Condorcet-Style Paradoxes for Majority Rule with Infinite Candidates

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Abstract

I present two possibility results and one impossibility result about a situation with three voters under a pairwise majoritarian aggregation function voting on a countably infinite number of candidates. First, from individual orders with no maximal or minimal element, it is possible to generate an aggregate order with a maximal or minimal element. Second, from dense individual orders, it is possible to generate a discrete aggregate order. Finally, I show that, from discrete orders with a particular property, namely the finite-distance property, it is not possible to generate a dense aggregate order.

Keywords: Condorcet Paradox, Preference Aggregation, Majority Rule, Infinite Candidates

1 Introduction

This paper concerns preference aggregation according to majority rule voting.[1] Majority rule is one way of generating group preferences from the preferences of individuals. As we will discuss it here, for any two candidates, the group preference depends on the individual preferences with respect to those candidates. Roughly, according to majority rule, one candidate is collectively preferred to another candidate when the number of individuals who

[1]A classic characterization of majority rule is given by Kenneth May in [19].

prefer the former is at least as great as the number of individuals who prefer the latter.

Consider the following example. We have three people, Ann, Bob, and Chris, forming a group preference over vanilla, chocolate, or strawberry ice cream using pairwise majority voting. For each pair of possibilities, each person is asked whether they prefer one to the other. So, to pick one person, Ann is asked whether she prefers vanilla to chocolate, whether she prefers vanilla to strawberry, and whether she prefers strawberry to chocolate. Say each of the individuals in our example is asked these questions and the answers are that Ann and Bob prefer strawberry to vanilla, Bob and Chris prefer chocolate to vanilla, and Ann and Chris prefer strawberry to chocolate. The group then prefers strawberry to chocolate to vanilla.

More abstractly, let $I = \{a, b, c\}$ be a set of individuals and $X = \{x, y, z, ...\}$ be a set of candidates. Each individual $i \in I$ has a *preference ordering*, $\succ_i$, over X. We will treat $\succ_i$ as a complete, anti-symmetric, and transitive binary relation on X. Complete means that for any two candidates, $x, y \in X$, either $x \succ_i y$ or $y \succ_i x$. Anti-symmetric means that for all $x, y \in X$ if $x \succ_i y$ and $y \succ_i x$ then $x = y$. Transitive means that for all $x, y, z \in X$ if $x \succ_i y$ and $y \succ_i z$, then $x \succ_i z$. We then interpret $\succ_i$ as follows. For any $x, y \in X$, $x \succ_i y$ means that individual i prefers x to y. Each preference ordering is then a linear order, which we denote with subscripts, $\succ_a, \succ_b, \succ_c$. Call a combination of preference orderings across the individuals $\succ = (\succ_a, \succ_b, \succ_c)$ a *profile*. A preference aggregation function is a function, f, that assigns to each profile, in some domain of admissible profiles, a group preference relation $\succ_G = f(\succ)$ on X. Our function is *pairwise majority voting*, denoted by f^M. It says that, for any profile $\succ$ and any $x, y \in X$, $x \succ_G y$ if and only if at least as many individual preference orderings have $x \succ_i y$ as have $y \succ_i x$.

At first glance, majority rule is the most intuitive way to generate group preferences. But some important properties of the individual orders are lost in majority rule transformation. The most famous example of this is *Condorcet's paradox*, and concerns the loss of transitivity.[2] By slightly rearranging the preferences of the individuals, we can use our simple ice cream case

[2]The original formulation is in [6]. For discussion of the paradox, see [14]. While Condorcet considers a specific voting method, namely majoritarian voting, in [2], Kenneth Arrow famously introduced a general approach to preference aggregation that shows how a class of possible aggregation functions fails to satisfy some plausible axioms. For discussion of how Condorcet's paradox relates to this and other important results about aggregation mechanisms, see [16].

to create an example of Condorcet's paradox. Say Ann prefers vanilla to strawberry, strawberry to chocolate, and vanilla to chocolate. Bob prefers strawberry to chocolate, chocolate to vanilla, and strawberry to vanilla. And Chris prefers chocolate to vanilla, vanilla to strawberry, and chocolate to strawberry. Each of these preferences is transitive. When we aggregate according to majority rule, however, the group preference is not. Two of the three people prefer vanilla to strawberry (Ann and Chris), two of the three people prefer strawberry to chocolate (Ann and Bob) and yet two of the three people prefer chocolate to vanilla (Bob and Chris).

The question this paper considers has a similar form, namely what properties of individual orders are changed in the group preference generated by majority rule, but a different content. The first change in content is that instead of cases with a finite number of candidates, we will look at preferences over a countably infinite number of candidates. The second change is that instead of transitivity, we will look at other properties of the individual orders, in particular *not having a minimal element*, *not having a maximal element*, *not having a maximal and a minimal element*, *density*, and *discreteness*.

Specifically, we will look at the following three questions about our majority rule aggregation function with an infinite set of candidates:

(i) From a set of individual orders with no minimal or maximal element, is it possible to generate an order with a minimal or a maximal element?

(ii) From a set of dense orders, is it possible to generate a discrete order?

(iii) From a set of discrete orders, is it possible to generate a dense order?

Each of these questions asks, about relations over countably infinite sets, whether there is some property of the individual orders that is not a property of the aggregate order. Answering these questions contributes to the literature in two complementary ways.

First, preservation results are a key component of the analysis of aggregation rules in social choice theory and order theory.[3] Take an early example that discusses them in model theoretic terms. In [4], Donald Brown characterizes an important aim of social choice theory as giving a "complete characterization of all first order properties of individual preferences, which are preserved by aggregation procedures" (p. 3-4) that meet certain basic

[3] Many thanks to a reviewer for this journal for helping me develop this point.

requirements. One example of an aggregation procedure that meets all the specified requirements is majority rule. This paper extends that general discussion of all first-order properties by considering properties of infinite orders that have not yet been considered.

Second, philosophers and social scientists have found various uses for these properties. Most importantly, the possibility of a maximal or minimal element, addressed in the first question, tells us about the group's most or least favourite element. This property in particular is of considerable social choice theoretic interest. When modelling political situations, we don't just want to know a group's preference ordering, we want to know which alternative the group *chooses*. On many interpretations of that kind of choice, it depends on the existence of an element such that no elements are preferred to it in the group preference ordering. That is, it depends on the existence of at least one maximal element (a formal definition is given below). As a result, there is a substantial literature in social choice theory that concerns the conditions on the presence of a maximal element.[4] This paper also contributes to that project. As we will see, surprisingly, it is *not* a condition on having a group preference order with a maximal element that the individual preference orderings have a maximal element.

Moreover, since we don't always know in advance which properties are going to be important, the better we understand how preservation results work in general, the better off we will be when we attempt to use these properties for mathematical, logical[5], philosophical, or social-scientific purposes.

[4]See, for example, [1] in which the authors state, "the existence theorems of maximal elements are useful and important tools to prove the existence of non-empty choice sets" (p. 8), or [3], which uses a measure-theoretic approach to define solution concepts using the existence of a maximal element in the group preference relation with an infinite set of alternatives. Other examples include [13] and [12].

[5]There have been several exciting developments in approaches to aggregation rules from a logical perspective in the literature on judgment aggregation. This approach generalizes preference aggregation to all kinds of judgments, which involve acceptances or rejections of (multiple, logically-interconnected) propositions. Aggregating preferences is then a special case since one way to understand preferences is that an agent prefers x to y if and only if they judge that x is preferable to y. Indeed, many foundational results of social choice theory have been recast in the judgment aggregation framework [8] [11] [10]. Several important theorems answer questions with the same form as those of this paper, namely what properties of the individual judgments are preserved under various aggregation functions, but again with a different content. The central property of concern for judgment aggregation is logical consistency. Informally stated, under a version of

I address question (i) in the next section, and questions (ii) and (iii) in the section after that.

2 Maximal and Minimal Elements

Let us begin with the definition of an order type. Two ordered sets, $\langle S, \leq\rangle$ and $\langle T, \leq\rangle$, have the same order type iff they are *order isomorphic*. Those sets are order isomorphic if there is a function $f\colon S \to T$, such that, for all $u, v \in S$:

$$u \leq_S v \iff f(u) \leq_T f(v)$$

Next we have the definition of minimal and maximal elements. If $\leq$ is a linear order on a set S, element u is *minimal* if there is no element $v \neq u$ such that $v \leq u$. Similarly, we say that element u is *maximal* if there is no $v \neq u$ such that $u \leq v$. For convenience, we can then speak of the order type $\langle \mathbb{N}, \leq\rangle$, or, for short, of type $\mathbb{N}$, and order type $\langle \mathbb{Z}, \leq\rangle$, or type $\mathbb{Z}$, where $\mathbb{N} = \{0, 1, 2...\}$ and $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$. The reason it is convenient for our purposes is that $\mathbb{N}$ is order isomorphic to corresponding preference orderings with a minimal element and without a maximal element, while $\mathbb{Z}$ is order-isomorphic to corresponding preference orderings without a minimal or maximal element. Finally, a profile $\succ$ is said to be of type $\mathbb{A}$ if each $\succ_i$ is an order of type $\mathbb{A}$.

With these definitions, we can translate the questions above about preference aggregation using majority rule with three voters into questions about orders of type $\mathbb{N}$ and $\mathbb{Z}$. Since in question (i) we are aiming to go from a set without minimal or maximal elements to a set with minimal or maximal elements, we can formulate an initial translated version: can we find an instance of a profile of type $\mathbb{Z}$ resulting in an aggregate order of type $\mathbb{N}$?

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majority voting, it is sometimes the case that all the individual judgments sets are logically consistent, while the collective judgment set is not [23]. This result has been formalized and generalized to show impossibility results about classes of aggregation rules that meet certain conditions, and the most attractive ways of relaxing the conditions to generate possibility results have also been discussed [17][18]. Further, while results are sometimes formulated in classical propositional logic [20][9], extensions of this approach have been developed for many-valued [21], predicate [15], modal [24][22], conditional, and fuzzy-valued logics [7]. To my knowledge, questions concerning the aggregation of judgments on infinitely many propositions have not yet been addressed.

Theorem 1. *Assume that there are three individuals. There exists a profile* $\succ = (\succ_a, \succ_b, \succ_c)$ *of type* $\mathbb{Z}$ *such that* $f^M(\succ)$ *is an order of type* $\mathbb{N}$.

Proof. Consider the following three orderings, $\succ_a$, $\succ_b$, $\succ_c$. We represent the candidates in each using the same symbols as the natural numbers (even though each preference ordering will be order isomorphic to the integers). The relation between two candidates is given by their placement in the table below. For any two elements x and y, if x is above y in the table, then $x \succ_i y$. We construct $\succ_a$ by putting all the symbols of the multiples of 3 below 0, and the symbols for the rest of the natural numbers above. We construct $\succ_b$ by putting all the symbols for natural numbers congruent with 1 mod 3 below 0 and the symbols for the rest of the natural numbers above. We construct $\succ_c$ by putting all the symbols for natural numbers of 2 mod 3 below 0 and the rest of the natural numbers above. The result is that all the symbols above 0 are ordered in the usual way, while all the symbols below 0 reverse the usual order. This gives us the following table:

$\succ_a$	$\succ_b$	$\succ_c$
.	.	.
.	.	.
.	.	.
4	5	6
2	3	3
1	2	1
0	0	0
3	1	2
6	4	5
.	.	.
.	.	.
.	.	.

These three preference orderings have no maximal or minimal element and are order isomorphic to $\mathbb{Z}$. So, $\succ = (\succ_a, \succ_b, \succ_c)$ is of type $\mathbb{Z}$. Also, $f^M(\succ) = \succ_G$, gives us:

$\succ_G$
.
.
.
3
2
1
0

This order is simply $\langle \mathbb{N}, \leq \rangle$, if we treat the symbols as representing what they usually do, and so is clearly order isomorphic to $\mathbb{N}$, according to our shorthand. For a brief explanation why, consider the profiles. In the profile, when comparing '1' and '0', there are two votes for '1' and one vote for '0'. When comparing '2' and '1', there are two votes for '2' and one vote for '1'. When comparing '3' and '2', there are two votes for '3' and one vote for '2', and so on. In general this pattern results because, when asked questions about pairs of candidates, none of the individuals agree on any of the symbols below 0 in the order and for each symbol n two individuals agree that the symbol for $n+1$ is preferable. We therefore have three individuals with preference orderings with no minimal elements and an aggregate order with a minimal element.

□

To answer whether we can do the same thing with a maximal element, it will be convenient to define $\mathbb{N}^*$ as $\mathbb{N}$ upside down. That is, $\mathbb{N}^*$ has all the same elements as $\mathbb{N}$, but in reverse order, so that 0 is the maximal element.

Corollary 1. *Assume that there are three individuals. There exists a profile* $\succ = (\succ_a, \succ_b, \succ_c)$ *of type* $\mathbb{Z}$ *such that* $f^M(\succ)$ *is an order of type* $\mathbb{N}^*$.

We start with the three orders from above, $\succ_a, \succ_b, \succ_c$, and take their inverses, $\succ_a^{-1}, \succ_b^{-1}, \succ_c^{-1}$. For each order and for any two elements in that order, x and y, if $x \succ_i y$, then $y \succ_i^{-1} x$. To illustrate, this is $\succ_a^{-1}$:

$$\begin{array}{c} \succ_a^{-1} \\ \hline \cdot \\ \cdot \\ \cdot \\ 6 \\ 3 \\ 0 \\ 1 \\ 2 \\ 4 \\ \cdot \\ \cdot \end{array}$$

As before, this is an order of type $\mathbb{Z}$, as are $\succ_b^{-1}, \succ_c^{-1}$. So, $\succ^{-1} = (\succ_a^{-1}, \succ_b^{-1}, \succ_c^{-1})$ is of type $\mathbb{Z}$. Clearly, $f^M(\succ^{-1}) = \succ_g^{-1}$ is an order of type $\mathbb{N}^*$, since all the relations in the profile are simply reversed. That is, in the profile, when comparing '0' and '1', there are two votes for '0' and one vote for '1'. When comparing '1' and '2', there are two votes for '1' and one vote for '2'. When comparing '2' and '3', there are two votes for '2' and one vote for '3', and so on. So, '0' is the maximal element in $\succ_G^{-1}$ and there is no minimal element. Therefore, it is possible, using a majority rule aggregation function with three voters, to generate an aggregate order with a maximal element from individual orders with no maximal or minimal element.

□

We can extend this point further by combining these two results and considering what happens when each individual votes over two copies of $\mathbb{Z}$. In general, if $\mathbb{A}$ and $\mathbb{B}$ are linear orders, then $\mathbb{A}+\mathbb{B}$ is the linear order resulting from putting a copy of $\mathbb{A}$ below a copy of $\mathbb{B}$, so that everything in the $\mathbb{A}$ copy is below every element in the $\mathbb{B}$ copy, but with each copy the same as before. Note that $\mathbb{Z}+\mathbb{Z}$, like $\mathbb{Z}$, has neither a maximal nor a minimal element. In this case, we can start from individual orderings such that none of them have either a maximal element or a minimal element, and generate an aggregate order that has both.

Corollary 2. *Assume that there are three individuals. There exists a profile $\succ = (\succ_a, \succ_b, \succ_c)$ of type $(\mathbb{Z}+\mathbb{Z})$ such that $f^M(\succ)$ is an order of type $(\mathbb{N}+\mathbb{N}^*)$.*

Proof. Suppose that individuals a, b, and c have preferences over two copies $\mathbb{Z}_j + \mathbb{Z}_k$. Over $\mathbb{Z}_j$ they have the same pairwise relations as $\succ_i$, and over $\mathbb{Z}_k$ they have the pairwise relations as $\succ_i^{-1}$, again represented in the table below. I'll denote the composite preference orders with the corresponding uppercase letter $(\succ_A, \succ_B, \succ_C)$. We can denote the individual objects in $\mathbb{Z}_j$ using j with a subscript for the symbol corresponding to a natural number and the individual objects over $\mathbb{Z}_k$ using k with a subscript for the symbols corresponding to a natural number. We can then represent the order as follows:

$\succ_A$	$\succ_B$	$\succ_C$
.	.	.
.	.	.
.	.	.
k_6	k_4	k_5
k_3	k_1	k_2
k_0	k_0	k_0
k_1	k_2	k_1
k_2	k_3	k_3
.	.	.
.	.	.
.	.	.
j_2	j_3	j_3
j_1	j_2	j_1
j_0	j_0	j_0
j_3	j_1	j_2
j_6	j_4	j_5
.	.	.
.	.	.
.	.	.

Since $\succ_A, \succ_B, \succ_C$ are all order isomorphic to $\mathbb{Z} + \mathbb{Z}$, the profile $\succ_I = (\succ_A, \succ_B, \succ_C)$ is too. $f^M(\succ_I) = \succ_G$, gives us:

$$\begin{array}{c} \succ_G \\ \hline k_0 \\ k_1 \\ k_2 \\ . \\ . \\ . \\ j_2 \\ j_1 \\ j_0 \end{array}$$

This results because the aggregate results over $\mathbb{Z}_j$ are the same as in **Theorem 1**, and the aggregate results over $\mathbb{Z}_k$ are the same as in **Corollary 1**. And $\succ_G$ is order isomorphic to $\mathbb{N} + \mathbb{N}^*$, since it has a minimal element, a maximal element, and countably infinite elements. So, from individual orders with neither a minimal nor maximal element it is possible to generate an order with both a minimal and maximal element.

□

A few remarks about these results are in order before we move on to the next section. First, they do not hold for all orders of the types in question. Clearly, there are also cases of individuals orders of type $\mathbb{Z}$ that aggregate to an order of type $\mathbb{Z}$, as does, for example, the profile consisting of three instances of $\succ_a$ from **Theorem 1**. And this is the case for both corollaries as well. Second, although my concern, in keeping with the simplicity of Condorcet's paradox, is only with cases of three voters, these three results are extendable to any number $n \in \mathbb{N}$ of voters using the same set of devices. To do so, construct the individual orders corresponding to **Theorem 1** by putting the the symbols for the multiples of n below 0 for the first order, 1 mod n, for the second, 2 mod n for the third, and so on, until $(n-1)$ mod n. Then, taking the inverses gives the result corresponding to **Corollary 1** and copying them gives the result corresponding to **Corollary 2**. Finally, a question I leave open for further research is the following: what are the conditions on the individual orders such that these results arise, that is when can individuals without favourite or least favourite candidates converge to a group favourite?

3 Density and Discreteness

Questions (ii) and (iii) deal with a different kind of order type. One feature that distinguishes this order type from order types isomorphic to the natural numbers and the integers with the greater-than-or-equal-to relation is that it is dense, rather than discrete. A *dense* strict linear order is one for which, for any two elements, x and y and relation $<$, whenever $x < y$, there is an element z such that $x < z < y$. We can contrast this with the *discrete* linear orders we have been dealing with, for which whenever $x < y$, there is a z with $x < z$ and no elements between x and z, and also an element w with $w < y$ and no elements between w and y. One instance of a dense order type is the rational numbers, $\langle \mathbb{Q}, < \rangle$, which I'll refer to as order type $\mathbb{Q}$ for short. One important feature of $\mathbb{Q}$, proved by Cantor, is that every countable, dense linear order without endpoints is isomorphic to $\mathbb{Q}$.[6] This makes it convenient to use in what follows. Our translated questions then become, under our majority rule aggregation function, (ii) can we move from a profile of type $\mathbb{Q}$ to a aggregate order of type $\mathbb{N}$, and (iii) can we move from a profile of type $\mathbb{N}$ to an aggregate order of type $\mathbb{Q}$?

Theorem 2. *Assume that there are three individuals. There exists a profile* $\succ = (\succ_a, \succ_b, \succ_c)$ *of type* $\mathbb{Q}$ *such that* $f^M(\succ)$ *is an order of type* $\mathbb{N}$.

Proof. We are aiming at a group preference order, $\succ_g$, which is order isomorphic to $\mathbb{N}$. Again we can represent this using the symbols of the natural numbers and a table in which, for any two elements x and y, if x is above y in the table, then $x \succ_g y$:

$\succ_g$
.
.
.
3
2
1
0

One way to achieve that aim is to develop a procedure that generates individual orders that result in this aggregate order. The intuitive idea behind

[6]See [5].

one such procedure is the following. Our profile is $\succ = (\succ_a, \succ_b, \succ_c)$. In $\succ$, when each number symbol is compared to the number symbol below it in the aggregate order, it needs only two votes to be above that lower number symbol. So, in two individual orders, it must also be above the symbol it is above in the aggregate order. This leaves the instance in the other individual order "free", in the sense that, wherever it is in that individual order it will have the place it does in the aggregate order, since it will still have two votes in the profile. We can then put this symbol in between any other symbols in the individual order in question to ensure density.

Here is one way of executing of that procedure. Start with number symbols '0' and '1':

$\succ_a$	$\succ_b$	$\succ_c$
1	1	1
0	0	0

Put number symbols between the '0' row and the '1' row. For example, put '2' above '1' for $\succ_a$ and $\succ_b$, ensuring its place in $\succ_g$, and between '0' and '1' for $\succ_c$

$\succ_a$	$\succ_b$	$\succ_c$
2	2	1
1	1	2
0	0	0

Put '3' above '2' for $\succ_b$ and '1' for $\succ_c$ and between '0' and '1' for $\succ_a$.

$\succ_a$	$\succ_b$	$\succ_c$
2	3	3
1	2	1
3	1	2
0	0	0

To complete the process of putting symbols between all the '0's and '1's with which we started, put '4' above '2' for $\succ_a$ and '3' for $\succ_c$ and between '0' and '1' for $\succ_b$.

$\succ_a$	$\succ_b$	$\succ_c$
4	3	4
2	2	3
1	1	1
3	4	2
0	0	0

But since $\mathbb{Q}$ is also endless, we need to make sure the individual orders continue downwards without a minimal element. We can do so with the next three symbols, '5', '6', and '7'. Using the same process as above gives us:

$\succ_a$	$\succ_b$	$\succ_c$
7	6	7
5	5	6
4	3	4
2	2	3
1	1	2
3	4	1
0	0	0
6	7	5

Since there are a countably infinite number of symbols corresponding to all n in $\mathbb{N}$ we can continue this process, ensuring that each new symbol is introduced in such a way that all the current rows end up with a number in between, that the current highest row ends up with a row on top, and that the current lowest row ends up with a row below it. The resulting table representing the individual orders is then (with a single ' . ' representing infinite elements):

$\succ_a$	$\succ_b$	$\succ_c$
.	.	.
7	6	7
.	.	.
5	5	6
.	.	.
4	3	4
.	.	.
2	2	3
.	.	.
1	1	2
.	.	.
3	4	1
.	.	.
0	0	0
.	.	.
6	7	5
.	.	.

Each of $\succ_a$, $\succ_b$, $\succ_c$ is a dense linear order without endpoints, that is, of type $\mathbb{Q}$. So, $\succ = (\succ_a, \succ_b, \succ_c)$ is of type $\mathbb{Q}$. And, $f^M(\succ) = \succ_g$, gives us our desired order:

$$\begin{array}{c} \succ_g \\ \hline . \\ . \\ . \\ 3 \\ 2 \\ 1 \\ 0 \\ \hline \end{array}$$

For no number symbol are there two instances below '0', and for each symbol corresponding to an $n \in \mathbb{N}$ starting with '1' there are two individual orders in which it is higher than the symbol corresponding to $n-1$. Thus, from a profile of type $\mathbb{Q}$, we generate an aggregate order of type $\mathbb{N}$. In other words, from dense individual orders, we generate a discrete group order.

□

So far, the questions have been answered in the affirmative. In the case of (iii), we get an impossibility result. It turns out that there is no set of individual preference orders that satisfy the conditions.

The proof will make use of the *finite-distance property.* A linear order, l, is *finite-distant* iff $\forall\, x, y \in l \;\; \{z \mid x <_l z <_l y\}$ is finite. That is, any two elements have only finitely many elements between them. Notice that $\langle \mathbb{N}, \leq \rangle$, $\langle \mathbb{N}^*, \leq \rangle$, and $\langle \mathbb{Z}, \leq \rangle$, as well as finite orders, all have this property, but $\langle \mathbb{Q}, \leq \rangle$ does not. As an aside, the orders dealt with in the proof for **Corollary 2** also do not have this property because there are infinitely many elements between the copies. A profile $\succ$ is said to be finite-distant if each $\succ_i$ is finite-distant.

Theorem 3. *Assume that there are three individuals. There is no profile $\succ = (\succ_a, \succ_b, \succ_c)$ of type $\mathbb{N}$ such that $f^M(\succ)$ is an order of type $\mathbb{Q}$.*

Proof. The first step is to show that a group order aggregated from three individual orders with the finite-distance property by majority rule also has the finite distance property.

Lemma 3.1. *Assume that there are three individuals. If $\succ = (\succ_1, \succ_2, \succ_3)$ is finite-distant, then $f^M(\succ) = \succ_g$ is finite-distant.*

Proof. The claim is that $\succ_g$ is finite-distant. To see this, assume that, for two elements x and y, $x \succ_g y$. If $x \succ_g z \succ_g y$, then two voters think that $x \succ_i z$ and two voters think that $z \succ_i y$. So, at least one voter thinks that $x \succ_i z \succ_i y$. This means that z is from $\{u \mid x \succ_1 u \succ_1 y\} \cup \{u \mid x \succ_2 u \succ_2 y\} \cup \{u \mid x \succ_3 u \succ_3 y\}$. Since we have specified that all the individual orders are finite-distant, we know that $\{u \mid x \succ_1 u \succ_1 y\}$, $\{u \mid x \succ_2 u \succ_2 y\}$, and $\{u \mid x \succ_3 u \succ_3 y\}$ are all finite, and so their union is finite. Therefore, there is a finite distance between $x \succ_g y$.

□

We can now finish the proof. Assume that $\succ_g$ has order type $\mathbb{Q}$. $\mathbb{Q}$ is dense, so it is not finite distant. That is, for any two elements, x and y there will be another element z between them, and another element w between z and y and so on. So, the distance between x and y is infinite. But, according to lemma 3.1, if $\succ = (\succ_1, \succ_2, \succ_3)$ is finite-distant, and we are assuming that it is, so is $\succ_g$. This is a contradiction. So, it is not possible to move from a profile of type $\mathbb{N}$ to an aggregate order of type $\mathbb{Q}$ using a majority rule aggregation function with three individuals.

□

4 Conclusion

In answer to our questions then, we can say that for majoritarian rule preference aggregation functions with three voters, whose preferences range over a countably infinite number of candidates:

(ia) There is a set of individual orders with no minimal element that aggregates to an order with a minimal element.

(ib) There is a set of individual orders with no maximal element that aggregates to an order with a maximal element.

(ic) There is a set of individual orders with no maximal or minimal element that aggregates to an order with both a maximal and a minimal element.

(ii) There is a set of dense orders that aggregates to a discrete order.

(iii) There is no set of discrete orders with the finite distance property that aggregates to a dense order.

Not only does majority rule not preserve transitivity, as Condorcet's paradox shows, it does not preserve the properties of *not having a minimal element*, *not having a maximal element*, *not having a maximal and a minimal element*, and *density*. But, for finite-distant orders, it does preserve *discreteness*.

These results reveal two striking features of majority rule. First, individuals who do not have a favourite candidate can converge on a group winner, and vice versa. If we're in a situation in which what matters is only what the group most prefers, perhaps because we're deciding on a course of action or leader, none of us need have a most preferred option in order for the group to make a decision. Second, in an intuitive sense, simplicity is maintained or induced. We can induce endpoints on orders without them, or finite distances between elements on orders with infinite distances between elements, but we can't generate infinite distances between elements from orders without them.

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