notin X$ (transcendence)
   (b) $\delta(X) \in \Omega$ (closure)

Take the Liar paradox as an example: Let Ω be the set of all true sentences. Let $\psi(X)$ be the property of being definable (in the sense that X has a name). Let δ be a function that takes a subset X of Ω to a sentence that says of itself that it is not in X. Now suppose that $\delta(X)$ is in X. Since it is in X, it is in

Ω, and hence, it is true. But $\delta(X)$ is the sentence that says that is not in X. So by the T-Schema, $\delta(X)$ is not in X—a contradiction. Hence, $\delta(X)$ is not in X (transcendence). By the T-Schema, '$\delta(X)$ is not in X' is true. But '$\delta(X)$ is not in X' is $\delta(X)$, hence $\delta(X)$ is true. In other words, $\delta(X) \in \Omega$—closure. The Liar occurs in the limit case when we let $X = \Omega$. As a result we get the contradiction: $\delta(\Omega) \in \Omega$ and $\delta(\Omega) \notin \Omega$.

Other paradoxes such as Russell's paradox, Grelling-Nelson's paradox, and, a more controversial case, the Sorites paradox [18], exhibit the same inclosure schema as the Liar, or so Priest claims. This raises the question: what about the Curry?

1.2 The Too Narrow Argument

1.2.1 The Liar and Curry are Patently Related

Curry's paradox and the Liar share very significant features. They both are self-referential, employ the truth predicate, rely on the T-schema, and crucially involve ungroundedness. As Burgis and Bueno put it, "When it comes to all versions of both paradoxes, the problem revolves around ungrounded truth talk that leads to triviality when it is fed into the Biconditional Truth Schema" ([7], p. 11).

Even an apparent difference—that the Curry is negation-free while the Liar is not—is illusory. More specifically, $\rightarrow \varphi$ is very negation-y. To see how negation figures in Curry sentences, consider what Greg Restall calls *Negative Modalities*:

> A one place connective m is a negative modality if and only if the rule
>
> $$\frac{A \vdash B}{mB \vdash mA}$$
>
> is valid for all formulae A and B ([21], p. 59).

It is not hard to see that negation is a negative modality:

$$\dfrac{\dfrac{A \vdash B}{\neg B, A \vdash}\,\neg\vdash}{\neg B \vdash \neg A}\,\vdash\neg$$

Similarly, $\rightarrow \varphi$ for any φ is a negative modality ([21], p. 60):[1]

$$\dfrac{\dfrac{A \vdash B \qquad \varphi \vdash \varphi}{B \rightarrow \varphi, A \vdash \varphi}\,\rightarrow\vdash}{B \rightarrow \varphi \vdash A \rightarrow \varphi}\,\vdash\rightarrow$$

[1]Restall uses different but equivalent conditional rules to show $\rightarrow \varphi$ is a negative modality:

$$\dfrac{\dfrac{A \vdash B \qquad B \rightarrow \varphi \vdash B \rightarrow \varphi}{B \rightarrow \varphi, A \vdash \varphi}\,\rightarrow \mathrm{E}}{B \rightarrow \varphi \vdash A \rightarrow \varphi}\,\rightarrow \mathrm{I}$$

We will return to negative modalities later when we provide our schema. But for the moment we can see that $\to \varphi$ is not really negation-free—it is a negative modality after all.

There is further evidence that supports the claim that $\to \varphi$ is negation-y (i.e., Curry is not really negation-free):[2]

1. $\to \varphi$ obeys De Morgan's law:

   $(A \vee B) \to \varphi \dashv\vdash (A \to \varphi) \wedge (B \to \varphi)$

   $(A \wedge B) \to \varphi \dashv\vdash (A \to \varphi) \vee (B \to \varphi)$

2. $\to \varphi$ behaves in the way that triple negation does:

   $((A \to \varphi) \to \varphi) \to \varphi \dashv\vdash A \to \varphi$

3. $\to \varphi$ obeys the law of excluded middle:

   $\vdash A \vee (A \to \varphi)$

4. $\to \varphi$ partially behaves in the way that double negation does:

   $A \vdash (A \to \varphi) \to \varphi$

So, $\to \varphi$ shares nearly all of the same properties that negation has. The only exception is that $\to \varphi$ is missing double negation elimination (i.e., $(A \to \varphi) \to \varphi \vdash A$). However, not all negations have double negation elimination. The negation in intuitionistic logic lacks double negation elimination, but we would not want to say that intuitionistic logic lacks a negation.

1.2.2 The Curry and the Inclosure Schema

Given the similarities between the Liar and the Curry, the Curry should be of the same family as the Liar, and hence, should fit the Inclosure Schema. However, Weber et al. [30] claim that the Curry is not an inclosure paradox. That is because the Curry instantiates the Inclosure Schema when the consequent of the Curry sentence is false (or absurd), but not when the consequent is true. To clarify, let us see how a Curry sentence with a false consequent fits the Inclosure Schema. Let s_1 be "if s_1 is true, then it is not the case that snow is white":

Let $\varphi(y) :=$ 'Ty'
$\psi(x) :=$'x is definable'
$\delta(X) := s_1$

[2] We owe Shay Logan for the following observations (via personal communication on February 9th, 2021). Similar observations are discussed in ([27], p. 1240) and hinted at in ([21], p. 201).

> where s_1 is a sentence of the form $\langle s_1 \in X \to \neg A \rangle$.
> Let $\Omega = \{y : y \text{ is true}\}$
>
> The function $\delta(x)$ is clearly defined when $\psi(x)$ holds. The set Ω exists since it is a set of sentences of some (countable) language, and it is obvious that $\psi(\Omega)$. Take an X such that $X \subseteq \Omega$. Suppose that $\delta(X) = s_1$ and $s_1 \in X$. Then, it follows that s_1 is true, and so $s_1 \in X \to \neg A$, which gives $\neg A$ by modus ponens. Discharging the supposition, $s_1 \in X \to \neg A$; that is, Ts_1; that is $s_1 \in \Omega$ (closure). But since $s_1 \in X \to \neg A$ is true, if $s_1 \in X$ then $\neg A \in \Omega$, which it is not [since snow is white]. Hence, by contraposition, $s_1 \notin X$ (transcendence). The limit contradiction arises when $X = \Omega$. Then $s_1 \in \Omega$ and $s_1 \notin \Omega$ where $\delta(\Omega) = s_1$ ([30], p. 822-823).[3]

Weber et al. point out that when we have a Curry sentence with a true consequent, we cannot make the contraposition move above since the negation of "snow is white" is not in Ω, and hence, cannot achieve transcendence for a Curry sentence with a true consequent. They claim that since a Curry sentence with a true consequent does not fit the Inclosure Schema, Curry's paradox should not be considered an inclosure paradox. Additionally, they claim that "Curry's paradox has nothing essentially to do with negation: it would arise even if the language were entirely positive" ([30], p. 823-824). It is true that one can construct a Curry's paradox in a negation-free language. However, as we have already shown, it is not true that Curry's paradox has nothing intrinsically to do with negation.

There is indeed a problem of fit between the Curry and the Inclosure Schema. But it is not the one Weber et al. suggest. The real problem is the need to import outside empirical information.[4] Suppose we allow this. Then we can, pace Weber et al., fit the true-consequent Curry into the Schema. Here's how: Let s_2 be $\langle s_2 \in X \to A \rangle$ ("If s_2 is true, then snow is white"). We start by carrying out the Inclosure Schema for the Curry sentence with a negated consequent of s_2, namely, s_1 as displayed above. From the reasoning about s_1, we have closure for s_1, namely $s_1 \in \Omega$, and we also have 'if $s_1 \in \Omega$ then $\neg A \in \Omega$'. By modus ponens, we get $\neg A \in \Omega$, and hence $A \notin \Omega$, and thus we can use this in s_2 (along with 'if $s_2 \in X$ then $A \in \Omega$') in order to make the contraposition move and get transcendence—$s_2 \notin X$. Therefore, Curry sentences with true consequents can fit the Inclosure Schema as long as we work with the Curry sentences with false consequents first. In both proofs we rely on "outside" information. In the first proof, we relied on the information that 'snow is white' is true to achieve transcendence, while in the second proof, we used $\neg A \in \Omega$ from the first proof in order to achieve transcendence.

[3] This quote has been modified to avoid use/mention confusions. For example, they let s be $\langle s \in \dot{x} \to \neg A \rangle$ where $\dot{x}$ is a name for x. The $\dot{x}$ involves some use/mention confusion.

[4] Priest acknowledges the problem of importing outside information in ([20], p. 119), and hence he rejects the claim that the Curry with false consequents fit the Inclosure Schema.

This result should not be surprising. After all, we can get the empty sequent from two Curry sentences. For instance, given s_1, we can prove that snow is not white ($\vdash \neg A$) via Curry's paradox. Similarly, given s_2, we can prove that snow is white ($\vdash A$) via Curry's paradox. Putting the two proofs together, we can use Cut to get to the empty sequent:

$$\dfrac{\dfrac{\vdots}{\vdash A} \qquad \dfrac{\dfrac{\vdots}{\vdash \neg A}}{A \vdash}{\scriptstyle \neg\vdash}}{\vdash}{\scriptstyle \mathrm{Cut}}$$

This further supports the claim that the Curry and the Liar should be considered as paradoxes of the same family. Yes we do need two Curry sentences in order to get to the empty sequent, but so do loop-liars. Loop-liars need at least two sentences to get to the empty sequent, yet we still count them as Liar-like paradoxes.

Nonetheless, this does not look good for the Inclosure Schema since the only way for the Curry to fit the Inclosure Schema—and hence conform to our intuitions on the relationship between the Liar and the Curry—is by requiring extra empirical assumptions. However, empirical truths have no place in logic, or in a theory of paradox. Neither a logical system nor a theory of paradox should be able to distinguish whether a sentence, or a consequent of a conditional in this case, is true or false.

Now, a proponent of the Inclosure Schema is left with three options:

1. Reject the claim that Curry's paradox fits the Inclosure Schema, even when the Curry sentence has a false consequent. Priest seems to lean that way ([20], p. 119).

2. Claim that Curry's paradox fits the Inclosure Schema, and bite the bullet on the ad hoc importation of outside information.

3. Reject the Inclosure Schema altogether.

The best option is to reject the Inclosure Schema altogether: If the Inclosure Schema does not accommodate Curry's paradox, even when we have a false consequent, then the Inclosure Schema is miscalibrated since the Liar and Curry are closely related. If Curry sentences with false consequents fit the Inclosure Schema, then Curry sentences with true consequents must fit the Inclosure Schema as well (as we have just shown). But the only way for a Curry sentence of either kind to fit the Inclosure Schema, is by importing outside assumptions, and that is a costly ask. To anticipate, the Recipe does not make any outside (or extra irrelevant) assumptions. The Recipe deals with the Curry even when the truth value of the consequent is left entirely unspecified.

One could argue that there is one more option available for the proponent of the Inclosure Schema, namely what Beall calls a sub-paradoxical account of (inclosure) paradoxicality.[5] The sub-paradoxical account says that "a sentential scheme is an inclosure paradox iff *some* instances are (inclosure) paradoxical" ([3], p. 843). Since Curry's paradox with false consequents fit the Inclosure Schema (and hence, *some* instances of the Curry fit the Inclosure Schema), we conclude that the Curry fits the Inclosure Schema. However, this is not a different option from option 2. In order for the "bad" Currys to fit the Inclosure Schema, we have to import outside information (i.e., we rely on the truth of certain empirical sentences). Plus, if "bad" Currys fit the inclosure, then so do "good" Currys as we have just shown. So then we might as well take the super-paradoxical approach—where we require all instances be inclosure paradoxes—since, indeed, all instances fit the Inclosure Schema if we allow importation of outside information.

A proponent of the Inclosure Schema might push back a little. They could grant that the only way for a Curry with a false consequent to fit the Inclosure Schema is by importing empirical information, and that is unacceptable. Hence, they could claim that we do not need Currys with false consequents to fit the inclosure Schema in order to call the Curry an inclosure paradox. After all, all we need is one instance of Curry to fit the Inclosure, and we have one available, namely what Beall calls "Neo-Curry" ([3], p. 844-846). Neo-Curry (e.g., $j := Tr(\ulcorner j \urcorner) \rightarrow (A \wedge \neg A)$) fits the Inclosure Schema without the need to import any empirical information. Hence, some instances of the Curry fit the Schema, and by embracing a sub-paradoxical account, we can argue that the Curry fits the Inclosure Schema (again, without importing any empirical information).

However, just because we are not importing any *empirical* information, it does not mean that we are not importing *extra irrelevant* information—that our logical system proves the negation of the consequent in j (i.e., it proves $\neg(A \wedge \neg A)$). As Priest notes, when we construct a Curry argument to reach an arbitrary conclusion, we do not utilize such extra irrelevant information ([20], p. 119). Even if the consequent of a Curry sentence is $(A \wedge \neg A)$, we do not utilize the fact that our logical system proves its negation – $\neg(A \wedge \neg A)$. In other words, when we construct the Curry, we do not utilize any facts (whether empirical or not) about the consequent in the Curry sentence. So why do we import such extra irrelevant information when we analyze whether the Curry argument fits the Inclosure Schema? The fact that we do not use such extra information in constructing the Curry indicates that the consequent of a Curry sentence is irrelevant to the Curry argument; it is merely a parameter as Priest ([20], p. 118) and Weber et al. ([30], p. 823) point out. Proving the consequent of a Curry sentence out of thin air is the problem. It is a problem even if that consequent was true.

Moreover, the sub-paradoxical approach trivializes the Inclosure Schema. Through its lens, the sentence-schema $B \rightarrow (A \wedge \neg A)$ is an inclosure paradox since $j := Tr(\ulcorner j \urcorner) \rightarrow (A \wedge \neg A)$ is an instance that fits the Inclosure Schema.[6]

[5] Thanks to an anonymous reviewer for emphasizing the need to consider this further option.

[6] Not only the sentence-schema $B \rightarrow (A \wedge \neg A)$ would count as an inclosure paradox, but also the sentence-schema with a tautology as a consequent (e.g., $B \rightarrow (A \vee \neg A)$) would count as

This renders the Inclosure Schema to admit way more sentence-schemas than it should. To escape this problem, one might want to modify the definition of sub-paradoxicality so that all instances of the sentence-schema must first be paradoxical. But then you run into having a double standard: for a sentence scheme to count as paradoxical, one must appeal to super-paradoxicality (*all* instances of the sentence-schema must be paradoxical), yet for a [paradoxical] sentence-scheme to count as inclosure paradoxical, we appeal to sub-paradoxicality—only *some* instances must fit the Inclosure Schema. But this is surely ad hoc.

The aforementioned issue shows one of the problems with thinking that a paradox is a sentence-schema. We agree with Priest on this one; a paradox is an argument, not a sentence.[7] There are paradoxical sentences such as a liar sentence, a Curry sentence, and so on, but these paradoxical sentences are not the paradoxes. They are harmless by themselves. The arguments that employ these paradoxical sentences to reach unwarranted conclusions are the paradoxes.

1.3 The Too Broad Argument

On the other end of the spectrum, we will argue that the Inclosure Schema is too broad. One indication that it is too broad is provided by Priest himself—the Sorites fits the Inclosure Schema, yet the Sorites seems far removed from the Liar and the other paradoxes of self-reference.

1.3.1 The Sorites Paradox and the Inclosure Schema

To see how the Sorites arguably fits into the Inclosure Schema, let P be a vague predicate (e.g., 'is a heap' or 'is bald'). Let $\Omega = \{x : Px\}$ and $a_0, ..., a_n$ be a sequence of objects where at least a_0 is P and at least a_n is not P. If $X \subseteq \Omega$ then there must be a first member of the sequent not in X; let this be $\delta(X)$. Therefore, we have transcendence, since we defined $\delta(X)$ to be the first member not in X (i.e., $\delta(X) \notin X$). Now since all members of X are P and $\delta(X)$ comes right after them, $\delta(X)$ is P due to the principle of tolerance—if a member of a successive sequence satisfy P, then its successor also satisfies P. Thus, $\delta(X) \in \Omega$ (closure). Hence, we have an inclosure contradiction of the form $\delta(\Omega) \in \Omega$ and $\delta(\Omega) \notin \Omega$ when $X = \Omega$ ([18], p. 70-71, [19], p. 368-369 [20], p. 122).

There is a debate on whether the Sorites actually fit the Inclosure Schema. For example, Beall argues that if the Curry is not an inclosure paradox because true-consequent Currys do not fit the Inclosure Schema (or so Weber et al. claim), then the Sorites should not count as an inclosure paradox for similar reasons ([3], p. 847).[8] He concludes that,

an inclosure paradox. We simply use the method discussed above where we start by reaching transcendence and closure for sentence j, and then we carry out the second argument for the sentence $h := Tr(\ulcorner h \urcorner) \rightarrow (A \vee \neg A)$.

[7] For further arguments on why paradoxes are arguments and not sentences, see ([20], p. 115-119).

[8] Thanks to an anonymous reviewer for suggesting that we include this disclaimer.

> After all, what is bad about the sorites is not that it lands in absurdity; rather, it allows us to prove whatever we want from an appropriate soritical series—including the truth that Dave Ripley is tall, that 10K grains of sand is a heap, that Marcus Rossberg is bald (on top), that there are too few women in philosophy, and so on. To 'prove' as much, one merely needs to start with the right soritical series (i.e. staying at the 'safe side' of a typical soritical series)[...]But, then, applying the [Weber et al.'s] reply in the case of the sorites, we conclude that what is troubling about the sorites is not essentially the absurd conclusions that one can derive from the given reasoning (and appropriate starting class of objects); it is rather that one can prove anything—true or not—depending on which 'instances' (collections of objects) one starts with. But, then, this undercuts the inclosure argument for the sorites at precisely the same point at which [Weber et al.] arrest the corresponding argument for Currys paradox: namely, contraposition.

It is not clear that contraposition is used in the argument that purports to establish that the Sorites fit the Inclosure Schema. However, maybe Beall meant that in both cases, transcendence cannot be achieved when we look at the "good" versions of the paradox. Nevertheless, this is a false analogy. What is paradoxical about a Curry with true consequent is that we can prove the consequent out of thin air. On the other hand, what is paradoxical about the Sorites is reaching a contradiction or an unacceptable conclusion. To clarify, proving that Dave Ripley is tall using a soritical argument does not come out of thin air. We rely on certain empirical facts (or highly probable assumptions) to reach that conclusion (e.g., that LeBron James is tall, if LeBron James is tall, then Virgil van Dijk is tall, if Virgil van Dijk is tall, then Dave Ripley is tall...etc.). In other words, the proof is not purely logical as in the case of the Curry; in the Curry we can prove that Dave Ripley is tall in a purely logical proof without appealing to any of the aforementioned empirical facts.[9] Moreover, a proponent of the Inclosure Schema could argue that if you choose an object where the predicate does indeed apply to it (i.e., the predicate is true of that object), then you must run the argument from the opposite direction. Hence, the set Ω must be $\{x : \neg Px\}$. You can, then, get closure and transcendence. Therefore, what is paradoxical here is starting with the premise that Dave Ripley is tall yet we can prove that he is not tall.

In any case, we will not pursue the issue whether the Sorites fit the Inclosure Schema further. We will take the opening paragraph of this section (§1.3.1) at face value, and assume with Priest that the Sorites fits the Inclosure Schema. But this is clearly a disadvantage for the Inclosure Schema. The Sorites does not involve a truth predicate or the T-schema, and more importantly, it does not rely on self-referential or ungrounded sentences. This is a preliminary indication

[9] This, of course, applies to the Liar as well; we can prove anything in a purely logical proof (out of thin air) using the Liar argument. This is an initial indicator that the Sorites should not be counted as the same family of paradoxes as the Liar.

that the Inclosure Schema is too broad.

The too-broad problem can be seen in a more general way. In fact, the Inclosure Schema can be trivialized in two different, though related, ways.

1.3.2 The First Trivializing Case

In ([16], p. 148), Priest mentions a general case that could trivialize the Inclosure Schema: Take any contradiction of the form of $Pa \wedge \neg Pa$ where 'a' is a singular term referring to a particular object. Let Ω be the set of $\{y : Py\}$ and let $\psi(x)$ be $x = \Omega$. Let our δ be a constant function where $\delta(\Omega) = a$. Because of the first conjunct of the contradiction $Pa \wedge \neg Pa$, a is P, and hence, $\delta(\Omega) \in \Omega$ (closure). Similarly, because of the second conjunct of $Pa \wedge \neg Pa$, a is not P, and hence, $\delta(\Omega) \notin \Omega$ (transcendence). This example shows that any contradiction can be made to fit the Schema and hence the Schema doesn't capture the underlying structure of paradoxes.

Priest responds to this trivializing case as follows:

> "We want not just any old pattern, but the essential pattern...[For] genuine satisfaction of the Schema we need the fact that a contradiction fits the pattern to *explain* why the contradiction arises.[...And in the present case:] That the pattern is satisfied can hardly, therefore, be used to explain why the contradiction arises"([16], p. 149).

This suggests that fitting the schema is not enough to be counted as a paradox of the same type. So, by Priest's own admission, the schema itself does not distinguish between different patterns, many of which are unrelated to the target family of paradoxes. Priest requires more than the schema itself. One has to further check whether the fact that the contradiction fits the pattern explains why the contradiction arises. But that is a serious lacuna—one wants the schema to provide enough structure so that, by itself, it explains why the contradiction arises, and thereby sheds some light on the structure of paradoxes.

1.3.3 The Second Trivializing Case (Zeno's Paradox)

In the aforementioned general trivializing case, we did not utilize any subset X of Ω—we jumped straight to using Ω as an argument for our function δ. However, we can present another general trivializing case where we retain the subsets X of Ω. The trick is to let our function δ pick out a specific object O regardless of the input. In other words, let δ be a constant function (i.e., $\delta(X) =$ O for any X). Additionally, we need a set S and two prima facie convincing arguments, where one argument says O is in S and the other argument says O is not in S. Fulfilling these conditions will trivialize the roles that δ and X play in the Inclosure Schema. To illustrate this trivializing case, consider Zeno's paradox:

Take any version of Zeno's paradox, say, the Dichotomy version. Suppose there is a runner who wants to run a 100-meter race. In order to cover the whole 100-meter track, she must first cover the first half of the track (50 meters). In order to cover the first 50 meters, she must cover the first quarter of the track (25 meters), and so on. So for the runner to cover any distance, she must first cover half that distance, and so on, ad infinitum—the runner could not even begin to move. This suggests that motion is impossible, while there is another argument, an argument from common sense, which suggests that motion is possible and she would be able begin her run.

Now let Ω be the set of all moving objects. Let a be our 100-meter runner. For any $X \subseteq \Omega$, let $\delta(X) = a$. Now, the argument above—that motion is impossible—suggests that our runner is motionless. Hence, we have transcendence $\delta(X) \notin X$. However, there is another argument (from common sense) that shows that our runner can indeed begin to run. Therefore, $\delta(X) \in \Omega$ (closure). Thus, we have an inclosure contradiction of the form $\delta(\Omega) \in \Omega$ and $\delta(\Omega) \notin \Omega$ at the limit case when $X = \Omega$. This shows that the Inclosure Schema is too broad to capture a specific structure of paradox.

1.3.4 Zeno's Revenge

Both of the trivializing cases above have the feature where δ is a constant function, hence, it does not depend on its arguments. A proponent of the Inclosure Schema might claim that the value of the function δ must depend on the argument. Priest hints at the importance of that dependence in ([16], footnote 18, p. 149). But until we have an account of this dependence, the schema itself will not provide the structure of the target family of paradoxes. It is also worth noting that Priest himself takes the issue of dependence here to be "tricky and unresolved"([16], footnote 18, p. 149).

But having a dependency between the value of δ and its argument is not enough; there is still a way for Zeno's paradox to fit into the Inclosure Schema. Let Ω be the set of all non-moving objects. Let $X \subseteq \Omega$. Let $\delta(X)$ be the heaviest object outside of X. It is not an interesting or a relevant dependence, but the value of δ depends on its argument. Now, $\delta(X) \notin X$ by the definition of our function δ (transcendence). Now, $\delta(X)$ is either a non-moving object, hence, $\delta(X) \in \Omega$, or $\delta(X)$ is a moving object. If it is a moving object, then since Zeno's argument above applies to every object, $\delta(X)$ is motionless, and thus, $\delta(X) \in \Omega$. Via argument by cases, we get closure.[10] The upshot here is that not any dependency between the value of δ and its argument will do. The nature of that dependency would need to be spelled out. But in this, the Inclosure Schema stays silent.[11]

Formally speaking, there is nothing in the Inclosure Schema itself that requires the value of the δ function to depend on its argument, just as there is nothing that requires a contradiction fitting the pattern to explain why the contradiction arises. Priest himself requires more than the schema can supply by

[10] The Omnipotence Paradox can fit the Inclosure Schema in a similar fashion.

[11] In ([16], p. 143), Priest calls the δ function a diagonaliser—though he briskly points out that "a diagonaliser need not be defined literally by diagonalization" ([16], p. 143). The reason is that Priest takes both the Sorites and the Burali-Forti paradoxes as inclosure paradoxes, yet neither of them involves diagonalization.

itself. Until these requirements are built into the schema itself, the Inclosure Schema will remain too broad to capture the structure of a specific family of paradoxes. One would have hoped for more structure to be built into the Inclosure Schema itself. As we will shortly see, our recipe for paradox does exactly that—it builds in the kind of structure that limits the pattern to the family of paradox in question. For Priest, we would have to plug the right kind of paradox into the Schema before we can discern its structure. It is the job of the theory of paradox to tell us why and how something is paradoxical, not the other way around.

Given that there are not enough requirements built into the Inclosure Schema to prevent "any old pattern" from creeping in, and given that the Inclosure Schema cannot accommodate the Curry without an ad hoc importation of outside information, the best option is to reject the Inclosure Schema and use a different tool that does directly capture the underlying structure of a family of paradoxes.

2 The Recipe

2.1 A Partially Transparent Predicate

In order to obtain a recipe for paradox, let us start with two of the most familiar paradoxes: The Liar and the Curry. We do not need full transparency for the truth predicate in order to construct the Liar (nor it is needed for Curry). All we need is partial transparency provided by the following two rules:[12]

(T1) If $\vdash \varphi$ then $\vdash Tr(\ulcorner \varphi \urcorner)$
(T2) $Tr(\ulcorner \varphi \urcorner) \vdash \varphi$

Note that we do not have full transparency because the following derivation is not possible using (T1)+(T2) if either Γ or Δ are non-empty:

$$\frac{\Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta, Tr(\ulcorner \varphi \urcorner)}$$

We have nothing against going full transparency, but we want to work with the minimal requirements on the truth predicate that will allow the construction of the Liar and the Curry. In this way we can capture other paradoxes that use predicates where full transparency is not available. Nonetheless, we can see objections related to full transparency specifically to the rule above. For instance, suppose that color theory proves that snow is white ($CT \vdash Ws$). Given

[12] This is the reason why some logicians consider the Liar as a weaker version of the Knower. The Knower uses Necessitation which is equivalent to our (T1), while we usually use Tr-I when we construct the liar. Tr-I is, of course, stronger than Necessitation. For more on how the Liar is a weaker version of the Knower, see Murzi's [13].

full transparency, we can conclude that color theory proves that the sentence 'snow is white' is true ($CT \vdash Tr(\ulcorner Ws \urcorner)$). However, color theory does not concern itself with how the truth predicate behaves or with linguistic matters.

Given the partial transparency of the truth predicate (i.e., the rules (T1) and (T2)), we can construct the Liar and the Curry:

Liar

Let λ be $\neg Tr(\ulcorner \lambda \urcorner)$:

\begin{prooftree}
\AxiomC{}
\RightLabel{T2}
\UnaryInfC{$Tr(\ulcorner\lambda\urcorner) \vdash \lambda$}
\RightLabel{$\vdash \neg$}
\UnaryInfC{$\vdash \neg Tr(\ulcorner\lambda\urcorner), \lambda$}
\RightLabel{def of λ}
\UnaryInfC{$\vdash \lambda, \lambda$}
\RightLabel{Contraction}
\UnaryInfC{$\vdash \lambda$}
\AxiomC{}
\RightLabel{T2}
\UnaryInfC{$Tr(\ulcorner\lambda\urcorner) \vdash \lambda$}
\RightLabel{$\vdash \neg$}
\UnaryInfC{$\vdash \neg Tr(\ulcorner\lambda\urcorner), \lambda$}
\RightLabel{def of λ}
\UnaryInfC{$\vdash \lambda, \lambda$}
\RightLabel{Contraction}
\UnaryInfC{$\vdash \lambda$}
\RightLabel{T1}
\UnaryInfC{$\vdash Tr(\ulcorner\lambda\urcorner)$}
\RightLabel{$\neg \vdash$}
\UnaryInfC{$\neg Tr(\ulcorner\lambda\urcorner) \vdash$}
\RightLabel{def of λ}
\UnaryInfC{$\lambda \vdash$}
\RightLabel{Cut}
\BinaryInfC{$\vdash$}
\end{prooftree}

Hence, the empty sequent is reached by having a sentence that says of itself that it is not true. Similarly, we can prove anything out of thin air by having a sentence that says something along the lines 'if this sentence is true, then [plug in whatever you want to prove]':

Curry

Let κ be $Tr(\ulcorner \kappa \urcorner) \to \bot$:

\begin{prooftree}
\AxiomC{}
\RightLabel{T2}
\UnaryInfC{$Tr(\ulcorner\kappa\urcorner) \vdash \kappa$}
\RightLabel{Weakening}
\UnaryInfC{$Tr(\ulcorner\kappa\urcorner) \vdash \bot, \kappa$}
\RightLabel{$\vdash\to$}
\UnaryInfC{$\vdash Tr(\ulcorner\kappa\urcorner) \to \bot, \kappa$}
\RightLabel{Def of κ}
\UnaryInfC{$\vdash \kappa, \kappa$}
\RightLabel{Contraction}
\UnaryInfC{$\vdash \kappa$}
\end{prooftree}

Call this derivation D_0

\begin{prooftree}
\AxiomC{D_0}
\UnaryInfC{$\vdash \kappa$}
\AxiomC{D_0}
\UnaryInfC{$\vdash \kappa$}
\RightLabel{T1}
\UnaryInfC{$\vdash Tr(\ulcorner\kappa\urcorner)$}
\AxiomC{$\bot \vdash \bot$}
\RightLabel{$\to\vdash$}
\BinaryInfC{$Tr(\ulcorner\kappa\urcorner) \to \bot \vdash \bot$}
\RightLabel{Def of κ}
\UnaryInfC{$\kappa \vdash \bot$}
\RightLabel{Cut}
\BinaryInfC{$\vdash \bot$}
\end{prooftree}

In order to explain what causes the paradoxes, we need to revisit the notion of a negative modality.

2.2 Partial Negative Modalities

We introduced negative modalities earlier as follows:

Let m be a one place connective. m is said to be a negative modality if and only if for any A and any B:

$$\frac{A \vdash B}{mB \vdash mA}$$

is valid ([21],p. 59).

For our purposes, as will become clearer below, we do not need the full definition of a negative modality. We are only concerned with the bolded portion of a negative modality:

$$\frac{\boldsymbol{A} \vdash B}{mB \vdash \boldsymbol{mA}}$$

In other words, we are only concerned with:

$$\frac{A \vdash}{\vdash mA}$$

From now on, if $\vdash mA$ is validly deducible from $A \vdash$, we will call m a *partial* negative modality.

2.3 Spelling out The Recipe

The previous results show us exactly what we need to construct a paradox in a classical setting.[13] Here is what we need:
(1) A partially transparent predicate P. That is, a predicate P that follows two rules:[14]

[13]Note that even though all of the paradoxes mentioned in this paper are constructivisable, we still presume that we are working within a classical setting here. That is because, pace Tennant [29], there are unconstructivisable paradoxes; there are strictly classical paradoxes that fulfill the Recipe yet cannot be constructed in an intuitionistic setting (see manuscript on Unconstructivisable Paradoxes).

[14]Note that the converses:

(a*) If $\vdash P(\ulcorner\varphi\urcorner)$ then $\vdash \varphi$

(b*) $\varphi \vdash P(\ulcorner\varphi\urcorner)$

would also give rise to paradoxes. Thanks to an anonymous reviewer for pointing this out. David Ripley has also made similar remarks (via personal communication on April 14th, 2021). However, these two rules are a lot harder to motivate. We have yet to encounter a predicate that justifiably fit these two rules (with the exception of the truth predicate, since we generally take it to be fully transparent). As we will see, it is hard to justify (b*), for example, when we consider the provability and the knowability predicate.

(a) If $\vdash \varphi$ then $\vdash P(\ulcorner\varphi\urcorner)$

(b) $P(\ulcorner\varphi\urcorner) \vdash \varphi$

Alternatively, we can write the second rule (b) as follows:

$$\frac{\Gamma, \varphi \vdash \Delta}{\Gamma, P(\ulcorner\varphi\urcorner) \vdash \Delta}$$

(2) A partial negative modality m that uses such a predicate P. That is, a use of 'P' occurs in m:

$$\frac{A \vdash}{\vdash mA}$$

(3) A diagonal lemma that establishes a sentence S containing a use of 'P' where S is equivalent to its own partial negative modality. In other words, given the diagonal lemma, there is a sentence S where S is its own partial negative modality.[15]

To explain (2) and (3), let us see how the Liar and Curry fit in the Recipe. First, they both make use of the truth predicate where the truth predicate fulfills step (1). The partial negative modality used in the Liar can be found in the last step of the following proof:[16]

$$\dfrac{\dfrac{\lambda \vdash}{Tr(\ulcorner\lambda\urcorner) \vdash}\,\text{T2}}{\vdash \neg Tr(\ulcorner\lambda\urcorner)}\,\vdash\neg$$

Our A here is λ and our mA is $\neg Tr(\ulcorner\lambda\urcorner)$. So our m here is the predicate $\neg Tr(\ulcorner\urcorner)$.

While the partial negative modality used in Curry's can be found in the last step of the following proof:

$$\dfrac{\dfrac{\dfrac{\kappa \vdash}{Tr(\ulcorner\kappa\urcorner) \vdash}\,\text{T2}}{Tr(\ulcorner\kappa\urcorner) \vdash \bot}\,\text{Weakening}}{\vdash Tr(\ulcorner\kappa\urcorner) \to \bot}\,\vdash\to$$

[15] Keith Simmons, in [24] and [25], also ties the structure of paradox to diagonalization. Though Simmons, of course, does not develop the other ingredients of our recipe, or deal with the wide variety of paradoxes with which we deal.

[16] These proofs are not trying to prove something in our system, but rather, they are used to search for partial negative modalities.

Our A here is κ and our mA is $Tr(\ulcorner\kappa\urcorner) \rightarrow \bot$. So our m here is the predicate $Tr(\ulcorner\urcorner) \rightarrow \bot$. Note that using Weakening, we can put anything we want in the consequent and not just $\bot$. It would still turn out to be paradoxical. So, as we anticipated earlier, the Recipe deals with Curry sentences even if the truth-value of the consequent is left entirely unspecified.

This fulfills step (2) for the Liar and Curry.

Step (3): Using the diagonal lemma, let λ be its partial negative modality $\neg Tr(\ulcorner\lambda\urcorner)$ and let κ be its partial negative modality $Tr(\ulcorner\kappa\urcorner) \rightarrow \bot$.

Before we explain *why* these ingredients give rise to the Liar and the Curry, we will show how these ingredients are present in various semantic paradoxes.

2.4 Applying the Recipe to Various Paradoxes

We will now demonstrate that the Recipe applies to a bundle of paradoxes and not just paradoxes involving the truth predicate. As a start, let us take a look at the provability predicate and the knowability predicate.

2.4.1 Provability and Knowability

We will assume that the provability predicate is governed by the following rules:[17]

(P1) If $\vdash \varphi$ then $\vdash Bew(\ulcorner\varphi\urcorner)$
(P2) $Bew(\ulcorner\varphi\urcorner) \vdash \varphi$

and the predicate 'is known/is knowable' is governed by the following rules:

(K1) If $\vdash \varphi$ then $\vdash Knw(\ulcorner\varphi\urcorner)$
(K2) $Knw(\ulcorner\varphi\urcorner) \vdash \varphi$

(P1) says if a sentence is provable in our system, then it is reasonable to say *within* our system that it is provable. While (P2) says if something is provable, then it had better be true. Similarly, (K1) says if a sentence is proved to be true, then it is known, and (K2) says if a sentence is known, then it is true.

It is difficult to accept full transparency for the provability and the knowability predicates. For instance, if we have full transparency for the provability predicate, then we have $\varphi \vdash Bew(\ulcorner\varphi\urcorner)$. As a result, this suggests that every true sentence is provable. For example, if Goldbach's conjecture is true, then it is provable. However, Goldbach's conjecture might very well be unprovable. Similarly, if we have full transparency for the knowability predicate, then we would have to accept that every truth is known or knowable.

[17] See Boolos et al. [6], p. 233-235.

As with the Liar and the Curry, we can find partial negative modalities for the provability and the knowability predicates by replacing (T2) with (P2)/(K2). Using the diagonal lemma, we can let ϑ be its partial negative modality $\neg Bew(\ulcorner\vartheta\urcorner)$ for a Provability Liar and let γ be its partial negative modality $Bew(\ulcorner\gamma\urcorner) \to \bot$ for a Provability Curry.[18] Similarly, we can let σ be its partial negative modality $\neg Knw(\ulcorner\sigma\urcorner)$ for the Knower's paradox and let δ be its partial negative modality $Knw(\ulcorner\delta\urcorner) \to \bot$ for the Knowability Curry.[19]

The paradoxical proofs of the Provability Liar and the Knower's paradox mirror the Liar but with different clothes. For example, to construct the Provability Liar, use similar moves as the Liar, and simply replace all truth predicates with provability predicates and any occurrence of (T1) and (T2) with (P1) and (P2). Similarly, the Knowability Curry and the Provability Curry mirror Curry's paradox.

2.4.2 Validity

Although the validity predicate is two-place, the treatment of the validity Curry fits naturally into the Recipe:

Step 1

Let the validity predicate have the following rules:

(V1) If $\vdash \alpha \to \beta$ then $\vdash Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner)$
(V2) $Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner) \vdash \alpha \to \beta$

In fact (V1) and (V2) do not differ from Beall and Murzi's (VP) and (VD)[5]. We just pushed α towards β to form a conditional. In other words, (V1) and (V2) follow directly from (VP) and (VD) using $\vdash\to$.

Step 2

Let us find a partial negative modality that uses the validity predicate:

$$\dfrac{\dfrac{\dfrac{v \vdash}{v \vdash p}\text{ Weakening}}{\vdash v \to p}\vdash\to}{\vdash Val(\ulcorner v\urcorner, \ulcorner p\urcorner)}\text{ V1}$$

[18]The sentence ϑ is the same sentence that is used in the proof of Gödel's First Incompleteness Theorem. However, in the proof of the first incompleteness theorem we reason without (P2). Additionally, a similar sentence to γ is used in the proof of Löb's theorem. Albeit, (P2) is not available in that proof but only one instance of it—the assumption that $\vdash Bew(\ulcorner\bot\urcorner) \to \bot$.

[19]As with regular Curry, the consequents in the Provability Curry and Knowability Curry do not have to be $\bot$—they can be any sentences we like.

Step 3

Let v be its own partial negative modality $Val(\ulcorner v \urcorner, \ulcorner p \urcorner)$

Validity Curry

Let v be $Val(\ulcorner v \urcorner, \ulcorner p \urcorner)$:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{V2}
\UnaryInfC{$Val(\ulcorner v \urcorner, \ulcorner p \urcorner) \vdash v \to p$}
\RightLabel{Def of v}
\UnaryInfC{$v \vdash v \to p$}
\RightLabel{Weakening}
\UnaryInfC{$v \vdash p, v \to p$}
\RightLabel{$\vdash\to$}
\UnaryInfC{$\vdash v \to p, v \to p$}
\RightLabel{Contraction}
\UnaryInfC{$\vdash v \to p$}
\RightLabel{V1}
\UnaryInfC{$\vdash Val(\ulcorner v \urcorner, \ulcorner p \urcorner)$}
\RightLabel{Def of v}
\UnaryInfC{$\vdash v$}
\end{prooftree}
$$

Call this derivation D_0

$$
\begin{prooftree}
\AxiomC{D_0}
\UnaryInfC{$\vdash v$}
\AxiomC{D_0}
\UnaryInfC{$\vdash v$}
\AxiomC{$p \vdash p$}
\RightLabel{$\to\vdash$}
\BinaryInfC{$v \to p \vdash p$}
\RightLabel{V2}
\UnaryInfC{$Val(\ulcorner v \urcorner, \ulcorner p \urcorner) \vdash p$}
\RightLabel{Def of v}
\UnaryInfC{$v \vdash p$}
\RightLabel{Cut}
\BinaryInfC{$\vdash p$}
\end{prooftree}
$$

Hence, given the sentence v where v says "the argument from v to p is valid", we can prove p. Since p is arbitrary, we can prove any sentence via Validity Curry.

2.4.3 Heterological and Membership

Unsurprisingly, the Grelling-Nelson Paradox and Russell's Paradox in terms of extensions share the same proof, with one exception: the former uses a two-place truth predicate 'is true of', while the latter uses a two-place membership predicate 'is in the extension of'.

Step 1

Let our two-place truth predicate have the following rules:
(H1) If $\vdash \varphi(\alpha)$ then $\vdash Tr(\ulcorner \varphi \urcorner, \alpha)$

(H2) $Tr(\ulcorner \varphi \urcorner, \alpha) \vdash \varphi(\alpha)$

Thus, (H1) says if you can prove that α is φ, then it is safe to say that the predicate φ is true of α. On the other hand, (H2) says if φ is true of α, then α

is φ.

Similarly, let $\epsilon(\alpha, ext(\ulcorner\varphi\urcorner))$ to express 'α is in the extension of φ'. We often state the membership-schema as $\epsilon(\alpha, ext(\ulcorner\varphi\urcorner))$ iff $\varphi(\alpha)$. However we can provide a slightly weaker membership-schema through the following two rules:

(M1) If $\vdash \varphi(\alpha)$ then $\vdash \epsilon(\alpha, ext(\ulcorner\varphi\urcorner))$

(M2) $\epsilon(\alpha, ext(\ulcorner\varphi\urcorner)) \vdash \varphi(\alpha)$

Step 2

For the sake of brevity, we will only focus on Russell's Paradox. Since membership uses predicates instead of our usual sentences, we will try to find a partial negative modality for some random predicate $\Re$ applied to an arbitrary α:

$$\dfrac{\dfrac{\Re(\alpha) \vdash}{\epsilon(\alpha, ext(\ulcorner\Re\urcorner)) \vdash}\,\text{M2}}{\vdash \neg\epsilon(\alpha, ext(\ulcorner\Re\urcorner))}\,\vdash\neg$$

Step 3

Let $\Re(\alpha)$ be its own partial negative modality $\neg\epsilon(\alpha, ext(\ulcorner\Re\urcorner))$

Russell's Paradox in terms of extensions

Let $\Re(\alpha)$ be $\neg\epsilon(\alpha, ext(\ulcorner\Re\urcorner))$. Remember α works as a place holder and we can put anything we want in its place. So we can put $ext(\ulcorner\Re\urcorner)$ in its place:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\in(ext(\ulcorner\Re\urcorner), ext(\ulcorner\Re\urcorner)) \vdash \Re(ext(\ulcorner\Re\urcorner))}\,\text{M2}}{\vdash \neg\in(ext(\ulcorner\Re\urcorner), ext(\ulcorner\Re\urcorner)), \Re(ext(\ulcorner\Re\urcorner))}\,\vdash\neg}{\vdash \Re(ext(\ulcorner\Re\urcorner)), \Re(ext(\ulcorner\Re\urcorner))}\,\text{Def of }\Re}{\vdash \Re(ext(\ulcorner\Re\urcorner))}\,\text{Contraction}$$

Call this derivation D_0

$$\dfrac{\dfrac{D_0}{\vdash \Re(ext(\ulcorner\Re\urcorner))} \qquad \dfrac{\dfrac{\dfrac{\dfrac{D_0}{\vdash \Re(ext(\ulcorner\Re\urcorner))}}{\vdash \in(ext(\ulcorner\Re\urcorner), ext(\ulcorner\Re\urcorner))}\,\text{M1}}{\neg\in(ext(\ulcorner\Re\urcorner), ext(\ulcorner\Re\urcorner)) \vdash}\,\neg\vdash}{\Re(ext(\ulcorner\Re\urcorner)) \vdash}\,\text{Def of }\Re}{\vdash}\,\text{Cut}$$

We reached the empty sequent by proving that the extension of $\Re$ is a member of itself and not a member of itself. The same proof works for the Grelling-Nelson Paradox. We just let $\eta(\alpha)$ be its own partial negative modality $\neg Tr(\ulcorner\eta\urcorner, \alpha)$ and use the rules (H1) and (H2) instead of (M1) and (M2). We arrive at the empty sequent by proving that 'heterological' is heterological and not heterological.

2.5 An Explanation

So far, we provided the ingredients that give rise to paradoxes, but we have not explained *why* they give rise to paradoxes. In other words, why does having a partially transparent predicate along with letting a sentence be its own partial negative modality create a paradox? Take the Liar as an example. If we look at the proof of the paradox, then it is not surprising that the aforementioned conditions would result in a paradox. We used (T2) and ($\vdash \neg$) when we wanted to find a partial negative modality. That tells us that we can make the same moves in our proof of the Liar paradox. The only difference is that we have an extra λ on the right-hand side in the proof of the Liar. However, given the diagonal lemma, we can reduce the complex sentence $\neg Tr(\ulcorner\lambda\urcorner)$ to its simple form λ. So in a sense, we teleported λ from the left-hand side to the right-hand side without adding or removing connectives (e.g., negation). To see this visually:

$$\frac{\lambda \vdash \lambda}{\vdash \lambda, \lambda}\ \text{T2} + \vdash \neg + \text{def of } \lambda \text{ (via diagonal lemma)}$$

From there, we can use contraction to get λ on its own. In other words, we got a proof of λ because λ is its own partial negative modality. Using (T1)+($\neg \vdash$)+ def of λ, we can teleport λ from the right-hand side of the sequent to the left-hand side. Finally, the Cut rule would get us to the empty sequent. All of the paradoxes mentioned in this paper, including Curry, work in a similar fashion. Defining a sentence by its partial negative modality along with partially transparent predicate rules allow us to teleport sentences from one side of the sequent to the other.

The partial negative modalities can have all different kinds of connectives and different complexities. The paradoxes do not hinge on specific connectives. What is required for all of these paradoxes are the structural rules—Cut, Contraction, and Identity. Any partial negative modality that uses (b) $P(\ulcorner\varphi\urcorner) \vdash \varphi$ (e.g., T2, P2, K2...etc.) would guarantee the sentence defined by it (its simple form) to teleport from left to right. With the help of contraction, we would have a proof of said sentence.[20]

[20]This guarantee is within the classical warranty and not necessarily transferable to other logics. For example, in Constructive logic, you will need to teleport from right to left; this does not work out with all paradoxes (See manuscript on Unconstructivisable Paradoxes).

The ability to teleport is reflected informally by exiting a supposition with a proof of the assumed sentence. For example, in the Curry, we start by assuming κ, and with a few moves, we exit the supposition with a proof of κ. The recipe shows that this kind of move would happen with any sentence defined by its own partial negative modality. That is because, by definition, a partial negative modality is an outcome of a supposition. So once we let a sentence be its own partial negative modality, the outcome of a supposition is a proof of the supposition itself—hence the teleportation. Once such a sentence teleports, then given the harmony of the operational rules (i.e., the balance between left and right operational rules in Classical Logic) and the rule (a) If $\vdash \varphi$ then $\vdash P(\ulcorner\varphi\urcorner)$, we can rebuild the partial negative modality on the antecedent side to reach a contradiction or to reach an unwanted outcome with the help of Cut.

3 The lessons

In this final section, we aim to address the usefulness of the Recipe. The Recipe not only shows us that the aforementioned paradoxes share the same structure, but it also (i) provides a guide to new or unexplored paradoxes, (ii) explains why Truth-tellers are not paradoxical and when can loops be paradoxical, (iii) explains why the Sorites is not in the family, (iv) paves a path to uniform solutions, and (v) helps to adjudicate between uniform solutions.

3.1 Unexplored Paradoxes

3.1.1 Alternative Paradoxes

Let us look at what we call an alternative Liar. Instead of τ saying “I am not true”, it says something along the lines of “my negation is true”:

(τ) The negation of τ is true.

To see how this is a paradox informally, suppose τ is true. It follows that $\neg\tau$ is true. Given that we are assuming classical reasoning, τ is not true, hence, a contradiction. Suppose τ is not true. It follows that $\neg\tau$ is not true. Hence τ is true; a contradiction. Let us see if our recipe can accommodate the alternative Liar:

Step 1

Use the truth predicate.

Step 2

For the alternative Liar, instead of using partial negative modalities, we have to use what we will call *Inverse Partial Negative Modalities*: Let m be a one place connective. m is said to be an inverse partial negative modality if and only if for any A:

$$\frac{\vdash A}{mA \vdash}$$

is valid. We can find the inverse partial negative modality of the alternative Liar as follows:

$$\dfrac{\dfrac{\vdash \tau}{\neg\tau \vdash}\,\neg\vdash}{Tr(\ulcorner\neg\tau\urcorner) \vdash}\,\text{T2}$$

Step 3

We want to let τ be its partial negative modality $Tr(\ulcorner\neg\tau\urcorner)$. However, that is not possible using our diagonal lemma. Instead, we want a different diagonal function that would give us the lemma $\vdash G \leftrightarrow B(\ulcorner\neg G\urcorner)$. This lemma is available to us with a few tweaks to the proof of the original diagonal lemma.[21]

Given our modified diagonalization lemma, we can let τ be $Tr(\ulcorner\neg\tau\urcorner)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{Tr(\ulcorner\neg\tau\urcorner) \vdash \neg\tau}\,\text{T2}}{\tau \vdash \neg\tau}\,\text{Def of }\tau}{\vdash \neg\tau, \neg\tau}\,\vdash\neg}{\vdash \neg\tau}\,\text{Contraction}$$

Call this derivation D_0

$$\dfrac{\dfrac{D_0}{\vdash \neg\tau} \qquad \dfrac{\dfrac{\dfrac{\dfrac{D_0}{\vdash \neg\tau}}{\vdash Tr(\ulcorner\neg\tau\urcorner)}\,\text{T1}}{\vdash \tau}\,\text{Def of }\tau}{\neg\tau \vdash}\,\neg\vdash}{\vdash}\,\text{Cut}$$

Presumably, we can provide alternative paradoxes for all of the aforementioned paradoxes. For instance, instead of having a Curry sentence that says "If I am true, then absurdity follows", it would say "It is true that absurdity follows from me" (i.e., let $\varkappa$ be $Tr(\ulcorner\varkappa \to \bot\urcorner)$). Again, we would need a different diagonal-like function to allow us to let $\varkappa$ be $Tr(\ulcorner\varkappa \to \bot\urcorner)$. We can also provide arguably alternative forms of Curry's Paradox using our original diagonal lemma. For example, both $\neg Tr(\ulcorner\kappa\urcorner) \vee \bot$ and $\neg(Tr(\ulcorner\kappa\urcorner) \wedge \neg\bot)$ are partial negative modalities, and thus paradoxical.

[21] For reasons of space, this adjusted proof is omitted.

3.1.2 "Mix-and-Match" Paradoxes

We do not have to stop there; we can mix and match the predicates that we have. For instance, let us find a partial negative modality that uses 'is known/is knowable', 'is provable', and 'is true':

$$\dfrac{\dfrac{\dfrac{\dfrac{\varpi \vdash}{Tr(\ulcorner\varpi\urcorner) \vdash}\,\text{T2}}{\vdash \neg Tr(\ulcorner\varpi\urcorner)}\,\vdash\neg}{\vdash Bew(\neg Tr(\ulcorner\varpi\urcorner))}\,\text{P1}}{\vdash Knw(Bew(\neg Tr(\ulcorner\varpi\urcorner)))}\,\text{K1}$$

Now let ϖ be its own partial negative modality $Knw(Bew(\neg Tr(\ulcorner\varpi\urcorner)))$ so that ϖ says something along the lines of: "it is known that it is provable that I am not true". From there, we can construct a Liar-like paradox that uses (T1), (T2), (P1), (P2), (K1), and (K2).

3.1.3 Other Predicates

It is not hard to find other predicates that are prone to paradoxes. As long as we can tell a story why a predicate is partially transparent, we will be able to find new paradoxes. For example, the following predicates can be justified to have rules similar to (a) and (b): "undeniable", "indisputable", "unquestionable", "irrefutable", "inarguable", "evident", "obvious", "definite", "assertable", "acceptable", "admissible", "derivable from", "deducible from", "incompatible with"..., and so on. It is not surprising that such predicates would have rules similar to (a) and (b) because all of these predicates share some properties with truth or factitude.[22]

To test it out, let us consider "undeniable". If we can prove, say, that snow is white, then 'snow is white' is undeniable because we have a proof for it. If 'snow is white' is undeniable, then snow is indeed white:
(U1) If $\vdash \varphi$ then $\vdash Und(\ulcorner\varphi\urcorner)$
(U2) $Und(\ulcorner\varphi\urcorner) \vdash \varphi$

From there, we can let ϱ be its partial negative modality $\neg Und(\ulcorner\varrho\urcorner)$ so that ϱ says something along the lines that "it is not the case that I am undeniable". We can then construct a Liar-like argument to show the paradoxicality of ϱ. Similar approaches can be carried out for the other aforementioned predicates.

3.2 Truth-Tellers and Loops

Truth-tellers and Truth-teller-like sentences, on the other hand, are not paradoxical because they cannot be constructed as partial negative modalities. For instance, we cannot go from $A \vdash$ to $\vdash Tr(\ulcorner A\urcorner)$. Hence, we cannot "teleport"

[22] In ([12], p. 320-321), Littmann and Simmons use the predicates "assertable" and "acceptable" as Liar-like arguments against dialetheist approaches. Although they use full transparency for these predicates, full transparency is not required to construct the paradoxes.

Truth-tellers from one side of the turnstile to another. Nevertheless, we can find *partial positive modalities* by moving from $A \vdash$ to $mA \vdash$ where m includes a partially transparent predicate such as 'is true', 'is provable', 'is known'...etc. If we let sentences be their partial positive modalities, then we will find other truth-teller-like sentences. In other words, the Recipe captures our intuitions regarding Truth-tellers and their relation to Liar-like paradoxes. The Recipe divides pathologies into two categories: the paradoxical using partial negative modalities, and the non-paradoxical using partial positive modalities. This is another advantage of the Recipe compared to the Inclosure Schema since the Inclosure Schema stays silent regarding non-paradoxical pathologies such as the Truth-tellers. That is because the Inclosure Schema is only concerned with contradictions of the form $\delta(\Omega) \in \Omega$ and $\delta(\Omega) \notin \Omega$. It can only tell you that the truth-teller, for example, is not an inclosure paradox. It cannot differentiate between non-paradoxical pathologies and non-pathological sentences.

We can also extend the Recipe to accommodate paradoxical loops. Instead of having sentences be their own partial negative/positive modalities, we can let two sentences be each other's partial negative/positive modalities. For example, we can let A be B's partial negative modality (e.g., $\neg Tr(\ulcorner B \urcorner)$), and let B be A's partial positive modality (e.g., $Tr(\ulcorner A \urcorner)$). From there, we can construct a paradox.[23]

3.3 Sorites Paradox

Unlike the Inclosure Schema, our recipe does not count the Sorites paradox as of the same family of paradoxes as the Liar. The vague predicates on which the Sorites turns do not satisfy *any* of the clauses of the Recipe: The rules (a) and (b) do not apply to a vague predicate such as 'bald' or 'tall'. Moreover, unsurprisingly, the Sorites paradox does not involve the diagonal lemma, since the Sorites does not involve self-reference. Our proposed schema is more in line with our intuitions compared to the Inclosure Schema. The same can be said about the trivializing cases, namely, an arbitrary contradiction and Zeno's paradox (and Zeno's revenge). None of these cases employ partially transparent predicates, nor do they involve any diagonalization. Hence, these cases do not trivialize the Recipe as they trivialize the Inclosure Schema.[24]

3.4 Possible Uniform Solutions

An adequate solution to the semantic paradoxes must adhere to what Priest calls the principle of uniform solution: "same kind of paradox, same kind of solution"([16], p. 183). One lesson we can learn from the Recipe is that Curry's paradox must have the same solution as the Liar. And, obviously, modifying

[23]Note that having two sentence be each other's partial negative modalities is equivalent to a partial positive modality (analogous to a double negation). That would result in a non-paradoxical pathological loop.

[24]Thanks to an anonymous reviewer for suggesting that we tie it back to the trivializing cases.

the T-schema alone will not provide a uniform solution. Other paradoxes of the same family do not rely on the T-schema or the truth predicate. So if we want to modify the T-schema, say, by targeting (T1), and still adhere to the principle of uniform solution, then we would have to target every instance of the rule (a) (If $\vdash \varphi$ then $\vdash P(\ulcorner\varphi\urcorner)$), namely, (P1), (K1), (M1)...etc. Similarly, we would not be providing a uniform solution if we took a hierarchical approach or a singularity approach for just the truth predicate. Either approach would need also to be applied to each predicate.

Thus, the Recipe suggests the following possible uniform solutions:

1. Targeting rule (a) or rule (b) (or both) of a partially transparent predicate.
2. Targeting all methods for allowing sentences be their partial negative modalities.
3. Targeting a structural rule such as Contraction or Cut.

But not all possible uniform solutions are created equal. Option 3—targeting a structural rule—has further theoretical virtues. For instance, option 3 provides a single solution—e.g., restrict Cut—to solve all the paradoxes at a stroke rather than providing similar solutions to different paradoxes—e.g., placing similar restrictions on each of (T1), (P1), (V1)...etc. Further, option 3 requires a single justification whereas a stratifying solution, for example, must justify the stratification of each partially transparent predicate. Similarly, rejecting the rule (a) would require a justification for rejecting each instance of rule (a) (i.e., a justification for rejecting (T1), a justification for rejecting (P1), and so on).

As a response to the principle of uniform solution, Nicholas J.J. Smith in ([26], p. 118) claims that "two objects can be of the same kind at some level of abstraction and of different kinds at another level of abstraction." To put it another way, why do we think that there is a fact of the matter regarding whether a paradox is in the same family? It might seem that sameness of family is just interest-relative (or relative to a level of abstraction).[25] As an example, one might say that only the Liar and the Curry are of the same family because they both use the truth predicate while other paradoxes do not. But while it is true that the Curry and the Liar share the truth predicate, the Recipe shows that there is nothing special about the truth predicate compared to other partially transparent predicates. If we provide a solution by targeting the truth predicate alone, then we are only "attacking peripherals, not essentials" (Priest [17], p. 124). The difference between the Liar and the Provability Liar, for example, is only of appearance. The Recipe shows how the Liar and the Provability Liar are the exact same paradox but with different clothes; thus, separating these two paradoxes on an irrelevant metric is a mistake. As Priest

[25] We owe William Lycan and Lionel Shapiro for this question (via personal communication on September 10th, 2021)

puts it, "the correct level of abstraction for an analysis of the paradoxes of self-reference is not one which depends upon the presence of certain words ('set', 'true', etc.), but the level of the underlying structure that generates and causes the contradictions" ([17], p. 125). The point of finding the underlying common structure of paradoxes is not just to find anything that two paradoxes have in common. Rather it is to find the crucial ingredients that are causing these paradoxes. Once this common structure is located, not only can we discern which solutions would count as uniform, but also which solutions would be optimal.[26]

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