Semantic Incompleteness of del Cerro and Herzig's Hilbert System for a Combination of Classical and Intuitionistic Propositional Logic

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Abstract

This paper shows that the Hilbert system $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$, given by del Cerro and Herzig (1996), is semantically incomplete. This system is proposed as a proof theory for a combination of classical and intuitionistic propositional logic, which is called $\mathbf{C}+\mathbf{J}$. The Kripke semantics for this combination is obtained by adding the semantic clause of classical implication directly to the Kripke semantics for intuitionistic propositional logic. The Hilbert system $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$ does not contain the classical modus ponens, although it contains the intuitionistic modus ponens as a rule, which makes this system semantically incomplete. This paper gives an argument that the system $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$ is semantically incomplete because of the absence of the classical modus ponens. Our way to prove this is based on the logic of paradox, which is a paraconsistent logic proposed by Priest (1979).

1 Introduction

This paper shows the semantic incompleteness of the Hilbert system $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$, given by del Cerro and Herzig [23]. This system was provided for a combination of classical and intuitionistic propositional logic, denoted by $\mathbf{C}+\mathbf{J}$. Let us define that a logic is a *combination* of classical and intuitionistic logic if it is a conservative extension of both logics. Roughly speaking, there are three ways to obtain a combination of classical and intuitionistic logic. The first one is to make use of linear logic, an approach that is taken in [27, 34, 35, 36]. The second one is to add classical propositional variables to intuitionistic logic, an approach that is taken in [32, 33, 34, 43, 53].[1] The third one is to introduce classical and intuitionistic operators, an approach that is taken in [13, 14, 20, 21, 22, 23, 28, 29, 37, 41, 42, 47, 48, 49, 50, 54].[2] The combination $\mathbf{C}+\mathbf{J}$, which is studied in [22, 23, 28, 37, 54], is based on the third way and is one of the most studied combinations of classical and intuitionistic logic.

The syntax for $\mathbf{C}+\mathbf{J}$ has two implications: an intuitionistic implication, denoted by "$\to_{\mathtt{i}}$", and a classical implication, denoted by "$\to_{\mathtt{c}}$". Humberstone [28] proposed the semantics for

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[1] Other studies related to this second way are [30, 31, 38, 40], although they do not provide a combination of classical and intuitionistic logic in our sense.

[2] Another study related to this third way is [25], although it does not provide a combination of classical and intuitionistic logic in our sense.

this combination, which is obtained by adding to intuitionistic Kripke semantics the following satisfaction relation for a formula whose main connective is "$\rightarrow_{\mathtt{c}}$":

$$w \models_M A \rightarrow_{\mathtt{c}} B \quad \text{iff} \quad w \models_M A \text{ implies } w \models_M B,$$

where $M = (W, R, V)$ is a Kripke model for intuitionistic propositional logic and w is a possible world in W. The notion of validity in this Kripke semantics is defined as in an ordinary modal logic, as follows (cf. [12]): a formula A is valid if $w \models_M A$ for any Kripke model $M = (W, R, V)$ and any possible world $w \in W$.

Although the notion of validity is defined as in an ordinary modal logic, how to axiomatize the set of valid formulas in $\mathbf{C} + \mathbf{J}$ is far from trivial. Even though classical and intuitionistic implications are introduced, a proof theory for this combination cannot be obtained by directly combining the axioms and the rules of both logics. If the axioms and the rules of both logics are combined directly, the formula $A \rightarrow_{\mathtt{i}} B$ will be derivable from $A \rightarrow_{\mathtt{c}} B$, and vice versa. This means that intuitionistic and classical implications will be collapsed into one. This problem is called "the collapsing problem," which was pointed out in [11, 26, 55]. Humberstone [28] proposed a natural deduction system for $\mathbf{C} + \mathbf{J}$, which successfully avoids the collapsing problem by imposing a restriction on the rule called $(RAA_{\neg_{\mathtt{i}}})$. Lucio [37] employed the notion of a "structured sequent" in order to avoid the problem and successfully proposed a single-succedent structured sequent calculus for $\mathbf{C} + \mathbf{J}$, denoted by $\mathsf{FO}^{\supset}$.[3] In [54], a multi-succedent sequent calculus for $\mathbf{C} + \mathbf{J}$ is proposed. This proof theory employs only an ordinary notion of a sequent, which is possible because of a restriction on the right rule for intuitionistic implication. These proof theories for $\mathbf{C} + \mathbf{J}$, which are all sound and semantically complete to the Kripke semantics for $\mathbf{C} + \mathbf{J}$, are explained in Section 4 in detail.

The Hilbert system $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$, proposed by del Cerro and Herzig [23], also successfully avoids the collapsing problem by using ideas of conditional logics. However, the Hilbert system $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$ is semantically incomplete. The semantic completeness is one of the most fundamental properties of a proof theory since it, together with the soundness, ensures that both the semantic and the proof-theoretic approaches capture the same objects, i.e., the set of theorems is the same as the set of all valid formulas. Therefore, the failure of the semantic completeness of $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$ implies that it fails to capture the set of valid formulas in $\mathbf{C} + \mathbf{J}$, i.e., there exists a valid formula such that it is not a theorem of $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$. It should be noted that De and Omori [22] provided a Hilbert system for $\mathbf{C} + \mathbf{J}$ that is slightly different from $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$, and this system also successfully avoids the collapsing problem. Their way to avoid the collapsing problem is almost the same as that in $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$. It should also be remarked that even though $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$ is semantically incomplete, De and Omori's Hilbert system is semantically complete.

The failure of the semantic completeness of $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$ is due to the absence of the classical modus ponens, which may just be an unfortunate typo. We show that the formula $(p \wedge (p \rightarrow_{\mathtt{c}} q)) \rightarrow_{\mathtt{i}} q$ is valid in the Kripke semantics for $\mathbf{C} + \mathbf{J}$, but not a theorem of $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$, by employing the logic of paradox, provided by Priest [51]. We make use of the fact that the set of valid formulas of the logic of paradox is the same as that of classical logic, while the sets of consequence relations of these two logics diverge, as noted in [51].

[3] To be more precise, Lucio [37] proposed a first-order expansion of $\mathbf{C} + \mathbf{J}$, and $\mathsf{FO}^{\supset}$ is a calculus for this expansion. However, $\mathsf{FO}^{\supset}$ contains the calculus for the propositional $\mathbf{C} + \mathbf{J}$ as a fragment, and in this paper we limit our attention to this propositional fragment. It should also be noted that the multi-succedent sequent calculus, proposed in [54], is also the one for the Lucio's first-order expansion.

This paper proceeds as follows. Section 2 introduces the syntax and the Kripke semantics and explains the Hilbert system $\mathsf{H}(\mathbf{C}+\mathbf{J})^{-}$. Section 3 shows that the Hilbert system $\mathsf{H}(\mathbf{C}+\mathbf{J})^{-}$ is semantically incomplete (Theorem 1). We show this by arguing that the formula $(p\wedge(p\to_{\mathtt{c}} q))\to_{\mathtt{i}} q$ is valid in the Kripke semantics for $\mathbf{C}+\mathbf{J}$ but is not derivable in $\mathsf{H}(\mathbf{C}+\mathbf{J})^{-}$. Section 4 describes how proof theories for $\mathbf{C}+\mathbf{J}$ other than $\mathsf{H}(\mathbf{C}+\mathbf{J})^{-}$ avoid the collapsing problem and how the classical modus ponens is expressed in these proof theories.

2 Syntax, Kripke Semantics, and the Hilbert System $\mathsf{H}(\mathbf{C}+\mathbf{J})^{-}$

The syntax $\mathcal{L}$ consists of a countably infinite set Prop of propositional variables and the following logical connectives: falsum $\bot$, disjunction $\vee$, conjunction $\wedge$, intuitionistic implication $\to_{\mathtt{i}}$, and classical implication $\to_{\mathtt{c}}$. We denote by $\mathcal{L}_{\mathbf{C}}$ (the syntax for the classical logic) and $\mathcal{L}_{\mathbf{J}}$ (the syntax for the intuitionistic logic) the resulting syntax dropping $\to_{\mathtt{i}}$ and $\to_{\mathtt{c}}$ from $\mathcal{L}$, respectively.

The set Form of all formulas in the syntax $\mathcal{L}$ is defined inductively as follows:

$$A ::= p \mid \bot \mid A \vee A \mid A \wedge A \mid A \to_{\mathtt{i}} A \mid A \to_{\mathtt{c}} A,$$

where $p \in \mathsf{Prop}$. We denote by $\mathsf{Form}_{\mathbf{C}}$ and $\mathsf{Form}_{\mathbf{J}}$ the set of all formulas in $\mathcal{L}_{\mathbf{C}}$ and the set of all formulas in $\mathcal{L}_{\mathbf{J}}$, respectively. The sets of propositional variables in $\mathsf{Form}_{\mathbf{C}}$ and $\mathsf{Form}_{\mathbf{J}}$ are the same, i.e., $\mathsf{Prop} \subseteq \mathsf{Form}_{\mathbf{C}}$ and $\mathsf{Prop} \subseteq \mathsf{Form}_{\mathbf{J}}$. We define $\top := \bot \to_{\mathtt{i}} \bot$, $\neg_{\mathtt{c}} A := A \to_{\mathtt{c}} \bot$, and $\neg_{\mathtt{i}} A := A \to_{\mathtt{i}} \bot$.

Let us move to the semantics for the syntax $\mathcal{L}$. The semantic we introduce was first proposed by Humberstone [28].

Definition 1 (Kripke Model)**.** A *Kripke model* is a tuple $M = (W, R, V)$, where

- W is a non-empty set of possible worlds,
- R is a preorder on W, i.e., R satisfies reflexivity and transitivity,
- $V : \mathsf{Prop} \to \mathcal{P}(W)$ is a valuation function satisfying the following *heredity* condition: $w \in V(p)$ and wRv jointly imply $v \in V(p)$ for all possible worlds $w, v \in W$.

Definition 2. Given a Kripke model $M = (W, R, V)$, a possible world $w \in W$, and a formula A, the *satisfaction relation* $w \models_M A$ is inductively defined as follows:

$$\begin{array}{lll} w \models_M p & \text{iff} & w \in V(p), \\ w \not\models_M \bot, & & \\ w \models_M A \wedge B & \text{iff} & w \models_M A \text{ and } w \models_M B, \\ w \models_M A \vee B & \text{iff} & w \models_M A \text{ or } w \models_M B, \\ w \models_M A \to_{\mathtt{i}} B & \text{iff} & \text{for all } v \in W, (wRv \text{ and } v \models_M A \text{ jointly imply } v \models_M B), \\ w \models_M A \to_{\mathtt{c}} B & \text{iff} & w \models_M A \text{ implies } w \models_M B. \end{array}$$

From Definition 2, the following satisfaction relations for a formula whose main connective is "$\neg_{\mathtt{c}}$" or "$\neg_{\mathtt{i}}$" is obtained:

$$\begin{array}{lll} w \models_M \neg_{\mathtt{c}} A & \text{iff} & w \not\models_M A, \\ w \models_M \neg_{\mathtt{i}} A & \text{iff} & \text{for all } v \in W, (wRv \text{ implies } v \not\models_M A). \end{array}$$

Definition 3 (Validity)**.** A formula A is *valid*, written as $\models A$, if $w \models_M A$ for any Kripke model $M = (W, R, V)$ and any possible world $w \in W$.

We proceed to some arguments about the property called *heredity*, which is an important property in intuitionistic logic. First of all, we define the notion of heredity.

Definition 4 (Heredity)**.** A formula A satisfies *heredity* if for any model M and $w, v \in W$, wRv and $w \models_M A$ jointly imply $v \models_M A$.

In ordinary intuitionistic logic, any formula satisfies heredity, and this fact is preserved only when the formulas in $\mathsf{Form}_{\mathbf{J}}$ are considered.

Proposition 5. Any formula $A \in \mathsf{Form}_{\mathbf{J}}$ satisfies heredity.

Proposition 5 can be shown by induction on the construction of a formula. However, if we take a formula in $\mathsf{Form}_{\mathbf{C}}$ into consideration, there exists a formula that does not satisfy heredity. In order to see this, let us consider the model $M_1 = (W, R, V)$ such that $W = \{w, v\}, R = \{(w, w), (w, v), (v, v)\}$, and $V(p) = V(q) = \{v\}$.

Proposition 6. A formula $\neg_{\mathtt{c}} p$ does not satisfy heredity.

Proof. In the model M_1, wRv and $w \models_{M_1} \neg_{\mathtt{c}} p$, but $v \not\models_{M_1} \neg_{\mathtt{c}} p$. ∎

Moreover, the following proposition also holds.

Proposition 7. Both $\neg_{\mathtt{c}} p \to_{\mathtt{c}} (q \to_{\mathtt{i}} \neg_{\mathtt{c}} p)$ and $\neg_{\mathtt{c}} p \to_{\mathtt{i}} (q \to_{\mathtt{i}} \neg_{\mathtt{c}} p)$ are *invalid.*

Proof. Let us take the model M_1 again. Since wRv, $v \models_{M_1} q$, and $v \not\models_{M_1} \neg_{\mathtt{c}} p$, $w \not\models_{M_1} q \to_{\mathtt{i}} \neg_{\mathtt{c}} p$ in the model M_1. This, together with $w \models_{M_1} \neg_{\mathtt{c}} p$, implies $w \not\models_{M_1} \neg_{\mathtt{c}} p \to_{\mathtt{c}} (q \to_{\mathtt{i}} \neg_{\mathtt{c}} p)$ in the model M_1. In almost the same way, it is shown that $w \not\models_{M_1} \neg_{\mathtt{c}} p \to_{\mathtt{i}} (q \to_{\mathtt{i}} \neg_{\mathtt{c}} p)$. ∎

Proposition 7 implies that an intuitionistic theorem $A \to_{\mathtt{i}} (B \to_{\mathtt{i}} A)$ is no longer a theorem in this combination.

Let us move to the Hilbert system $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$, provided by del Cerro and Herzig [23]. Before giving the details of the axiomatization, we introduce the syntactic notion of *persistent formulas*. The set of all persistent formulas in Form is defined as follows:

$$E ::= \bot \mid p \mid A \to_{\mathtt{i}} A \mid E \wedge E \mid E \vee E,$$

where $p \in \mathsf{Prop}$ and $A \in \mathsf{Form}$. [4]

Definition 8. The Hilbert system $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$ consists of axioms (CL), (CK), (ID), (CMP), and (PER) of Table 1 and rules (MPI) and (RCN) of Table 1. The Hilbert system $\mathsf{H}(\mathbf{C} + \mathbf{J})$ is the extended system of $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$ with the rule (MPC) of Table 1.

An important axiom is (PER). Recall that the formula $\neg_{\mathtt{c}} p \to_{\mathtt{i}} (q \to_{\mathtt{i}} \neg_{\mathtt{c}} p)$ is not valid in the Kripke semantics, as stated in Proposition 7. In order for this formula to be underivable, an antecedent formula A of (PER) should be restricted to a persistent formula. It should also be noted that this axiom enables the Hilbert systems $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$ and $\mathsf{H}(\mathbf{C} + \mathbf{J})$ to avoid the collapsing problem. If this restriction did not exist, then $(A \to_{\mathtt{c}} B) \to_{\mathtt{c}} (A \to_{\mathtt{i}} B)$ would be derivable, which implies that the intuitionistic implication and the classical implication would collapse into one implication.

[4] Del Cerro and Herzig [23] did not define $\bot$ as a persistent formula. However, this is not an essential point, since $\bot$ is equivalent to $p \wedge \neg_{\mathtt{i}} p$, which is a persistent formula in the sense of [23]. This slight change allows us to state that all formulas in $\mathsf{Form}_{\mathbf{J}}$ are persistent.

Table 1 Hilbert Systems $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$ and $\mathsf{H}(\mathbf{C}+\mathbf{J})$

	Hilbert System $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$
(CL)	All instances of classical tautologies
(CK)	$(A \to_{\mathtt{i}} (B \to_{\mathtt{c}} C)) \to_{\mathtt{c}} ((A \to_{\mathtt{i}} B) \to_{\mathtt{c}} (A \to_{\mathtt{i}} C))$
(ID)	$A \to_{\mathtt{i}} A$
(CMP)	$(A \to_{\mathtt{i}} B) \to_{\mathtt{c}} (A \to_{\mathtt{c}} B)$
(PER)	$A \to_{\mathtt{c}} (B \to_{\mathtt{i}} A)^{\dagger}$ †: A is *persistent*.
(MPI)	From A and $A \to_{\mathtt{i}} B$, we may infer B
(RCN)	From A, we may infer $B \to_{\mathtt{i}} A$
	Hilbert System $\mathsf{H}(\mathbf{C}+\mathbf{J})$
	All the axioms and rules of $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$
(MPC)	From A and $A \to_{\mathtt{c}} B$, we may infer B

In the next section, we show the Hilbert system $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$ is semantically incomplete. This semantic incompleteness is the result of the absence of (MPC). The system $\mathsf{H}(\mathbf{C}+\mathbf{J})$ may be what del Cerro and Herzig [23] intended to provide, but (MPC) does not exist, which may be an unfortunate typo. It is reasonable that (MPC) is necessary, because $\mathsf{H}(\mathbf{C}+\mathbf{J})$ is based on an idea of an axiomatization of conditional logic, which adds axioms and rules to the conditional on top of classical tautologies and the rule of classical modus ponens (see, e.g., [17, 18, 44]).

The following proposition ensures that the rule (MPI) can be deleted from the Hilbert system $\mathsf{H}(\mathbf{C}+\mathbf{J})$.

Proposition 9. If we drop (MPI) from $\mathsf{H}(\mathbf{C}+\mathbf{J})$, (MPI) is derivable in the resulting system.

Proof. Suppose that A and $A \to_{\mathtt{i}} B$ are theorems in the resulting system. By (CMP) and (MPC), $A \to_{\mathtt{c}} B$ is obtained from $A \to_{\mathtt{i}} B$. By applying (MPC) to A and $A \to_{\mathtt{c}} B$, B is obtained, as is desired. ■

Thus, in order to obtain $\mathsf{H}(\mathbf{C}+\mathbf{J})$ from $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$, replacing (MPI) with (MPC) is sufficient.

3 Semantic Incompleteness of the Hilbert System $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$

In this section, we show that the Hilbert system $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$ is semantically incomplete. We provide a formula C such that C is valid in the semantics described in Definitions 1 and 2 but C is not a theorem of $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$. Our candidate for C is $(p \land (p \to_{\mathtt{c}} q)) \to_{\mathtt{i}} q$. The following proposition is easy to establish.

Proposition 10. The formula $(p \land (p \to_{\mathtt{c}} q)) \to_{\mathtt{i}} q$ is valid in Kripke semantics defined in Definitions 1 and 2.

Thus, we need to show that $(p \land (p \to_{\mathtt{c}} q)) \to_{\mathtt{i}} q$ is not a theorem of $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$. For this purpose, we need to find a non-standard semantics to which $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$ is sound but in which the formula $(p \land (p \to_{\mathtt{c}} q)) \to_{\mathtt{i}} q$ is not valid.

For our purpose, we utilize the three-valued semantics for a paraconsistent logic provided by Priest (cf. [52]), i.e., the logic of paradox [51], which allows the third truth value of "both true and false" $\{0,1\}$, in addition to the values $\{0\}$ ("false only") and $\{1\}$ ("true only").

Definition 11. A valuation v is a mapping from Prop to $\{\{0\},\{0,1\},\{1\}\}$. A valuation v is uniquely extended to a function $\overline{v}$ from the set Form of all formulas to $\{\{0\},\{0,1\},\{1\}\}$, as follows:

$$\begin{array}{rcl} 1 \in \overline{v}(\bot) & & \text{Never,} \\ 0 \in \overline{v}(\bot) & & \text{Always,} \\ 1 \in \overline{v}(A \wedge B) & \text{iff} & 1 \in \overline{v}(A) \text{ and } 1 \in \overline{v}(B), \\ 0 \in \overline{v}(A \wedge B) & \text{iff} & 0 \in \overline{v}(A) \text{ or } 0 \in \overline{v}(B), \\ 1 \in \overline{v}(A \vee B) & \text{iff} & 1 \in \overline{v}(A) \text{ or } 1 \in \overline{v}(B), \\ 0 \in \overline{v}(A \vee B) & \text{iff} & 0 \in \overline{v}(A) \text{ and } 0 \in \overline{v}(B), \\ 1 \in \overline{v}(A \to_{\mathtt{c}} B) & \text{iff} & 0 \in \overline{v}(A) \text{ or } 1 \in \overline{v}(B), \\ 0 \in \overline{v}(A \to_{\mathtt{c}} B) & \text{iff} & 1 \in \overline{v}(A) \text{ and } 0 \in \overline{v}(B), \\ 1 \in \overline{v}(A \to_{\mathtt{i}} B) & \text{iff} & 1 \notin \overline{v}(A) \text{ or } 1 \in \overline{v}(B), \\ 0 \in \overline{v}(A \to_{\mathtt{i}} B) & \text{iff} & 1 \in \overline{v}(A) \text{ and } 0 \in \overline{v}(B). \end{array}$$

A consequence relation $\Sigma \models_3 A$ is defined as follows: if $1 \in \overline{v}(B)$ holds for all $B \in \Sigma$, then $1 \in \overline{v}(A)$. We say that a formula A is 3-*valid* if $\models_3 A$ holds.

Proposition 12. For every valuation $v : \mathsf{Prop} \to \{\{1\},\{0,1\},\{0\}\}$ and every $A \in \mathsf{Form}$, $1 \in \overline{v}(A)$ or $0 \in \overline{v}(A)$.

Proof. Fix any valuation $v : \mathsf{Prop} \to \{\{1\},\{0,1\},\{0\}\}$. By induction on A, we can obtain the desired statement. ■

Remark 13. Let us denote $\{1\},\{0,1\},\{0\}$ by $\mathbf{t},\mathbf{b},\mathbf{f}$, respectively. Then, the semantics described above provides the three-valued truth table, described in Table 2, where the values $\mathbf{t}$ and $\mathbf{b}$ are defined as "designated values."

Table 2 Three-Valued Truth Tables

$\wedge$	$\mathbf{t}$	$\mathbf{b}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{b}$	$\mathbf{f}$
$\mathbf{b}$	$\mathbf{b}$	$\mathbf{b}$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$

$\vee$	$\mathbf{t}$	$\mathbf{b}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
$\mathbf{b}$	$\mathbf{t}$	$\mathbf{b}$	$\mathbf{b}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{b}$	$\mathbf{f}$

$\to_{\mathtt{c}}$	$\mathbf{t}$	$\mathbf{b}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{b}$	$\mathbf{f}$
$\mathbf{b}$	$\mathbf{t}$	$\mathbf{b}$	$\mathbf{b}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$

$\to_{\mathtt{i}}$	$\mathbf{t}$	$\mathbf{b}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{b}$	$\mathbf{f}$
$\mathbf{b}$	$\mathbf{t}$	$\mathbf{b}$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$

	$\bot$
$\mathbf{t}$	$\mathbf{f}$
$\mathbf{b}$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{f}$

By recalling that $\neg_{\mathtt{c}} A := A \to_{\mathtt{c}} \bot$ and $\neg_{\mathtt{i}} A := A \to_{\mathtt{i}} \bot$, we can also obtain the following satisfaction relations for a formula whose main connective is a negation:

$$\begin{array}{rcl} 1 \in \overline{v}(\neg_{\mathtt{c}} A) & \text{iff} & 0 \in \overline{v}(A), \\ 0 \in \overline{v}(\neg_{\mathtt{c}} A) & \text{iff} & 1 \in \overline{v}(A), \\ 1 \in \overline{v}(\neg_{\mathtt{i}} A) & \text{iff} & 1 \notin \overline{v}(A), \\ 0 \in \overline{v}(\neg_{\mathtt{i}} A) & \text{iff} & 1 \in \overline{v}(A). \end{array}$$

The satisfaction relations for a formula whose main connective is "$\neg_{\mathtt{c}}$" ensures that $\neg_{\mathtt{c}}$ is De Morgan negation.[5] We can also get the truth table for $\neg_{\mathtt{c}}$ and $\neg_{\mathtt{i}}$, described in Table 3.

[5] Following [52], we define a negation, denoted by "$\sim$", as a *De Morgan negation* if it satisfies all the following three conditions here: (1) $\sim (A \wedge B)$ and $\sim A \vee \sim B$ have the same truth value, (2) $\sim (A \vee B)$ and $\sim A \wedge \sim B$ have the same truth value, and (3) $\sim\sim A$ and A have the same truth value. The satisfaction relation for a formula whose main connective is "$\neg_{\mathtt{c}}$" ensures that this negation satisfies all the three conditions described above.

Table 3 Truth Table for Negations

A	$\neg_{\mathtt{c}} A$
$\mathbf{t}$	$\mathbf{f}$
$\mathbf{b}$	$\mathbf{b}$
$\mathbf{f}$	$\mathbf{t}$

A	$\neg_{\mathtt{i}} A$
$\mathbf{t}$	$\mathbf{f}$
$\mathbf{b}$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{t}$

The set of the primitive connectives of the propositional part of the logic of paradox [51, 52] is $\{ \wedge, \vee, \neg_{\mathtt{c}} \}$. The truth tables for these connectives, described in Table 2, are the same as those in the semantics of the logic of paradox. The truth table for "$\to_{\mathtt{c}}$", described in Table 2, is the same as the one provided by Asenjo [4]. Since the truth value of $A \to_{\mathtt{c}} B$ is the same as that of $\neg_{\mathtt{c}} A \vee B$, the connective "$\to_{\mathtt{c}}$" can be defined in the original syntax of the logic of paradox. It is remarked, however, that $\bot$ cannot be defined in terms of $\{ \wedge, \vee, \neg_{\mathtt{c}} \}$ (since B would return the value $\mathbf{b}$ for a valuation sending all propositional variables to $\mathbf{b}$, if a formula B defined $\bot$, it would be a contradiction). Therefore, in terms of the three-valued semantics above, $\{ \wedge, \vee, \to_{\mathtt{c}}, \bot \}$, i.e., the syntax of $\mathcal{L}_{\mathbf{C}}$ is stronger than $\{ \wedge, \vee, \neg_{\mathtt{c}} \}$, i.e, the syntax for the logic of paradox. The truth table for "$\to_{\mathtt{i}}$" is the same as that of an implication introduced by da Costa [19]. In [6], this connective is called "internal implication." This connective is also studied in [1, 2, 3, 5, 7, 8, 9, 10, 15, 16, 24, 39, 45, 46].

Lemma 14. $p, p \to_{\mathtt{c}} q \not\models_3 q$

Proof. Take a valuation v such that $v(p) = \{ 0, 1 \}$ and $v(q) = \{ 0 \}$. Then, $\overline{v}(p \to_{\mathtt{c}} q) = \{ 0, 1 \}$. But, $1 \notin v(q)$. Therefore, $p, p \to_{\mathtt{c}} q \not\models_3 q$. ■

While the logic of paradox [51, 52] has a different consequence relation from that of classical logic (as shown in Lemma 14), it is well known that the logic of paradox has the same theorems as those of classical logic (see, e.g., [52, p.310]). We may also extend this fact to $\{ \wedge, \vee, \to_{\mathtt{c}}, \bot \}$, i.e., the syntax of $\mathcal{L}_{\mathbf{C}}$, as follows.

Proposition 15. Let $A \in \mathsf{Form}_{\mathbf{C}}$. Then, A is a tautology in classical logic iff A is 3-valid.

Proof. The proof from right to left is trivial, since $v : \mathsf{Prop} \to \{ 0, 1 \}$ is regarded as a three-valued valuation by regarding 0 and 1 with $\{ 0 \}$ and $\{ 1 \}$, respectively. Conversely, assume that A is a tautology in classical logic and fix any valuation $v : \mathsf{Prop} \to \{\{0\}, \{0, 1\}, \{1\}\}$. Our goal is to show that $1 \in \overline{v}(A)$. Define a valuation v_1 from v by changing all outputs $\{0, 1\}$ of v to $\{1\}$. We regard v_1 as a two-valued valuation function by regarding $\{ 0 \}$ and $\{ 1 \}$ with 0 and 1, respectively. It is easy to see that $v_1(p) \subseteq v(p)$ for all $p \in \mathsf{Prop}$. We also have $\overline{v}_1(B) \subseteq \overline{v}(B)$ for all $B \in \mathsf{Form}_{\mathbf{C}}$ by Definition 11 (recall that $C \to_{\mathtt{c}} D$ is equivalent to $\neg_{\mathtt{c}} C \vee D$). Since A is a classical tautology, then $1 \in \overline{v}_1(A)$. By $\overline{v}_1(A) \subseteq \overline{v}(A)$, we conclude that $1 \in \overline{v}(A)$. ■

Definition 16. Given a function $\sigma : \mathsf{Prop} \to \mathsf{Form}$ and $A \in \mathsf{Form}_{\mathbf{C}}$, the resulting formula $\tilde{\sigma}(A)$, substituting all occurrences of each propositional variable p in A uniformly by $\sigma(p)$, is inductively defined as follows:

$$\begin{array}{lcl}\tilde{\sigma}(p) & := & \sigma(p),\\ \tilde{\sigma}(\bot) & := & \bot,\\ \tilde{\sigma}(B \wedge C) & := & \tilde{\sigma}(B) \wedge \tilde{\sigma}(C),\\ \tilde{\sigma}(B \vee C) & := & \tilde{\sigma}(B) \vee \tilde{\sigma}(C),\\ \tilde{\sigma}(B \to_{\mathtt{c}} C) & := & \tilde{\sigma}(B) \to_{\mathtt{c}} \tilde{\sigma}(C),\\ \tilde{\sigma}(B \to_{\mathtt{i}} C) & := & \tilde{\sigma}(B) \to_{\mathtt{i}} \tilde{\sigma}(C).\end{array}$$

When no confusion arises, we simply say that $\sigma : \mathsf{Prop} \to \mathsf{Form}$ is a *uniform substitution*.

Lemma 17. Let $A \in \mathsf{Form}_{\mathbf{C}}$. If $\models_3 A$, then $\models_3 \tilde{\sigma}(A)$ for all uniform substitutions $\sigma : \mathsf{Prop} \to \mathsf{Form}$.

Proof. Assume that $\models_3 A$. Fix any valuation $v : \mathsf{Prop} \to \{\{1\}, \{0, 1\}, \{0\}\}$ and any uniform substitution $\sigma : \mathsf{Prop} \to \mathsf{Form}$. The goal is to show that $1 \in \overline{v}(\tilde{\sigma}(A))$. Define $v' : \mathsf{Prop} \to \{\{1\}, \{0, 1\}, \{0\}\}$ as follows:

$$\begin{array}{lcl}1 \in v'(p) & \text{iff} & 1 \in \overline{v}(\sigma(p)),\\ 0 \in v'(p) & \text{iff} & 0 \in \overline{v}(\sigma(p)),\end{array}$$

for all $p \in \mathsf{Prop}$. By the assumption, we have $1 \in \overline{v'}(A)$. By induction on a formula B, we can establish:

$$\begin{array}{lcl}1 \in \overline{v'}(B) & \text{iff} & 1 \in \overline{v}(\tilde{\sigma}(B)),\\ 0 \in \overline{v'}(B) & \text{iff} & 0 \in \overline{v}(\tilde{\sigma}(B)).\end{array}$$

Here, we deal only with the case in which B is of the form $C \to_{\mathtt{i}} D$:

$$\begin{array}{lll}1 \in \overline{v'}(C \to_{\mathtt{i}} D) & \text{iff} & 1 \notin \overline{v'}(C) \text{ or } 1 \in \overline{v'}(D),\\ & \text{iff} & 1 \notin \overline{v}(\tilde{\sigma}(C)) \text{ or } 1 \in \overline{v}(\tilde{\sigma}(D)), \text{ by induction hypothesis,}\\ & \text{iff} & 1 \in \overline{v}(\tilde{\sigma}(C) \to_{\mathtt{i}} \tilde{\sigma}(D)),\\ & \text{iff} & 1 \in \overline{v}(\tilde{\sigma}(C \to_{\mathtt{i}} D)).\end{array}$$

$$\begin{array}{lll}0 \in \overline{v'}(C \to_{\mathtt{i}} D) & \text{iff} & 1 \in \overline{v'}(C) \text{ and } 0 \in \overline{v'}(D),\\ & \text{iff} & 1 \in \overline{v}(\tilde{\sigma}(C)) \text{ and } 0 \in \overline{v}(\tilde{\sigma}(D)), \text{ by induction hypothesis,}\\ & \text{iff} & 0 \in \overline{v}(\tilde{\sigma}(C) \to_{\mathtt{i}} \tilde{\sigma}(D)),\\ & \text{iff} & 0 \in \overline{v}(\tilde{\sigma}(C \to_{\mathtt{i}} D)).\end{array}$$

Since we have $1 \in \overline{v'}(A)$, we conclude that $1 \in \overline{v}(\tilde{\sigma}(A))$, as required. ■

Lemma 18. Let $A \in \mathsf{Form}$. If A is a theorem of $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$, then A is 3-valid.

Proof. It suffices to show tha each axiom of $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$ is 3-valid and that each rule of the system preserves 3-validity.

- `(CL)` Let A be an instance of a classical tautology, i.e., A is of the form $\tilde{\sigma}(A')$, where $A' \in \mathsf{Form}_{\mathbf{C}}$ is a classical tautology and $\sigma : \mathsf{Prop} \to \mathsf{Form}$ is a uniform substitution. Recall that Form is the set of all formulas of the syntax for $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$. Our goal is to show that $\models_3 \tilde{\sigma}(A')$. By Lemma 17, it suffices to show that $\models_3 A'$. Since $A' \in \mathsf{Form}_{\mathbf{C}}$, Proposition 15 tells us that we need to establish that A' is a classical tautology. But, this is our assumption.
- `(ID)` This is trivial, since $1 \in \overline{v}(A \to B)$ iff $1 \in \overline{v}(A)$ implies $1 \in \overline{v}(B)$.

- (CK) We show that $\models_3 (A \to_{\mathtt{i}} (B \to_{\mathtt{c}} C)) \to_{\mathtt{c}} ((A \to_{\mathtt{i}} B) \to_{\mathtt{c}} (A \to_{\mathtt{i}} C))$. Fix any valuation $v : \mathsf{Prop} \to \{\{1\}, \{0,1\}, \{0\}\}$. Our goal is to show that $1 \in \overline{v}((A \to_{\mathtt{i}} (B \to_{\mathtt{c}} C)) \to_{\mathtt{c}} ((A \to_{\mathtt{i}} B) \to_{\mathtt{c}} (A \to_{\mathtt{i}} C)))$. It suffices to show that $0 \notin \overline{v}(A \to_{\mathtt{i}} (B \to_{\mathtt{c}} C))$ implies that $1 \in \overline{v}((A \to_{\mathtt{i}} B) \to_{\mathtt{c}} (A \to_{\mathtt{i}} C))$. Suppose that $0 \notin \overline{v}(A \to_{\mathtt{i}} (B \to_{\mathtt{c}} C))$. This implies that $1 \notin \overline{v}(A)$ or $0 \notin \overline{v}(B \to_{\mathtt{c}} C)$. For each case, we establish that $1 \in \overline{v}((A \to_{\mathtt{i}} B) \to_{\mathtt{c}} (A \to_{\mathtt{c}} B))$, i.e., $0 \in \overline{v}(A \to_{\mathtt{i}} B)$ or $1 \in \overline{v}(A \to_{\mathtt{c}} C)$. If $1 \notin \overline{v}(A)$ holds, then $1 \in \overline{v}(A \to_{\mathtt{i}} C)$ holds; hence, $1 \in \overline{v}((A \to_{\mathtt{i}} B) \to_{\mathtt{c}} (A \to_{\mathtt{i}} C))$. If $0 \notin \overline{v}(B \to_{\mathtt{c}} C)$ holds, then $1 \notin \overline{v}(B)$ or $0 \notin \overline{v}(C)$ holds. Without loss of generality, we can also assume that $1 \in \overline{v}(A)$. If $1 \notin \overline{v}(B)$ holds, we derive from Proposition 12 that $0 \in \overline{v}(B)$. With $1 \in \overline{v}(A)$, it implies that $0 \in \overline{v}(A \to_{\mathtt{i}} B)$. Therefore, we can obtain $1 \in \overline{v}((A \to_{\mathtt{i}} B) \to_{\mathtt{c}} (A \to_{\mathtt{i}} C))$. If $0 \notin \overline{v}(C)$ holds, then we deduce from Proposition 12 that $1 \in \overline{v}(C)$. By this, we can obtain that $1 \in \overline{v}(A \to_{\mathtt{i}} C)$. Therefore, $1 \in \overline{v}((A \to_{\mathtt{i}} B) \to_{\mathtt{c}} (A \to_{\mathtt{i}} C))$ holds. This finishes our argument by cases.
- (CMP) We show that $\models_3 (A \to_{\mathtt{i}} B) \to_{\mathtt{c}} (A \to_{\mathtt{c}} B)$. Fix any valuation $v : \mathsf{Prop} \to \{\{1\}, \{0,1\}, \{0\}\}$. Our goal is to show that $1 \in \overline{v}((A \to_{\mathtt{i}} B) \to_{\mathtt{c}} (A \to_{\mathtt{c}} B))$. It suffices to show that $0 \notin \overline{v}(A \to_{\mathtt{i}} B)$ implies that $1 \in \overline{v}(A \to_{\mathtt{c}} B)$. Suppose that $0 \notin \overline{v}(A \to_{\mathtt{i}} B)$. This implies that $1 \notin \overline{v}(A)$ or $0 \notin \overline{v}(B)$. If $1 \notin \overline{v}(A)$ holds, then, by Proposition 12, $0 \in \overline{v}(A)$ holds; hence, $1 \in \overline{v}(A \to_{\mathtt{c}} B)$. If $0 \notin \overline{v}(B)$, then we deduce from Proposition 12 that $1 \in \overline{v}(B)$; hence, $1 \in \overline{v}(A \to_{\mathtt{c}} B)$. For both cases, we have established that $1 \in \overline{v}(A \to_{\mathtt{c}} B)$.
- (PER) For this validity, we do not have to impose any restriction on A. We show that $\models_3 A \to_{\mathtt{c}} (B \to_{\mathtt{i}} A)$. Fix any valuation $v : \mathsf{Prop} \to \{\{1\}, \{0,1\}, \{0\}\}$. Our goal is to show that $1 \in \overline{v}(A \to_{\mathtt{c}} (B \to_{\mathtt{i}} A))$. It suffices to show that $0 \notin \overline{v}(A)$ implies that $1 \in \overline{v}(B \to_{\mathtt{i}} A)$. Suppose that $0 \notin \overline{v}(A)$. By Proposition 12, we get $1 \in \overline{v}(A)$; hence, $1 \in \overline{v}(B \to_{\mathtt{i}} A)$, as required.
- (MPI) We show that $\models_3 A$ and $\models_3 A \to_{\mathtt{i}} B$ imply that $\models_3 B$. Suppose that $\models_3 A$ and $\models_3 A \to_{\mathtt{i}} B$. Our goal is to show that $\models_3 B$. Fix any valuation $v : \mathsf{Prop} \to \{\{1\}, \{0,1\}, \{0\}\}$. We show that $1 \in \overline{v}(B)$. By the supposition, we have $1 \in \overline{v}(A)$ and $1 \in \overline{v}(A \to_{\mathtt{i}} B)$. Since $1 \in \overline{v}(A)$ and $1 \notin \overline{v}(A)$ are not compatible, we can deduce from $1 \in \overline{v}(A \to_{\mathtt{i}} B)$ that $1 \in \overline{v}(B)$, as desired.
- (RCN) We show that $\models_3 A$ implies that $\models_3 B \to_{\mathtt{i}} A$. Suppose that $\models_3 A$. Our goal is to show that $\models_3 B \to_{\mathtt{i}} A$. Fix any valuation $v : \mathsf{Prop} \to \{\{1\}, \{0,1\}, \{0\}\}$. We show that $1 \in \overline{v}(B \to_{\mathtt{i}} A)$. By the supposition, $1 \in \overline{v}(A)$ holds. Therefore, we can obtain $1 \in \overline{v}(B \to_{\mathtt{i}} A)$ straightforwardly by Definition 11. ∎

Lemma 19. $\models_3 (p \wedge (p \to_{\mathtt{c}} q)) \to_{\mathtt{i}} q$ iff $p, p \to_{\mathtt{c}} q \models_3 q$.

Proof. This follows from the following equivalence: $1 \in \overline{v}(A \to_{\mathtt{i}} B)$ iff $1 \in \overline{v}(A)$ implies $1 \in \overline{v}(B)$. ∎

Theorem 1. The formula $(p \wedge (p \to_{\mathtt{c}} q)) \to_{\mathtt{i}} q$ is not a theorem in $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$.

Proof. Suppose that $(p \wedge (p \to_{\mathtt{c}} q)) \to_{\mathtt{i}} q$ is a theorem in $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$. By Lemma 18, $(p \wedge (p \to_{\mathtt{c}} q)) \to_{\mathtt{i}} q$ is 3-valid. By Lemma 19, $p, p \to_{\mathtt{c}} q \models_3 q$ should hold. This contradicts Lemma 14. ∎

Corollary 1. The Hilbert system $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$ is not semantically complete, i.e., there exists a formula C such that C is not a theorem of $\mathsf{H}(\mathbf{C} + \mathbf{J})^-$ but C is valid in the Kripke semantics in Definition 2.

Proof. By Proposition 10 and Theorem 1. ■

The argument described above implies, in order to obtain the completeness theorem, that the rule (MPC) is necessary. If (MPC) is added, Theorem 1 will no longer hold. This is because Lemma 18 does not hold for $\mathsf{H}(\mathbf{C}+\mathbf{J})$ ($\mathsf{H}(\mathbf{C}+\mathbf{J})^-$ plus the classical modus ponens (MPC)), since (MPC) does not preserve 3-validity, which is a well-known feature of the logic of paradox, as noted in [51].

4 Related Work

This section reviews proof theories for $\mathbf{C}+\mathbf{J}$ that are different from $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$ and explains how the collapsing problem is avoided and how the classical modus ponens, whose absence is a source of the semantic incompleteness of $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$, is expressed in these proof theories.

Humberstone [28] proposed a natural deduction system for $\mathbf{C}+\mathbf{J}$. Instead of classical and intuitionistic implications, Humberstone's syntax has classical and intuitionistic negations, denoted by "$\neg_{\mathtt{c}}$" and "$\neg_{\mathtt{i}}$," respectively, as primitive connectives. As a proof theory, he provided a natural deduction system for this logic, which contains the following two rules:

$$\frac{\Gamma, A \vdash B \wedge \neg_{\mathtt{c}} B}{\Gamma \vdash \neg_{\mathtt{c}} A}\ (RAA_{\neg_{\mathtt{c}}}) \quad \frac{\Gamma, A \vdash B \wedge \neg_{\mathtt{i}} B}{\Gamma \vdash \neg_{\mathtt{i}} A}\ (RAA_{\neg_{\mathtt{i}}})^{\dagger},$$

where † means that all occurrences of "$\neg_{\mathtt{c}}$" in some formulas in Γ are in the scope of "$\neg_{\mathtt{i}}$." In this natural deduction system, $A \to_{\mathtt{c}} B \vdash A \to_{\mathtt{i}} B$ does not hold because of the restriction imposed on $(RAA_{\neg_{\mathtt{i}}})$. Moreover, this natural deduction system has an ordinary elimination rule for disjunction, and the classical modus ponens is derived as a rule from this elimination rule, together with $(RAA_{\neg_{\mathtt{c}}})$. Humberstone [28] showed that this natural deduction system is sound and complete to the Kripke semantics for $\mathbf{C}+\mathbf{J}$.

Lucio [37] proposed a cut-free single-succedent calculus called $\mathsf{FO}^{\supset}$ for $\mathbf{C}+\mathbf{J}$. This calculus handles a "structured sequent," which has the following form: $\Delta \Rightarrow A$, where Δ is a finite sequence of finite sets of formulas and A is a formula. In the following, we use Γ, probably with subscripts or superscripts, to denote a finite set of formulas and Δ, Δ', and Δ'' to denote a finite sequence of finite sets of formulas. The semicolon sign is used to split a finite sequence of finite sets of formulas into its component finite sets of formulas, and the comma sign is used to split the sets of formulas into its elements. For example, $\Delta; B, \Gamma; \Delta' \Rightarrow A$ is a structured sequent, and its antecedent is the finite sequence beginning with the finite sequence Δ of finite sets of formulas, followed by the finite set $\{B\} \cup \Gamma$ of formulas, and ending with the finite sequence Δ' of finite sets of formulas. The calculus $\mathsf{FO}^{\supset}$ has the following left and right rules for classical and intuitionistic implications:

$$\frac{\Delta; \Gamma, A \Rightarrow B}{\Delta; \Gamma \Rightarrow A \to_{\mathtt{c}} B}\ (\Rightarrow \to_{\mathtt{c}}) \quad \frac{\Delta; \Gamma \Rightarrow A \quad \Delta; \Gamma, B; \Delta' \Rightarrow C}{\Delta; \Gamma, A \to_{\mathtt{c}} B; \Delta' \Rightarrow C}\ (\to_{\mathtt{c}} \Rightarrow)$$

$$\frac{\Delta; \{A\} \Rightarrow B}{\Delta \Rightarrow A \to_{\mathtt{i}} B}\ (\Rightarrow \to_{\mathtt{i}}) \quad \frac{\Delta; \Gamma; \Delta'; \Gamma' \Rightarrow A \quad \Delta; \Gamma; \Delta'; \Gamma', B; \Delta'' \Rightarrow C}{\Delta; \Gamma, A \to_{\mathtt{i}} B; \Delta'; \Gamma'; \Delta'' \Rightarrow C}\ (\to_{\mathtt{i}} \Rightarrow).$$

The collapsing problem is avoided by making use of the notion of a structured sequent. The rules for intuitionistic and classical implications can be distinguished since the antecedent of a sequent has a structure. This prevents the structured sequent $\{A \to_{\mathtt{c}} B\} \Rightarrow A \to_{\mathtt{i}} B$ from being derivable in $\mathsf{FO}^{\supset}$. The classical modus ponens is expressed as $(\Rightarrow \rightarrow_{\mathtt{c}})$ in this calculus, as is in LK, which is the ordinary sequent calculus for classical logic.

The notion of validity of a single-succedent structured sequent is defined as follows: a single-succedent structured sequent $\Gamma_0; \Gamma_1; \ldots; \Gamma_n \Rightarrow A$ is valid if for all Kripke models $M = (W, R, V)$ and for all sequences of possible worlds $w_0, w_1, \ldots, w_n \in W$ such that $w_{i-1} R w_i$ for any $1 \leq i \leq n$, if $w_i \models_M C$ for any $C \in \Gamma_i$ and any $0 \leq i \leq n$, then $w_n \models_M A$. Based on this notion, Lucio showed that the cut-free single-succedent sequent calculus $\mathsf{FO}^{\supset}$ is sound and complete to the Kripke semantics for $\mathbf{C}+\mathbf{J}$.

De and Omori [22] provided a Hilbert system that is slightly different from $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$ by expanding subintuitionistic logic with classical negation. This Hilbert system contains an axiom that is essentially the same as the axiom (PER) in $\mathsf{H}(\mathbf{C}+\mathbf{J})^-$, which prevents us from deriving $(A \rightarrow_{\mathtt{c}} B) \rightarrow_{\mathtt{c}} (A \rightarrow_{\mathtt{i}} B)$. Moreover, this system has the following axiom and rule:

(DNE)	$\neg_{\mathtt{c}}\neg_{\mathtt{c}} A \rightarrow_{\mathtt{i}} A$
(D-Antilogism)	From $((A \wedge B) \rightarrow_{\mathtt{i}} \neg_{\mathtt{c}} C) \vee D$, we may infer $((A \wedge C) \rightarrow_{\mathtt{i}} \neg_{\mathtt{c}} B) \vee D$.

The classical modus ponens is derivable as a rule in this system by using (DNE) and (D-Antilogism), as noted in [22, Proposition 4.9].

In [54], a multi-succedent sequent calculus is proposed as a proof theory for $\mathbf{C}+\mathbf{J}$. This calculus uses an ordinary notion of sequent, in which the antecedent and succedent of a sequent are multisets. This calculus is obtained by adding the following left and right rules for intuitionistic implications to the classical sequent calculus LK:

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Sigma \Rightarrow \Pi}{A \rightarrow_{\mathtt{i}} B, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\rightarrow_{\mathtt{i}} \Rightarrow)$$

$$\frac{A, C_1 \rightarrow_{\mathtt{i}} D_1, \ldots, C_m \rightarrow_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow B}{C_1 \rightarrow_{\mathtt{i}} D_1, \ldots, C_m \rightarrow_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow A \rightarrow_{\mathtt{i}} B}\ (\Rightarrow \rightarrow_{\mathtt{i}}),$$

where $p_k (1 \leq k \leq n)$ is a propositional variable. By the restriction on $(\Rightarrow \rightarrow_{\mathtt{i}})$, the sequent $A \rightarrow_{\mathtt{c}} B \Rightarrow A \rightarrow_{\mathtt{i}} B$ is not derivable in this sequent calculus. The classical modus ponens is expressed as the right rule for classical implication, as in $\mathsf{FO}^{\supset}$. This calculus is cut-free and sound and complete to the Kripke semantics for $\mathbf{C}+\mathbf{J}$.

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