A note on signed occurrences of propositional variables

Thomas Randriamahazaka

Department of Philosophy
University of St Andrews
tr52@st-andrews.ac.uk

Abstract

This note concerns the positive and negative occurrences of propositional variables. Just as the theory of infectious truth-values provides an algebraic understanding of the position according to which identity of subject-matter between two formulas can approximated syntactically by the identity of propositional variables occurring in these formulas, we develop an algebraic understanding of the similar position which considers signed occurrence (i.e., positive or negative) instead of mere occurrence. We apply our framework to classical logic, yielding this first (to our knowledge) semantic characterisation of the logic called SCL by Hornischer [15]. Moreover, we settle two conjectures by Humberstone [17] which use signed occurrences to study the equational logic of the power algebra of the two-element Boolean algebra.

Introduction: Parry's and Angell's Conditions

Since the publication of Yablo's book *Aboutness* [25], research on formal treatments of subject-matter has been burgeoning. During the last decade, theories of subject-matter have been studied for their own sake (e.g., [11, 14, 21, 26]) but also in relation to truthmaker semantics (e.g., [5, 10]), universal algebra (e.g., [4, 5, 12]) and potential applications to intentional attitudes (e.g., [2, 13, 27]). In particular, a recent proposal in philosophical logic (e.g., [3, 15, 25]) has put forward the idea according to which two statements have the same content if (a) they have the same truth-conditions and (b) they have the same subject-matter. It has been argued by Hornischer [15] that, in the context of developing a logic of content identity between formulas, the second condition could be approximated by a syntactic criterion based on the propositional variables occurring in these formulas. There are, however, some disagreement as to what the correct syntactic criterion is. Two proposals stand out.

According to the first one, two formulas can be treated as having the same subject-matter if the same propositional variables occur in them. This criterion will be called the Parry Condition and is exemplified in [20], [3] and [6]. In particular, the Parry Condition states that a formula and its negation always have the same subject-matter.

The second proposal disagrees and demands that propositional variables are treated differently depending on whether they occur positively or negatively in the formula. It cares about the signed (i.e., positive or negative) occurrences of variables rather than their mere occurrences. For present purposes, a propositional variable occurs positively (respectively, negatively) in a formula if it occurs in it under the scope of an even (respectively, odd) number of negations.[1] The second criterion treats two formulas as having the same subject-matter if the same propositional variables occur positively (respectively, negatively) in the both of them. It will be called the Angell Condition and is exemplified in [1], [9] and [24].

There has been recently a growing interest (e.g., [6, 8, 19]) in an algebraic understanding of the Parry Condition. When the condition (a) of identity of truth-conditions is understood algebraically via the equational logic of an algebra $\mathcal{A}$ of truth-values, a method has been developed to modify $\mathcal{A}$ into a new algebra whose equational logic corresponds to the logic of content identity built on top of $\mathcal{A}$ using the Parry condition. This method is based on the theory of infectious truth-values and the construction consists in adding an infectious truth-value to $\mathcal{A}$.

[1] As noted by an anonymous referee, there are some contexts in which we might want to characterise the occurrence of a propositional variable as negative even though it does not occur on the scope of a negation. For instance, the classical definition of the material condition seems to commit us to say that p occurs negatively in $p \supset q$ and positively in $\neg(p \supset q)$. In this paper, however, we will only be concerned with contexts in which negation is the only connective to modify the polarity of an occurrence of a propositional variable.

This note aims to develop a similar algebraic understanding for the Angell Condition. We start in Section 1 by sketching a general theory of signed infectious extension. Our main result is that the equational logic of the signed infectious extension of an algebra $\mathcal{A}$ is the largest fragment of the equational logic of $\mathcal{A}$ respecting the Angell Condition. In Section 2, we apply our framework to classical logic and settle two conjectures by Humberstone [17] – one negatively and one positively – relating the signed occurrences of propositional variables in formulas and the equational logic of the power algebra of the two-element Boolean algebra.

1 General Case

1.1 Technicalities

Let $\sigma = \langle V, n \rangle$ be a non-nullary type, i.e., V is a non-empty set of logical connectives and $n : V \to \mathbb{N}$ is an arity function such that $n(\dagger) \neq 0$ for all $\dagger \in V$. An algebra of type σ is a pair $\mathcal{A} = \langle A, (.)^{\mathcal{A}} \rangle$ where A is non-empty set and $(.)^{\mathcal{A}}$ is a function such that, for all $\dagger \in V$, we have $\dagger^{\mathcal{A}} : A^{n(\dagger)} \to A$. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two algebras of type σ, a homomorphism is a function $h : A_1 \to A_2$ such that, for all $\dagger \in V$ and $a_1, ..., a_{n(\dagger)}$ in A_1, we have $h(\dagger^{\mathcal{A}_1}(a_1, ..., a_{n(\dagger)})) = \dagger^{\mathcal{A}_2}(h(a_1), ..., h(a_{n(\dagger)}))$.

Let *Prop* be a countably infinite set of propositional variables. The set F_σ of formulas of type σ is defined recursively as follows:

- if $p \in Prop$ then $p \in F_\sigma$,
- for all $\dagger \in V$ and $\varphi_1, ..., \varphi_{n(\dagger)}$ in F_σ, we have $\dagger(\varphi_1, ..., \varphi_{n(\dagger)}) \in F_\sigma$.

The formula algebra of type σ is $\mathcal{F}_\sigma = \langle F_\sigma, (.)^{\mathcal{F}_\sigma} \rangle$ where, for all $\dagger \in V$, we have $\dagger^{\mathcal{F}_\sigma} : \langle \varphi_1, ..., \varphi_{n(\dagger)} \rangle \mapsto \dagger(\varphi_1, ..., \varphi_{n(\dagger)})$. For all algebras $\mathcal{A}$ of type σ, a homomorphism $v : \mathcal{F}_\sigma \to \mathcal{A}$ is called a $\mathcal{A}$-valuation. It is well-known that $\mathcal{A}$-valuations are entirely determined by their behaviour on propositional variables. If φ_1 and φ_2 are two formulas of then $\varphi_1 \approx \varphi_2$ is called an equivalence. An equivalence $\varphi_1 \approx \varphi_2$ is valid in an algebra $\mathcal{A}$ if $v(\varphi_1) = v(\varphi_2)$ for all $\mathcal{A}$-valuations v. The set of equivalences valid in an algebra $\mathcal{A}$ is called the equational logic of $\mathcal{A}$

1.2 Infectious extensions

The set $var(\varphi)$ of propositional variables of a formula φ is recursively defined as follows:

- if $p \in Prop$ then $var(p) = \{p\}$,
- for all $\dagger \in V$ and $\varphi_1, ..., \varphi_{n(\dagger)}$ in F_σ, we have $var(\dagger(\varphi_1, ..., \varphi_{n(\dagger)})) = var(\varphi_1) \cup ... \cup var(\varphi_{n(\dagger)})$.

Let $\mathcal{A}$ be an algebra of type σ. The *infectious extension* of $\mathcal{A}$ is the algebra $\mathcal{A}^i = \langle A^i, (.)^{\mathcal{A}^i}\rangle$ of type σ where $A^i = A \cup \{i\}$ (for $i \notin A$) and, for all $\dagger \in V$ and $a_1, ..., a_{n(\dagger)}$ in A^i, we have $\dagger^{\mathcal{A}^i}(a_1, ..., a_{n(\dagger)}) = \dagger^{\mathcal{A}}(a_1, ..., a_{n(\dagger)})$ if $\{a_1, ..., a_{n(\dagger)}\} \subseteq A$ and $\dagger^{\mathcal{A}^i}(a_1, ..., a_{n(\dagger)}) = i$ otherwise.

Proposition 1. *Let $\mathcal{A}$ be an algebra of type σ and let φ_1 and φ_2 be two formulas of type σ. The equivalence $\varphi_1 \approx \varphi_2$ is valid in $\mathcal{A}^i$ if and only if it is valid in $\mathcal{A}$ and $var(\varphi_1) = var(\varphi_2)$.*

Proof. Suppose that $\varphi_1 \approx \varphi_2$ is valid in $\mathcal{A}^i$. Since $\mathcal{A}$ is a subalgebra of $\mathcal{A}^i$, this equivalence is valid in $\mathcal{A}$. Suppose moreover that $var(\varphi_1) \neq var(\varphi_2)$. Without loss of generality, we can assume that there is some $p \in var(\varphi_1) \setminus var(\varphi_2)$. Let v be a $\mathcal{A}^i$-valuation such that $v(p) = i$ and $v(q) \in A$ for all $q \in Prop \setminus \{p\}$. Clearly, $v(\varphi_1) = i$ and $v(\varphi_2) \in A$, which contradicts our initial assumption. So $var(\varphi_1) = var(\varphi_2)$.

Now suppose that $\varphi_1 \approx \varphi_2$ is valid in $\mathcal{A}$ and $var(\varphi_1) = var(\varphi_2)$. Let v be a $\mathcal{A}^i$-valuation. If there is some $p \in var(\varphi_1)$ such that $v(p) = i$ then $v(\varphi_1) = v(\varphi_2) = i$. Otherwise, let v' be a $\mathcal{A}$-valuation such that $v \upharpoonright var(\varphi_1) = v' \upharpoonright var(\varphi_1)$. Clearly, $v(\varphi_1) = v'(\varphi_1) = v'(\varphi_2) = v(\varphi_2)$. So $\varphi_1 \approx \varphi_2$ is valid in $\mathcal{A}^i$. □

1.3 The algebra of signed occurrences of propositional variables

From now on, we suppose that $\neg \in V$ and that $n(\neg) = 1$. The sets $var^+(\varphi)$ and $var^-(\varphi)$ of positive and negative propositional variables of a formula φ are defined recursively as follows:

- if $p \in Prop$ then $var^+(p) = \{p\}$,
- if $p \in Prop$ then $var^-(p) = \emptyset$,

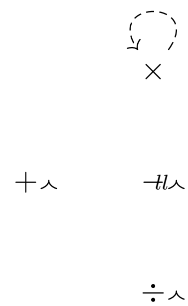
- for all $\dagger \in V \backslash \{\neg\}$ and $\varphi_1, ..., \varphi_{n(\dagger)}$ in F_σ, we have $var^+(\dagger(\varphi_1, ..., \varphi_{n(\dagger)})) = var^+(\varphi_1) \cup ... \cup var^+(\varphi_{n(\dagger)})$,
- for all $\dagger \in V \backslash \{\neg\}$ and $\varphi_1, ..., \varphi_{n(\dagger)}$ in F_σ, we have $var^-(\dagger(\varphi_1, ..., \varphi_{n(\dagger)})) = var^-(\varphi_1) \cup ... \cup var^-(\varphi_{n(\dagger)})$,
- if $\varphi \in F_\sigma$ then $var^+(\neg\varphi) = var^-(\varphi)$,
- if $\varphi \in F_\sigma$ then $var^-(\neg\varphi) = var^+(\varphi)$.

For all formulas $\varphi \in F_\sigma$, we define $var^s(\varphi) = \langle var^+(\varphi), var^-(\varphi)\rangle$.

Let $S_4 = \{+, -, \times, \div\}$ and let $\mathcal{S}_4 = \langle S_4, \sqcup, \iota\rangle$ such that $\sqcup : (S_4)^2 \to S_4$ and $\iota : S_4 \to S_4$ are defined as follows:[2]

$\sqcup$	$+$	$-$	$\times$	$\div$
$+$	$+$	$\times$	$\times$	$+$
$-$	$\times$	$-$	$\times$	$-$
$\times$	$\times$	$\times$	$\times$	$\times$
$\div$	$+$	$-$	$\times$	$\div$

	ι
$+$	$-$
$-$	$+$
$\times$	$\times$
$\div$	$\div$



Notice that, for all $a, b \in S_4$, we have $\iota(\iota(a)) = a$ and $\iota(a \sqcup b) = \iota(a) \sqcup \iota(b)$. We define the algebra $\mathcal{S}_4^\sigma = \langle S_4, (.)^{\mathcal{S}_4^\sigma}\rangle$ of type σ such that $\neg^{\mathcal{S}_4^\sigma} = \iota$ and, for all $\dagger \in V \setminus \{\neg\}$ and $s_1, ...s_{n(\dagger)}$ in S_4, we have $\dagger^{\mathcal{S}_4^\sigma}(s_1, ..., s_{n(\dagger)}) = s_1 \sqcup ... \sqcup s_{n(\dagger)}$. For all $\dagger \in V \setminus \{\neg\}$ and formulas $\varphi_1, ..., \varphi_{n(\dagger)}$, the equivalence $\neg\dagger(\varphi_1, ..., \varphi_{n(\dagger)}) \approx \dagger(\neg\varphi_1, ..., \neg\varphi_{n(\dagger)})$ is valid in $\mathcal{S}_4^\sigma$.

Proposition 2. *Let φ_1 and φ_2 be formulas of type σ. The equivalence $\varphi_1 \approx \varphi_2$ is valid in $\mathcal{S}_4^\sigma$ if and only if $var^s(\varphi_1) = var^s(\varphi_2)$.*

Proof. A quick induction shows that, for all $\varphi \in F_\sigma$ and $\mathcal{S}_4^\sigma$-valuations v, we have $v(\varphi) = \bigsqcup_{p \in var^+(\varphi)} v(p) \sqcup \bigsqcup_{p \in var^-(\varphi)} v(\neg p)$. It directly follows that $var^+(\varphi_1) = var^+t(\varphi_2)$ and $var^-(\varphi_1) = var^-(\varphi_2)$ entail that $\varphi_1 \approx \varphi_2$ is valid in $\mathcal{S}_4^\sigma$.

Conversely, suppose that $var^s(\varphi_1) \neq var^s(\varphi_2)$. For all $p \in Prop$, let v_p be the $\mathcal{S}_4^\sigma$-valuation such that $v_p(p) = +$ and $v_p(q) = \div$ for all $q \in Prop \setminus \{p\}$. The following table sums up the value of $v_p(\varphi)$ depending on whether or not p is in $var^+(\varphi)$ and $var^-(\varphi)$:

[2]This structure has been used along similar lines in [16] and [9].

$v_p(\varphi) =$	$p \in var^+(\varphi)$	$p \notin var^+(\varphi)$
$p \in var^-(\varphi)$	$\times$	$-$
$p \notin var^-(\varphi)$	$+$	$\div$

Without loss of generality, we know that there is some $p \in Prop$ such that $p \in var^+(\varphi_1) \setminus var^+(\varphi_2)$ or $p \in var^-(\varphi_1) \setminus var^-(\varphi_2)$. In the first case, $v_p(\varphi_1) = \times$ or $v_p(\varphi_1) = +$ and $v_p(\varphi_2) = -$ or $v_p(\varphi_2) = \div$. In the second case, $v_p(\varphi_1) = \times$ or $v_p(\varphi_1) = -$ and $v_p(\varphi_2) = +$ or $v_p(\varphi_2) = \div$. In any case, $v_p(\varphi_1) \neq v_p(\varphi_2)$. By contraposition, it follows that if $\varphi_1 \approx \varphi_2$ is valid in $\mathcal{S}_4^\sigma$ then $var^s(\varphi_1) = var^s(\varphi_2)$. □

Corollary 1. *Let $\mathcal{A}$ be an algebra of type σ and let φ_1 and φ_2 be two formulas of type σ. The equivalence $\varphi_1 \approx \varphi_2$ is valid in the cartesian product $\mathcal{A} \times \mathcal{S}_4^\sigma$ if and only if it is valid in $\mathcal{A}$ and $var^s(\varphi_1) = var^s(\varphi_2)$.*

So if an equational logic corresponds to the equational logic of a n-element algebra, then its largest fragment respecting the Angell Condition corresponds to the equational logic of a $4n$-element algebra. In fact, the number of elements can be reduced to $n+3$, as shown by the Proposition 3 below.

1.4 Signed infectious extension

Let $\mathcal{A}$ be an algebra of type σ, where we suppose that $+$, $-$ and $\times$ are not members of A. The *signed infectious extension* of $\mathcal{A}$ is the algebra $\mathcal{A}^s = \langle A^s, (.)^{\mathcal{A}^s} \rangle$ of type σ where $A^s = A \cup \{+, -, \times\}$ and, for all $\dagger \in V$ and $a_1, ..., a_{n(\dagger)}$ in A^i, we have $\dagger^{\mathcal{A}^s}(a_1, ..., a_{n(\dagger)}) = \dagger^{\mathcal{A}}(a_1, ..., a_{n(\dagger)})$ if $\{a_1, ..., a_{n(\dagger)}\} \subseteq A$ and $\dagger^{\mathcal{A}^i}(a_1, ..., a_{n(\dagger)}) = \bigsqcup \{a_1, ..., a_{n(\dagger)}\} \cap S_4$ otherwise. The algebra A^s is the result of plugging the $\{+, -, \times\}$-subalgebra of $\mathcal{S}_4^\sigma$ on top of $\mathcal{A}$. Similarly, it can be viewed as the result of replacing $\div$ by $\mathcal{A}$ in $\mathcal{S}_4^\sigma$.The truth-values $+, -$ and $\times$ behave infectiously when combined with truth-values of $\mathcal{A}$ and according to $\mathcal{S}_4^\sigma$ when combined together.[3]

Proposition 3. *Let $\mathcal{A}$ be an algebra of type σ and let φ_1 and φ_2 be two formulas of type σ. The equivalence $\varphi_1 \approx \varphi_2$ is valid in $\mathcal{A}^s$ if and only if it is valid in $\mathcal{A}$ and $var^s(\varphi_1) = var^s(\varphi_2)$.*

Proof. Since $\mathcal{A}^s$ is a homomorphic image of $\mathcal{A} \times \mathcal{S}_4^\sigma$, if $\varphi_1 \approx \varphi_2$ is valid in $\mathcal{A}$ and $var^s(\varphi_1) = var^s(\varphi_2)$ then it is valid in $\mathcal{A}^s$.

[3] A similar algebra has been used in [9], building from the four-valued logic FDE to get a seven-valued semantics for the logic AC.

Since $\mathcal{A}$ is a subalgebra of $\mathcal{A}^s$, if $\varphi_1 \approx \varphi_2$ is valid in $\mathcal{A}^s$ then it is valid in $\mathcal{A}$.

Now suppose that $var^s(\varphi_1) \neq var^s(\varphi_2)$. For all $p \in Prop$, let v_p be a $\mathcal{A}^s$-valuation such that $v_p(p) = +$ and $v_p(q) \in A$ for all $q \in Prop \setminus \{p\}$. The following table sums up the value of $v_p(\varphi)$ depending on whether or not p is in $var^+(\varphi)$ and $var^-(\varphi)$:

$v_p(\varphi)$	$p \in var^+(\varphi)$	$p \notin var^+(\varphi)$
$p \in var^-(\varphi)$	$\times$	$-$
$p \notin var^-(\varphi)$	$+$	$\in A$

Without loss of generality, we know that there is some $p \in Prop$ such that $p \in var^+(\varphi_1) \setminus var^+(\varphi_2)$ or $p \in var^-(\varphi_1) \setminus var^-(\varphi_2)$. In the first case, $v_p(\varphi_1) = \times$ or $v_p(\varphi_1) = +$ and $v_p(\varphi_2) = -$ or $v_p(\varphi_2) \in A$. In the second case, $v_p(\varphi_1) = \times$ or $v_p(\varphi_1) = -$ and $v_p(\varphi_2) = +$ or $v_p(\varphi_2) \in A$. In any case, $v_p(\varphi_1) \neq v_p(\varphi_2)$. By contraposition, it follows that if $\varphi_1 \approx \varphi_2$ is valid in $\mathcal{A}^s$ then $var^s(\varphi_1) = var^s(\varphi_2)$. $\square$

It follows from Proposition 3 that the largest fragment of the equational logic of an n-element algebra respecting the Angell Condition is the equational logic of an $n+3$-element algebra.

2 Application to classical logic

2.1 The signed infectious extension of classical logic

Let θ be the type $\langle \{\wedge, \vee, \neg\}, n \rangle$ where $n(\wedge) = n(\vee) = 2$ and $n(\neg) = 1$. Let $\mathbf{2}$ be the algebra $\langle \{0, 1\}, (.)^{\mathbf{2}} \rangle$ of type θ where:

$\wedge^{\mathbf{2}}$	1	0
1	1	0
0	0	0

$\vee^{\mathbf{2}}$	1	0
1	1	1
0	1	0

	$\neg^{\mathbf{2}}$
1	0
0	1

10⅄

If an equivalence is valid in $\mathbf{2}$, it is said to be classical.

The infectious extension of $\mathbf{2}$, the algebra $\mathbf{2}^i$, corresponds to the three-element algebra defined by the Weak Kleene truth-tables. As mentioned, its equational logic corresponds to the logic of content identity where identity of truth-conditions is understood as classical equivalence and identity of subject-matter is understood via the Parry Condition. This logic is considered by Hornischer [15] as a candidate for the correct logic of synonymy and is called SCA in this context. It also corresponds to first-degree equivalence in Parry's logic AI of Analytic Implication [20] and to first-degree equivalence in Epstein's logic D of Dependence [7].

The signed infectious extension of $\mathbf{2}$, the algebra $\mathbf{2}^s$, is given by the following truth-tables:

$\wedge^{\mathbf{2}^s}$	1	0	$+$	$-$	$\times$
1	1	0	$+$	$-$	$\times$
0	0	0	$+$	$-$	$\times$
$+$	$+$	$+$	$+$	$\times$	$\times$
$-$	$-$	$-$	$\times$	$-$	$\times$
$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

$\vee^{\mathbf{2}^s}$	1	0	$+$	$-$	$\times$
1	1	1	$+$	$-$	$\times$
0	1	0	$+$	$-$	$\times$
$+$	$+$	$+$	$+$	$\times$	$\times$
$-$	$-$	$-$	$\times$	$-$	$\times$
$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

	$\neg^{\mathbf{2}^s}$
1	0
0	1
$+$	$-$
$-$	$+$
$\times$	$\times$

Its equational logic corresponds to the logic of content identity where identity of truth-conditions is understood by classical equivalence and identity of subject-matter via the Angell Condition. This logic has also been considered by Hornischer [15] as a candidate for the correct logic of synonymy and is called SCL in this context. Interestingly, this logic also corresponds to a very natural weakening of the notion of isomorphism in the category of formulas and classical proofs studied by Restall [24]. These five-valued truth-tables are, to our knowledge, the first purely semantic characterisation of this logic.

2.2 Humberstone's two conjectures

In [17], Humberstone studies the power algebra[4] of $\mathbf{2}$, namely the algebra $\mathcal{P}(\mathbf{2}) = \langle \mathcal{P}(\{0,1\}), (.)^{\mathcal{P}(\mathbf{2})} \rangle$ where:

$\wedge^{\mathcal{P}(\mathbf{2})}$	$\{1\}$	$\{0\}$	$\{0,1\}$	$\emptyset$
$\{1\}$	$\{1\}$	$\{0\}$	$\{0,1\}$	$\emptyset$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\emptyset$
$\{0,1\}$	$\{0,1\}$	$\{0\}$	$\{0,1\}$	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

$\vee^{\mathcal{P}(\mathbf{2})}$	$\{1\}$	$\{0\}$	$\{0,1\}$	$\emptyset$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\emptyset$
$\{0\}$	$\{1\}$	$\{0\}$	$\{0,1\}$	$\emptyset$
$\{0,1\}$	$\{1\}$	$\{0,1\}$	$\{0,1\}$	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

	$\neg^{\mathcal{P}(\mathbf{2})}$
$\{1\}$	$\{0\}$
$\{0\}$	$\{1\}$
$\{0,1\}$	$\{0,1\}$
$\emptyset$	$\emptyset$

One notices[5] that $\mathcal{P}(\mathbf{2})$ is isomorphic to the infectious extension of the

[4] For a general presentation of power set algebras, see [17].

[5] This has been noted, though in different terms, by Priest [22].

Strong Kleene three-element algebra $\mathcal{K}_3$, defined by the following truth-tables:

$\wedge^{\mathcal{K}_3}$	1	a	0
1	1	a	0
a	a	a	0
0	0	0	0

$\vee^{\mathcal{K}_3}$	1	a	0
1	1	1	1
a	1	a	a
0	1	a	0

	$\neg^{\mathcal{K}_3}$
1	0
a	a
0	1

Humberstone formulates two conjectures concerning the equational logic of $\mathcal{P}(\mathbf{2})$, both involving the positive and negative occurrences of propositional variables.

His second conjecture, the more ambitious, is the following:

Conjecture 2. *Let φ_1 and φ_2 be formulas of type θ. If $\varphi_1 \approx \varphi_2$ is valid in $\mathbf{2}$ and $var^s(\varphi_1) = var^s(\varphi_2)$ then it is valid in $\mathcal{P}(\mathbf{2})$.*

Unfortunately, one can find counterexamples.

Proposition 4. *The equivalence*

$$(p \wedge \neg p) \wedge ((p \wedge \neg p) \vee (q \wedge \neg q)) \approx (q \wedge \neg q) \wedge ((q \wedge \neg q) \vee (p \wedge \neg p))$$

is a counterexample to Conjecture 2.

Proof. First, notice that

$$\begin{aligned}\{p, q\} &= var^+((p \wedge \neg p) \wedge ((p \wedge \neg p) \vee (q \wedge \neg q)))\\ &= var^-((p \wedge \neg p) \wedge ((p \wedge \neg p) \vee (q \wedge \neg q)))\\ &= var^+((q \wedge \neg q) \wedge ((q \wedge \neg q) \vee (p \wedge \neg p)))\\ &= var^-((q \wedge \neg q) \wedge ((q \wedge \neg q) \vee (p \wedge \neg p)))\end{aligned}$$

So $var^s((p\wedge\neg p)\wedge((p\wedge\neg p)\vee(q\wedge\neg q))) = var^s((q\wedge\neg q)\wedge((q\wedge\neg q)\vee(p\wedge\neg p)))$.

Moreover, for all formulas φ_1 and φ_2, the equivalence $\varphi_1 \approx \varphi_1 \wedge (\varphi_1 \vee \varphi_2)$ is valid in $\mathbf{2}$. Since $p\wedge\neg p \approx q\wedge\neg q$ is valid in $\mathbf{2}$, it follows that the equivalence

$$(p \wedge \neg p) \wedge ((p \wedge \neg p) \vee (q \wedge \neg q)) \approx (q \wedge \neg q) \wedge ((q \wedge \neg q) \vee (p \wedge \neg p))$$

is valid in $\mathbf{2}$.

Now suppose that this equivalence is valid in $\mathcal{P}(\mathbf{2})$. So it follows that it is valid in $\mathcal{K}_3^i$ and therefore in $\mathcal{K}_3$. Similarly, for all formulas φ_1 and φ_2, the equivalence $\varphi_1 \approx \varphi_1 \wedge (\varphi_1 \vee \varphi_2)$ is valid in $\mathcal{K}_3$. So it follows that the equivalence $p \wedge \neg p \approx q \wedge \neg q$ is valid in $\mathcal{K}_3$. This, however, is false since the valuation $v : p \mapsto 1, q \mapsto a$ is such that $v(p \wedge \neg p) = 0$ and $v(q \wedge \neg q) = a$. $\square$

This refutes Humberstone's second conjecture. His first conjecture, however, is less ambitious and goes as follows:

Conjecture 1. *Let φ_1 and φ_2 be formulas of type θ. If $\varphi_1 \approx \varphi_2$ is valid in $\mathbf{2}$ and $var(\varphi_1) = var(\varphi_2)$ and $(var^+(\varphi_1) \cup var^+(\varphi_2)) \cap (var^-(\varphi_1) \cup var^-(\varphi_2)) = \emptyset$ then this equivalence is valid in $\mathcal{P}(\mathbf{2})$.*

In fact, we will show that a stronger claim holds. Let us say that a formula φ is well-polarised if $var^+(\varphi) \cap var^-(\varphi) = \emptyset$.

Proposition 5. *Let φ_1 and φ_2 be formulas of type θ. If $\varphi_1 \approx \varphi_2$ is valid in $\mathbf{2}$ and $var(\varphi_1) = var(\varphi_2)$ and φ_1 and φ_2 are well-polarised then this equivalence is valid in $\mathcal{P}(\mathbf{2})$.*

Proof. It suffices to show that if $\varphi_1 \approx \varphi_2$ is classical and φ_1 and φ_2 are well-polarised then this equivalence is valid in $\mathcal{K}_3$. The result will then follow from this isomorphism between $\mathcal{K}_3^i$ and $\mathcal{P}(\mathbf{2})$ and Proposition 1.

We adapt proof techniques from [18] and we use the following facts about $\mathcal{K}_3$. For every formula φ, there is a formula φ^c in conjunctive normal form such that $var^s(\varphi) = var^s(\varphi^c)$ and $\varphi \approx \varphi^c$ is valid in $\mathcal{K}_3$ and there is a formula φ^d in disjunctive normal form such that $var^s(\varphi) = var^s(\varphi^d)$ and $\varphi \approx \varphi^d$ is valid in $\mathcal{K}_3$.

Suppose $\varphi_1 \approx \varphi_2$ is not valid in $\mathcal{K}_3$. So there is some $v : \mathcal{F}_\theta \to \mathcal{K}_3$ such that $v(\varphi_1) \neq v(\varphi_2)$. Without loss of generality, we can assume that $v(\varphi) \in \{0, 1\}$ and $v(\varphi_2) = a$.

Suppose $v(\varphi_1) = 0$. So for all disjuncts χ in φ_1^d, we have $v(\chi) = 0$ and so there is some $p_\chi \in var^+(\chi)$ such that $v(p_\chi) = 0$ or some $q_\chi \in var^-(\chi)$ such that $v(q_\chi) = 1$. Moreover, there is some disjunct ζ in φ_2^d such that $v(\zeta) = a$. Let v' be the $\mathbf{2}$-valuation such that $v'(p) = v(p)$ if $v(p) \in \{0, 1\}$ and $v'(p) = 1$ if $p \in var^+(\zeta)$ and $v'(p) = 0$ if $p \in var^-(\zeta)$. This is well-defined because φ_2 is well-polarised. Clearly, $v'(\zeta) = 1$ and so $v'(\varphi_2) = 1$. It follows that $v'(\varphi_1) = 1$. So there is some disjunct χ of φ_1 such that $v'(\chi) = 1$. But $v(p_\chi) = 0$ or $v(q_\chi) = 1$ and so $v'(p_\chi) = 0$ or $v'(q_\chi) = 1$. This entails that $v'(\chi) = 0$. So it cannot be the case that $v(\varphi_1) = 0$.

Suppose $v(\varphi_1) = 1$. So for all conjunct χ in φ_1^c, we have $v(\chi) = 1$ and so there is some $p_\chi \in var^+(\chi)$ such that $v(p_\chi) = 1$ or some $q_\chi \in var^-(\chi)$ such that $v(q_\chi) = 0$. Moreover, there is some conjunct ζ in φ_2^c such that $v(\zeta) = a$. Let v' be the $\mathbf{2}$-valuation such that $v'(p) = v(p)$ if $v(p) \in \{0, 1\}$ and $v'(p) = 0$ if $p \in var^+(\zeta)$ and $v'(p) = 1$ if $p \in var^-(\zeta)$. This is well-defined because

φ_2 is well-polarised. Clearly, $v'(\zeta) = 0$ and so $v'(\varphi_2) = 0$. It follows that $v'(\varphi_1) = 0$. So there is some conjunct χ of φ_1 such that $v'(\chi) = 0$. But $v(p_\chi) = 1$ or $v(q_\chi) = 0$ and so $v'(p_\chi) = 1$ or $v'(q_\chi) = 0$. This entails that $v'(\chi) = 1$. So it cannot be the case that $v(\varphi_1) = 1$. $\square$

It allows us to confirm Conjecture 1. Indeed, one easily sees that $(var^+(\varphi_1) \cup var^+(\varphi_2)) \cap (var^-(\varphi_1) \cup var^-(\varphi_2)) = \emptyset$ entails that both φ_1 and φ_2 are well-polarised. Proposition 5 is however strictly stronger than Conjecture 1, since it entails that equivalences like $p \wedge (p \vee q) \approx p \wedge (p \vee \neg q)$ are valid in $\mathcal{P}(\mathbf{2})$.

Proposition 5 does not provide a full characterisation of the equational logic of $\mathcal{P}(\mathbf{2})$. Indeed, the equivalence considered in Proposition 4 is a counterexample to the converse of Proposition 5. A necessary and sufficient syntactic characterisation of classical equivalences that are valid in $\mathcal{P}(\mathbf{2})$ is still, to our knowledge, an open problem.

Conclusion

In this note, we have developed an algebraic understanding of the notion of sameness of subject-matter encapsulated in the Angell Condition. Just like infectious extensions provide a purely semantic representation of the syntactic Parry Condition, signed infectious extensions allow us to understand semantically the syntactic Angell Condition. This led us to give the first semantics, to our knowledge, for the logic SCL put forward in [15] as a candidate for the logic of propositional synonymy. Moreover, we disproved a conjecture by Humberstone [17] linking the Angell Condition and the equational logic of the power algebra of the two-element Boolean algebra. We however proved another related conjecture of his by proving a stronger result about well-polarised formulas.

Left for future work is the project of deepening our understanding of the equational logic of the power algebra of the two-element Boolean algebra and its link with the occurrences (signed or not) of propositional variables. This will allow us to link our theories of subject-matter to another recent research program in philosophical logic, namely paraconsistency and paracompleteness via plurivalence (i.e. power algebras) [23]. Moreover, it will be necessary to develop a philosophical interpretation of signed infectious extensions. Indeed, the formal theory of infectious extensions is linked to the philosophical development of logics of nonsense [8] and the infectious truth-value i can be interpreted as the meaningless truth-value. This provides a philosophical ground for the Parry Condition. Therefore, developing a philosophical interpretation of the signed infectious truth-values $+, -$ and $\times$ will be crucial to consolidating the Angell Condition as a viable alternative to the Parry Condition.

Acknowledgments: I am grateful to Franz Berto for his valuable comments on an earlier draft. The paper also greatly benefited from the suggestions made by an anonymous referee.

References

[1] Richard B. Angell. Deducibility, entailment and analytic containment. In Jean Norman and Richard Sylvan, editors, *Directions in relevant logic*, pages 119–143. Springer, 1989.

[2] Francesco Berto. Aboutness in imagination. *Philosophical Studies*, 175 (8):1871–1886, 2018.

[3] Franz Berto. Adding 4.0241 to TLP. In Gabriele M. Mras, Paul Weingartner, and Bernhard Ritter, editors, *Philosophy of Logic and Mathematics: Proceedings of the 41st International Ludwig Wittgenstein Symposium*, pages 415–428. De Gruyter, 2019.

[4] Stefano Bonzio and Michele Pra Baldi. Containment logics: Algebraic completeness and axiomatization. *Studia Logica*, 109(5):969–994, 2021.

[5] Benjamin Brast-McKie. Identity and aboutness. *Journal of Philosophical Logic*, 50(6):1471–1503, 2021.

[6] Roberto Ciuni, Damian Szmuc, and Thomas Macaulay Ferguson. Relevant logics obeying component homogeneity. *The Australasian Journal of Logic*, 15(2):301–361, 2018.

[7] Richard L. Epstein. *The Semantic Foundations of Logic Volume 1: Propositional Logics*. Springer, 1990.

[8] Thomas Macaulay Ferguson. Logics of nonsense and Parry systems. *Journal of Philosophical Logic*, 44(1):65–80, 2015.

[9] Kit Fine. Angellic content. *Journal of Philosophical Logic*, 45(2):199–226, 2016.

[10] Kit Fine. A theory of truthmaker content II: Subject-matter, common content, remainder and ground. *Journal of Philosophical Logic*, 46(6):675–702, 2017.

[11] Kit Fine. Yablo on subject-matter. *Philosophical Studies*, 177(1):129–171, 2020.

[12] Jeremy Goodman. Agglomerative algebras. *Journal of Philosophical Logic*, 48(4):631–648, 2019.

[13] Peter Hawke. Questions, topics and restricted closure. *Philosophical Studies*, 173(10):2759–2784, 2016.

[14] Peter Hawke. Theories of aboutness. *Australasian Journal of Philosophy*, 96(4):697–723, 2018.

[15] Levin Hornischer. Logics of synonymy. *Journal of Philosophical Logic*, 49(4):767–805, 2020.

[16] Lloyd Humberstone. False though partly true–an experiment in logic. *Journal of Philosophical Logic*, 32(6):613–665, 2003.

[17] Lloyd Humberstone. Power matrices and Dunn–Belnap semantics: Reflections on a remark of Graham Priest. *The Australasian Journal of Logic*, 11(1):1–32, 2014.

[18] John N. Martin. A syntactic characterization of Kleene's strong connectives with two designated values. *Mathematical Logic Quarterly*, 21(1):181–184, 1975.

[19] Hitoshi Omori and Damian Szmuc. Conjunction and disjunction in infectious logics. In *International Workshop on Logic, Rationality and Interaction*, pages 268–283. Springer, 2017.

[20] William T. Parry. Analytic implication; its history, justification and varieties. In Jean Norman and Richard Sylvan, editors, *Directions in relevant logic*, pages 101–118. Springer, 1989.

[21] Matteo Plebani and Giuseppe Spolaore. Subject matter: A modest proposal. *The Philosophical Quarterly*, 71(3):605–622, 2021.

[22] Graham Priest. Hyper-contradictions. *Logique et Analyse*, 27(107):237–243, 1984.

[23] Graham Priest. Plurivalent logics. *The Australasian Journal of Logic*, 11(1):1–12, 2014.

[24] Greg Restall. Isomorphisms in a category of proofs, presented at CUNY Graduate Center in the Logic and Metaphysics seminar on 9 April 2018. Handout available at: http://consequently.org/handouts/isomorphisms-handout-cuny-2018.pdf, (last checked 16 February 2022), 2018.

[25] Stephen Yablo. *Aboutness*. Princeton University Press, 2014.

[26] Stephen Yablo. Reply to Fine on aboutness. *Philosophical Studies*, 175 (6):1495–1512, 2018.

[27] Seth Yalcin. Belief as question-sensitive. *Philosophy and Phenomenological Research*, 97(1):23–47, 2016.