BCI-Algebras and Related Logics

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Abstract

Kabzinski in [6] first introduced an extension of BCI-logic that is isomorphic to BCI-algebras. Kashima and Komori in [7] gave a Gentzen-style sequent calculus version of this logic as well as another sequent calculus which they proved to be equivalent. They used the second to prove decidability of the word problem for BCI-algebras. The decidability proof relies on cut elimination for the second system, this paper provides a fuller and simpler proof of this. Also supplied is a new decidability proof and proof finding algorithm for their second extension of BCI-logic and so for BCI-algebras.

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1 BCI-Algebra and Logic

BCI-algebras were first defined by Iseki in [5]. His work mentions the connection between theorems of the algebras and the axioms of BCI-logic, which are the types of the combinators B,C and I. (For more on combinators and implicational logics see Bunder [2]). The following formulation, which is equivalent to that of [5], but uses the notation of Kashima and Komori [7], having $a \to b$ for the $b * a$ and 1 for the 0 of [5], brings this out most clearly.

Definition 1 A BCI-algebra is an algebra $\langle A; \to, 1\rangle$ of type $(2, 0)$ such that for every $a, b, c \in A$ the following hold:
(1) $(a \to b) \to (c \to a) \to c \to b = 1$
(2) $(a \to b \to c) \to b \to a \to c = 1$
(3) $a \to a = 1$
(4) $a \to b = b \to a = 1 \implies a = b$

The formulas of the implicational logic formed by omitting the $= 1$ in (1), (2) and (3) are the types of the combinators B, C and I and the Hilbert-style axioms of BCI-logic.

$1 \to a = 1 \implies a = 1$, which is derivable from these axioms, and the unstated substitution of equality rule give the algebraic counterpart to the modus ponens or $\to_e$

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rule. Interestingly, Iseki had a different axiom (2) (corresponding to the type of the combinator CI), but still called his algebras BCI-algebras, even though he only had the (2) above as a theorem.

Given the $\to$ notation, the general notion of an isomorphism between an implicational logic and an algebra (for any set A), given in Bunder [3], simplifies to:

A logic L is isomorphic to an algebra $\langle A; \to, 1\rangle$ if $\vdash \alpha$ iff $\alpha = 1$.

Note that the above only considers theorems of the logic and equations of the form $\alpha = 1$ of the algebra that are provable from the axioms, so isomorphism is independent of A. We could also say that such a logic L is isomorphic to the class of all such algebras that have $1 \in A$. Note too that this use of the word isomophic is an extension of the standard one which is between mathematical structures of the same type.

It was shown in [3] that BCI-logic and algebra are not isomorphic but that BCK-logic, which has the extra axiom $\vdash a \to b \to a$, is isomorphic to BCK-algebra, which is BCI-algebra with the extra axiom $a \to 1 = 1$. Also, in [3], it was shown that BCI-logic, with the extra axiom $\vdash a \to a \to b \to b$ is isomorphic to BCI-algebra with the extra axiom $a \to a \to 1 = 1$. These were thought to be the minimal strength isomorphic logic and algebra.

In the same year however, Kabzinski in [6] improved upon this by showing that BCI-algebra is strongly adequate, i.e. a model for BCI-logic with the extra rule:

$$\to^+ \qquad \vdash a, \ \vdash b \quad \Longrightarrow \quad \vdash a \to b$$

The addition of this rule makes the extended BCI-logic isomorphic to BCI-algebra.

BCI-logic, with the rule $\to^+$, we will call BCI$^+$, as it is in Humberstone [4]. In [4] Humberstone shows that $\to^+$ can be replaced by $\vdash a \implies \vdash a \to b \to b$. Work in [4] and in Kowalski and Butchard [9], shows that BCI$^+$ is equivalent to BCI-logic with the extra axiom $\vdash (a \to a) \to b \to b$. The latter is probably the simplest form of a logic isomorphic to BCI-algebra.

Some years after [3] and [6], in [1], Blok and Pigozzi introduced algebraizable logics, which can include connectives other than $\to$. They show that a class of algebras is the equivalent algebraic semantics of a logic if the consequence operator $\vdash$ of the logic corresponds to the equational consequence operator of the class of algebras.(An extension of) BCI-logic being algebraizable is equivalent to being isomorphic to (an extension of) BCI-algebra, in the sense defined above.

In Section 2 of this paper we outline the sequent-style calculus version of BCI$^+$, due to Kashima and Komori [7], called BCI-pw. This uses a provably weakening rule, similar to the one later used by Humberstone in [4]. They also proved BCI$^+$ to be isomorphic to BCI-algebra.

In Section 3 we outline the sequent calculus BCI-bw of Kashima and Komori [7], which has, a seemingly stronger, balanced weakening rule. They prove BCI-bw to be equivalent to BCI-pw and use the former to prove the decidability of BCI$^+$. Their decidability proof relies on cut elimination being provable in BCI-bw (it is not provable in BCI-pw). For the proof of cut elimination they refer to the cut elimination proof for LJ given in Takeuti [9]. It is however not fully obvious how this applies when the bw rule is used.

In Section 4 we give a new proof of cut elimination for BCI-bw that takes account of the bw rule and is also simpler than the Takeuti proof.

In Section 5 we give a decision procedure for BCI^+ (and so for BCI-algebras), which is also a proof finding algorithm.

2 Kashima and Komori's Logic BCI-pw

Kashima and Komori, in [7], give the following Gentzen-style sequent calculus formulation for BCI^+.

Definition 2 BCI-pw
Axioms: $\alpha => \alpha, \qquad => 1$
Rules of Inference:

$$\frac{\Gamma, \alpha, \beta, \Delta => \gamma}{\Gamma, \beta, \alpha, \Delta => \gamma} exchange$$

$$\frac{\Gamma => \alpha \quad \alpha, \Delta => \beta}{\Gamma, \Delta => \beta} cut$$

$$\frac{\Gamma, \alpha => \beta}{\Gamma => \alpha \to \beta} \to right$$

$$\frac{\Gamma => \alpha \quad \beta, \Delta => \gamma}{\alpha \to \beta, \Gamma, \Delta => \gamma} \to left$$

$$\frac{\Gamma => \alpha \quad => \beta}{\Gamma, \beta => \alpha} provably\ weakening$$

They show that $=> \alpha$ if and only if $\alpha = 1$ holds for every BCI-algebra , i.e. that BCI^+ and the class of BCI-algebras are isomorphic.

Below we will use the following natural deduction formulation of BCI-pw or BCI^+. The counterpart of $=> 1$ is omitted as 1 can be defined as $a \to a$ for any particular variable a.

Definition 3 BCI-pwnd
Axiom: $\alpha \vdash \alpha$
Rules of Inference:

$$\frac{\Gamma, \alpha, \beta, \Delta \vdash \gamma}{\Gamma, \beta, \alpha, \Delta \vdash \gamma} exchange$$

$$\frac{\Gamma \vdash \alpha \quad \Delta \vdash \alpha \to \beta}{\Gamma, \Delta \vdash \beta} \to_e$$

$$\frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \to \beta} \to_i$$

$$\frac{\Gamma \vdash \alpha \quad \vdash \beta}{\Gamma, \beta \vdash \alpha} pw$$

BCI-pw and BCI-pwnd are equivalent because the $\to left$ rule of BCI-pw can be derived in BCI-pwnd using $\alpha \to \beta \vdash \alpha \to \beta$, $\to_e$, $\to_i$, and another use of $\to_e$.

Note that if the provably weakening rule is strengthened to

$$\frac{\Gamma \vdash \alpha \qquad \Delta \vdash \beta}{\Gamma, \Delta, \beta \vdash \alpha}$$

the resultant natural deduction system is theorem equivalent to the Hilbert-style BCI-logic with the extra axiom $\vdash a \to a \to b \to b$ of Bunder [3] mentioned above.

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3 Kashima and Komori's Logic BCI-bw

In [7], Kashima and Komori also introduce another sequent style extension of BCI logic, BCI-bw. This uses the notion of a balanced formula, defined below.

Definition 4 If $\cup$ is multiset union,
(a) $pos(a) = \{a\}$, $neg(a) = \emptyset$.

(b) $pos(\alpha \to \beta) = neg(\alpha) \cup pos(\beta)$.

$neg(\alpha \to \beta) = pos(\alpha) \cup neg(\beta)$.

(c) $pos(\{\alpha_1, \ldots, \alpha_n\}) = pos(\alpha_1) \cup \ldots \cup pos(\alpha_n)$,

$neg(\{\alpha_1, \ldots, \alpha_n\}) = neg(\alpha_1) \cup \ldots \cup neg(\alpha_n)$.

Definition 5
(a) A multiset of implicational formulas Δ is balanced if $neg(\Delta) = pos(\Delta)$.
(b) $\alpha_1, \ldots, \alpha_n \vdash \alpha$ is balanced if $\alpha_1 \to \ldots \to \alpha_n \to \alpha$ is balanced.

It follows that $pos(\Delta \vdash \alpha) = neg(\Delta) \cup pos(\alpha)$, We can now define BCI-bw.

Definition 6 BCI-bw is BCI-pw with the provably weakening rule replaced by the balanced weakening rule:

$$\frac{\Delta => \alpha \qquad \Gamma\ balanced}{\Delta, \Gamma => \alpha}\, balanced\ weakening.$$

BCI-bwnd is BCI-pwnd with the pw rule replaced by:

$$\frac{\Delta \vdash \alpha \qquad \Gamma\ balanced}{\Delta, \Gamma \vdash \alpha}\, bw$$

As in Kashima and Komori [7] we have:

Theorem 1 If $\Delta \vdash \alpha$ then $\Delta \vdash \alpha$ is balanced.

and

Theorem 2 BCI-pw and BCI-bw, and so BCI-pwnd and BCI-bwnd, are equivalent.

In any of these extensions of a BCI proof we can ignore any uses of the exchange rule, as any Δ in a $\Delta \vdash \alpha$ or $\Delta => \alpha$ can be treated as a multiset. So any Δ, α, Γ will be the same as Δ, Γ, α in the use of the $\to_i$ rule. By subset of a multi set, we will below, always mean submultiset.

In a branch of a BCI-bwnd proof that (not counting exchange rules) only has $\to_e$ and bw steps, the bw rules can be postponed to the end of the branch as

$$\dfrac{\dfrac{\Delta \vdash \alpha \qquad \Gamma_1 \ \textit{balanced}}{\Delta, \Gamma_1 \vdash \alpha}\,bw \qquad \Gamma_2 \ \textit{balanced}}{\Delta, \Gamma_1, \Gamma_2 \vdash \alpha}\,bw$$

can be replaced by

$$\dfrac{\Delta \vdash \alpha \qquad \Gamma_1, \Gamma_2 \ \textit{balanced}}{\Delta, \Gamma_1, \Gamma_2 \vdash \alpha}\,bw$$

and

$$\dfrac{\dfrac{\Delta \vdash \alpha \to \beta \qquad \Gamma \ \textit{balanced}}{\Delta, \Gamma \vdash \alpha \to \beta}\,bw \qquad \Delta' \vdash \alpha}{\Delta, \Delta', \Gamma \vdash \beta}\to\mathrm{e}$$

can be replaced by

$$\dfrac{\dfrac{\Delta \vdash \alpha \to \beta \qquad \Delta' \vdash \alpha}{\Delta, \Delta' \vdash \beta}\to_e \qquad \Gamma \ \textit{balanced}}{\Delta, \Delta', \Gamma \vdash \beta}\,bw$$

In the following case a bw step before a $\to_i$ step can also be postponed.

$$\dfrac{\dfrac{\Delta, \alpha \vdash \beta \qquad \Gamma \ \textit{balanced}}{\Delta, \Gamma, \alpha \vdash \beta}\,bw}{\Delta, \Gamma \vdash \alpha \to \beta}\to_i$$

can be replaced by

$$\dfrac{\dfrac{\Delta, \alpha \vdash \beta}{\Delta \vdash \alpha \to \beta}\to_i \qquad \Gamma \ \textit{balanced}}{\Delta, \Gamma \vdash \alpha \to \beta}\,bw$$

If $\alpha \notin \Delta$, in

$$\dfrac{\dfrac{\Delta \vdash \beta \qquad \Gamma, \alpha \ \textit{balanced}}{\Delta, \Gamma, \alpha \vdash \beta}\,bw}{\Delta, \Gamma \vdash \alpha \to \beta}\to_i$$

the bw step cannot be postponed.

We will call a proof an *amended proof* if it has no bw steps that can be postponed.

In an amended natural deduction proof a cut is a sequence of steps of the form:

$$\dfrac{\dfrac{\Delta_1, \alpha \vdash \beta}{\Delta_1 \vdash \alpha \to \beta}\to_i \qquad \Delta_2 \vdash \alpha}{\Delta_1, \Delta_2 \vdash \beta}\to_e$$

The formula $\alpha \to \beta$ is called the cut formula.

We now prove cut elimination for BCI$^+$.

Theorem 3 All cuts can be eliminated from a BCI$^+$ proof.

Proof Consider the first cut in the proof as shown above.

If the α is first introduced to the left of the $\vdash$ by means of the axiom $\alpha \vdash \alpha$, the proof can be modified by replacing $\alpha \vdash \alpha$ by the proof of $\Delta_2 \vdash \alpha$ and replacing one α on the left of the $\vdash$ by Δ_2 in the steps up to $\Delta_1, \alpha \vdash \beta$ which becomes $\Delta_1, \Delta_2 \vdash \beta$.

The new proof is now shorter by two steps, the axiom step and the final $\to_e$ step, even if, as a result of the change, α is now a cut formula.

The only other way of introducing α to the left of the $\vdash$ is in a bw step from $\Delta_3 \vdash \delta$ to $\Delta_3, \Delta_4, \alpha \vdash \delta$. We then have, up to and including $\Delta_1, \alpha \vdash \beta$:

$$\dfrac{\Delta_3 \vdash \delta \qquad \Delta_4, \alpha \ \textit{balanced}}{\Delta_3, \Delta_4, \alpha \vdash \delta}$$

$$\vdots$$

$$\Delta_1, \alpha \vdash \beta$$

Now as the proof is in BCI$^+$, $\Delta_2 \vdash \alpha$ is balanced, so we have

$$pos(\Delta_2) \cup neg(\alpha) = neg(\Delta_2) \cup pos(\alpha)$$

and

$$pos(\Delta_4) \cup pos(\alpha) = neg(\Delta_4) \cup neg(\alpha).$$

i.e. $\quad pos(\Delta_2, \Delta_4) \cup neg(\alpha) \cup pos(\alpha) = neg(\Delta_2, \Delta_4) \cup pos(\alpha) \cup neg(\alpha).$

As these are multiset unions $pos(\Delta_2, \Delta_4) = neg(\Delta_2, \Delta_4)$ so Δ_2, Δ_4 is balanced.

The proof can now be rewritten starting with:

$$\dfrac{\Delta_3 \vdash \delta \qquad \Delta_2, \Delta_4 \ \textit{balanced}}{\Delta_2, \Delta_3, \Delta_4 \vdash \delta}$$

Now the steps changing Δ_3, Δ_4 to Δ_1 and δ to β follow giving $\Delta_1, \Delta_2 \vdash \beta$. The new proof is shorter by the steps leading to $\Delta_2 \vdash \alpha$ and the final $\to_e$ step.

So, in either case, if an amended proof contains a cut, it can be replaced by a shorter proof. So all cuts can be eliminated.

4 A Decision Procedure / Proof Finding Algorithm for BCI$^+$

We start this section with an analysis of the form that a cut free amended proof in BCI$^+$ must take. This will lead to the algorithm below and is also a demonstration that the algorithm is effective.

In this analysis we need the concept of the length of a formula and statement.

Definition 7 The length of a formula and statement are given by:
$l(a) = 1$, $l(\alpha \to \beta) = l(\alpha) + l(\beta) + 1$
$l(\alpha_1, \ldots, \alpha_n \vdash \alpha) = l(\alpha_1 \to \ldots \to \alpha_n \to \alpha)$.

The last steps in a cut free amended proof of

$$\Delta \vdash \alpha \tag{1}$$

must include, for some Δ^1 and β:

$$\Delta^1 \vdash \beta \tag{2}$$

which does not come from a $\to_i$ step and (2) is followed by:

$$\dfrac{\dfrac{\Delta^1 - \{\alpha_n\} \vdash \alpha_n \to \beta \qquad \Delta_1 \textit{ balanced}}{\Delta^2 \vdash \alpha_n \to \beta}\,bw}{\Delta^2 - \{\alpha_{n-1}\} \vdash \alpha_{n-1} \to \alpha_n \to \beta \qquad \Delta_2 \textit{ balanced}}\to_i$$
$$\vdots$$
$$\dfrac{\Delta^n - \{\alpha_1\} \vdash \alpha_1 \to \ldots \to \alpha_n \to \beta \qquad \Delta_n \textit{ balanced}}{\Delta \vdash \alpha}\,bw$$

where $n \geq 0, \alpha = \alpha_1 \to \ldots \to \alpha_n \to \beta$, $\Delta = \Delta^n - \{\alpha_1\} \cup \Delta_n$, and $\Delta^{i+1} = \Delta^i - \{\alpha_{n-i+1}\} \cup \Delta_i$ for $0 < i < n$. Note that if, for $0 < i < n$, $\alpha_i \notin \Delta_{n-i}$, the bw step involving Δ_{n-i} would have been postponed. In that case we will still use the above notation but with $\Delta_{n-i} = \emptyset$. So, for each $i, \alpha_i \in \Delta_{n-i}$ or $\Delta_{n-i} = \emptyset$.

There are now two options.

(a) (2) can come by a bw step from Γ balanced and $\Delta^0 \vdash \beta$ (which doesn't come by a $\to_i$ step or a bw step), where $\alpha_n \in \Gamma$, $\alpha_n \notin \Delta^0$ (as the proof is amended) and $\Delta^1 = \Delta^0, \Gamma$. We then have:

$$\Delta, \alpha_1, \ldots, \alpha_n = \Delta^0, \Gamma, \Delta_1, \ldots, \Delta_n. \tag{3}$$

As for each choice of $\beta, \Delta, \alpha_1, \ldots, \alpha_n$ are known, this gives a finite number of choices of $\Delta^0, \Gamma, \Delta_1, \ldots, \Delta_n$ where $\Gamma, \Delta_1, \ldots, \Delta_n$ and $\Delta^0 \vdash \beta$ must all be balanced and for $0 < i < n$, either $\alpha_i \in \Delta_{n-i}$ or $\Delta_{n-i} = \emptyset$.

(b) Otherwise (2) must come from

$$\Gamma_0 \vdash \delta_1 \to \ldots \to \delta_k \to \beta \tag{4}$$

which doesn't come by a $\to_e$ step, and $k(\geq 0) \to_e$ steps using $\Gamma_1 \vdash \delta_1, \ldots, \Gamma_k \vdash \delta_k$. As the proof is amended and cut free, $\Gamma_0 = \delta_1 \to \ldots \to \delta_k \to \beta$ i.e. (4) is the axiom

$$\delta_1 \to \ldots \to \delta_k \to \beta \vdash \delta_1 \to \ldots \to \delta_k \to \beta. \tag{5}$$

The instance of $\delta_1 \to \ldots \to \delta_k \to \beta$ introduced on the left of the $\vdash$ in (5) will either be one of the α_is or end up as an element of Δ.

We then have:

$$\Delta, \alpha_1, \ldots, \alpha_n = \Gamma_1, \ldots, \Gamma_k, \delta_1 \to \ldots \to \delta_k \to \beta, \Delta_1, \ldots, \Delta_n. \tag{6}$$

As, for each choice of β, $\Delta, \alpha_1, \ldots, \alpha_n$ are known, this gives a finite number of choices of $\delta_1 \to \ldots \to \delta_k \to \beta$ and then, for each of these, a finite number of choices of $\Gamma_1, \ldots, \Gamma_k, \Delta_1, \ldots, \Delta_n$ where for $0 < i < n$, either $\alpha_i \in \Delta_{n-i}$ or $\Delta_{n-i} = \emptyset$.

If there is to be a proof of (1) there must be, in case (b), cut free amended proofs of $\Gamma_1 \vdash \delta_1, \ldots, \Gamma_k \vdash \delta_k$ for at least one of these choices or, in case (a), such a proof of $\Delta^0 \vdash \beta$.

Clearly each possible $\Gamma_i \vdash \delta_i$ or $\Delta^0 \vdash \beta$ is shorter (i.e. of lesser length) than $\Delta \vdash \alpha$, so the finding of a proof for (1) can be reduced to the finding of proofs for a finite number of shorter statements. This process converges resulting in either a proof of (1) or a demonstration that there can be no proof.

The BCI$^+$ Decision Procedure / Proof Finding Algorithm

Aim To find a proof for

$$\Delta \vdash \alpha. \tag{7}$$

or show that it has no proof.

Step 1 If (7) is not balanced it has no proof.

Step 2 If $\Delta = \{\alpha\}$, (7) is the axiom.

If $\alpha \in \Delta$ and $\Delta - \{\alpha\}$ is nonempty, as (7) is balanced, $\Delta - \{\alpha\}$ is also balanced, so (7) is proved by:

$$\frac{\alpha \vdash \alpha \qquad \Delta - \{\alpha\}\ \mathit{balanced}}{\Delta \vdash \alpha}bw$$

Step 3 For every n such that $\alpha = \alpha_1 \to \ldots \to \alpha_n \to \beta$:

(a) Determine if there are $\Delta^0, \Gamma, \Delta_1, \ldots, \Delta_n$ which satisfy (3) and its restrictions and prove $\Delta^0 \vdash \beta$. If this can be done the rest of the proof of (2) is

$$\frac{\Delta^0 \vdash \beta \qquad \Gamma\ \mathit{balanced}}{\Delta^1 \vdash \beta}bw$$

which is followed by the n $\to_i$ steps and up to n bw steps to prove (7).

(b) For each selection of appropriate $\Gamma_1, \ldots, \Gamma_k, \delta_1, \ldots, \delta_k, \Delta_1, \ldots, \Delta_n$, which satisfy (6) and its restrictions, prove $\Gamma_1 \vdash \delta_1, \ldots, \Gamma_k \vdash \delta_k$.

If there is such a set of k proofs the proof of (7) is:

$$\dfrac{\Gamma_1 \vdash \delta_1 \qquad \delta_1 \to \ldots \to \delta_k \to \beta \vdash \delta_1 \to \ldots \to \delta_k \to \beta}{\Gamma_1, \delta_1 \to \ldots \to \delta_k \to \beta \vdash \delta_2 \to \ldots \to \delta_k \to \beta} \to_e$$

$$\vdots$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_k \vdash \delta_k \qquad \Gamma_1, \ldots, \Gamma_{k-1}, \delta_1 \to \ldots \to \delta_k \to \beta \vdash \delta_k \to \beta}{\Delta^1 \vdash \beta} \to_e}{\Delta^1 - \{\alpha_n\} \vdash \alpha_n \to \beta} \to_i \qquad \Delta_1 \textit{ balanced}}{\Delta^2 \vdash \alpha_n \to \beta} bw$$

$$\vdots$$

$$\dfrac{\Delta^n - \{\alpha_1\} \vdash \alpha_1 \to \ldots \to \alpha_n \to \beta \qquad \Delta_n \textit{ balanced}}{\Delta \vdash \alpha} bw$$

If there is no set of proofs $\Gamma_i \vdash \delta_i$ for terms satisfying (6) and no proof of $\Delta^0 \vdash \beta$ for terms satisfying (3), there is no proof of (7).

Note 1 If $\Delta = \emptyset, \Delta_n = \emptyset$ and also $n > 0$. (If $\Delta = \emptyset$ and $n = 0$, (6) can't hold and if (3) holds we have $\Gamma = \emptyset$, which is impossible as we must have $\alpha_n \in \Gamma$).

Note 2 This algorithm can be reduced to a BCI algorithm by leaving out the options involving the bw rule, i.e. the second part of Step 2 and the part of Step 3 involving equation (3). When using equation (6), $\Delta_1 = \ldots = \Delta_n = \emptyset$.

In the examples below we omit, the by now obvious, indications of the rules that are used.

Example 1 To find all amended cut free proofs of $\vdash \alpha$, where

$$\alpha = (a \to a) \to (a \to a) \to a \to a.$$

Step 1 $\vdash \alpha$ is balanced.

Step 2 $\{\alpha\} \neq \emptyset$ and $\alpha \notin \emptyset$.

Step 3 Case 1 If $n = 1$, $\alpha_1 = a \to a$ and $\beta = (a \to a) \to a \to a$, so (6) cannot be satisfied.

If (3) holds, $\Gamma = a \to a,\ \Delta^0 = \Delta_1 = \emptyset$, so we must prove $\vdash (a \to a) \to a \to a$, by this same algorithm. However any cut free amended proof of this ends in a $\to_i$ step, which is excluded by the conditions on (3).

Case 2 If $n = 2, \beta = \alpha_1 = \alpha_2 = a \to a$. (6) then holds only if $k = 0$ and $\Delta_1 = a \to a$. The proof is then:

$$\dfrac{\dfrac{\dfrac{a \to a \vdash a \to a}{\vdash (a \to a) \to a \to a} \qquad a \to a \textit{ balanced}}{a \to a \vdash (a \to a) \to a \to a}}{\vdash (a \to a) \to (a \to a) \to a \to a}$$

If (3) holds with $\Gamma = a \to a, a \to a,\ \Delta^0 = \Delta_1 = \emptyset$ or $\Gamma = \Delta_1 = a \to a$ and $\Delta^0 = \emptyset$ we need a cut free amended proof of $\vdash a \to a$ not ending with a $\to_i$ step, which is impossible.

We also can't have $\Gamma = \ \Delta^0 = a \to a$ and $\Delta_1 = \emptyset$, as in that case the bw step involving Γ could have been postponed.

Case 3 If $n = 3, \beta = \alpha_3 = a, \alpha_1 = \alpha_2 = a \to a$. (3) doesn't hold as it requires $a \in \Gamma$ where Γ is balanced. (6) holds, with $k = 1$, $\delta_1 = a$, $\Gamma_1 = a \to a, a$ and $\Delta_1 = \Delta_2 = \emptyset$, so the proof is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a \vdash a \qquad a \to a \vdash a \to a}{a \to a, a \vdash a} \qquad a \to a \vdash a \to a}{a \to a, a \to a, a \vdash a}}{a \to a, a \to a \vdash a \to a}}{a \to a \vdash (a \to a) \to a \to a}}{\vdash (a \to a) \to (a \to a) \to a \to a}$$

Note that the proof in Case 2 is a BCI$^+$ proof, but that in Case 3 is (the only) cut free, amended, BCI proof.

Example 2 To find all minimal length amended cut free proofs of $\vdash \alpha$ where $\alpha = a \to (b \to b) \to a$.

Step 1 $\vdash \alpha$ is balanced.

Step 2 $\Delta = \emptyset \neq \{\alpha\}$.

Step 3 If $n = 1, \alpha_1 = a, \beta = (b \to b) \to a$, (3) can't hold as Γ must be balanced and contain $\alpha_1 = a$. (6) also can't hold.

If $n = 2, \alpha_1 = a, \alpha_2 = b \to b$ and $\beta = a$. If (6) holds $k = 0$ and $\Delta_1 = b \to b$. But then $\Delta^1 \vdash \beta$ is $a \vdash a$ and must be followed by an immediate $\to_i$ step, which doesn't give the result.

(3) holds with $\Delta^0 = a$ and $\Gamma = b \to b$. So the proof comes by a bw step after $a \vdash a$ and two $\to_i$ steps.

Note that this, the only amended cut free BCI$^+$ proof of $a \to (b \to b) \to a$, is not a BCI proof.

The final example shows that not all balanced theorems of $H_\to$ are theorems of BCI$^+$.

Example 3 To find all minimal length amended cut free proofs of $\vdash \alpha$ where $\alpha = (\gamma \to \gamma \to b) \to \gamma \to b$ and $\gamma = ((a \to c) \to c) \to a$

Step 1 $\vdash \alpha$ is balanced.

Step 2 $\Delta = \emptyset \neq \{\alpha\}$.

Step 3 If $n = 1, \alpha_1 = \gamma \to \gamma \to b$ and $\beta = \gamma \to b$. (3) would require and $\alpha_1 = \Gamma$, so it can't hold as Γ must be balanced. (6) also can't hold as it would require $\delta_1 = \gamma$, $\Gamma_1 = \emptyset$ and $\vdash \gamma$, which is not balanced.

If $n = 2, \alpha_1 = \gamma \to \gamma \to b$, $\alpha_2 = \gamma$ and $\beta = b$.

If (6) holds, $k = 2$, $\delta_1 = \delta_2 = \gamma$ and either $\Gamma_1 = \gamma$ and $\Gamma_2 = \emptyset$ or $\Gamma_2 = \gamma$ and $\Gamma_1 = \emptyset$. For α to have a BCI$^+$ proof we also need $\vdash \gamma$, which is not possible as it is not even a theorem of $H_\to$

(3) requires $\Delta^0 = \gamma \to \gamma \to b$ and $\Gamma = \gamma$. So a BCI$^+$ proof of α requires $\gamma \to \gamma \to b \vdash b$.

To test this we need to reuse the algorithm. Steps 1 and 2 are satisfied, so we need a new Step 3 with $n = 0$. As neither (3) nor (6) can hold neither $\gamma \to \gamma \to b \vdash b$ nor $\vdash \alpha$ are provable.

5 References

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