A NOTE ON R-MINGLE AND THE DANGER OF SAFETY

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ABSTRACT. Dunn has recently argued that the logic **R**-Mingle (or **RM**) is a good, at least good enough, choice for many purposes in relevant and paraconsistent logic. Dunn gives an argument that the validity of the *Safety* principle in **RM**, according to which one may infer an arbitrary instance of the law of excluded middle from an arbitrary contradiction, is not a problem because it doesn't allow one to infer anything *new* from a contradiction. The only consequences one can derive from a contradiction in **RM** besides what one can derive in **R** are theorems of **RM**. In this paper, I argue that while it is plausible at the level of the logic that Safety is, indeed, safe, this is not the case when we consider *theories* closed under the logic. This fact, I suggest, should give pause to relevantists, and at least some paraconsistentists, when considering whether **RM** is adequate for their purposes.

1. **R**-MINGLE, ITS THEORIES, AND VARIETIES OF SAFETY

Dunn [4, §7.7] has given a *consumer report* style checklist-argument for why the logic **RM**[1] is a better choice, for some applications in paraconsistent and relevant logic, than the standard relevant logics like **R**. He notes that **RM** has many nice properties, including decidability (unlike **R** and many other of the standard relevant logics) and having a simple semantics which extends straightforwardly to the quantifiers (which is, again, also unlike systems like **R**, quantified extensions of which require a more complex treatment). In addition, **RM** is, indeed, a paraconsistent logic in that the explosion principle $(A \wedge \neg A) \rightarrow B$ is not valid. More to the point, **RM** can support *non-trivial* but *inconsistent* theories, as is desirable in a paraconsistent logic.[2] So, Dunn argues, those drawn to paraconsistency and relevant logics ought to consider **RM** as a nice, *good enough*, option for many of their purposes.

Having said this, and as Dunn discusses, **RM** is not a *relevant logic*: it fails to satisfy the *variable sharing property*, as is shown in [2, §29.4]. That is, fixing $at(A)$ as the set of atomic subformulas of A, it is not the case that $\vdash_{\mathbf{RM}} A \rightarrow B$ implies that $at(A) \cap at(B) \neq \varnothing$, unlike in standard relevant logics.[3] The following chain of

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[1] The expansion of the relevant logic **R** by the *mingle* axiom $A \rightarrow (A \rightarrow A)$; further information about **RM** can be found in [2], especially in the sections cited below.

[2] In [6], this property is suggested as being definitive of a logic's being paraconsistent.

[3] I use the turnstile $\vdash_{\mathbf{RM}}$ to denote *theoremhood* in **RM**.

provable implications in **RM** showcases the reason:

$$(A \wedge \neg A) \to \neg(A \to A) \to (B \to B) \to (B \vee \neg B)$$

That is, the implication between a formula and any formula to the right of it in the above chain is valid in **RM**. The implication between the extremal elements of this sequence, $(A \wedge \neg A) \to (B \vee \neg B)$, is what Dunn calls *Safety*, though he treats the implication between the interior pair of formulas, $\neg(A \to A) \to (B \to B)$, as another form of Safety. Many of the formulas mentioned above are of interest for my purposes here, and so, following Dunn, I'll use the term *Safety* to refer to them somewhat indiscriminately (though context will make clear which I mean in cases where it matters). Note that the theoremhood of the various forms of Safety is enough to ensure that **RM** does not satisfy the letter of the relevant law. While this the case, there is a related fact about **RM** which may give the relevantist some solace:

Proposition 1. If $\vdash_{\mathbf{RM}} A \to B$ then either (1) $at(A) \cap at(B) \neq \varnothing$ or (2) $\vdash_{\mathbf{RM}} \neg A$ and $\vdash_{\mathbf{RM}} B$.

This was shown by Meyer [2, §29.3.3]: it indicates that while **RM** does not satisfy the variable sharing property, it does satisfy a *weaker* version thereof: while an implication's validity does not require shared content, the only cases where this doesn't hold concern implications to logically true formulas from negations thereof. The Safety principles provide examples of case (2) above: indeed, these are the only kinds of principles which keep **RM** from satisfying the variable sharing property.

Dunn [4] argues that Safety (along with its variants) shouldn't give undue alarm to the relevantist, or paraconsistentist, because it "unlike Explosion leads to nothing new" [4, p. 161]. The basic idea here is that if we are reasoning from some premises which we discover to be inconsistent, or to involve rejecting some law of logic, this will lead to some irrelevant consequences, but it won't lead to *all* consequences. Indeed, what it leads to are already theorems of the logic. But, Dunn suggests, these are not *new* in some sense: being part of the logic, we already had them.

I want to argue here that Safety is, in fact, unsafe – not from the point of view of the logic, but from the point of view of the set of *theories* of the logic. The class of **RM** theories seem to have some rather strange and, I think, undesirable properties, and Safety is the culprit. I'll start by discussing one rather serious problem, noted by Meyer, before moving on to discuss some other problems which I'll note and discuss in more detail.

To make my point, let me state some definitions of the central notions here (all of these are standard except, possibly, the notion of *funky* theory: to my knowledge, this concept does not have a standard name in the literature):

Definition 1. Call a set of formulas Γ an **RM**-theory just in case whenever there are some formulas $A_1, \ldots, A_n \in \Gamma$ such that $\vdash_{\mathbf{RM}} \bigwedge_{i \leq n} A_i \to B$, then $B \in \Gamma$. The following are some salient subclasses of **RM**-theories:

- Γ is *prime* just in case whenever $A \vee B \in \Gamma$ then either $A \in \Gamma$ or $B \in \Gamma$.
- Γ is *inconsistent* just in case there is some formula A s.t. $A \in \Gamma$ and $\neg A \in \Gamma$; otherwise Γ is *consistent*.
- Γ is *incomplete* just in case there is some formula A s.t. $A \notin \Gamma$ and $\neg A \notin \Gamma$; otherwise Γ is *complete*.[4]
- Γ is *trivial* just in case it includes every formula; otherwise it is *non-trivial*.
- Γ is *regular* just in case whenever $\vdash_{\mathbf{RM}} A$, $A \in \Gamma$.
- Γ is *funky* just in case there is some formula A s.t. $\vdash_{\mathbf{RM}} A$ and $\neg A \in \Gamma$.

Let's note some facts about **RM**. First up is an already fairly damning fact proved by Meyer [2, pp.417–418]:

Proposition 2. $\vdash_{\mathbf{RM}} ((A \wedge \neg A) \wedge (B \wedge \neg B)) \to (A \leftrightarrow B)$. Therefore, if Γ is an inconsistent **RM** theory, then if A, B are contradictory according to Γ, then $A \leftrightarrow B \in \Gamma$.

This fact indicates that inconsistent **RM** theories say of their contradictions that they are indistinguishable, in a sense. This is quite bad, and indicates a reason why a paraconsistentist might have good reason to be suspicious of **RM**: usually interesting inconsistent theories are not those which collapse their inconsistencies in this manner (consider, for instance, inconsistent mathematical theories of the kind discussed in [12]).

Having noted this, there are other, somewhat subtler, reasons why we should be skeptical of **RM**, which I'll discuss below: some of the reasons concern properties of **RM**, particularly as a result of having Safety, which hold of other systems, and indeed of subsystems, in which the fact about **RM** above may not hold. To that end, let's note a few facts:

Proposition 3. The following are true of the set of **RM** theories:

(1) Every inconsistent **RM** theory is regular.
(2) Every funky **RM** theory is regular (and thus inconsistent).
(3) Every regular prime **RM** theory is complete.
(4) There are prime, negation consistent **RM** theories which are incomplete.
(5) There are prime **RM** theories which are not regular.

Proof. For point (1), let's flesh out the argument a bit, though it winds up following from Safety. Note that, with safety, we have $\vdash_{\mathbf{RM}} (A \wedge \neg A) \to (B \to B)$. Following Anderson and Belnap's method [1] for representing the Ackermann **t** constant (also employed in [11]), we can thereby show that for any theorem B, $\vdash_{\mathbf{RM}} (A \wedge \neg A) \to B$: in effect, this just relies on the fact that $\vdash_{\mathbf{RM}} \bigwedge\{p \to p \mid p \in at(B)\} \to B$[5], and so

[4] The notions of completeness and consistency at work here are often called "negation incompleteness" and "negation inconsistency". Given that these are the only versions of these concepts I'll be interested in here, I'll drop the qualifier "negation" throughout.

[5] Note that the conjunction $\bigwedge\{p \to p \mid p \in at(B)\}$ is finite. The reason this fact obtains is that, in general, we can prove that $\vdash_{\mathbf{RM}} \bigwedge\{p \to p \mid p \in at(B)\} \to (B \to B)$, and thus, assuming that $\vdash_{\mathbf{RM}} B$, we obtain the desired result from permutation.

since, because of Safety, we have that $\vdash_{\mathbf{RM}} (A \wedge \neg A) \to \bigwedge\{p \to p \mid p \in at(B)\}$, it follows that $\vdash_{\mathbf{RM}} (A \wedge \neg A) \to B$ holds for every theorem B, proving the point.

Point (2) follows from considerations in the proof of point (1). For note that for any theorem B of **RM** we have that $\vdash_{\mathbf{RM}} \bigwedge\{p \to p \mid p \in at(B)\} \to B$, it follows that $\vdash_{\mathbf{RM}} \neg B \to \bigvee\{\neg(p \to p) \mid p \in at(B)\}$, and so since $\vdash_{\mathbf{RM}} \neg(A \to A) \to (C \to C)$, we will have that $\vdash_{\mathbf{RM}} \neg B \to (C \to C)$ for any theorem B, and so, following on the reasoning in point (1), we'll have that for any theorems B, C, we have that $\vdash_{\mathbf{RM}} \neg B \to C$. Thus, any funky theory will also contain the **RM**-theorem witnessing its funkiness, resulting in inconsistency.[6]

Point (3) is due to the fact that $\vdash_{\mathbf{RM}} A \vee \neg A$.

For point (4), we construct such a theory using the *pair extension* theorem (see [8, §5.2] for definitions and details). I'll continue to use lower-case letters from the Latin alphabet as propositional variables and upper-case letters from that alphabet as metavariables. Consider the **RM**-theory $\{A \mid \vdash_{\mathbf{RM}} p \to A\}$ and the set $\{q \vee \neg q\}$. We can easily check to see that this is a pair, in the sense of [8]: for suppose that there were $A_1, \ldots, A_n$ s.t. $\vdash_{\mathbf{RM}} p \to A_i$ holds for every $i \leq n$ and that $\vdash_{\mathbf{RM}} \bigwedge\limits_{i \leq n} A_i \to (q \vee \neg q)$. It's immediate that $\vdash_{\mathbf{RM}} p \to \bigwedge\limits_{i \leq n} A_i$ and so it follows that $\vdash_{\mathbf{RM}} p \to (q \vee \neg q)$, but this does not hold if p, q are distinct propositional variables (by the fact that **RM** satisfies the weak variable sharing property). Thus, by Theorem 5.17 of [8], it follows that we can construct a prime **RM**-theory Γ such that $\{A \mid \vdash_{\mathbf{RM}} p \to A\} \subseteq \Gamma$ and $q \vee \neg q \notin \Gamma$. Furthermore, since this Γ is prime, we know that neither q nor $\neg q$ is contained in Γ, and so Γ is incomplete; also, by (1) and (3), Γ is negation consistent.

Note that Γ also suffices for a non-regular, prime **RM** theory, and hence for point (5).

□

These results spell out some odd features of the set of **RM** theories, beyond Meyer's noted point. Indeed, some of them rely directly on the principle $(A \wedge \neg A) \to (B \vee \neg B)$: a consequence of which is that *any inconsistent prime theory is complete* (note this is a consequence of (1) and (3) above). This will be the case even in first-degree logics which satisfy this version of Safety, such as the first-degree fragment of **RM**.[7] Similarly, if we retain the property that a contradiction entails every validity, even in systems other than **RM**, there are, as we'll see, reasons to be suspicious. So if your system admits the rule "from B infer $(A \wedge \neg A) \to B$", there are, I'll argue, reasons to be suspicious of the paraconsistent credentials of such a system. Furthermore, a logic may satisfy this property without having the theorem

[6] Note that points (1) and (2) can easily be seen by considering the algebras appropriate for **RM** (for discussion, see [2, §29.3.2]), where theorems always take values of 0 or above, and so the negations thereof take values of 0 or below, and so the value of any non-theorem is below that of any theorem. Similarly, the value of a contradiction will always be 0 or below, and the value of an instance of excluded middle will always be 0 or above.

[7] Such systems have been studied by Yaroslav Shramko, a talk by whom on the topic motivated this paper.

of **RM** noted by Meyer. So the upshots I'll give below do concern **RM**, they are also of interest in problematising the properties themselves. Their close relation to Safety does rely on some properties of **RM**, but are available in other systems as well.

The point is that the badness of **RM**, and of Safety, is somewhat overdetermined; the properties mentioned Proposition 3 are, I'll argue, enough to make any relevantist or paraconsistentist wary of any logic which satisfies them. To that end, let's consider the reasons for wariness is some detail.

2. Upshots of Safety

Let's consider two problems that arise because of safety principles in **RM**.

2.1. **Problems with Funkiness and Regularity.** In **RM** there is a curious, though local, *blurring* of a distinction important in relevant logic; that between a theory *being closed under* and a theory *containing* logic. Being closed under logic is something all theories share, given the definition above; on the other hand, containing the logic is distinctive of regular theories, and theories are generally not required to satisfy this property. This distinction tracks the idea that we may study logically well behaved theories (as all theories are) which nonetheless do not *in any sense* concern logic themselves as their subject matter.

Beall [3] has recently leaned on the behaviour of theories to provide a philosophy of logic: he understands logic as providing the *closure properties* for theories. Our theories, so the story goes, are deductively closed according to some logical consequence relation, and the weakest such relation, in common between all theories, is logic (the differences come in with extralogical assumptions invoked in particular theories). A part of this story is that the theories themselves need not be *about* logic, but just need to be closed under it – in fact, Beall's preferred logic **FDE** has no theorems, and so the question of regularity for theories is moot in his system. Having said that, a similar approach to the relation between logic and theory building has been proposed by Logan [5], dealing in systems with theorems, and for his project the possibility of non-regular, and, perhaps, even of *funky* theories is essential. For him, logic plays a role as a certain kind of *background* theory, which, when applied to another theory, generates the deductive closure of the latter. That some but not all theories include the logic is, then, a key part of Logan's view.

Now **RM** does have non-regular theories and even funky theories: however, as we've seen, all of the latter are, in fact, regular, and none of the former are inconsistent. On the theory-building line, then, this, in effect, means that **RM** dictates that a theory may be about something other than logic, or it may be inconsistent, but it may not be *both*. But why think that the theory of, say, an inconsistent model of arithmetic must *also* contain the complete theory of logic?

This is a deeply weird consequence. On Logan's way of putting things, among the *background* theories which enforce logic we find every inconsistent theory and, indeed, every funky theory, even if, *prima facie*, these theories are not, in an intuitive sense, intended as theories of logic. At very least, this does indicate an important sense in which Safety principles do give us *something new*; not in the logic, but among its inconsistent and funky theories. If a paraconsistentist is interested in inconsistent theories, or a relevantist in funky theories (and there seems to be good reasons why they should be, as we'll see), then this should raise their suspicions.

2.2. **Asymmetry Between Inconsistency and Incompleteness.** Let's turn to a related point that mainly concerns *prime* theories. Recall that (1) and (3) imply that every inconsistent, prime **RM** theory is complete. This fact, in combination with (4), indicates that while some prime **RM** theories may, in general, be incomplete, they may *not* be so if they are inconsistent. Speaking metaphorically, in **RM** one may, in general, refrain from answering some yes-or-no questions, but not if one answers *any* such question "yes and no". Stepping away from the metaphor, let's consider some reasons to think this is an undesirable result.

One reason concerns an early use of theories, especially inconsistent and incomplete theories, in doxastic relevant logic (though the kind of considerations here applies more broadly). Sylvan (né Routley) and Plumwood (née Routley) [9] argue that odd, logically ill-behaved theories, such as the inconsistent and funky theories, play an important role in the context of doxastic logic. The main point concerns the familiar fact that theories of what an agent believes (or perhaps of what they are committed to) should make room for genuinely inconsistent beliefs (commitments). In such a setting, if we take *prime* theories to correctly model beliefs or commitments (there are good reasons not to demand primeness, but let's do so for a moment), then it seems incredibly bizarre to say that whenever an agent believes (is committed to) a contradiction, then they believe (are committed to) one of $A, \neg A$ for any proposition A.

This seems extremely implausible as a constraint on the would-be inconsistent believer, especially given that it is *not* a constraint on the consistent believer! The use of prime theories to model belief (commitments) is trouble, as clearly one can believe $A \vee B$ without picking one of A, B: Sherlock could believe that either Moriarty committed the murder or that Queen Victoria did, without having the evidence to clinch the case. However, on this point, note that a requirement given by **RM**, which holds even in non-prime theories, is that while the consistent believer may, rationally, fail to believe some instance of the principle of excluded middle, the inconsistent believer is not, rationally, also so allowed. As a fanciful example, the consistent believer is allowed to be a constructivist, while this is not permitted for the inconsistent believer. While one may have doubts as to the straightforward application of theories to the logic of belief of the kind motivated by Sylvan and Plumwood, nonetheless the point serves, I think, to highlight why **RM** delivers undesirable results for modeling possibly inconsistent collections of propositions.

Returning to [3], Beall's aim there is to argue for subclassical logic on the grounds that it provides the means for us to study a wider range of theories than does classical logic. It is from this wider range that we can, hopefully, pluck the best candidates for the true theory. Imposing further restrictions, he argues, may lead us to rule out of court a true, or potentially useful, theory. According to Beall's approach, we should take the theories we're interested in to all be *prime* (though this may be a questionable assumption), and to obey certain other properties concerning the behaviour of $\wedge$ and $\neg$, but otherwise to be unconstrained. The constraint that a theory can be inconsistent or incomplete, *but not both* seems quite powerful. This rules out a fairly wide range of potential theories, and, in line with Beall's argument, doing so places us in some epistemic risk for throwing out some good ones.

While not exactly Beall's view, a related position has a long history among relevant logicians. Sylvan included a related discussion, building on Plumwood's notion[8] that logical deducibility should be an *absolute sufficiency* relation, writing as follows:

> Sufficiency is a go-anywhere notion, which is not limited by the fact that the situation in which it operates is somehow classically incoherent, e.g. inconsistent or paradoxical. If A is sufficient for B then it does not matter what else goes on; logical laws may go haywire but nothing subtracts from As sufficiency.
>
> [10, p. 8]

Sylvan is here talking about *situations*, which form the basis of a standard interpretation of the ternary relation semantics, but the same point seems to apply to *theories*: indeed, we should expect that for every situation there is a theory including all and only the propositions made true by the situation. With this in mind, Sylvan motivates the tolerance of inconsistency and incompleteness, but also motivates the need for situations which seem best modeled by funky theories (those where logical laws 'go haywire').

The reason I bring this up is because, first, the **RM** requirement that inconsistent theories (situations) are complete is just as bizarre here as anywhere else. Second, however, this ill-fit relates to Sylvan's appeal to the logically ill-behaved situations, such as inconsistent and funky situations. According to **RM**, any situation which is logically ill-behaved enough to be funky must, in fact, actually have been a situation supporting all the truths of logic after all. Hopefully at this point the reader agrees that this is a distorting, and undesirable, constraint to place on the wide space of situations Sylvan invokes here.

As one more argument, this fact brings along an odd asymmetry between inconsistency and incompleteness among prime theories. The treatment of incomplete theories in **RM** is really *quite* different from the treatment of inconsistent (and nontrivial) theories. Incompleteness does not force (prime) theories to be inconsistent, but the converse implication does hold. While not, by itself, a big problem, it is somewhat aesthetically unsatisfying.

It should be noted that the treatment of inconsistent theories in **RM** will only appear as a problem for certain paraconsistentists, namely those who also want to allow paracompleteness (Beall is a notable example thereof, but not the only one). So, for instance, Priest [7] whose preferred paraconsistent logics (which extend **LP**)

[8] Defended in her "Some False Laws of Logic", to be published in the near future in this journal.

admit inconsistency, but not incompleteness, will not be moved by this consideration.[9]

Having said that, I think that these considerations should at least give pause to the relevantist and paraconsistentist. The problem with **RM** for relevantist purposes is not *just* that it fails variable sharing, but that the manner in which is fails it causes some untoward consequences in its treatment of theories. The paraconsistentist is, as well, forced by **RM** into adopting some fairly odd views about how their inconsistent theories work. So both such logicians ought to be wary of Safety after all.

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[9] An anonymous referee has suggested that variations on some of the considerations given above may provide reasons to think that the principle of excluded middle is dubious on relevance grounds: in any logic with this theorem regular prime theories will be complete. This is an interesting suggestion, but for now my only comment is that I'm not sure that the kinds of reasons I gave above problematise this result (for example, it's, perhaps, not *too* bizarre that the regularity of a regular, prime theory should enforce completeness).