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Minimally Nonstandard K3 and FDE

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Abstract

Graham Priest has formulated the minimally inconsistent logic of paradox (MiLP), which is paraconsistent like Priest's logic of paradox (LP), while staying closer to classical logic. We present logics that stand to (the propositional fragments of) strong Kleene logic (K3) and the logic of first-degree entailment (FDE) as MiLP stands to LP. That is, our logics share the paracomplete and the paraconsistent-cum-paracomplete nature of K3 and FDE, respectively, while keeping these features to a minimum in order to stay closer to classical logic. We give semantic and sequent-calculus formulations of these logics, and we highlight some reasons why these logics may be interesting in their own right.

Keywords: bilateralism, logic of paradox, paraconsistent logics; paracomplete logics; non-monotonic logics

1 Introduction

Priest (1991; 2006) has suggested a way to bring his logic of paradox (LP) closer to classical logic by a tweak that results in what Priest calls the "minimally inconsistent logic of paradox" (MiLP). In this paper, we will offer parallel tweaks for (the propositional fragments of) the paracomplete strong Kleene logic (K3) and the logic of first-degree entailment (FDE). The

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results are what we call minimally nonstandard K3 and FDE, i.e., MiK3 and MiFDE. These logics enjoy similar advantages over K3 and FDE, respectively, as MiLP enjoys over LP. In particular, these logics stay closer to classical logic than K3 and FDE, in a way that is analogous to how MiLP stays closer to classical logic than LP.

One motivation for this project is that MiLP is, as Rosenblatt (2021, fn 25) puts it, "Priest's preferred way of carrying out classical recapture." Now, K3 is the dual logic of LP and it is often seen as the logic that is for the paracomplete approach to the semantic paradoxes what LP is for the paraconsistent approach (Cobreros et al., 2020a,b). Hence, the question arises whether advocates of K3 can adopt Priest's preferred approach to classical recapture. And since FDE combines the paraconsistency of LP and the paracompleteness of K3 and is also advocated as a response to the paradoxes (Leitgeb, 1999; Beall, 2017),[1] the same question arises for FDE. Our aim in this paper is to show how advocates of K3 and FDE can adopt the approach to classical recapture exemplified by MiLP.

Advocates of K3 and FDE have presented various ways of carrying out kinds of recapture, such as Beall's (2018) use of 'shrieked' and 'shrugged' predicates. We will not compare our approach to these existing approaches. Rather, we will focus on how the MiLP approach can be carried over to K3 and FDE. Putting the technical details of this on the table will prove to be interesting enough in its own right. In particular, we will present sequent calculi for MiK3 and MiFDE that include some important innovations, which are inspired by a recently developed sequent calculus for MiLP by one of us (Golan, 2022), and also draws on Da Ré and Pailos (2022).

It is important to note at the outset, however, that one can mean different things by "classical recapture." The sense in which MiLP recaptures classical logic is that if there is a classical valuation that makes all the premises, Γ, true, then something follows from Γ in MiLP iff it follows from Γ in classical logic. The corresponding kind of recapture that we will offer for K3 is that if there is a classical valuation that makes all of the conclusions, Δ, false, then Δ follows from something in MiK3 iff it Δ follows from it in classical logic. And if there is a classical valuation that makes the

[1]One might allow gluts and gaps, e.g., because one wants to stay agnostic on whether paradoxical sentences are gluts or gaps (Restall, 2017) or to say that the Liar is a glut but the Truth-Teller is a gap (Beall, 2019).

premises true and a classical valuation that makes the conclusions false, then the inference is valid in MiFDE iff it is classically valid. Note that just like K3 so MiK3 has no logical truths, i.e., no sentence ϕ such that $\emptyset \models \phi$. Hence, the way in which MiK3 is closer to classical logic than K3 is not in its logical truths but merely in its valid inferences. This is the dual of what holds in MiLP, which, like LP, has no unsatisfiable sentences, i.e., no sentence ϕ such that $\phi \models \emptyset$. Thus, MiLP is closer to classical logic than LP only in its valid inferences and not what is logically true or unsatisfiable.

We will work in a SET-SET framework, and our philosophical interpretation of that framework will follow the bilateralist conception of consequence suggested by Restall (2005). In general, bilateralism is the view that the two sides of concept-pairs like truth and falsity, assertion and denial, or proof and refutation should each be considered on a par and neither side as more basic or fundamental. There are several versions of this general idea (see Rumfitt, 2000; Wansing, 2017). According to the version of bilateralism advocated by Restall (2005) and Ripley (2013; 2015), $\Gamma \models \Delta$ should be understood as saying that it is out-of-bounds to assert all the sentences in Γ and also deny all the sentences in Δ, i.e., this combination of assertions and denials is ruled out by coherence norms governing assertions and denials. This version of bilateralism, which we will simply call "bilateralism" below for brevity, offers an intuitive interpretation of multiple-conclusion consequence relations. Moreover, as will become clear below, bilateralism also allows us to interpret Priest's MiLP as looking at only assertions (i.e. premises) to determine which models are minimal in the relevant sense. In contrast, we will also look at denials (i.e. conclusions).

Our project is related to existing generalizations of MiLP to four-valued settings and adaptive logics (Geibinger and Tompits, 2020; Skurt, 2017; Crabbé, 2011; Batens, 2001; Arieli and Avron, 1998). As will become clear in due course, however, our approach differs markedly from all such extant approaches because, for assessing validity, we don't just look at minimal models of the premises of an argument but also at minimal countermodels of the conclusions.[2] This is motivated by our bilateralist under-

[2]To the best of our knowledge, this is in sharp contrast with all extant versions of "minimally nonstandard"-style semantics. As a result, we will pursue our project at some distance from extant "minimally nonstandard"-style approaches. We don't think there is anything wrong with these approaches, given their aims. But we want to explore another way forward.

standing of the SET-SET framework and by the goal of allowing the duality between K3 and LP to carry over to the minimal setting.

The paper is organized thus: In the next section, we set the stage by introducing MiLP. In Section 3, we present MiK3 and MiFDE model theoretically. We present sequent calculi for both logics in Section 4. And Section 5 concludes.

2 Background: Minimally Inconsistent LP

Our aim with MiK3 and MiFDE is to do for K3 and FDE what MiLP does for LP. In this section, we set the stage by explaining how one can tweak LP to get MiLP. To define LP and MiLP, let $\mathscr{L}$ be a language that results from adding $\neg$, $\vee$, and $\wedge$ to a countable stock of atomic sentences in the usual way.[3] Let a valuation, v, be a function that assigns to each sentence of $\mathscr{L}$ a subset of $\{0, 1\}$ such that (Priest, 2006, 75)[4]:

(1a) $1 \in v(\neg\phi)$ iff $0 \in v(\phi)$

(1b) $0 \in v(\neg\phi)$ iff $1 \in v(\phi)$

(2a) $1 \in v(\phi \wedge \psi)$ iff $1 \in v(\phi)$ and $1 \in v(\psi)$

(2b) $0 \in v(\phi \wedge \psi)$ iff $0 \in v(\phi)$ or $0 \in v(\psi)$

(3a) $1 \in v(\phi \vee \psi)$ iff $1 \in v(\phi)$ or $1 \in v(\psi)$

(3b) $0 \in v(\phi \vee \psi)$ iff $0 \in v(\phi)$ and $0 \in v(\psi)$

For LP and MiLP, we require that for all sentences ϕ, $v(\phi) \neq \emptyset$. Hence, $v(\phi)$ is either $\{0\}$ or $\{1\}$ or $\{0, 1\}$. Valid inferences preserve truth in the sense of containing 1: $\Gamma \models_{LP} \Delta$ iff for every valuation, v, if $\forall\gamma \in \Gamma\,(1 \in v(\gamma))$, then $\exists\delta \in \Delta\,(1 \in v(\delta))$. We say that a valuation v is a "classical model" iff there is no sentence A such that $v(A) = \{0, 1\}$. It is easy to

[3]We restrict ourselves to propositional logic throughout.

[4]As Priest (2006, 75) notes, while we can think of evaluations as relations between sentences and truth-values, "it is technically simpler to think of an evaluation, equivalently, as a function which maps a formula to the set of truth values to which the formula is related." We follow Priest in this. Throughout, we assume classical set-theory and classical logic in our meta-theory.

see that this happens just in case no atomic sentence is assigned the value $\{0, 1\}$.

Now, LP has the same logical truths as classical logic but deems some classically valid inferences invalid, most prominently *ex contradictione quodlibet*, also known as "explosion": $\phi, \neg\phi \models \psi$. This rejection of explosion has wide-reaching effects that are not always welcome. LP invalidates, e.g., the rule of disjunctive syllogism, namely $\neg\phi, \phi \vee \psi \models \psi$. As Priest (2006, pp. 221-222) acknowledges, uses of disjunctive syllogism are unproblematic when reasoning in consistent domains, and we can often (defeasibly) assume that we are reasoning in such a domain. The goal of MiLP is to capture this idea.

As a first step toward defining MiLP, let us consider the following strict partial ordering of valuations (of the same language).

Definition 1. *Consistency ordering*: $v_1 <_c v_2$ iff $\{p \mid v_1(p) = \{0, 1\}\} \subsetneq \{p \mid v_2(p) = \{0, 1\}\}$ (where p ranges over atomic sentences).

Using this ordering, we can define minimal models of sets of premises, which in turn allows us to define consequence in MiLP.

Definition 2. *Minimal model*s *(c-minimal models)*: A valuation v is a minimal LP-model of the set of sentences Γ iff (i) v is an LP-model of Γ, i.e., $\forall\gamma \in \Gamma\,(1 \in v(\gamma))$, and (ii) for all $v' <_c v$, v' is *not* a LP-model of Γ, i.e., there is some $\gamma \in \Gamma$ such that $1 \notin v'(\gamma)$.

Definition 3. MiLP-*Validity*: $\Gamma \models_{MiLP} \Delta$ iff for every minimal LP-model, v, of Γ there is some $\delta \in \Delta$ such that $1 \in v(\delta)$.

Since every MiLP model is an LP model, every LP-valid inference is MiLP-valid. In particular, it follows that all logical truths of classical logic are logical truths of MiLP. However, MiLP is "more classical" than LP in its consequence relation (Priest, 2006, 224). In particular, in consistent situations, the consequence relation of MiLP coincides with classical consequence. For example, the minimal models of $\{p, \neg p \vee q\}$ (where p and q are atoms) are those where p and q are assigned 1. Thus, in contrast to LP, the corresponding instance of disjunctive syllogism is MiLP-valid: $p, \neg p \vee q \models_{MiLP} q$.

As Priest (2006, 224-25) points out, MiLP is a non-monotonic logic. For example, $r \vee (p \wedge \neg p) \models_{MiLP} r$, where p and r are atoms. However, $p \wedge$

$\neg p, r \vee (p \wedge \neg p) \not\models_{MiLP} r$. After all, if $1 \in v(p \wedge \neg p)$, then $v(p) = \{0, 1\}$. So, if we let $v(r) = \{0\}$, we have a counter-example.

Interpreted in a bilateralist way, MiLP says that it is out-of-bounds to assert everything in Γ and deny everything in Δ iff the most consistent cases in which everything in Γ has values that include 1 are also cases in which something in Δ has a value that includes 1. In other words, MiLP interprets our assertions (premises) in such a way that they commit us to a minimum of inconsistencies among atomic sentences. This makes sense because it is *ceteris paribus* problematic to assert something that commits one to instances of $\phi \wedge \neg\phi$. Thus, MiLP attributes such problematic commitments (for the purpose of assessing validity) only if this is required to make all the assertions have values that include 1. On the other hand, there is nothing problematic about denying instances of $\phi \wedge \neg\phi$. So we don't need to minimize commitments to denials of instances of $\phi \wedge \neg\phi$ that come with our denials (conclusions). Thus, only minimal models of the premises matter, when it comes to assessing whether an inference is MiLP-valid. This bilateralist way of looking at MiLP will prove illuminating in comparisons with our logics below.

3 Minimally Nonstandard K3 and FDE

We will now transfer Priest's construction of MiLP to K3 and FDE. This will yield logics that we call "minimally nonstandard" K3 and FDE, respectively labeled MiK3 and MiFDE. What it means to be "minimally nonstandard" is different for the two logics. For K3, it means that we minimize indeterminacy in a particular respect. Hence, we also call the resulting logic "minimally indeterminate K3." In this section, we characterize MiK3 and MiFDE model theoretically; we present proof systems in the next section. We start with MiK3 and then turn to MiFDE.

Before we dive into the technical details, however, let's start with a philosophical way into our proposal for minimally indeterminate K3. An advocate of a paracomplete logic like K3 holds that paradoxical sentences like the liar sentence are neither true nor false but indeterminate, and this leads her to hold that some denials of sentences of the form $\phi \vee \neg\phi$ or, equivalently, joint denials of ϕ and $\neg\phi$ are not out-of-bounds, as the classical logician claims they are. Now, what stands to K3 as disjunctive syllogism stands to LP is the inference schema $\psi \models \neg\phi, \phi \wedge \psi$. On a bilateralist reading this says that it is out-of-bounds to assert ψ and also deny $\neg\phi$ and deny $\phi \wedge \psi$. This fails according to the advocate of K3 because the sentence that takes the place of ϕ might be indeterminate (neither true nor false) while ψ's place might be taken by a true sentence. However, an advocate of the paracomplete approach who wants to stay closer to classical logic might hold that we should not consider the possibility of sentences being indeterminate unless there is no other way to render the denials in a position under consideration accurate. If we can, e.g., render the denials of $\neg A$ and $A \wedge B$ accurate without assuming that any sentence is indeterminate, then we should ignore, by default, the possibilities in which any sentence is indeterminate. Hence, in such cases, we will say that $B \models \neg A, A \wedge B$ is valid. If, however, we must assume that some sentences are indeterminate in order to render the denials of of $\neg A$ and $A \wedge B$ accurate, then we will assume that there are only as few such indeterminate sentences as needed and no more and then assess the validity of our inference against this assumption. This is the line of thought that we will pursue in our formulation of minimally indeterminate K3.

Formulated in bilateralist terms, MiLP ignores the possibility of inconsistencies that are not needed in order to make assertions accurate. And MiK3 ignores the possibility of indeterminacies that are not needed in order to make denials accurate. And, by ignoring these possibilities, each such logic comes closer to classical logic than LP or K3, respectively, in the inferences it deems valid. We will show how both of these strategies can be combined in MiFDE. In MiFDE, assertions whose accuracy requires that some sentences are both truth and false may be considered accurate, and denials whose accuracy requires that some sentences are neither true nor false can be considered accurate. But we assume that this doesn't happen unless we are considering collections of assertions and denials that cannot be rendered accurate in any other way.

3.1 Minimally Indeterminate K3

Let us start our exposition of minimally indeterminate K3 with a look at ordinary K3, which has been advocated as a response to the semantic paradoxes, e.g., by Kripke (1975). We use the same general setup as for LP. The difference is that we now allow that there are sentences, ϕ, such that $v(\phi) = \emptyset$, but we require that there is no sentence such that $v(\phi) = \{0, 1\}$.

Thus, we allow for sentences that have neither truth-value but we forbid sentences that have both values. We say that $\Gamma \models_{K3} \Delta$ iff for all such valuations, v, if $\forall \gamma \in \Gamma\,(1 \in v(\gamma))$, then $\exists \delta \in \Delta\,(1 \in v(\delta))$. K3 invalidates the law of excluded middle: $\not\models_{K3} \phi \vee \neg\phi$. The model such that $v(p) = \emptyset$ is a counter-example because in that model $1 \notin v(p \vee \neg p) = \emptyset$. Notice that, just as LP validates all classical logical truths, so K3 agrees with classical logic on what is unsatisfiable, i.e., $\Gamma \models_{K3} \emptyset$ iff $\Gamma \models_{CL} \emptyset$ (where $\models_{CL}$ denotes classical consequence).

A natural first idea for transferring the classical recapture approach from MiLP to K3 is to look only at minimal K3-models of the premises, using an ordering over models similar to the one above. Unfortunately, this won't work, as can be seen from the following fact.

Fact 4. *If a* K3*-valuation assigns* $\{1\}$ *to every member of* Γ*, then there is a* K3*-valuation that assigns* $\{1\}$ *to every member of* Γ *and doesn't assign* $\emptyset$ *to any sentence, i.e., there is a classical model of* Γ*.*

Proof. We take the valuation, v, that assigns $\{1\}$ to every member of Γ and change it to v' thus: if p is atomic, then $v'(p) = \{1\}$ iff $v(p) = \{1\}$, and $v'(p) = \{0\}$ otherwise. A simple induction on sentence complexity shows that if $v(A) = \{1\}$, then $v'(A) = \{1\}$, for any sentence A. ■

It follows that if there is a minimal K3-model of the premises, then there is a classical model of the premises. So, every minimal K3-model of the premises would be a classical model, according to this idea. Which is to say, looking only at such models yields classical logic rather than a strengthened version of K3.

We can overcome this obstacle if we reflect on the fact that K3 is the dual logic of LP, i.e., $\Gamma \models_{LP} \Delta$ iff $\neg\Delta \models_{K3} \neg\Gamma$, where $\neg X = \{\neg A : A \in X\}$ (see e.g. Cobreros et al., 2020a). This suggests that in order to preserve the duality for the minimally nonstandard variants of these logics, we should not look at K3-models of the premises but at K3-counter-models of the conclusions, i.e., models that give all of the conclusions values other than $\{1\}$. Indeed we should look at the least indeterminate models of this kind. For that, we must first define a determinacy ordering over the K3-models.

Definition 5. *Determinacy ordering*: $v_1 <_d v_2$ iff $\{p \mid v_1(p) = \emptyset\} \subsetneq \{p \mid v_2(p) = \emptyset\}$ (where p ranges over atomic sentences).

We can now define minimal counter-models in terms of this determinacy ordering.

Definition 6. *Minimal counter-model*s *(d-minimal counter-models)*: A valuation v is a minimal K3-counter-model of the set of sentences Δ iff (i) v is a K3-counter-model of Δ, i.e., $\forall\delta \in \Delta\,(1 \notin v(\delta))$, and (ii) for all $v' <_d v$, v' is *not* a K3-counter-model of Δ, i.e., $\exists\delta \in \Delta(1 \in v'(\delta))$.

In order to define validity in MiK3, we think of valid arguments as those arguments such that, any counter-model of all the conclusions is a counter-model of at least one premise. This is equivalent to the usual definition of validity as absence of counter-models, but it allows us to look only at the least indeterminate counter-models of the conclusions. More precisely, we can define MiK3 thus:

Definition 7. MiK3-*Validity*: $\Gamma \models_{MiK3} \Delta$ iff for every v that is a minimal K3-counter-model of Δ there is some $\gamma \in \Gamma$ such that $1 \notin v(\gamma)$.

The idea behind MiK3 is that all the "least gappy" situations in which all the conclusions have values other than $\{1\}$ are ones in which at least one of the premises has value $\{1\}$. Put differently, if all the premises hold (i.e. have values of $\{1\}$), then the conclusions cannot all fail in the least indeterminate way possible, i.e. we cannot be in a case of a minimal K3-counter-model to the conclusions.

The bilateralist reading of consequence is illuminating here. On a bilateralist interpretation of MiK3, it is out-of-bounds to assert everything in Γ and deny everything in Δ iff the least indeterminate way to make the denied sentences false also make some assertion false. This makes sense because it is *ceteris paribus* problematic to engage in denials (conclusions) that commit one to truth-value gaps: after all, denying claims of the form "*p* or not-*p*" is problematic. On the other hand, there is nothing problematic about asserting instances of "*p* or not-*p*," and so MiK3 doesn't care about minimizing such instances on the left-hand side (premises). Thus, whereas MiLP interprets our assertions (premises) in such a way that they commit us to as few instances of $\phi \wedge \neg\phi$, for atomic ϕ, as possible, MiK3 interprets our denials (conclusions) in such a way that they commit us to as few rejections of $\phi \vee \neg\phi$, for atomic ϕ, as possible. Put differently, MiLP sees our assertions as involving only minimal commitments to gluts, while MiK3 sees our denials as involving only minimal commitments to gaps.

We now turn to an investigation of some of the properties of MiK3. Let's start with its relation to classical logic. As noted above, K3 shares its unsatisfiabilities with classical logic. That remains true for MiK3.

Proposition 8. $\Gamma \models_{CL} \emptyset$ *iff* $\Gamma \models_{MiK3} \emptyset$.

Proof. Since every model is a K3-counter-model of $\emptyset$, the minimal such models are those in which no sentence gets value $\emptyset$, i.e., classical models. So, both sides of our biconditional hold just in case there is no classical model of Γ. ■

We saw above that MiLP agrees with classical logic as long as there is a classical model of the premises. The dual is true of MiK3: it agrees with classical logic as long as there is a classical counter-model of the conclusions. This is the sense in which MiK3 recaptures classical logic; it is dual to the sense in which MiLP recaptures classical logic.

Proposition 9. *If there is a classical counter-model of* Δ*, then* $\Gamma \models_{CL} \Delta$ *iff* $\Gamma \models_{MiK3} \Delta$.

Proof. Suppose that there is a classical counter-model of Δ. Then every minimal K3-counter-model of Δ is a classical model. Now, for the left-to-right direction, suppose that $\Gamma \models_{CL} \Delta$. Since there is a classical counter-model of Δ, it follows that any such model is also a K3-counter-model of Γ. Hence, $\Gamma \models_{MiK3} \Delta$.

For the right-to-left direction, suppose that $\Gamma \models_{MiK3} \Delta$. So, all the minimal K3-counter-models of Δ are K3-counter-models of Γ. The classical models that aren't counter-models of Δ cannot (by that very fact) be counter-models of $\Gamma \models_{CL} \Delta$. And all the classical counter-models of Δ are minimal K3-counter-models of Δ. Hence, they are all counter-models of Γ and, therefore, not counter-examples for $\Gamma \models_{CL} \Delta$. Therefore, there are no such counter-examples. So $\Gamma \models_{CL} \Delta$. ■

This means that MiK3 allows us to reason classically as long as our conclusions can fail (jointly) without any truth-value gaps. For instance, if $\{A \wedge B, \neg B\}$ has a classical counter-model, then $A \models_{MiK3} A \wedge B, \neg B$, which fails in K3 because models in which $v(B) = \emptyset$ provide counter-examples. Note that in MiK3 this inference can be defeated by the addition of conclusions. This happens, e.g., if we add $B \vee \neg B$ as a conclusion: $A \not\models_{MiK3} A \wedge B, \neg B, B \vee \neg B$. After all, a minimal counter-model of

$\{A \wedge B, \neg B, B \vee \neg B\}$ assigns the value $\emptyset$ to B, which allows the model to be such that $1 \notin v(A \wedge B)$ even when $1 \in v(A)$. Hence, MiK3 is non-monotonic on the right side.

When considering the non-monotonicity of MiK3, it is helpful to note that MiLP belongs to a class of non-monotonic logics that are build on preferential models, in the so-called KLM-tradition (Kraus et al., 1990). This means that MiLP has the so-called KLM properties (Strasser and Antonelli, 2019), including Cautious Monotony: if $\phi \models \psi$ and $\phi \models \tau$, then $\phi, \psi \models \tau$. As we will see momentarily, MiK3 is the dual logic of MiLP, and it, hence, enjoys the properties that are dual to the KLM properties. The dual of Cautious Monotony is this: if $\psi \models \phi$ and $\tau \models \phi$, then $\tau \models \phi, \psi$. Another KLM property is cumulative transitivity, i.e. the additive version of cut: if $\phi, \psi \models \tau$ and $\phi \models \psi$, then $\phi \models \tau$. The dual of this property is: if $\tau \models \phi, \psi$ and $\psi \models \phi$, then $\tau \models \phi$. This and the analogous duals of the other KLM properties hold for MiK3. And as will become clear in Section 4.1 when we show that cut can be eliminated in MiK3, cumulative transitivity and its dual hold in MiK3 and MiLP.

These facts aren't too surprising: after all, if the duality of K3 and LP holds for their minimally nonstandard variants, then the non-monotonicity of MiLP on the left can be expected to have its dual in the non-monotonicity of MiK3 on the right. This, however, raises the question of whether MiK3 and MiLP are dual logics. To show that they are, the following lemma will prove useful.

Lemma 10. *There is a one-to-one mapping $*$ between* LP*-models and* K3*-counter-models, such that v is an* LP*-counter-model of Δ iff $v*$ is a* K3*-model of $\neg\Delta$, and v is a minimal* LP*-model of Γ iff $v*$ is a minimal* K3*-counter-model of $\neg\Gamma$.*[5]

Proof. We map LP-models and K3-counter-models to each other by leaving the values $\{0\}$ and $\{1\}$ unchanged but mapping $\{0,1\}$ in LP-models to $\emptyset$ in K3-counter-models and vice versa. As a quick induction on sentence complexity shows, we can do this for all sentences by mapping the values of atomic sentences in this way. Note that, under this mapping, our two orderings map into each other, i.e., $v' <_c v$ iff $v'* <_d v*$. Now, for the first conjunct, if there is no $\delta \in \Delta$ such that $1 \in v(\delta)$, then all $\neg\delta \in \neg\Delta$ are such that $v * (\neg\delta) = \{1\}$. After all, if $1 \notin v(\delta)$ in an LP-model, then $v(\delta) = \{0\}$. And the same reasoning works in reverse. For the second conjunct, the

[5]As above, negation of sets are defined thus: $\neg X = \{\neg A : A \in X\}$.

left-to-right direction is shown thus: Suppose that v is a minimal LP-model of Γ. So $\forall\gamma \in \Gamma\, (v(\gamma) = \{1\}$ or $v(\gamma) = \{0,1\})$. If $v(\gamma) = \{1\}$, then $v * (\gamma) = \{0\}$. If $v(\gamma) = \{0,1\}$, then $v * (\gamma) = \emptyset$. So, $\forall\neg\gamma \in \neg\Gamma\, (1 \notin v * (\neg\gamma))$. Hence, v is a K3-counter-model of $\neg\Gamma$. Moreover, for all $v' <_c v$, there is a $\gamma \in \Gamma$ such that $v'(\gamma) = \{0\}$ and, hence, $\exists\neg\gamma \in \neg\Gamma(1 \in v' * (\neg\gamma))$, i.e., $v'*$ is *not* a K3-counter-model of $\neg\Gamma$. So $v*$ is a minimal K3-counter-model of $\neg\Gamma$. Right-to-left: Suppose that $v*$ is a minimal K3-counter-model of $\neg\Gamma$. So $\forall\neg\gamma \in \neg\Gamma\, (v * (\neg\gamma) = \{0\}$ or $v * (\neg\gamma) = \emptyset)$. If $v * (\neg\gamma) = \{0\}$, then $v(\gamma) = \{1\}$. If $v * (\neg\gamma) = \emptyset$, then $v(\gamma) = \{0,1\}$. So (i) v is a LP-model of Γ, i.e., $\forall\gamma \in \Gamma\, (1 \in v(\gamma))$, and (ii) for all $v' <_c v$, v' is *not* an LP-model of Γ, i.e., there is some $\gamma \in \Gamma$ such that $v'(\gamma) = \{0\}$. So, v is a minimal LP-model of Γ. ∎

Proposition 11. MiLP *and* MiK3 *are dual logics, i.e.,* $\Gamma \models_{MiLP} \Delta$ *iff* $\neg\Delta \models_{MiK3} \neg\Gamma$.

Proof. Suppose that $\Gamma \models_{MiLP} \Delta$. So there is no minimal LP-model of Γ that is an LP-counter-model of Δ. By Lemma 10, it follows that there is no minimal K3-counter-model of $\neg\Gamma$ that is a K3-model of $\neg\Delta$. The same reasoning works in the other direction as well. ∎

Thus, the duality between LP and K3 carries over to MiLP and MiK3. The strategy of MiLP to allow classical reasoning by default hence has a dual, MiK3, that is available to the K3 advocate. While the MiLP advocate looks at the least nonstandard models of the premises, however, the MiK3 advocate looks at the least nonstandard counter-models of the conclusions. Thus, the MiK3 advocate should endorse the default assumption that if the conclusions of an inference fail, they do so in the "least gappy way" possible. This is the dual of the MiLP assumption that the premises hold in the "least glutty way" possible.

We would like to conclude the model-theoretic discussion of MiK3 with a few words on what Priest calls "reassurance." In general, there is a concern that classical recapture may come with a cost, which one may not be willing to pay. In the MiLP case, the fear is that looking only at minimal models would make this logic unnecessarily collapse into triviality. In other words, the concern is that there will be some sets Σ whose closure under classical logic and MiLP is trivial—namely, Σ "explodes" in the eyes of these logics—whereas the closure of Σ under LP isn't trivial. If that were the case, MiLP's theoretical legitimacy would have to be restricted, just as that of classical logic is. Fortunately, Priest (2006, p. 226) proves his "reassurance" result, according to which the concern is uunsubstantiated: for any Σ, if Σ isn't LP-explosive, it is not MiLP-explosive.[6]

How about an analogous concern in the MiK3 case? That is, a concern arises that there are some sets Σ such that both $\Rightarrow_{CL} \Sigma$ and $\Rightarrow_{\mathsf{MiK3}} \Sigma$, whereas Σ cannot be proved in K3 as the latter logic has no theorems. In particular, such Σ might consist of instances of the excluded middle, in which case MiK3 would commit us to accepting those instances, that may be rejected in K3. This would be problematic for proponents of the paracomplete approach, who do not want to commit themselves to any such instance.

However, it is an immediate consequence of Proposition 11 that whenever MiK3 proves an instance of the excluded middle of the form $\vdash A \vee \neg A$, MiLP proves that $A \wedge \neg A$ "explodes."[7] According to Priest's reassurance result, that happens iff LP already proves that $A \wedge \neg A$ explodes. The duality of LP and K3 guarantees, in turn, that the latter is the case iff K3 proves $A \vee \neg A$. Wrapping things up, we get that MiK3 proves an instance of the excluded middle iff K3 already proves that instance, and so we get a reassurance result for MiK3: it doesn't prove any instance of the excluded middle as a theorem, since K3 has no theorems.

3.2 Minimally Nonstandard FDE

Given that the strategy underlying MiLP can be applied to K3, it is natural to ask whether it can be generalized to FDE. To do so, we must first introduce FDE. Let the language $\mathscr{L}$ be as before, and let a valuation, v, be a function that assigns to each sentence of $\mathscr{L}$ a subset of $\{0,1\}$ in accordance with the clauses (1a)-(3b) above. This time, however, we forbid neither that sentences get the value $\emptyset$ nor that they get the value $\{0,1\}$. Thus, sentences can be neither true nor false, and they can be both true and false. Consequently, $v(\phi)$ can take all four values: $\emptyset$, $\{0\}$, $\{1\}$, $\{0,1\}$. We say that $\Gamma \models_{FDE} \Delta$ iff for every valuation, v, if $\forall\gamma \in \Gamma\,(1 \in v(\gamma))$, then $\exists\delta \in \Delta\,(1 \in v(\delta))$.

[6] Given the scope of the present paper, we confine the discussion to reassurance for propositional MiLP. There are analogous results for the first-order case, but we will not touch upon them here.

[7] The proof rests on the fact that the De Morgan laws hold in these logics. The details are left to the reader as an exercise.

In order to apply the strategy of MiLP again, we must define minimal models. However, since FDE has two nonstandard truth-values—namely $\emptyset$ and $\{0,1\}$, representing gaps and gluts—we have choices regarding which values we want to treat as ruled out by our default assumption: we could define minimality in terms of just $\{0,1\}$, just $\emptyset$, or both. Arieli and Avron (1998) have explored four-valued logics in this spirit (see also Geibinger and Tompits, 2020).[8] Or we could combine the consistency ordering and the determinacy ordering into a single ordering.[9] However, to the best of our knowledge, all such extant logics are defined in terms of minimal models of premises, not minimal counter-models of conclusions. Given the duality of MiLP and MiK3, and given that FDE can be seen as combining LP and K3 by allowing gluts and gaps, the above approaches strike us as lopsided. Our aim in this subsection is to apply the strategy of minimality to FDE in a more balanced way.

At this point, the bilateralist approach proves useful. Recall that in our bilateralist setting, we have to minimize assertions of gluts and denials of gaps, whereas on the other hand, there is nothing wrong with denying contradictions and asserting instances of excluded middle. This rationale, we argue, carries over to MiFDE. That is, when we assess collections of assertions and denials for out-of-boundness, MiFDE looks only at the least glutty way of understanding our assertions and the least gappy ways of understanding our denials.

Recalling that we have two orderings $<_c$ and $<_d$, it is clear that to make our assertions as glut-free as possible and our denials as gap-free as pos-

[8]These logics often include vocabulary that makes minimal K3-models different from classical models, contrary to what we have seen above. While we acknowledge that this is interesting and important, we think that having such vocabulary around makes it easy to miss what is required for maintaining the duality of the minimal versions of LP and K3.

[9]Such an approach would probably be broadly similar to the combination of circumscription and MiLP due to Lin (1996). Lin combines the ordering of circumscription, which minimizes the extensions of predicates, and the consistency ordering of MiLP into a single ordering of valuations.

One could also define the joint ordering thus: $v_1 <_j v_2$ iff $\{p \mid v_1(p) \in \{\emptyset, \{0,1\}\}\} \subsetneqq \{p \mid v_2(p) \in \{\emptyset, \{0,1\}\}\}$. Looking at models of premises that are minimal with respect to that order results in (the fragment over our language of) what Arieli and Avron (1998, 116) call $\models^4_{\mathcal{I}_2}$. Given our goal to formulate a more balanced combination of MiLP and MiK3, and given the above bilateralist motivation according to which there is nothing wrong with denying contradictions and asserting instances of excluded middle, we don't pursue Arieli and Avron's line of thought here.

sible, we have to impose two minimality conditions: one for gluts applying to premises and one for gaps applying to conclusions. Thus, we will be looking, respectively, at c-minimal models of premises and d-minimal counter-models of conclusions.

Definition 12. MiFDE *consequence*: $\Gamma \models_{MiFDE} \Delta$ iff (i) every c-minimal model of Γ is a model of Δ and (ii) every d-minimal counter-model of Δ is a counter-model of Γ.

It is common to think of FDE as combining LP and K3. Analogously, we want MiFDE to combine MiLP and MiK3. It is thus natural to use both orderings simultaneously in order to define MiFDE. As we saw above, Fact 4 implies that d-minimal models of premises are in effect classical models. Similarly, c-minimal counter-models of conclusions are in effect classical counter-models. Hence, if we want our two orderings to do some work, we must do what the above bilateralist interpretation suggests: use $<_c$ on the left as in MiLP and use $<_d$ on the right as in MiK3.

Let us look at some properties of MiFDE. As is well-known, explosion and excluded middle both fail in FDE, and FDE is not the intersection of LP and K3, i.e., the FDE-valid arguments are not those that are (in the same language) LP-valid and also K3-valid. For $\phi \wedge \neg\phi \models \psi \vee \neg\psi$ is an LP-valid and also K3-valid schema (i.e., in the intersection of LP and K3) but not FDE-valid. The following two propositions imply that the situation is similar for MiFDE.

Proposition 13. *Explosion and the law of excluded middle both fail in* MiFDE.

Proof. For explosion, note that $p \wedge \neg p \not\models_{MiFDE} q$. For, there is a c-minimal model of $p \wedge \neg p$ that isn't a model of q. Indeed, any model v in which $v(p) = \{0,1\}$ and $v(q) = \{0\}$ and all other atoms have classical truth-values is such a model.

For the law of excluded middle, note that $q \not\models_{MiFDE} p \vee \neg p$ because there is an d-minimal counter-model of $p \vee \neg p$ that isn't a counter-model of q. Indeed, any model in which $v(p) = \emptyset$ and $v(q) = \{1\}$ and all other atoms have classical truth-values is such a model. ■

Proposition 14. MiFDE *is not the intersection of* MiLP *and* MiK3*, i.e., the* MiFDE*-valid arguments aren't those that are* MiLP*-valid and also* MiK3*-valid.*

Proof. The counterexample for the parallel claim about FDE carries over. For example, $p \wedge \neg p \models q \vee \neg q$ is valid in MiLP and also in MiK3, but $p \wedge \neg p \not\models_{MiFDE} q \vee \neg q$. This is because any model such that $v(p) = \{0, 1\}$ and $v(q) = \emptyset$ is a c-minimal model of $p \wedge \neg p$ and is not a model of $q \vee \neg q$. ■

So MiFDE combines the paraconsistency of MiLP and the paracompleteness of MiK3 without being their intersection. And the reasoning that shows this is the reasoning familiar from the relation between FDE, LP, and K3. We take this to be a virtue of MiFDE. It occupies the place that it should intuitively occupy, namely as related to FDE as MiLP is related to LP, and MiK3 is to K3, and hence related to MiLP and MiK3 as FDE is related to LP and K3.[10]

Given that MiFDE is weaker than the intersection of MiLP and MiK3, one might wonder whether MiFDE still enjoys the crucial advantage of MiLP over LP, namely that it allows us to reason classically by default. And indeed, MiFDE agrees with classical logic for all inferences whose premises are classically satisfiable and whose conclusions are classically falsifiable.

Proposition 15. *If there is a classical model of* Γ *and a classical counter-model of* Δ*, then* $\Gamma \models_{MiFDE} \Delta$ *iff* $\Gamma \models_{CL} \Delta$*.*

Proof. Suppose there is a classical model of Γ and a classical counter-model of Δ. Note that these models will be a c-minimal model of Γ and a d-minimal counter-model of Δ, respectively. Left-to-right: Suppose that $\Gamma \models_{MiFDE} \Delta$. Since all classical models of Γ are c-minimal, this implies that they are all models of Δ. And we know that there are such classical models. Similarly, since all classical counter-models of Δ are d-minimal, we know that they are all counter-models of Γ. So, $\Gamma \models_{CL} \Delta$.

Right-to-left: Suppose that $\Gamma \models_{CL} \Delta$. Since there is a classical model of Γ and a classical counter-model of Δ, the minimal models and counter-models of Γ and Δ, respectively, must all be classical. Thus, (i) every c-minimal model of Γ is a model of Δ and (ii) every d-minimal counter-model of Δ is a counter-model of Γ. So, $\Gamma \models_{MiFDE} \Delta$. ■

[10]This is not true of the logics presented by Arieli and Avron (1998), which we do not mean as a criticism, since this was not their aim. But this fact is noteworthy for placing our results in the extant literature.

Proposition 16. MiFDE *validates disjunctive syllogism and its dual in "standard" situations; i.e., if there is a classical model of the premises and a classical counter-model of the conclusions, then* $A, \neg A \wedge B \models_{MiFDE} B$ *and* $B \models_{MiFDE} A, \neg A \wedge B$ *hold.*

Proof. Immediate from the previous proposition. ■

These results illustrate that we can think of MiFDE as codifying the default assumption that if the premises are true, they are only true and not true and also false; and if the conclusions are not true, they are false and not neither true nor false. That is, we assess validity by looking only at models that come as close as possible to this assumption, namely c-minimal models of the premises and d-minimal counter-models of the conclusions. To put it in bilateralist terms: According to MiFDE, it is out-of-bounds to assert everything in Γ and deny everything in Δ iff the "least glutty" ways for the assertions to be all correct are ways for at least one denial to be incorrect and the "least gappy" ways for the denials to be all correct are ways for at least one assertion to be incorrect.

This concludes our model-theoretic presentation and discussion of MiK3 and MiFDE. In the next section, we present proof systems for both logics.

4 Sequent Calculi

In this section, we offer sequent systems for MiK3 and MiFDE. Our approach is based on recent work of one of us: Golan's (2022) sequent system for MiLP. In effect, what we are going to do is simply adjust to MiK3 and MiFDE the techniques and proofs Golan (2022) uses for MiLP. The basic idea here is, roughly, that we allow classical reasoning for sequents that are "less standard" than our target sequent. To implement this idea, we introduce the following notation for the subset of a given set that includes all the literals—i.e., atoms and their negations—for which both the atom and its negation are in the original set.

Definition 17. For any given set Δ, we write $\Delta^{\cap At!}$ for $\{p \mid p, \neg p \in \Delta\} \cup \{\neg p \mid p, \neg p \in \Delta\}$.

The idea will be that if, e.g., our target sequent includes Δ on the right side, then any sequent that properly includes $\Delta^{\cap At!}$ on the right side is "less standard" than our target sequent. And we will allow such sequents to behave classically. For example, we can derive instances of excluded middle that properly include $\Delta^{\cap At!}$ on the right side.

The feature of the sequent calculi below that allows us to capture the idea that we reason classically by default is that the calculi allow us to assume and discharge sequents. In the calculus for MiK3 below, for example, given that we begin our derivation with some axiom, we can assume and discharge any sequent that (i) would be guaranteed to hold in classical logic by excluded middle and (ii) whose conclusions are such that their joint failure requires a more nonstandard situation than the one required for the joint failure of the axiom's conclusions. We take our two logics in turn.

4.1 Sequent Calculus for MiK3

The proof system presented below results from making certain changes to the K3 sequent calculus as presented in Da Ré and Pailos (2022).[11] First, the axioms are restricted to sequents that involve only literals.[12] Second, we drop Weakening on the right, since it fails in MiK3, as explained above. Crucially, however, we also add a (natural deduction-style) rule for assuming and discharging sequents, which we call "CD" for "conjunction discharge." Here is the sequent system for MiK3:

Axioms:
With all the sentences that occur as premises or conclusions being literals:

ID: $\Gamma, p \Rightarrow p, \Delta$

LNC: $\Gamma, p, \neg p \Rightarrow \Delta$

Structural Rules:
Left Weakening:

$$\frac{\Gamma \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta}\,\text{K}$$

[11]The K3 sequent calculus presented by Da Ré and Pailos includes rules for truth constants, which we omit here.

[12]This doesn't change which sequents are provable, as can easily be proven by induction on proof height.

Cut:

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{Cut}$$

Operational Rules:
Operational Rules:

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ \text{L\&} \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ \text{R\&}$$

$$\frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\ \text{Lv} \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ \text{Rv}$$

DeMorgan Rules:

$$\frac{\Gamma, \neg A \vee \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta}\ \text{Ln\&} \qquad \frac{\Gamma \Rightarrow \Delta, \neg A \vee \neg B}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)}\ \text{Rn\&}$$

$$\frac{\Gamma, \neg A \wedge \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta}\ \text{Lnv} \qquad \frac{\Gamma \Rightarrow \Delta, \neg A \wedge \neg B}{\Gamma \Rightarrow \Delta, \neg(A \vee B)}\ \text{Rnv}$$

Double Negation:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \neg\neg A \Rightarrow \Delta}\ \text{Ldn} \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A}\ \text{Rdn}$$

Conjunction Discharge (where $\Sigma, \Pi, \Theta, \Lambda$ contain only literals, and p is an atom such that $p \notin \Pi^{\cap At!}$):

$$\frac{\dfrac{\dfrac{\Sigma \Rightarrow \Pi}{\vdots}}{\Gamma \Rightarrow A, \Delta} \qquad \dfrac{\dfrac{{}^{1:}[\Theta \Rightarrow \Lambda, p, \neg p, \Pi^{\cap At!}]}{\vdots}}{\Gamma \Rightarrow B, \Delta}}{\Gamma \Rightarrow A \wedge B, \Delta}\ \text{CD}\,[1]$$

Notice that, in CD, the assumption is presented inside square brackets, and the superscript "1" is not part of the syntax, but rather meant to keep track of the assumptions that need to be discharged. We do not allow vacuous discharges. But we allow multiple discharges of the same assumption, as well as a discharge of several assumptions at once.

A sequent $\Gamma \Rightarrow \Delta$ is derivable in MiK3 iff it can be derived via a proof-tree where (i) each step is licensed by one of the above rules and (ii) the proof tree is "closed," i.e., each assumption is discharged at some point in the tree.

Before moving on to soundness and completeness, let us explain the rationale for CD in a model-theoretic way.[13] Suppose that we begin a derivation with an axiom of the form $\Gamma \Rightarrow \Delta$. On the MiK3 "reading" of sequents, this means that every "minimally gappy" counter-model of Δ is a counter-model of at least one $\gamma \in \Gamma$. That is, when we begin a derivation with such a sequent, we don't really care about situations that are "more gappy" than $\Delta^{\cap At!}$. In other words, any model v such that $\Delta^{\cap At!} \subsetneq \{q \mid v(q) = \emptyset\}$ cannot provide a counterexample to whatever we would like to derive. Thus, we may safely *assume*—incorporate into our derivation—any sequent of the form $\Sigma \Rightarrow \Delta^{\cap At!}, p, \neg p, \Theta$ where $p \notin \Delta^{\cap At!}$ (and Σ, Θ are arbitrary). Such an assumption $\Sigma \Rightarrow \Delta^{\cap At!}, p, \neg p, \Theta$ will be *discharged* when a sequent whose subderivation makes use of it serves as a premise along a sequent whose subderivation makes use of the axiom that licenses the assumption, namely $\Gamma \Rightarrow \Delta$.

With this intuitive motivation for the calculus in place, we move on to the soundness and completeness proofs. Since the CD rule is special, we'll start with soundness for the other rules and establish the soundness of CD separately.

Lemma 18. *All the rules except for CD are sound. Moreover, a model v is a minimal counter-model of the conclusion-sequent of any such rule only if v is a minimal counter-model of at least one premise-sequent of that rule.*

Proof. By induction on proof height. For height 1, notice that the axioms are satisfied by all K3-models, and so they must be satisfied by all the minimal counter-models of their succedents.

For the left-rules, notice that the succedent of the conclusion-sequent is also the succedent of all premise-sequents. Hence, a model v is a minimal

[13] For a proof-theoretic motivation along these lines, see Golan (2022).

counter-model of the succedent of the conclusion-sequent of a left-rule iff v is a minimal counter-model of the succedents of all premise-sequents. Suppose that there is such a minimal counter-model of the succedent of the conclusion-sequent in which all premises have value $\{1\}$, i.e., a couter-model of the conclusion-sequent. By the clauses for the connectives, for some premise-sequent of the rule, all sentences on the left have value $\{1\}$, i.e., the model is a counter-model of at least one premise-sequent.

For Cut, assume to the contrary that $\Gamma, A \models_{MiK3} \Delta$ and $\Gamma \models_{MiK3} A, \Delta$, but $\Gamma \not\models_{MiK3} \Delta$. Hence, there is some v that is a minimal counter-model of Δ: $1 \notin v(\delta)$ for all $\delta \in \Delta$, but $1 \in v(\gamma)$ for all $\gamma \in \Gamma$. Since $\Gamma, A \models_{MiK3} \Delta$, it follows that $1 \notin v(A)$. It follows that v must be a minimal counter-model of $\Delta \cup \{A\}$. For, it is a minimal counter-model of Δ, which means that for every $v' < v$ there is some $\delta \in \Delta$ s.t. $1 \in v'(\delta)$, and so, in particular, any such v' cannot be a counter-model of $\Delta \cup \{A\}$. Since $\Gamma \models_{MiK3} A, \Delta$, we get that there is some $\gamma \in \Gamma$: $1 \notin v(\gamma)$, contradicting our assumption.

For the right-rules, let's first show that all one-premise right rules are sound. By the semantic clauses, the set of counter-models of the conclusion-sequent of each such rule is exactly the set of counter-models of the premise-sequent. Hence, the minimal counter-models are also identical. For example, consider Rdn. Let's assume that the premise-sequent holds, i.e., that in every minimal counter-model of $\Delta \cup \{A\}$ there is some $\gamma \in \Gamma$ that is assigned either $\{0\}$ or $\emptyset$. But v is a minimal counter-model of $\Delta \cup \{A\}$ iff it is a minimal counter-model of $\Delta \cup \{\neg\neg A\}$, and so the conclusion-sequent holds as well. Similar considerations show that *Rv*, *Rn*&, and *Rnv* are all sound.

It remains to deal with *R*&. Assume that $\Gamma \not\models_{MiK3} A \wedge B, \Delta$. So, there is a minimal counter-model of $\Delta \cup \{A \wedge B\}$ s.t. $1 \in v(\gamma)$ for all $\gamma \in \Gamma$. Since $1 \notin v(A \wedge B)$, either $1 \notin v(A)$ or $1 \notin v(B)$. It follows that v is a minimal counter-model either of $\Delta \cup \{A\}$ or of $\Delta \cup \{B\}$: Otherwise, there is some $v' <_d v$ that is a minimal counter-model of, say, $\Delta, \cup\{A\}$, which must also be a minimal counter-model of $\Delta \cup \{A \wedge B\}$, and we get a contradiction. Thus, either $\Gamma \not\models_{MiK3} A, \Delta$ or $\Gamma \not\models_{MiK3} B, \Delta$, as required. ■

In order to show that CD is also sound, the following notation will prove useful:

Definition 19. *Minimal Counterexamples* (*MC*): $MC(\Gamma \Rightarrow \Delta)$ is the set of all MiK3-counterexample to $\Gamma \models_{MiK3} \Delta$, i.e., $MC(\Gamma \Rightarrow \Delta) = \{v \mid v \text{ is a minimal K3-counter-model of } \Delta \text{ but } 1 \in v(\gamma) \text{ for all } \gamma \in \Gamma\}$.

With this notation, we can introduce the following corollary:

Corollary 20. *Assume that* $\Gamma \Rightarrow \Delta$ *can be derived from* $\Sigma_1 \Rightarrow \Pi_1, ..., \Sigma_n \Rightarrow \Pi_n$*, where the latter sequents are not necessarily axioms. Then* $MC(\Gamma \Rightarrow \Delta) \subseteq \bigcup_{i=1}^{n} MC(\Sigma_i \Rightarrow \Pi_i)$.

Proof. This follows from Lemma 18. By soundness of the non-CD-rules, for these rules, every minimal counterexample of $\Gamma \Rightarrow \Delta$ is a minimal counterexample of at least one premise-sequent in the last step of the derivation, which is also a minimal counterexample of at least one premise-sequent in the previous step, and so on. This result holds also in cases where the derivation makes use of CD. This is because CD is just $R\&$ along with the possibility of inserting an invalid sequent as one of the premises, and that is exactly what it means to say that $\Sigma_1 \Rightarrow \Pi_1, ..., \Sigma_n \Rightarrow \Pi_n$ need not necessarily be axioms. ■

Lemma 21. *Assume that we derive* $\Gamma \Rightarrow \Delta, A \wedge B$ *by CD, where (i)* $\Gamma \Rightarrow \Delta, A$ *is the discharging sequent, (ii)* $\Gamma \Rightarrow \Delta, B$ *is the discharged sequent, and (iii) CD is applicable since the subderivation of* $\Gamma \Rightarrow \Delta, A$ *makes use of the axiom* $\Sigma \Rightarrow \Pi$ *and the subderivation of* $\Gamma \Rightarrow \Delta, B$ *makes use of the assumption* $\Lambda \Rightarrow \Pi^{\cap At!}, p, \neg p, \Theta$ *licensed by* $\Sigma \Rightarrow \Pi$*. Then* $MC(\Gamma \Rightarrow \Delta, A \wedge B) \subseteq (MC(\Gamma \Rightarrow \Delta, A) \cup CM(\Gamma \Rightarrow \Delta, B)) \setminus MC(\Lambda \Rightarrow \Pi^{\cap At!}, p, \neg p, \Theta)$.

Remark. This lemma guarantees that we need not care about counter-models of underivable assumptions, provided that such assumptions are discharged at some point. As explained above, when we begin an MiK3 derivation with $\Gamma \Rightarrow \Delta, A$, we actually assume that any situation that is "more gappy" than $(\Delta \cup \{A\})^{\cap At!}$—in particular, any counter-model of $\Delta \cup \{B\}$—is a situation that in principle cannot provide counterexamples to whatever we would like to derive on the basis of $\Gamma \Rightarrow \Delta, A$.

Proof. By Corollary 20, $MC(\Gamma \Rightarrow \Delta, A \wedge B) \subseteq MC(\Gamma \Rightarrow \Delta, A) \cup MC(\Gamma \Rightarrow \Delta, B)$. So it remains to show that if $v \in MC(\Lambda \Rightarrow \Pi^{\cap At!}, p, \neg p, \Theta)$, then $v \notin MC(\Gamma \Rightarrow \Delta, A \wedge B)$. Assume that $v \in MC(\Lambda \Rightarrow \Pi^{\cap At!}, p, \neg p, \Theta)$. By definition, for every atom q, if $q \in \Pi^{\cap At!} \cup \{p\}$ then $v(q) = \emptyset$. Now, consider any v' that is a minimal counter-model of Π. By definition, $v'(q) = \emptyset$ iff $q \in \Pi^{\cap At!}$. Therefore, $v' < v$. Moreover, v' is clearly a K3-counter-model of $\Delta \cup \{A \wedge B\}$.[14] Thus, v cannot be a a minimal counter-model of $\Delta, A \wedge B$ because $v' < v$. So, $v \notin MC(\Gamma \Rightarrow \Delta, A \wedge B)$, as required. ■

Theorem 22. *Soundness: If* $\Gamma \Rightarrow \Delta$ *is derivable then* $\Gamma \models_{MiK3} \Delta$*.*

Proof. If the proof doesn't make any use of CD, then $\Gamma \models_{MiK3} \Delta$ by Lemma 18. It remains to deal with proof trees that do make use of CD, discharging underivable assumptions. The idea behind the proof is that, by Lemma 21, each time we discharge an assumption, the counterexamples to the assumed sequent are not counterexamples to the conclusion. Since $\Gamma \Rightarrow \Delta$ is derivable, the proof-tree is closed, i.e., any assumption in the proof tree is discharged at some point. Hence, the counterexamples of $\Gamma \Rightarrow \Delta$ are a subset of the counterexample of the axioms of the proof-tree. But there are no such counterexamples, and so $\Gamma \models_{MiK3} \Delta$.

To see how this reasoning plays out in more detail, assume then that the proof of $\Gamma \Rightarrow \Delta$ makes use of the assumptions $\Sigma_1 \Rightarrow \Pi_1, ..., \Sigma_n \Rightarrow \Pi_n$. Without loss of generality, we can assume that any assumption is made and discharged only once in the tree, and that $\Pi_i{}^{\cap At!} \not\subseteq \Pi_j{}^{\cap At!}$ for all $i \neq j$ $(1 \leq i, j \leq n)$. (Otherwise, we need only care about assumptions that are "minimal" because non-minimal assumptions don't count, and for each such assumption find the "lowest" point in the tree where it is discharged, at which point we get rid of the assumption's counterexample models.)

Now, let $\Lambda_i \Rightarrow \Theta_i$ be the conclusion-sequent of the application of CD that discharges $\Sigma_i \Rightarrow \Pi_i$ (for some $1 \leq i \leq n$). By Lemma 21, for every $v \in MC(\Sigma_i \Rightarrow \Pi_i)$,we have $v \notin MC(\Lambda_i \Rightarrow \Theta_i)$. Moreover, $\Gamma \Rightarrow \Delta$ can clearly be derived from $\Lambda_i \Rightarrow \Theta_i$ together with $\Sigma_1 \Rightarrow \Pi_1, ..., \Sigma_{i-1} \Rightarrow \Pi_{i-1}, \Sigma_{i+1} \Rightarrow \Pi_{i+1}, ..., \Sigma_n \Rightarrow \Pi_n$ (along with some axioms). So by Lemma 21, $v \notin MC(\Gamma \Rightarrow \Delta)$.[15] To sum up, for all $1 \leq i \leq n$, if $v \in MC(\Sigma_i \Rightarrow \Pi_i)$ then $v \notin MC(\Gamma \Rightarrow \Delta)$. Since the axioms of our sequent calculus have no counter-

[14] This fact is established by the completeness proofs for K3 given in Da Ré and Pailos (2022): a counter-model of some premise-sequent is a model of any conclusion-sequent derived from it. (That is the "soundness in reverse" aspect in which such a proof consists.)

[15] At this point, we draw on the assumption that all of $\Sigma_1 \Rightarrow_m \Pi_1, ..., \Sigma_n \Rightarrow_m \Pi_n$ are "minimal," and that each such assumption is made and discharged only once in the tree: it is for this reason that v isn't a counterexample to any of $\Sigma_1 \Rightarrow \Pi_1, ..., \Sigma_{i-1} \Rightarrow \Pi_{i-1}, \Sigma_{i+1} \Rightarrow \Pi_{i+1}, ..., \Sigma_n \Rightarrow \Pi_n$. But as we implied, this assumption can be dispensed with, since we need only care about assumptions that are "minimal," and for each such assumption find the "lowest" point in the tree where it is discharged (below which point its counterexample models stop counterexemplifying the sequents in the tree).

models, Lemma 21 gives us $MC(\Gamma \Rightarrow \Delta) \subseteq \bigcup_{i=1}^{n} MC(\Sigma_i \Rightarrow \Pi_i)$, and so $MC(\Gamma \Rightarrow \Delta) = \emptyset$, as required. ■

For simplicity, we shall now drop Left Weakening and Cut, and prove completeness without them. That will also allow us to obtain admissibility results for these rules.

Theorem 23. *Completeness: If* $\Gamma \models_{MiK3} \Delta$ *then* $\Gamma \Rightarrow \Delta$ *is derivable in* MiK3.

Proof. We use a slight variation of the standard method for constructing a counter-model from an unsuccessful proof search. Given an underivable sequent $\Gamma \Rightarrow \Delta$, we first decompose it by applying the rules of MiK3 "in reverse." For all the rules except Lv, $R\&$, and CD, such an application results in a single sequent with one less-complex formula. For Lv, $R\&$, and CD, on the other hand, we have to bifurcate the proof tree into two branches, one for each premise-sequent. Since $\Gamma \cup \Delta$ is finite, the decomposition process must come to an end after finitely many steps. At that point, we have a tree whose leaves are all sequents that contain only literals.[16]

As a second step, we consider the leaves in the resultant tree, looking for instances of the axioms of MiK3. Call a branch "closed" if it ends with a leaf that is an axiom, and "unclosed" otherwise.

As a third step, we consider the unclosed leaves, looking for assumptions that may be discharged by any axiom pinpointed in the previous step. Recall that, given such an axiom $\Gamma \Rightarrow \Delta$, the assumptions it may discharge are of the form $\Sigma \Rightarrow \Delta^{\cap At!}, p, \neg p, \Pi$, where $p \notin \Delta^{\cap At!}$ and Σ, Π are arbitrary. Call a branch that ends with such an assumption "semi-closed."

Now, if all the branches in our tree are either closed or semi-closed, we have a proof of $\Gamma \Rightarrow \Delta$. But we've just assumed that this sequent is underivable, and so there must be at least one branch in the tree that is neither closed nor semi-closed. Therefore, there must be some branch that ends with a sequent $\Gamma_0 \Rightarrow \Delta_0$ such that, for any other branch $\Sigma \Rightarrow \Pi$ that ends with an axiom, we have: $\Pi^{\cap At!} \not\subseteq {\Delta_0}^{\cap At!}$, or otherwise $\Gamma_0 \Rightarrow \Delta_0$ would have been declared an assumption that $\Sigma \Rightarrow \Pi$ may discharge in the third step. For the same reason, $\Pi^{\cap At!} \not\subseteq {\Delta_0}^{\cap At!}$ for any assumption $\Sigma \Rightarrow \Pi$ pinpointed in the third step, because any such assumption is licensed by some axiom, say, $\Lambda \Rightarrow \Theta$, where $\Theta^{\cap At!} \subsetneq \Pi^{\cap At!}$. But we already know that

[16]Observe that this procedure is not independent of the order in which we apply the rules, yet the leaves with which we end up are going to be the same, regardless of the order.

$\Theta^{\cap At!} \not\subseteq {\Delta_0}^{\cap At!}$, and so it must be the case that $\Pi^{\cap At!} \not\subseteq {\Delta_0}^{\cap At!}$. We may thus assume (and safely so!) that $\Gamma_0 \Rightarrow \Delta_0$ is minimal with respect to $<$: there is no other leaf $\Sigma \Rightarrow \Pi$ such that $\Pi^{\cap At!} \not\subseteq {\Delta_0}^{\cap At!}$.

As explained above, all the members of $\Gamma_0 \cup \Delta_0$ are literals, and since $\Gamma_0 \Rightarrow \Delta_0$ is not an axiom, $\Gamma_0 \cap \Delta_0 = \emptyset$, and there is no p such that $p, \neg p \in \Gamma_0$. Thus, the model v defined by

$$v(p) = \begin{cases} \{0\} & p \in \Delta, \neg p \notin \Delta \\ \emptyset & p, \neg p \in \Delta \\ \{1\} & \textit{otherwise} \end{cases}$$

is a minimal counter-model for Δ that is also a counterexample to $\Gamma_0 \Rightarrow \Delta_0$, i.e., $v \in MC(\Gamma_0 \Rightarrow \Delta_0)$.

Next, we have to show that $v \in MC(\Gamma \Rightarrow \Delta)$. We do that by proving that v is a minimal counter-model of each sequent in the branch that ends with $\Gamma_0 \Rightarrow \Delta_0$. The proof is done by backward induction on the length of the branch of $\Gamma_0 \Rightarrow \Delta_0$. In the base case, namely, where the branch is of length 1, $\Gamma \Rightarrow \Delta$ is simply $\Gamma_0 \Rightarrow \Delta_0$, and we're done. If $\Gamma \Rightarrow \Delta$ is derived by *Ldn*, then Γ is of the form $\Gamma', \neg\neg A$ and the sequent above $\Gamma', \neg\neg A \Rightarrow \Delta$ in the tree is $\Gamma', A \Rightarrow \Delta$. By the inductive hypothesis, $v \in MC(\Gamma', A \Rightarrow \Delta)$; that is, v is a minimal counter-model of Δ such that $1 \in v(\gamma)$ for all $\gamma \in \Gamma' \cup \{A\}$. In particular, $v(A) = \{1\}$ and so $v(\neg\neg A) = \{1\}$. Consequently, $v \in MC(\Gamma', \neg\neg A \Rightarrow \Delta)$, as required. The proofs for *Ln*&, *Lnv*, and *L*& are analogous.

If $\Gamma \Rightarrow \Delta$ is derived by *Lv* then $\Gamma = \Gamma', A \vee B$, and $\Gamma \Rightarrow \Delta$ is derived from $\Gamma', A \Rightarrow \Delta$ and $\Gamma', B \Rightarrow \Delta$. Without loss of generality, assume that $\Gamma', A \Rightarrow \Delta$ is in the branch of $\Gamma_0 \Rightarrow \Delta_0$. By the inductive hypothesis, $v \in MC(\Gamma', A \Rightarrow \Delta)$ and so for all $\delta \in \Delta$ we have $1 \notin v(\delta)$, but for all $\gamma \in \Gamma' \cup \{A\}$: $1 \in v(\gamma)$. In particular, $1 \in v(A)$ and so $1 \in v(A \vee B)$. Consequently, $v \in MC(\Gamma, A \vee B \Rightarrow \Delta)$, as required.

If $\Gamma \Rightarrow \Delta$ is derived by *Rdn*, then Δ is of the form $\Delta', \neg\neg A$ and $\Gamma \Rightarrow \Delta$ is derived from $\Gamma \Rightarrow \Delta', A$. By the inductive hypothesis, $v \in MC(\Gamma \Rightarrow \Delta', A)$. In that case, clearly $v \in MC(\Gamma \Rightarrow \Delta', \neg\neg A)$, as required. The proofs for the cases of *Rnv*, *Rn*&, and *Rv* are analogous.

It remains to prove the case where $\Gamma \Rightarrow \Delta$ is derived by *R*& or by CD. In that case, Δ is of the form $\Delta', A \wedge B$, and $\Gamma \Rightarrow \Delta$ is derived from $\Gamma \Rightarrow \Delta', A$ and $\Gamma \Rightarrow \Delta', B$. Without loss of generality, assume that $\Gamma \Rightarrow \Delta', A$ is in the branch of $\Gamma_0 \Rightarrow \Delta_0$. By the inductive hypothesis, $v \in MC(\Gamma \Rightarrow \Delta', A)$;

that is, v is a minimal counter-model of $\Delta' \cup \{A\}$ such that for all $\gamma \in \Gamma$: $1 \in v(\gamma)$.

Now, it is quite clear that v is a K3 counter-model of $\Delta' \cup \{A \wedge B\}$, but it is less clear that v is a minimal such model. Indeed, it could be the case that v is only a minimal counter-model of $\Delta' \cup \{A\}$, but not a minimal counter-model of $\Delta' \cup \{B\}$, in which case v wouldn't be a minimal counter-model of $\Delta' \cup \{A \wedge B\}$.

However, it is rather easy to verify that for any v' that is a minimal counter-model of $\Delta' \cup \{B\}$ there is a leaf in the tree $\Sigma \Rightarrow \Pi$ (among the leaves that take part in the subderivation of $\Gamma \Rightarrow \Delta', B$) such that for every atom p: $v'(p) = \emptyset$ iff $p \in \Pi^{\cap At!}$.[17] In addition, we already saw that for any such Π: $\Pi^{\cap At!} \nsubseteq {\Gamma_0}^{\cap At!}$, since $\Gamma_0 \Rightarrow \Delta_0$ is minimal with respect to $<_d$ regarding all the leaves in the tree. To sum up, (i) $v'(p) = \emptyset$ iff $p \in \Pi^{\cap At!}$, and (ii) $\Pi^{\cap At!} \nsubseteq {\Gamma_0}^{\cap At!}$. Now, (i) together with (ii) imply that $v' \not<_d v$. In other words, there is no v' such that $v' <_d v$, *and* v' is a minimal counter-model of $\Delta' \cup \{B\}$. Consequently, v is a minimal counter-model not only of $\Delta' \cup \{A\}$, but also of $\Delta' \cup \{A \wedge B\}$. But as we saw, $1 \in v(\gamma)$ for all $\gamma \in \Gamma$, and so $v \in MC(\Gamma \Rightarrow \Delta', A \wedge B)$, as required. ■

Corollary 24. *Left Weakening and Cut are admissible.*

Proof. As explained above, the completeness proof doesn't make use of these rules. But these rules are sound. So they are admissible. ■

The proof-system for MiK3 that we have presented in this subsection is not only sound and complete, it also brings out the core idea behind MiK3, namely that we can assume that the law of excluded middle holds everywhere except where a failure of it is required to make our conclusions fail at all. This comes out in the sequent calculus in the rule CD, which allows us to appeal freely to sequents that hold because of excluded middle (for atoms), as long as they require more truth-value gaps than the conclusions in which we are ultimately interested do. Hence, it seems to us that our sequent calculus captures MiK3 in an illuminating way.

[17] The proof is similar to that of Corollary 20: decompose only the formulas in the succedents of each sequent, while taking into account that our sequent rules are of such a nature that for each such rule, the set of minimal counter-models of the conclusion-sequent is a subset of, or equal to, the set that is the union of the sets of minimal counter-models of the premise-sequents.

4.2 Sequent Calculus for MiFDE

Recall that MiFDE-validity is defined by imposing two minimality conditions: one on models of the premises, and one on counter-models of the conclusions. Accordingly, to get a sequent calculus for MiFDE we will need two discharge rules: one for premises and one for conclusions. Moreover, MiFDE is non-monotonic for premises and conclusions alike, and so we will have to abandon weakening altogether. As for axioms, we have only instances of identity, since neither explosion nor excluded middle hold in MiFDE.

So the sequent calculus for MiFDE results from deleting the LNC axiom and the Left Weakening rule from the sequent calculus for MiK3 and adding the following Disjunction Discharge rule, called "DD":

> (DD) Where $\Sigma, \Pi, \Theta, \Lambda$ contain only literals, and p is an atom such that $p \notin \Pi^{\cap At!}$:

$$
\dfrac{\dfrac{\dfrac{\Sigma \Rightarrow \Pi}{\vdots}}{\Gamma, A \Rightarrow \Delta} \qquad \dfrac{\dfrac{{}^{1:}[\Sigma^{\cap At!}, p, \neg p, \Theta \Rightarrow \Lambda]}{\vdots}}{\Gamma, B \Rightarrow \Delta}}{\Gamma, A \vee B \Rightarrow \Delta}\ \mathrm{DD}\,[1]
$$

Soundness and completeness are proven for MiFDE in a similar way to the MiK3 case, with due adjustments. The main idea behind the soundness proof is that, as in the MiK3 case, each time we discharge an assumption, the counterexamples to the assumed sequent are not counterexamples to the conclusion. As opposed to MiK3, we have two discharge rules in MiFDE, and so we simply have to repeat the proof twice. This is straightforward, and we hence leave this as an exercise to the reader.

Regarding completeness, we sketch the proof below, focusing on the aspects in which it deviates from the completeness proof for MiK3. As before, we prove completeness for the system without Cut, whereby we also get an admissibility result for Cut.

Theorem 25. *Completeness: If* $\Gamma \models_{MiFDE} \Delta$ *then* $\Gamma \Rightarrow \Delta$ *is derivable in* MiFDE.

Proof. As in the MiK3 case, given an underivable sequent $\Gamma \Rightarrow \Delta$, we first decompose it by applying the rules of MiFDE "in reverse." As a result, we get a tree whose leaves are all sequents that contain only literals.

As a second step, we consider the leaves in the resultant tree, looking for instances of the axioms of MiFDE. Call a branch "closed" if it ends with a leaf with an axiom and "unclosed" otherwise.

As a third step, we consider the unclosed leaves, looking for assumptions that may be discharged by any axiom pinpointed in the previous step. Given such an axiom $\Gamma \Rightarrow \Delta$, the assumptions it may discharge (in the MiFDE case) are either of the form $\Gamma^{\cap At!}, p, \neg p, \Sigma \Rightarrow \Pi$ where $p \notin \Gamma^{\cap At!}$, or of the form $\Sigma \Rightarrow \Delta^{\cap At!}, p, \neg p, \Pi$ where $p \notin \Delta^{\cap At!}$, where in both cases Σ, Π are arbitrary. Call a branch that ends with such an assumption "semi-closed."

Now, if all the branches in our tree are either closed or semi-closed, we have a proof of $\Gamma \Rightarrow \Delta$. But we've just assumed that this sequent is underivable, and so there must be at least one branch in the tree that is neither closed nor semi-closed. Suppose that $\Gamma_0 \Rightarrow \Delta_0$ is the sequent at the end of this branch. Notice that for any axiom $\Sigma \Rightarrow \Pi$ pinpointed in the previous step, $\Sigma^{\cap At!} \nsubseteq {\Gamma_0}^{\cap At!}$ and $\Pi^{\cap At!} \nsubseteq {\Delta_0}^{\cap At!}$, otherwise $\Gamma_0 \Rightarrow \Delta_0$ would have been declared an assumption that $\Sigma \Rightarrow \Pi$ may discharge in the third step. Therefore, there must also be minimal such sequents with respect to the orderings $<_c, <_d$. In particular, we may safely assume without loss of generality that $\Gamma_0 \Rightarrow \Delta_0$ is minimal with respect to $<_c$. That is, we may safely assume that for any sequent $\Sigma \Rightarrow \Pi$ at the end of any branch in our tree, $\Sigma^{\cap At!} \nsubseteq {\Gamma_0}^{\cap At!}$.

Now, $\Gamma_0 \Rightarrow \Delta_0$ is not derivable, and so it cannot be an axiom, that is, $\Gamma_0 \cap \Delta_0 = \emptyset$. Thus, the model v defined by

$$v(p) = \begin{cases} \{1\} & p \in \Gamma, \neg p \notin \Gamma \\ \{0,1\} & p, \neg p \in \Gamma \\ \emptyset & p, \neg p \in \Delta_0 \\ \{0\} & \textit{otherwise} \end{cases}$$

is clearly a c-minimal model of Γ that poses a counterexample to $\Gamma_0 \Rightarrow \Delta_0$.

Next, we have to show that v is a c-minimal model of Γ that poses a counterexample to $\Gamma \Rightarrow \Delta$: if so, then v is a counterexample model of the latter sequent, as required. The proof is analogous to the MiK3 case, namely, we show by backward induction on the length of the branch of $\Gamma_0 \Rightarrow \Delta_0$ that (i) v is a c-minimal model of any antecedent of a sequent in that branch, and (ii) v poses a counterexample to that sequent. We shall discuss here only only the less-obvious cases, i.e., Lv, DD, $R\&$, and CD.

Assume $\Gamma \Rightarrow \Delta$ is derived either by $R\&$ or by CD. Thus, Δ is of the form $\Delta', A \wedge B$ and $\Gamma \Rightarrow \Delta$ is derived from $\Gamma \Rightarrow \Delta', A$ and $\Gamma \Rightarrow \Delta', B$. Without loss of generality, assume that $\Gamma \Rightarrow \Delta', A$ is in the branch of $\Gamma_0 \Rightarrow \Delta_0$. By the inductive hypothesis, v is a c-minimal model of Γ and $1 \notin v(\delta) \in$ for all $\delta \in \Delta \cup \{A\}$. So in particular $1 \notin v(A)$, and so $1 \notin v(A \wedge B)$. We thus get that v is a c-minimal model of Γ that poses a counterexample to $\Gamma \Rightarrow \Delta$, as required.

It remains to prove the case where $\Gamma \Rightarrow \Delta$ is derived by Lv or by CD. In that case, Γ is of the form $\Gamma', A \vee B$ and $\Gamma \Rightarrow \Delta$ is derived from $\Gamma', A \Rightarrow \Delta$ and $\Gamma', B \Rightarrow \Delta$. Assume, without loss of generality, that $\Gamma', A \Rightarrow \Delta$ is in the branch of $\Gamma_0 \Rightarrow \Delta_0$. By the inductive hypothesis, v is a c-minimal model of $\Gamma' \cup \{A\}$ s.t. $1 \notin v(\delta)$ for all $\delta \in \Delta$. Moreover, it is rather easy to verify that for any v' that is a c-minimal model of $\Gamma' \cup \{B\}$, there is a leaf in the tree $\Sigma \Rightarrow \Pi$ (among the leaves that take part in the subderivation of $\Gamma', B \Rightarrow \Delta$) such that for every atom p, $v'(p) = \{0, 1\}$ iff $p \in \Sigma^{\cap At!}$. But we already saw that for any such Σ, $\Sigma^{\cap At!} \not\subseteq {\Gamma_0}^{\cap At!}$, as $\Gamma_0 \Rightarrow \Delta_0$ is c-minimal regarding all the other leaves in the tree. Therefore, for any such v', it is not the case that $v' <_c v$. Consequently, we get that v is not only a c-minimal model of $\Gamma' \cup \{A\}$, but also a c-minimal model of $\Gamma' \cup \{A \vee B\}$, i.e., a c-minimal model of Γ s.t. $1 \notin v(\delta)$ for all $\delta \in \Delta$, as required. ■

Corollary 26. *Cut is admissible.*

Proof. As explained above, the completeness proof doesn't make use of Cut. Since the rule is sound, it is admissible. ■

Given that FDE was first formulated and studied in the context of relevance logics, it is natural to ask whether MiFDE is a relevance logic as well. One condition that relevance logics are typically required to meet is the variable sharing property, i.e., that for every valid argument at least one atomic sentence must occur (embedded or otherwise) in at least one premise and at least one conclusion. MiFDE is a relevance logic as measured by this criterion.

Proposition 27. MiFDE *has the variable sharing property: for any derivable sequent $\Gamma \Rightarrow \Delta$, there is at least one propositional variable that is shared by at least one formula in Γ and one formula in Δ.*

Proof. By induction on derivation height. The axioms clearly have that property, and it is also clearly preserved by any sequent rule except for

Cut. Yet, we just saw that Cut is eliminable, and so we can safely assume that it does not figure in the proof at hand. ■

To sum up, MiFDE is paraconsistent, paracomplete, has the variable sharing property, and agrees with classical logic on all inferences whose premises have a classical model and whose conclusions have a classical counter-model. Similarly to the calculus for MiK3, our sequent calculus for MiFDE captures the idea that we can assume that explosion and excluded middle hold for sequents that are more nonstandard than those we are ultimately interested in. Hence, the sequent calculus for MiFDE strikes us as an illuminating perspective on the strategy of MiFDE, i.e., of the idea that we should assume as a defeasible default assumption that we can reason classically.

5 Conclusion

Our goal in this paper was to show how advocates of K3 and FDE can adopt the strategy familiar from Priest's MiLP to sanction classical reasoning by a defeasible default assumption. We have seen that this adoption is not only possible but yields two logics, MiK3 and MiFDE respectively, that strike us as independently worthy of investigation. MiK3 is the dual logic to MiLP, and this has the interesting consequence that it looks only at minimal counter-models of conclusions, thus giving up weakening on the right side. And MiFDE is a deductively strong relevance logic that stands to MiK3 and MiLP as FDE stands to K3 and LP, respectively.

Given all the connections just mentioned and the recent interest in the version of non-transitive logic ST based on the strong Kleene truth-tables (Cobreros et al., 2012; Barrio et al., 2019, 2015; Dicher and Paoli, 2019), it is natural to ask whether there is a logic that stands to MiLP, MiK3, and MiFDE as the strong Kleene version of ST stands to LP, K3, and FDE, which we could call MiST. If so, MiLP would, e.g., capture the valid meta-inferences of MiST. We must leave such tantalizing ideas for another occasion.

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