Threshold-based belief change: Rankings and semiorders

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Abstract

In this paper we study changes of beliefs in a ranking-theoretic setting using non-extremal implausibility thresholds for belief. We represent implausibilities as ranks and introduce natural rank changes subject to a minimal change criterion. We show that many of the traditional AGM postulates for revision and contraction are preserved, except for the postulate of Preservation which is invalid. The diagnosis for belief contraction is similar, but not exactly the same. We demonstrate that the one-shot versions of both revision and contraction can be represented as revisions based on semiorders, but in two subtly different ways. We provide sets of postulates that are sound and complete in the sense that they allow us to prove representation theorems. We show that, and explain why, the classical duality between revision and contraction, as exhibited by the Levi and Harper identities, is partly broken by threshold-based belief changes. We also study the logic of iterated threshold-based revision and contraction. The traditional Darwiche-Pearl postulates for iterated revision continue to hold, as well as two additional postulates that characterize ranking-based revision as a restricted 'improvement' operator. We investigate the dual notion of iterated threshold-based belief contraction and provide a new set of postulates for it, characterizing contraction as a restricted 'degrading' operator.

Keywords: belief revision, belief contraction, thresholds for belief, ranking theory, semiorders, plausibility

1 Introduction

In this paper we study a model for belief revision and contraction which arises naturally in the context of ranking theory. The goal of a *revision* by A is to make A a belief. The goal of a *contraction* by A is to make A a non-belief. Two minimal-change ideas underlying many approaches to belief change are these: (a) If it is not necessary to make any changes,

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then don't change anything. (b) If it is necessary to make changes, make the smallest changes possible to achieve your goal. The idea behind the ranking-theoretic approach to belief change is this: (c) If you have to make changes regarding A, then don't change the plausibility distances within the A-worlds and don't change the plausibility distances between the $\overline{A}$-worlds.[1] In this paper, we will heed maxims (a) and (c). It can be argued that (c) is a means of heeding maxim (b), but we will allow non-minimal forms of belief change that violate (b) in other respects. Our aim is to identify the logic of threshold-based belief change in a ranking-theoretic setting. In order to do this, we also study belief change based on semiorders.

Our model brings together two ideas. On the one hand, we define a family of minimal change mechanisms to revise and contract sets of beliefs, using resources of the ranking theory developed by Spohn (2012). On the other hand, we use an operator which maps a ranking function to an associated belief set. This operator is parametrized by a threshold. We call the resulting model *threshold-based belief change*. Although the idea of threshold-based belief was mentioned by Spohn (2012; 2014), its consequences for belief change have not been studied yet.

An essential difference to classical AGM belief revision (Alchourrón, Gärdenfors and Makinson, 1985) is that the postulate of Preservation fails if the threshold is set to values greater than 0. Thus the stronger postulate of Rational Monotonicity fails as well. Instead two weakenings hold—Disjunctive Rationality and a new postulate that we call Semitransitivity (and then Cautious Monotonicity follows). Correspondingly, the essential difference to classical AGM belief contraction is that the postulate of Conjunctive Inclusion fails (if the ranking contraction is not minimal). Instead two weaker postulates hold here, too.

We will introduce a new distinction between two ways of generating a belief revision and contraction from orders—the canonical and the normal way. We then prove two representation theorems for revision in this paper: (1) A belief revision satisfying our new postulates is representable by a (canonical) semiorder belief revision, as studied by Peppas and Williams (2014) (but for the normal case). (2) Semiorder belief revision is essentially equivalent to threshold-based ranking belief revision and does not satisfy Preservation. Threshold-based revisions in the ranking-theoretic setting thus elegantly represent semiorder belief revisions, and in addition they provide the means for iterated revisions.

For contraction we prove two similar representation theorems: (3) Belief contractions satisfying our new contraction postulates are representable as semiorder belief contractions, as studied by Rott (2001; 2014), when the rep-

[1] Besides revision and contraction with respect to A, we might also consider the operations of *improving* and *degrading* A which are characterized by increasing and decreasing the difference $\kappa(\overline{A}) - \kappa(A)$, respectively. See Konieczny, Medina Grespan and Pino Pérez (2010).

resentation is 'normal' (instead of being 'canonical'). (4) Normal semiorder belief contraction is essentially equivalent to threshold-based ranking belief contractions. Again, threshold-based contractions in the ranking-theoretic setting elegantly represent normal semiorder belief contractions, and in addition they provide the means for iterated contractions.

We also show that (5) the Levi and Harper identities fail in our model, and we explain why. Essentially this is due to the fact that the minimal change criterion forces us to adopt canonical revision which fails to satisfy Preservation, but at the same time it forces us to adopt normal contraction which satisfies Vacuity. Furthermore, (6) we prove a representation theorem for iterated revision and (7) one for iterated contraction, thereby providing the first study of iterated revision and contraction in the weaker semiorder setting. We relate these results to iterated threshold-based revision and contraction.

Threshold-based belief change is motivated by the idea that different contexts may fix different levels of plausibility demarcating the attitude of belief, and that we may acknowledge differences between the plausibilities of possible worlds within an agent's belief core (her strongest believed proposition) just as we acknowledge such differences outside this core (Rott 2009). On the other hand, semiorders were introduced by Luce (1956) as a natural and realistic structure representing the perceptual differences between different objects, with the characteristic property that the relation of indistinguishability is not transitive (when properties can only be 'imperfectly discriminated'). We will exhibit a connection between these ideas by showing that threshold-based belief change in a ranking-theoretic setting can be represented by belief change based on semiorders.

The plan of the paper is as follows. Section 2 introduces our model for threshold-based belief change, presenting both threshold-based belief revision and contraction. Section 3 provides the basics of semiorders that we will use in our analysis. Section 4 proves the representation results for semiorder belief revision. Section 5 proves analogous representation results for semiorder belief contraction. Section 6 discusses the Levi and Harper identities. In Section 7, we discuss three new iterated revision postulates, prove representation results for iterated revision and show that our ranking revisions are restricted improvement operators. In Section 8, we discuss seven new iterated contraction postulates, prove representation results for iterated contraction and show that our ranking revisions are restricted degrading operators. The appendix collects proofs of the observations and theorems.

2 Rankings and thresholds

In this section, we introduce the ranking revisions and contractions on which we base our model for threshold-based belief change. We use the long-arrow notation $f : X \longrightarrow Y$ to indicate that f is a total function from X to Y.

Let W be a non-empty set of worlds. We assume W to be finite.[2] Propositions are subsets of W, i.e., elements of the powerset algebra $\wp(W)$ over W. We denote by $\overline{A} = W \setminus A$ the complement of A with respect to W.

Definition 1. A *belief-state change model* is a quadruple $\langle \mathbb{D}, W, \circ, \mathrm{Bel} \rangle$, where $\mathbb{D}$ and W are non-empty sets, and $\circ$ and Bel are functions $\circ : \mathbb{D} \times \wp(W) \longrightarrow \mathbb{D}$, and $\mathrm{Bel} : \mathbb{D} \longrightarrow \wp(W)$.

$\mathbb{D}$ is a set of *doxastic states*, W a set of possible worlds, $\circ$ is a *state change operator* over doxastic states, and Bel is a *belief core operator*. When $\circ$ is a revision operator, we write $*$ and talk of *belief state revision*, and when $\circ$ is a contraction operator, we write $\dot{-}$ and talk of *belief state contraction*. We write Ψ for arbitrary doxastic states and $\Psi \circ A$ for $\circ(\Psi, A)$. $\mathrm{Bel}(\Psi)$ is the *belief core* associated to Ψ, i.e., A is believed in state Ψ iff $\mathrm{Bel}(\Psi) \subseteq A$. We call a belief state Ψ *consistent* iff its associated belief core $\mathrm{Bel}(\Psi)$ is non-empty. Within this framework for the changes of *belief states*, the question of iterated changes of *belief cores* is trivial: once we have a state change operator and a belief operator, we have a belief change operator:

$$(\mathrm{Bel}(\Psi), A) \mapsto \mathrm{Bel}(\Psi \circ A) \tag{1}$$

Thus iterated belief core revision just comes down to applying Bel several times, on top of the iterated belief state changes. We will often briefly write $\langle \circ, \mathrm{Bel} \rangle$ for $\langle \mathbb{D}, W, \circ, \mathrm{Bel} \rangle$ and use the abbreviations $\mathrm{Bel} \circ A := \mathrm{Bel}(\Psi \circ A)$ and $\mathrm{Bel} := \mathrm{Bel}(\Psi)$. It should, however, always be remembered that $\mathrm{Bel} \circ A$ arises from the change of the underlying doxastic state Ψ.[3]

We focus on particular states, particular belief operators, and particular state changes. Our states are ranking functions. K is a *world ranking* iff $\mathrm{K}\colon W \longrightarrow \mathbb{N}$ such that $\mathrm{K}^{-1}[0] \neq \emptyset$.[4] K induces a function κ over $\wp(W)$, defined by $\kappa(A) = \min_{w \in A} \mathrm{K}(w)$ for non-empty A and $\kappa(\emptyset) = \infty$. This κ is a *ranking*, i.e., a function $\kappa\colon \wp(W) \longrightarrow \mathbb{N} \cup \{\infty\}$ which satisfies the *ranking axioms*: $\kappa(W) = 0$, $\kappa(\emptyset) = \infty$ and $\kappa(\bigcup S) = \min_{A \in S} \kappa(A)$ for all $S \subseteq \wp(W)$, and additionally $\kappa(A) = \infty$ only for $A = \emptyset$.[5] Conversely, given a ranking,

[2]The theory can be generalized to infinite W. But, for belief revisions based on orders, we would then also need to assume that these orders are well founded, i.e., there are no infinite descending chains. We confine ourselves to the simpler finite case.

[3]We can have $\mathrm{Bel}(\Psi) = \mathrm{Bel}(\Psi')$ without $\mathrm{Bel}(\Psi \circ A) = \mathrm{Bel}(\Psi' \circ A)$. Thus (1) does not specify a function of $\mathrm{Bel}(\Psi)$ and A.

[4]This is a regular point ranking function in the sense of Spohn (2012, Definitions 5.5 and 5.27).

[5]This is a regular completely minimitive natural negative ranking function in the sense of Spohn (2012, Definitions 5.5, 5.9.k and 5.27).

there is a unique world ranking K which induces it—namely K defined by $\mathrm{K}(w) = \kappa(\{w\})$. Henceforth, we will identify world ranking and ranking and write $\kappa(w)$ for $\kappa(\{w\})$.

Next we define a parametrized notion of belief: *A is believed to degree z* iff $\kappa(\overline{A}) > z$. One can easily prove that the set of believed propositions, $\mathrm{BEL}_z(\kappa) = \{A \in \wp(W) : \kappa(\overline{A}) > z\}$, is a principal proper filter. Thus the agent's beliefs can be characterized by a single non-empty proposition that we denote by $\mathrm{Bel}_z(\kappa) := \bigcap \mathrm{BEL}_z(\kappa)$. We call $\mathrm{Bel}_z(\kappa)$ the agent's *z-belief core*. It is the strongest believed proposition given the state κ and the threshold z. By construction, A is believed to degree z in κ iff $\mathrm{Bel}_z(\kappa) \subseteq A$. Since $\mathrm{BEL}_z(\kappa)$ is proper, we also have that $\mathrm{Bel}_z(\kappa) \neq \emptyset$. Finally, it is easily shown that the belief core can be represented by:

$$\mathrm{Bel}_z(\kappa) = \{w \in W : \kappa(w) \leq z\}. \tag{2}$$

This is our threshold-based belief core operator.

To represent inconsistent belief, we introduce the *improper* ranking $\kappa_\perp$, defined by $\kappa_\perp(A) = \infty$ for all $A \in \wp(W)$. It is induced by the *improper* world ranking $\mathrm{K}_\perp(w) = \infty$ for all $w \in W$. Note that $\mathrm{BEL}_z(\kappa_\perp) = \wp(W)$ and $\mathrm{Bel}_z(\kappa_\perp) = \emptyset$. Under the improper ranking everything is believed, whatever the (finite) threshold $0 \leq z$. When we speak of rankings in the following, we mean the above proper rankings.

Our state-changes are constructed from a well known partial operation over rankings (Spohn 2012, pp. 78, 83):

Definition 2. Let κ be a ranking over $\wp(W)$ and $A \neq \emptyset, W$. The *n-conditionalization* of κ by A is induced by

$$\kappa_{A \to n}(w) := \begin{cases} \kappa(w) - \kappa(A) & \text{if } w \in A, \\ \kappa(w) - \kappa(\overline{A}) + n & \text{otherwise.} \end{cases} \tag{3}$$

Our ranking revisions and contractions are then defined as follows:

Definition 3. Let κ be a ranking over $\wp(W)$, and $z \geq 0$ fixed.

1. For any $A \subseteq W$:

$$\kappa^n_A := \begin{cases} \kappa_{A \to n} & \text{if } \mathrm{Bel}_z(\kappa) \not\subseteq A, A \neq \emptyset, \\ \kappa & \text{if } \mathrm{Bel}_z(\kappa) \subseteq A, A \neq \emptyset, \\ \kappa_\perp & \text{if } A = \emptyset. \end{cases} \tag{4}$$

   κ^n_A is the *n-revision of κ by A for threshold z* iff $n > z$.

2. For any $A \subseteq W$:

$$\overline{\kappa}^n_A = \begin{cases} \kappa_{A \to n} & \text{if } \mathrm{Bel}_z(\kappa) \subseteq A, A \neq W, \\ \kappa & \text{if } \mathrm{Bel}_z(\kappa) \not\subseteq A, A \neq W, \\ \kappa & \text{if } A = W. \end{cases} \tag{5}$$

$\overline{\kappa}^n_A$ is the *n-contraction of* κ *with respect to* A *for threshold* z iff $n \leq z$.

Disregarding the empty set, the n-revision κ^n_A recommends n-conditionalization if A is not already believed and else does nothing. Similarly, disregarding the full set, the n-contraction $\overline{\kappa}^n_A$ recommends n-conditionalization if A is believed and else does nothing. The n-revision by the empty set yields the improper ranking, whereas the n-contraction by the full set does not change anything. We adopt these two special cases, to be in line with the classical AGM assumptions, but other options are possible.[6]

Whereas the effect of an n-revision is to make A believed, the effect of an n-contraction is to remove A from the beliefs (except when $A = W$). Both operations (except for revisions by $\emptyset$) preserve rank-differences within the A-worlds and rank-differences within the $\overline{A}$-worlds. For $z \geq 0$ fixed, there are countably many n-revisions, whereas there are only finitely many n-contractions. Strictly speaking, our notation κ^n_A is incomplete; what we define is a function of z which would better be written as $\kappa^{n,z}_A$—indeed we generally have $\kappa^{n,z}_A \neq \kappa^{n,z'}_A$ when $z \neq z'$. However, we adopt the simplified notation to minimize the use of indices, but we remind the reader that κ^n_A is to be understood in the context of a fixed threshold z (similarly for $\overline{\kappa}^n_A$).

From our belief operator $\mathrm{Bel}_z(\cdot)$ and our n-revisions and n-contractions (for a given threshold z), we obtain belief core revisions and contractions:

$$(\mathrm{Bel}_z(\kappa), A) \mapsto \mathrm{Bel}_z(\kappa^n_A), \tag{6}$$

$$(\mathrm{Bel}_z(\kappa), A) \mapsto \mathrm{Bel}_z(\overline{\kappa}^n_A). \tag{7}$$

Our requirement $n > z$ for revision ensures revision success: $\mathrm{Bel}_z(\kappa^n_A) \subseteq A$. Our requirement $n \leq z$ for contraction ensures contraction-success: $\mathrm{Bel}_z(\overline{\kappa}^n_A) \nsubseteq A$ (for $A \neq W$). We assume that thresholds remain invariant under normal processes of belief revision and contraction.

We introduce paradigmatic revisions and contractions: In the context $z \geq 0$, we call κ^{z+1}_A the *minimal revision*, $\overline{\kappa}^z_A$ the *minimal contraction* and $\overline{\kappa}^0_A$ the *maximal contraction*.

We will use the abbreviation $\min^m_\kappa A$ for $\{w \in A : \kappa(w) \leq \kappa(A) + m\}$. Note that the belief core of a ranking revision can be captured in terms of the prior ranking function κ, i.e., for $n > z$:

$$\mathrm{Bel}_z(\kappa^n_A) = \min\nolimits^z_\kappa A, \tag{8}$$

and that thus for $n > z$, we have $\mathrm{Bel}_z(\kappa^n_A) = \mathrm{Bel}_z(\kappa^{z+1}_A)$. So the minimal revision can be seen as the paradigmatic revision. No similar identity is valid for contractions.[7] However, the belief core of a ranking contraction,

[6]For example, one may posit that revising by the empty set should do nothing, but then AGM's postulate of success (see AGM2 below) gets violated.

[7]There are ranking functions such that $\mathrm{Bel}_z(\overline{\kappa}^n_A) \neq \mathrm{Bel}_z(\overline{\kappa}^m_A)$ for $n, m \leq z$ with $n \neq m$.

too, can be captured in terms of the prior ranking function κ, i.e., for $n \leq z$ and $\mathrm{Bel} \subseteq A$:

$$\mathrm{Bel}_z(\overline{\kappa}^n_A) = \mathrm{Bel} \cup \min\nolimits^{z-n}_\kappa \overline{A}. \tag{9}$$

We will later see that any belief contraction resulting from a ranking contraction is representable by a maximal contraction. This is the reason why we take maximal contraction rather than minimal contraction as the paradigmatic case. To designate an arbitrary belief revision in the context $z \geq 0$, we write $\mathrm{Bel}(\kappa * A)$, meaning $\mathrm{Bel}_z(\kappa^n_A)$ for some arbitrary $n > z$. To designate an arbitrary contraction in the context $z \geq 0$, we write $\mathrm{Bel}(\kappa \dot{-} A)$, meaning $\mathrm{Bel}_z(\overline{\kappa}^n_A)$ for some arbitrary $n \leq z$.

We adopt the following notions of ranking representability:

Definition 4. A belief state revision $\langle \mathbb{D}, W, *, \mathrm{Bel} \rangle$ is *ranking representable* iff for every consistent $\Psi \in \mathbb{D}$ there are a ranking κ and integers n and z with $n > z \geq 0$ such that

$$\mathrm{Bel}(\Psi) = \mathrm{Bel}_z(\kappa), \tag{10}$$

$$\forall A \subseteq W, \quad \mathrm{Bel}(\Psi * A) = \mathrm{Bel}_z(\kappa^n_A). \tag{11}$$

Definition 5. A belief state contraction $\langle \mathbb{D}, W, \dot{-}, \mathrm{Bel} \rangle$ is *ranking representable* iff for every consistent $\Psi \in \mathbb{D}$, there are a ranking κ over $\wp(W)$ and integers z and n with $z \geq n \geq 0$ such that

$$\mathrm{Bel}(\Psi) = \mathrm{Bel}_z(\kappa), \tag{12}$$

$$\forall A \subseteq W, \quad \mathrm{Bel}(\Psi \dot{-} A) = \mathrm{Bel}_z(\overline{\kappa}^n_A). \tag{13}$$

The representability criterion of Definition 4 does not impose a definite threshold nor a definite revision parameter. But note that if the threshold is given, then we could always use κ^{z+1}_A instead of κ^n_A as the paradigmatic ranking revision, by (8), as long as we are only interested in the laws of belief-core revision.

Our model is motivated by an initial proposal of Spohn, but departs from it in some respects. Spohn (2012, p. 89) uses $\kappa_{A\to n}$ as ranking revision, with $n > 0$. Later on Spohn (2014, p. 103) uses $\kappa^{(n)}_A = \kappa_{A\to n}$ provided $\kappa(A) > 0$. In both cases, he restricts his attention to the standard threshold $z = 0$. Spohn's $\mathrm{Bel}_0(\kappa_{A\to n})$ for $n > 0$ agrees with our $\mathrm{Bel}_0(\kappa^1_A)$, Spohn's new proposal $\mathrm{Bel}_0(\kappa^{(n)}_A)$ agrees with our $\mathrm{Bel}_0(\kappa^n_A)$ only on its domain of definition, but is undefined when A has rank 0. Our account disagrees for non-standard thresholds, since when $z > 0$, $\mathrm{Bel}_z(\kappa_{A\to n})$ agrees with $\mathrm{Bel}_z(\kappa^n_A)$, only when $n > z$. Hence, our n-revisions generalize Spohn's proposal to non-standard thresholds with two essential modifications which are motivated as follows. Spohn's $\kappa_{A\to n}$ is not minimal: $\overline{A}$-worlds are always moved by some amount. Contrary to this, our n-revisions only improve A when success requires it, i.e., when A is not already believed. Additionally, Spohn's proposal moves $\overline{A}$-worlds downwards when A is n-believed ($\kappa(\overline{A}) > n$). And thus it sometimes degrades A. Our n-revisions never do this—they are partial improvement operators, which only improve A when required by success. Spohn's new proposal $\kappa_A^{(n)}$ looks almost like ours when the standard threshold $z = 0$ is applied. However, it remains undefined when $\kappa(A) = 0$. This means that when $\kappa(\overline{A}) = 0 = \kappa(A)$, Spohn's revision cannot bring about a belief in A. Contrary to this, our n-revisions always satisfy revision success.[8]

Spohn (2012, Def. 5.41) uses our maximal contraction $\overline{\kappa}_A^0$ for the standard threshold $z = 0$, which he calls 'central contraction'. The contraction studied by Spohn (2012, p. 96, 5.55; 2014, Def. 11) is thus identical with our $\mathrm{Bel}_0(\overline{\kappa}_A^0)$. While he focuses on the threshold $z = 0$, he hints at the generalization but recommends keeping central (i.e., maximal) contraction. The n-contractions generalize maximal contraction and are motivated as follows. Spohn's central contraction is minimal only for the standard threshold $z = 0$, since for $z = 0$ there is only one contraction! When $z > 0$, it ceases to be minimal and is in fact the maximal contraction. We make room for the option that sometimes more incisive changes are desirable, represented by the remaining spectrum. Contractions with $0 < n < z$ are less incisive than maximal contractions, since they don't improve $\overline{A}$ to the level of A. At the same time, they are more incisive than minimal contractions, since they improve $\overline{A}$ more than necessary. When $z > 0$, any of the non-maximal contractions is less risk seeking than maximal contraction, with minimal contraction being the least risk seeking. Thus, Spohn's neutrality requirement—both $\overline{A}$ and A end up at rank 0—goes hand in hand with the maximal risk of loosing beliefs. Whereas minimal contraction behaves always like the full standard AGM contraction, we will show that non-minimal contractions deviate from it in important respects.

Though we think that the view from ranking theory clearly commits us to the revision and contraction methods we have introduced in Definitions 4 and 5, other options are conceivable. An alternative way of revising belief sets is to go for the intersection of Bel and A when they are consistent, and an alternative way of contracting them is to go for the union of Bel with all the "close" $\overline{A}$-worlds even when A is not believed.

Definition 6. A belief state revision $\langle \mathbb{D}, W, *, \mathrm{Bel} \rangle$ is *non-standardly ranking representable* iff for every consistent $\Psi \in \mathbb{D}$ there are a ranking κ and an integer $z \geq 0$ such that (10) and

$$\forall A \subseteq W, \quad \mathrm{Bel}(\Psi * A) = \begin{cases} \min_\kappa^z A & \text{if } \mathrm{Bel} \cap A = \emptyset, \\ \mathrm{Bel} \cap A & \text{if } \mathrm{Bel} \cap A \neq \emptyset. \end{cases} \tag{14}$$

[8] For this reason we think that Spohn really had in mind our κ_A^n. In the context $z = 0$, we would thus need to replace his condition $\tau(A) < 0$ by $\tau(A) \leq 0$, and say that else $\kappa_A^n = \kappa$.

Definition 7. A belief state contraction $\langle \mathbb{D}, W, \dot{-}, \mathrm{Bel} \rangle$ is *non-standardly ranking representable* iff for every consistent $\Psi \in \mathbb{D}$, there are a ranking κ over $\wp(W)$ and integers z and n with $z \geq n \geq 0$ such that (12) and

$$\forall A \subseteq W, \quad \mathrm{Bel}(\Psi \dot{-} A) = \mathrm{Bel} \cup \min_{\kappa}^{z-n} \overline{A}. \tag{15}$$

We call the operations defined by (14) and (15) *non-standard ranking revision* and *non-standard ranking contraction*, respectively, which is short for 'non-standard belief revision induced by a ranking revision/contraction'.

3 Semiorders

We will relate threshold-based belief change in a ranking-theoretic setting to a purely qualitative approach to belief change. Here a list of purely structural properties that apply to relations of any kind.

Definition 8.

(a) A *strict preorder* is an irreflexive and transitive (and, thus, asymmetric) relation $\prec$.

(b) An *interval order* is a strict preorder that satisfies the

(Interval condition) If $x \prec y$ and $u \prec v$, then $x \prec v$ or $u \prec y$.

(c) A *semiorder* is an interval order that satisfies

(Semitransitivity) If $x \prec y$ and $y \prec v$, then $x \prec u$ or $u \prec v$.

(d) A strict preorder is *modular*[9] iff it satisfies

(Modularity) If $x \prec y$, then $x \prec u$ or $u \prec y$.

For asymmetric relations, Modularity implies both the Interval condition and Semitransitivity, and for irreflexive relations, each of the Interval condition and Semitransitivity in turn implies transitivity. The Interval condition and Semitransitivity, however, are logically independent. Let us define $x \sim y$ iff $x \not\prec y$ and $y \not\prec x$. The relation $\sim$ signifies indistinguishibility and is not in general transitive. A strict preorder that is modular can also be called *total*. For a strict total preorder, $\sim$ is an equivalence relation. In the work of AGM and many other authors, $\prec$ is indeed assumed to be a strict total preorder. We will generalize AGM's canonical representation by employing semiorders.

For the bridge between semiorder revision and threshold-based ranking revision, we will use a result by Scott and Suppes (1958), which links a semiorder to a non-negative rational-valued point function which we can then transform into a world ranking.

[9]Alternatives terms to 'modular' are 'almost connected', 'virtually connected' and 'negatively transitive'.

Theorem 9 (Scott-Suppes)**.** *Let W be a finite set and $\prec$ a semiorder over W. Then there is a function $f : W \longrightarrow \mathbb{Q}^+$ such that for all $v, w \in W$, $v \prec w$ iff $f(v) + 1 < f(w)$, and in addition $f^{-1}[0] \neq \emptyset$.*

To keep this paper reasonably self-contained, we reproduce the proof of Scott and Suppes (1958) in the appendix, making use of the presentation of Suppes and Zinnes (1963, pp. 29–34).

Because the function f mentioned in Theorem 9 starts at zero and takes its values in the non-negative rational numbers, we can 'normalise' it by a multiplier z to obtain a world ranking relative to which belief is expressed by the threshold z. We call a semiorder *bottom* if it satisfies the following condition: If $v \in \min_\prec W$ and $w \notin \min_\prec W$, implies $v \prec w$.

Corollary 10. *Let $\prec$ be a semiorder over a finite set W.*

(a) There is a world ranking $\kappa : W \longrightarrow \mathbb{N}$ and $z \in \mathbb{N}$ such that for all $v, w \in W$, $v \prec w$ iff $\kappa(v) + z < \kappa(w)$.

*(b) If $\prec$ is bottom, then κ can be chosen so that it is z-*gappy *in the following sense: for all $v, w \in W$, $\kappa(v) \leq z$ and $z < \kappa(w)$ implies that $\kappa(v) + z < \kappa(w)$.*

4 Representing revisions

In this section we start with the well-known AGM postulates for belief revisions (Alchourrón, Gärdenfors, and Makinson, 1985). AGM showed that revisions conforming to them are representable as total preorder belief revisions. The *canonical way* to generate a one-shot AGM belief revision is to use a total preorder $\prec$ over possible worlds and define the belief cores $\mathrm{Bel} = \min_\prec W$ and $\mathrm{Bel} * A = \min_\prec A$, where $\min_\prec X := \{w \in X : \neg\exists v \in X, v \prec w\}$.[10] AGM revisions are different from our threshold-based belief revisions which we will show to be representable as semiorder belief revisions.

Definition 11. A belief state revision $\langle \mathbb{D}, W, *, \mathrm{Bel} \rangle$ is *canonically semiorder representable* iff for all consistent $\Psi \in \mathbb{D}$ there is a semiorder $\prec_\Psi$ over W such that

$$\mathrm{Bel}(\Psi) = \min\nolimits_{\prec_\Psi} W, \tag{16}$$

$$\forall A \in \wp(W),\ \mathrm{Bel}(\Psi * A) = \min\nolimits_{\prec_\Psi} A. \tag{17}$$

We call $\langle \mathbb{D}, W, *, \mathrm{Bel} \rangle$ a *canonical semiorder revision.*

There is another way to generalize the strict total preorder representation:

[10]If W is infinite, one needs to assume that there are no infinite descending chains.

Definition 12. A belief state revision $\langle \mathbb{D}, W, *, \mathrm{Bel} \rangle$ is *normally semiorder representable* iff for all consistent $\Psi \in \mathbb{D}$ there is a semiorder $\prec_\Psi$ over W that satisfies equation (16) and

$$\forall A \in \wp(W),\ \mathrm{Bel}(\Psi * A) = \begin{cases} \mathrm{Bel}(\Psi) \cap A & \text{if } \mathrm{Bel} \not\subseteq \overline{A} \\ \min_{\prec_\Psi} A & \text{else.} \end{cases} \tag{18}$$

We call $\langle \mathbb{D}, W, *, \mathrm{Bel} \rangle$ a *normal semiorder revision.*

In the total preorder case, the two definitions are equivalent, since if $\mathrm{Bel} \not\subseteq \overline{A}$, then $\min_\prec A = \mathrm{Bel} \cap A$. In the semiorder case, the two are not equivalent. The normal representation is tantamount to accepting the (in)famous postulate

(AGM4) If $\mathrm{Bel} \cap A \neq \emptyset$ then $\mathrm{Bel} * A \subseteq \mathrm{Bel}$. (Preservation)

The canonical representation allows violations of Preservation. Thus the choice will depend on one's inclinations for or against Preservation. Guided by intuitions about ranking revision, we think that the *canonical* representation is the more adequate generalization. And thus, when we speak of representability in the revision case, we mean canonical representability. The goal of a revision is to make A believed by minimally altering the belief state. Regarding the belief cores, the intersection of the belief core with A seems to be the minimal move—hence the normal representation. But regarding the rankings, the minimal move is the minimal downwards move. When $z > 0$, this move may violate Preservation—hence the canonical representation. Canonical and normal representability are equivalent only when the semiorder is bottom. This means that minimal elements have neither ties nor incomparabilities with non-minimal elements—they are always more plausible than the latter. A semiorder that is bottom ensures Preservation and the possibility of a normal representation.

Representability by a total preorder is equivalent to the existence a faithful assignment in the sense of Katsuno and Mendelzon. Their result (Katsuno and Mendelzon 1991, Theorem 3.3) reads as follows in our terminology:

Theorem 13. *A belief state revision $\langle \mathbb{D}, W, *, \mathrm{Bel} \rangle$ is total preorder representable iff it satisfies the following postulates for all $\Psi \in \mathbb{D}$ and all $A, B \subseteq W$:*

(AGM0) $\mathrm{Bel} * W = \mathrm{Bel}$. (Weak Vacuity)

(AGM2) $\mathrm{Bel} * A \subseteq A$. (Success)

(AGM3) $\mathrm{Bel} \cap A \subseteq \mathrm{Bel} * A$. (Inclusion)

(AGM4) *If* $\mathrm{Bel} \cap A \neq \emptyset$, *then* $\mathrm{Bel} * A \subseteq \mathrm{Bel}$. (Preservation)

(AGM5) *If* $A \neq \emptyset$, *then* $\mathrm{Bel} * A \neq \emptyset$. (Consistency)

(AGM7) $(\mathrm{Bel} * A) \cap B \subseteq \mathrm{Bel} * (A \cap B)$. (Conditionalization)

(AGM8) *If* $(\mathrm{Bel} * A) \cap B \neq \emptyset$, *then* $(\mathrm{Bel} * (A \cap B)) \subseteq \mathrm{Bel} * A$. (Rational Monotonicity)

We call a belief state revision *fully AGM* iff it satisfies the revision postulates mentioned in Theorem 13.[11] Note that AGM0 and AGM5 imply that $\mathrm{Bel} \neq \emptyset$ for all belief cores Bel, which is a strengthening of AGM5. The addition of AGM0 strengthens the traditional account of AGM and Katsuno and Mendelzon slightly, but this will turn out to be well justified for our purposes.[12] AGM7 together with AGM0 implies AGM3, and AGM8 together with AGM0 implies AGM4. So the above axiomatization is not without redundancies, but we keep it because we will have to give up AGM4 and AGM8 later and we want to maintain a parallelism with our treatment of contractions below.

We are interested in weaker revision operators. For (canonical and normal) semiorder revisions, it is known that AGM8 is invalid. In fact even Preservation fails:

Observation 14. AGM4 *is invalid for canonical semiorder revision.*

AGM8 is invalid because AGM4 is invalid and AGM0 holds. We now replace AGM8 by two new postulates:

(AGM8d) $\mathrm{Bel} * A \subseteq \mathrm{Bel} * (A \cup B)$ or $\mathrm{Bel} * B \subseteq \mathrm{Bel} * (A \cup B)$. (Disjunctive rationality)

(AGM8s) If $(\mathrm{Bel} * (A \cup B)) \cap B = \emptyset$ and $(\mathrm{Bel} * (B \cup C)) \cap C = \emptyset$, then $(\mathrm{Bel} * (A \cup D)) \cap D = \emptyset$ or $(\mathrm{Bel} * (D \cup C)) \cap C = \emptyset$. (Semitransitivity)

AGM8d and AGM8s are independent of each other, and jointly weaker than AGM8.[13] We call a belief state revision a *semi-AGM revision* iff it satisfies the basic AGM revision postulates AGM0, AGM2, AGM3,[14] AGM5, AGM7, and instead of AGM8 it satisfies our new postulates AGM8d and AGM8s. It is called a *normal semi-AGM revision* iff it in addition satisfies AGM4. Semi-AGM revisions also satisfy

(AGM8c) If $\mathrm{Bel} * A \subseteq B$, then $\mathrm{Bel} * (A \cap B) \subseteq \mathrm{Bel} * A$. (Cautious montonicity)

[11] Using propositions rather than sentences and belief cores rather than belief sets, our semantic framework makes the traditional postulates AGM1 and AGM6 trivially true. In a way, they don't apply at all, since 'Cn' and '$\vdash$' don't operate on the level of sets.

[12] There is a problem here because belief-state change models allow iterated revisions. AGM2 implies that $\mathrm{Bel} * \emptyset = \emptyset$. By AGM0, then $(\mathrm{Bel} * \emptyset) * W = \mathrm{Bel} * \emptyset = \emptyset$, contradicting AGM5. The least incisive way to avoid this problem seems to be to adapt Definition 1 slightly and rule out the empty input A.

[13] This corresponds to the fact noted above that the Interval condition and Semitransitivity are independent of each other and jointly weaker than Modularity.

[14] Since AGM3 is redundant here, we will hardly mention it from now on.

Observation 15. AGM8c *follows from* AGM2*,* AGM5*,* AGM8d*.*

Whereas AGM8c and AGM8d are known from interval-order belief revision, AGM8s is new and corresponds to Semitransitivity. AGM8d is Rott's (2014) II$^+$ and provides a much nicer characterization of interval-order belief revisions than Axiom 3 of Jamison and Lau (1973) and Fishburn (1975). Rott hinted at semiorder revision, but only studied semiorder contraction, assuming that the former can be obtained from the latter by the Levi identity—an assumption that we do not make here (see section 6 below). He assumed contractions to satisfy AGM$\dot{-}$4 which yields AGM4 for his semiorder revisions; hence he really considers the normal representation, not the canonical one. Peppas and Williams (2014) also proposed an axiomatization of semiorder revision, based on Axioms 3 and 5 of Jamison and Lau (1973) and Fishburn (1975) which are counterparts to the Interval condition and Semitransitivity. Peppas and Williams's encoding of the Interval condition (A8) and Semitransitivity (A9, A10) is much more complicated than ours (compare the discussion in Rott 2014, pp. 377–379). Besides they also presuppose Preservation AGM4, which gives their results a more restricted applicability. Neither Peppas and Williams's nor Rott's work is general enough for the analysis of threshold-based belief revision in a ranking-theoretic setting.

We obtain our first main theorem about threshold-based belief change:

Theorem 16. *Let W be finite, and let $\langle \mathbb{D}, W, *, \mathrm{Bel} \rangle$ be a belief state revision model. The following claims are equivalent:*
(a) $\langle \mathbb{D}, W, *, \mathrm{Bel} \rangle$ *is a semi-*AGM *revision;*
(b) it is canonically semiorder representable;
(c) it is ranking representable;
(d) it is representable by a minimal ranking revision.

The following result is obtained by conjoining two steps in the proof of Theorem 16:

Corollary 17. *For a ranking κ and a fixed threshold z, we generate a binary relation $\prec$ over W from $\langle \kappa, z \rangle$ as follows:*

$$v \prec w \text{ iff } \kappa(v) + z < \kappa(w) \tag{19}$$

For the relation $\prec$ thus induced we get: (1) *$\prec$ is a semiorder and* (2) $\mathrm{Bel}_z(\kappa) = \min_\prec W$ *and* $\mathrm{Bel}_z(\kappa^n_A) = \min_\prec A$ *for $n > z$.*

The following result on non-standard ranking revision can also be seen as a corollary to Theorem 16:

Corollary 18. *Let W be finite, and let $\langle \mathbb{D}, W, *, \mathrm{Bel} \rangle$ be a belief state revision model. The following claims are equivalent:*
(a) $\langle \mathbb{D}, W, *, \mathrm{Bel} \rangle$ *is a normal semi-*AGM *revision;*

(b) *it is normally semiorder representable;*

(c) *it is non-standardly ranking representable;*

(d) *it is non-standardly representable by a minimal ranking revision.*

Our representation results show that the distinction between standard and non-standard ranking representability corresponds to that between canonical and normal semiorder representability.

Our threshold-based revisions are of a format that is different from that of semiorder revisions. The former are *belief state* revisions, the latter are not. Indeed, a semiorder revision in the sense of Definition 11 or 12 is based on the prior doxastic state (represented by a semiorder $\prec$), but it produces only a revised *belief core* and the prior state itself (the semiorder) is not subject to any revision. So without further provisions, no iterated changes can be performed. (We will encounter lists of such provisions in Sections 7 and 8.) In contrast, threshold-based revisions as specified by Definition 3 and equation (6) are based on ranking revisions and the belief operator. The doxastic states are the ranking functions, the state revision is ranking revision, and the belief operator induces a belief core revision resulting from the revised ranking. This is much more straightforward: belief revision is derived from a principled method of ranking revision. In this sense, our revision $\langle *, \mathrm{Bel} \rangle$ applied to rankings (augmented by $\kappa_\perp$) is a belief state revision which allows iterated revisions.

To conclude, we have seen that semiorder revisions and threshold-based ranking revision are equivalent for one-shot revisions of belief cores.[15] They satisfy the basic AGM axioms AGM0, AGM2, AGM5, AGM7, but instead of AGM8, they satisfy the weaker postulates AGM8d and AGM8s. AGM4 is violated under the canonical order representation and the standard ranking representation, but validated under the normal order representation and the non-standard ranking representation. AGM8 can only be validated if the threshold is set to 0. Thus the difference between semi-AGM revision and full AGM revision is just *a matter of thresholds*.

5 Representing contractions

In this section we discuss the well-known AGM postulates for belief contraction (Alchourrón, Gärdenfors, and Makinson 1985). It is known that contractions conforming to them are representable as total preorder belief contractions. They differ from our threshold-based belief contractions which we will show to be normally representable as semiorder belief contractions.

The canonical representation of a contraction based on a strict total preorder $\prec$ is $\mathrm{Bel} = \min_\prec W$ and $\mathrm{Bel} \dot{-} A = \mathrm{Bel} \cup \min_\prec \overline{A}$. Again there are

[15] This is a restricted equivalence, since, as we will see, ranking revisions contain more information when it comes to iterated revisions.

two ways of generalizing this—the canonical and the normal representation:

Definition 19. A belief state contraction $\langle\, \mathbb{D}, W, \dot{-}, \mathrm{Bel}\,\rangle$ is *canonically semiorder representable* iff for all consistent $\Psi \in \mathbb{D}$ there is a semiorder $\prec_\Psi$ over W such that

$$\mathrm{Bel}(\Psi) = \min_{\prec_\Psi} W, \tag{20}$$

$$\forall A \in \wp(W),\ \mathrm{Bel}(\Psi \dot{-} A) = \mathrm{Bel}(\Psi) \cup \min_{\prec_\Psi} \overline{A}. \tag{21}$$

We call $\langle\, \mathbb{D}, W, \dot{-}, \mathrm{Bel}\,\rangle$ a *canonical semiorder contraction.*

Definition 20. A belief state contraction $\langle\, \mathbb{D}, W, \dot{-}, \mathrm{Bel}\,\rangle$ is *normally semiorder representable* iff for all consistent $\Psi \in \mathbb{D}$ there is a semiorder $\prec_\Psi$ over W that satisfies equation (20) and

$$\forall A \in \wp(W),\ \mathrm{Bel}(\Psi \dot{-} A) = \begin{cases} \mathrm{Bel}(\Psi) & \text{if } \mathrm{Bel}(\Psi) \not\subseteq A, \\ \mathrm{Bel}(\Psi) \cup \min_{\prec_\Psi} \overline{A} & \text{else.} \end{cases} \tag{22}$$

We call $\langle\, \mathbb{D}, W, \dot{-}, \mathrm{Bel}\,\rangle$ a *normal semiorder contraction.*

In the total preorder case (i.e., when $\prec_\Psi$ is modular), normal and canonical representation are equivalent. This is not so in the semiorder case. A normal semiorder contraction satisfies

(AGM$\dot{-}$4) If $\mathrm{Bel} \not\subseteq A$, then $\mathrm{Bel} \dot{-} A \subseteq \mathrm{Bel}$. ($\dot{-}$ Vacuity)

by construction. A canonical semiorder contraction in general invalidates this postulate. Again, the two notions agree when the semiorder is *bottom.* And again, the choice between the two representation will depend on one's inclinations for or against $\dot{-}$Vacuity. Guided by intuitions about ranking contraction, we think that this time it is the *normal* representation which is the more adequate generalization. The goal of a contraction is to make a minimal change needed to ensure that A is not believed. But when A is not believed to begin with, there is nothing to do—hence $\dot{-}$Vacuity, and hence the normal representation.

On the level of representation, it is important to note the following:

Observation 21.

(a) *Every normal semiorder contraction is canonically representable (namely by a bottom semiorder, and only by a bottom semiorder).*

(b) *Every canonical bottom semiorder contraction is normally representable (namely by the same semiorder).*

It follows that normal semiorder representability and canonical bottom-semiorder representability are equivalent. Semiorder representability, whether canonical or normal, has as a consequence that $\mathrm{Bel} \dot{-} W = \mathrm{Bel}$ and $\mathrm{Bel} \dot{-} \emptyset = \mathrm{Bel}$.[16]

Again, canonical and normal representability by a strict total preorder are equivalent to the existence a faithful assignment for a contraction. The representation result for contractions (see Konieczny and Pino Pérez 2017, Theorem 3) reads as follows in our terminology:

Theorem 22. *A belief state contraction $\langle \mathbb{D}, W, \dot{-}, \mathrm{Bel} \rangle$ is total preorder representable iff it satisfies the following contraction postulates for all consistent Ψ and all $A, B \subseteq W$:*

(AGM$\dot{-}$0) $\mathrm{Bel} \dot{-} \emptyset = \mathrm{Bel}$. ($\dot{-}$ Weak Vacuity)

(AGM$\dot{-}$2) *If $A \neq W$, then* $\mathrm{Bel} \dot{-} A \not\subseteq A$. ($\dot{-}$ Success)

(AGM$\dot{-}$3) $\mathrm{Bel} \subseteq \mathrm{Bel} \dot{-} A$. ($\dot{-}$ Inclusion)

(AGM$\dot{-}$4) *If* $\mathrm{Bel} \not\subseteq A$*, then* $\mathrm{Bel} \dot{-} A \subseteq \mathrm{Bel}$. ($\dot{-}$ Vacuity)

(AGM$\dot{-}$5) $(\mathrm{Bel} \dot{-} A) \cap A \subseteq \mathrm{Bel}$. (Recovery)

(AGM$\dot{-}$7) $\mathrm{Bel} \dot{-} (A \cap B) \subseteq (\mathrm{Bel} \dot{-} A) \cup (\mathrm{Bel} \dot{-} B)$. (Conjunctive overlap)

(AGM$\dot{-}$8) *If* $\mathrm{Bel} \dot{-} (A \cap B) \not\subseteq A$*, then* $\mathrm{Bel} \dot{-} A \subseteq \mathrm{Bel} \dot{-} (A \cap B)$. (Conjunctive inclusion)

We say that a belief state contraction $\langle \dot{-}, \mathrm{Bel} \rangle$ is *fully AGM* iff it satisfies the postulates mentioned in Theorem 22.[17] Note that AGM$\dot{-}$0 and AGM$\dot{-}$2 imply that $\mathrm{Bel} \neq \emptyset$ for all belief cores Bel, and the same comments apply as in the case of revisions (except that Definition 1 does not have to be adapted, we just have $\mathrm{Bel} \dot{-} W = \mathrm{Bel}$).

Here we are interested in weaker contraction operators. For semiorder contractions it is known that AGM$\dot{-}$8 is invalid (Rott 2014). It is to be replaced by the following new postulates:

(AGM$\dot{-}$8d) $\mathrm{Bel} \dot{-} A \subseteq \mathrm{Bel} \dot{-} (A \cap B)$ or $\mathrm{Bel} \dot{-} B \subseteq \mathrm{Bel} \dot{-} (A \cap B)$. ($\dot{-}$ Disjunctive rationality)

(AGM$\dot{-}$8s) If $\mathrm{Bel} \dot{-} (A \cap B) \subseteq B$ and $\mathrm{Bel} \dot{-} (B \cap C) \subseteq C$, then $\mathrm{Bel} \dot{-} (A \cap D) \subseteq D$ or $\mathrm{Bel} \dot{-} (D \cap C) \subseteq C$. ($\dot{-}$ Semitransitivity)

AGM8$\dot{-}$d and AGM8$\dot{-}$s are logically independent and jointly weaker than AGM$\dot{-}$8. We say that a belief state contraction is a *semi-AGM contraction* iff it satisfies the contraction postulates AGM$\dot{-}$0, AGM$\dot{-}$2, AGM$\dot{-}$3,

[16] $\mathrm{Bel} \dot{-} W = \mathrm{Bel} \cup \min \emptyset = \mathrm{Bel}$ on both representations. For consistent states, we have $\mathrm{Bel} \dot{-} \emptyset = \mathrm{Bel} \cup \min W = \mathrm{Bel}$ on the canonical representation, and $\mathrm{Bel} \dot{-} \emptyset = \mathrm{Bel}$ on the normal one. For inconsistent states, we have $\mathrm{Bel} \dot{-} \emptyset = \mathrm{Bel} \cup \min W = \mathrm{Bel}$ on both representations.

[17] Again, the traditional contraction postulates AGM$\dot{-}$1 and AGM$\dot{-}$6 are trivially true in our semantic framework. Cf. footnote 11.

AGM$\dot{-}$5, AGM$\dot{-}$7 and instead of AGM$\dot{-}$8, it satisfies our new postulates AGM$\dot{-}$8d and AGM$\dot{-}$8s.[18] It is a *normal semi-AGM contraction* iff it also satisfies AGM$\dot{-}$4. A semi-AGM contraction also satisfies

(AGM$\dot{-}$8c) If $\mathrm{Bel}\dot{-}(A \cap B) \subseteq B$ then $\mathrm{Bel}\dot{-}A \subseteq \mathrm{Bel}\dot{-}(A \cap B)$. ($\dot{-}$ Cautious monotonicity)

Observation 23. AGM$\dot{-}$8c *follows from* AGM$\dot{-}$2 *and* AGM8$\dot{-}$d.

Whereas AGM$\dot{-}$8c and AGM8$\dot{-}$d are known from interval-order belief contraction (Rott 2014), AGM$\dot{-}$8s is a new postulate which we prove to correspond to Semitransitivity. Semiorder belief contraction has not been axiomatized. What has been axiomatized is normal semiorder belief contraction. But even there, there are differences. Rott's axiomatization of normal semiorder belief revision is slightly different. It also contains a redundancy (AGM8$\dot{-}$c) and has other axioms for the Interval condition and Semitransitivity.[19] Peppas and Williams's (2014) axiomatization of normal semiorder contraction proceeds from their normal semiorder revision using the Harper and Levi identities, and it is also more complicated.

Again, we obtain two distinct representation results, one for canonical representability and one for normal representability:

Theorem 24. *Let W be finite, and let $\langle \mathbb{D}, W, \dot{-}, \mathrm{Bel} \rangle$ be a belief state contraction model. The following claims are equivalent:*
*(a) $\langle \mathbb{D}, W, \dot{-}, \mathrm{Bel} \rangle$ is a normal semi-*AGM *contraction;*
(b) it is normally semiorder representable;
(c) it is ranking representable;
(d) it is representable by a maximal ranking contraction.

Theorem 24 provides our main argument for the suggestion that maximal ranking contractions should be taken to be the paradigmatic ranking contractions. Minimal ranking contractions ($n = z$), on the other hand, are fully AGM.[20]

The following result is obtained by conjoining two steps in the proof of Theorem 24. It is slightly more complex than the corresponding result for revisions (Corollary 17), but the derived semiorder is bottom and can thus be used with identical results for canonical and normal representation.

[18] In contrast to the case of revisions, postulate AGM$\dot{-}$3, or more precisely the weakened version $\mathrm{Bel} \cap A \subseteq \mathrm{Bel}\dot{-}A$, is not redundant here.

[19] One can show that Rott's K$\dot{-}$d follows from our AGM8$\dot{-}$d and is in fact equivalent given the remaining axioms. Rott (2014, Observation 7) shows that his K$\dot{-}$s corresponds to the semiorder condition, but for entrenchments over sentences, not over worlds. It remains an open question which axioms are needed to prove equivalence between his K$\dot{-}$s and our AGM$\dot{-}$8s.

[20] Because we can always represent such contractions by a contraction with threshold 0—see the proof of Theorem 24.

Corollary 25. *For a ranking κ, a fixed threshold z and a contraction parameter n with $0 \leq n \leq z$, we generate a binary relation $\prec$ over W from $\langle \kappa, z \rangle$, as follows:*

$$v \prec w \quad \textit{iff} \quad \kappa(v) \leq z < \kappa(w) \ \textit{or} \ z < \kappa(v) < \kappa(w) - (z - n) \qquad (23)$$

For the relation $\prec$ thus induced we get: (1) *$\prec$ is a bottom semiorder and* (2) $\mathrm{Bel}_z(\kappa) = \min_\prec W$ *and* $\mathrm{Bel}_z(\overline{\kappa}^n_A) = \mathrm{Bel} \cup \min_\prec \overline{A}$ *for $n \leq z$.*

Equation (23) is obtained from concatenating Equation (28) from the proof of Theorem 24 with Definition 3.

The following result on non-standard ranking contraction is similar to and complements Theorem 24.

Corollary 26. *Let W be finite, and let $\langle \mathbb{D}, W, \dot{-}, \mathrm{Bel} \rangle$ be a belief state contraction model. The following claims are equivalent:*

- *(a) $\langle \mathbb{D}, W, \dot{-}, \mathrm{Bel} \rangle$ is a semi-*AGM *contraction;*
- *(b) it is canonically semiorder representable;*
- *(c) it is non-standardly ranking representable;*
- *(d) it is non-standardly representable by a maximal ranking revision.*

Whereas our threshold-based contractions are belief state contractions and allow iterations, this is not true for semiorder contractions.

To conclude, we have seen that normal semiorder contractions are essentially equivalent and in fact representable by threshold-based belief contractions, in particular by a maximal ranking contraction. These satisfy the normal semi-AGM postulates AGM$\dot{-}$0, AGM$\dot{-}$2, AGM$\dot{-}$3, AGM$\dot{-}$4, AGM $\dot{-}$5, AGM$\dot{-}$7, but instead of AGM$\dot{-}$8, they satisfy the weaker axioms AGM$\dot{-}$8d and AGM$\dot{-}$8s. Canonical semiorder contractions on the other hand are equivalent to, and representable by, non-standard threshold-based belief contractions, and they satisfy the semi-AGM postulates (i.e., AGM$\dot{-}$4 becomes invalid). AGM$\dot{-}$8 can only be validated if the threshold is set to 0 or, more generally, if we adopt minimal contraction $n = z$. Thus the difference between (normal) semi-AGM revision and full AGM revision can be viewed as a matter of thresholds or, more generally, of the difference $z - n$ between the threshold and the contraction parameter.

6 The Levi and Harper identities

We are now going to explore the relation between threshold-based revisions and contractions. It is known that full AGM belief revision and contraction satisfy the well-known Levi and Harper identities:

(LI) $\quad \mathrm{Bel} * A = (\mathrm{Bel} \dot{-} \overline{A}) \cap A.$ (Levi identity)

(HI) $\quad \mathrm{Bel} \dot{-} A = \mathrm{Bel} \cup (\mathrm{Bel} * \overline{A}).$ (Harper identity)

This is due to the canonical representation and, for LI, to the fact that $(\min_{\prec} W) \cap A \subseteq \min_{\prec} A$ for a total preorder $\prec$.[21] Since the latter in fact holds for arbitrary relations, we obtain similar results in the semiorder case:

Observation 27.

(a) Canonical semiorder revision and contraction satisfy LI *and* HI.

(b) Normal semiorder revision and contraction satisfy LI *and* HI.

Thus LI and HI hold within each representation scheme. This harmony is lost when we make a cross comparison. The paradigmatic case of interest to us is the one where we compare a canonical revision with a normal contraction. Each of the Levi identity and the Harper identity implies the weaker condition

(RCI) $\mathrm{Bel} * A \subseteq \mathrm{Bel} \dot{-} \overline{A}$. (Revision-contraction inclusion)

The latter is violated in our framework, and thus we only get restricted versions of the Levi and Harper identities.

Observation 28. *Let $\prec$ be a semiorder, $*$ be the* canonical *revision and $\dot{-}$ the* normal *contraction based on $\prec$. Then they satisfy*

(RLI) $$\mathrm{Bel} * A = \begin{cases} \min A & \textit{if } \mathrm{Bel} \not\subseteq \overline{A}, \\ (\mathrm{Bel} \dot{-} \overline{A}) \cap A & \textit{otherwise.} \end{cases}$$

(RHI) $$\mathrm{Bel} \dot{-} A = \begin{cases} \mathrm{Bel} & \textit{if } \mathrm{Bel} \not\subseteq A, \\ \mathrm{Bel} \cup (\mathrm{Bel} * \overline{A}) & \textit{otherwise.} \end{cases}$$

But RCI *and thus* LI *and* HI *are generally violated.*

Now we provide a similar result for threshold-based belief changes in the ranking-theoretic framework.

Theorem 29. *Let $m > z \geq n \geq 0$, $A \notin \{\emptyset, W\}$ and consider* $\mathrm{Bel}(\Psi * A) = \mathrm{Bel}_z(\kappa_A^m)$ *and* $\mathrm{Bel}(\Psi \dot{-} A) = \mathrm{Bel}_z(\overline{\kappa}_A^n)$. *Then*

(a) $*$ *and* $\dot{-}$ *violate* RCI *and thus also* LI *and* HI;

(b) RCI, LI *and* HI *are satisfied if $z = 0$.*

The reason for the general failure of LI and HI is that threshold-based belief change corresponds to the canonical representation for revision and to the normal representation for contractions (Theorems 16 and 24). This combination gets strong motivation from the idea of minimal change as applied to rankings. In the revision case when A is compatible with the beliefs, a minimal-change ranking revision is a downwards move of A paired with an upwards move of the complement—which often results in a proper superset

[21] HI requires no such additional order property.

of the intersection of the belief core with A. Hence the violation of Preservation and hence the canonical representation. In the contraction case when A is not already believed, a minimal-change contraction by A has to do nothing at all. Hence the satisfaction of AGM$\dot{-}$4, and hence the normal representation.

For maximal contraction ($n = 0$), we recover something very much like the so-called Harper identity (to which we will turn in a moment), restricted to the case $\mathrm{Bel} \subseteq A$:

$$\mathrm{Bel}_z(\overline{\kappa}^0_A) = \mathrm{Bel} \cup \min^z_\kappa \overline{A} = \mathrm{Bel} \cup \mathrm{Bel}_z(\kappa^m_{\overline{A}}) \quad (\text{for } m > z). \tag{24}$$

We may of course also ask which ranking-change on the side of contraction would yield the Levi identity for our ranking revision. It is the operation $\kappa_{\overline{A}\to 0}$, for which we get for all $A \neq \emptyset, W$:

(LI^n) For $n > z$, $\mathrm{Bel}_z(\kappa^n_A) = \mathrm{Bel}_z(\kappa_{\overline{A}\to 0}) \cap A$.

The operation $\kappa_{\overline{A}\to 0}$ is close to the maximal contraction of κ by $\overline{A}$. As a replacement for the Harper identity, we have $z+1$ restricted Harper identities (compare RHI). For all $A \neq W$:

(RHI^n) For $n \leq z$, $\mathrm{Bel}_z(\overline{\kappa}^n_A) = \begin{cases} \mathrm{Bel}_z & \text{if } \mathrm{Bel}_z \not\subseteq A, \\ \mathrm{Bel}_z \cup \mathrm{Bel}_{z-n}(\kappa * \overline{A}) & \text{otherwise.} \end{cases}$

For maximal contraction ($n = 0$), the variation in the threshold on the right-hand side disappears, since $z - n$ gets replaced by z. The same is of course true for $z = 0$, but then we are back at full AGM where we obtain the unrestricted Harper identity.

The above discussion shows that ranking revision and contraction are not related according to the Levi and Harper identities. They are rather two partial operations which are similar but differ first in the conditions when an action is taken, and second in the amount and direction of change. This has been clear since the beginning, by our way of defining both κ^m_A partially from $\kappa_{A\to m}$ when $\kappa(\overline{A}) \leq z$, and $\overline{\kappa}^n_A$ partially from $\kappa_{A\to n}$ in the complementary case when $\kappa(\overline{A}) > z$. This is the similarity. But the difference is that $m > z$ and $n \leq z$. This implies a difference in the degree and direction of change. An m-revision by A moves $\overline{A}$-worlds *up* (and A-worlds down) in the implausibility ordering—if it moves anything. An n-contraction by A moves $\overline{A}$-worlds *down* in the implausibility ordering—if it moves anything. This duality is expressed by the fact that $\kappa^m_A = \kappa$ iff $\overline{\kappa}^n_A \neq \kappa$.

Generally speaking, there is an important difference between revision and contraction in the threshold-based ranking-theoretic model. Adopting a new belief A requires a lot more cognitive effort than withdrawing the belief $\overline{A}$. For the latter operation, one just needs to make A *sufficiently* plausible, but for the former, one needs to make A *maximally* plausible (and $\overline{A}$ sufficiently implausible, but this is less important here). This is the more fundamental reason why revision and contraction in our model do not validate the Levi and Harper identities. This asymmetry between revision and contraction is intuitively very natural, but it gets lost both with the simpler AGM belief change model and with Spohn's fixation of the belief threshold at $z = 0$.

7 Iterated revision

Consider the following iterated revision postulates: for $A \neq \emptyset$,[22]

(IR0) If $\mathrm{Bel}(\Psi) \subseteq A$, then $\mathrm{Bel}((\Psi * A) * B) = \mathrm{Bel}(\Psi * B)$.

(IR1) If $B \subseteq A$, then $\mathrm{Bel}((\Psi * A) * B) = \mathrm{Bel}(\Psi * B)$.

(IR2) If $B \subseteq \overline{A}$, then $\mathrm{Bel}((\Psi * A) * B) = \mathrm{Bel}(\Psi * B)$.

(IR3) If $\mathrm{Bel}(\Psi * B) \subseteq A$, then $\mathrm{Bel}((\Psi * A) * B) \subseteq A$.

(IR4) If $\mathrm{Bel}(\Psi * B) \nsubseteq \overline{A}$, then $\mathrm{Bel}((\Psi * A) * B) \nsubseteq \overline{A}$.

(IR5) If $\mathrm{Bel}(\Psi) \subseteq \overline{A}$ and $\mathrm{Bel}(\Psi * B) \nsubseteq \overline{A}$, then $\mathrm{Bel}((\Psi * A) * B) \subseteq A$.

(IR6) If $\mathrm{Bel}(\Psi) \nsubseteq A$ and $\mathrm{Bel}(\Psi * B) \nsubseteq \overline{A}$, then $\mathrm{Bel}((\Psi * A) * B) \subseteq A$.

(IR7) If $\mathrm{Bel}(\Psi * B) \nsubseteq \overline{A}$, then $\mathrm{Bel}((\Psi * A) * B) \subseteq A$.

IR1–IR4 correspond to the well known Darwiche-Pearl postulates C1–C4 (Darwiche and Pearl 1997, p. 11). IR7 is condition P of Booth and Meyer (2006, p. 134) and the Independence condition 'Ind' of Jin and Thielscher (2007, p. 8). IR0, IR5 and IR6 are new. IR0 says that if A is initially believed, then an interpolated revision by A will not change the beliefs resulting from a revision by any other proposition B. IR0 partially implies IR3 and IR4, namely for A initially believed. IR6 provides the other partial implication, so given AGM2 and AGM5, IR0 and IR6 together imply IR3 and IR4.[23] Finally, IR5 and IR6 are just weaker versions of IR7: the latter implies IR6 which in turn implies IR5 given AGM0.

As Darwiche and Pearl linked IR1–IR4 to order conditions in the context of full AGM revision, we link IR0–IR7 to order conditions in the weaker context of semi-AGM revision. We only assume $\prec$ to be a semiorder. We consider the following postulates, where $\prec$ is the semiorder representing (the revisions of) Ψ and $\prec_A$ is the semiorder representing (the revisions of) $\Psi * A$:

(IR0$\prec$) If $\mathrm{Bel} \subseteq A$, then: $v \prec w$ iff $v \prec_A w$.

[22] The restriction to $A \neq \emptyset$ is only important for IR2. When $A = \emptyset$, IR2 may fail for semiorder as for ranking-based revisions, since $\Psi * \emptyset$ may be different from Ψ.

[23] The case $\mathrm{Bel} \subseteq A$ is covered by IR0. So suppose $\mathrm{Bel} \nsubseteq A$. IR3: Assume $\mathrm{Bel} * B \subseteq A$. Thus, by AGM5, $\mathrm{Bel} * B \nsubseteq \overline{A}$ unless $B = \emptyset$, a case for which IR3 and IR4 are trivial due to AGM2. IR6 yields $\mathrm{Bel} * A * B \subseteq A$. IR4: Assume $\mathrm{Bel} * B \nsubseteq \overline{A}$. Thus $A \neq \emptyset$. By IR6, we get $(\mathrm{Bel} * A) * B \subseteq A$. Thus $(\mathrm{Bel} * A) * B \nsubseteq \overline{A}$ by AGM2 and AGM5.

(IR1$\prec$) If $v, w \in A$, then: $v \prec w$ iff $v \prec_A w$.

(IR2$\prec$) If $v, w \in \overline{A}$, then: $v \prec w$ iff $v \prec_A w$.

(IR3$\prec$) If $v \in A$ and $w \in \overline{A}$, then $v \prec w$ implies $v \prec_A w$.

(IR4$\prec$) If $v \in \overline{A}$ and $w \in A$, then $v \prec_A w$ implies $v \prec w$.

(IR5$\prec$) If Bel $\subseteq \overline{A}$, $w \in A$ and $v \in \overline{A}$, then $v \not\prec w$ implies $w \prec_A v$.

(IR6$\prec$) If Bel $\not\subseteq A$, $w \in A$ and $v \in \overline{A}$, then $v \not\prec w$ implies $w \prec_A v$.

(IR7$\prec$) If $w \in A$ and $v \in \overline{A}$, then $v \not\prec w$ implies $w \prec_A v$.

Our core postulates IR1$\prec$–IR4$\prec$ are the famous postulates CR1–CR4 investigated in the total preorder context by Darwiche and Pearl. The only difference is that we take strict relations as primitive and consider the weaker semiorder context, and hence IR1$\prec$ and IR2$\prec$ are formulated with $\prec$ and $\prec_A$.[24] Spohn (2012, p. 95) argued for a condition similar to IR6$\prec$. The remaining postulates are new. IR0 says that no change should occur when A is already believed. And IR5$\prec$–IR7$\prec$ are conditions of effective improvement, which recommend that A gets effectively improved relative to $\overline{A}$. As before, IR7$\prec$ implies IR6$\prec$ which implies IR5$\prec$. And taken together, IR0$\prec$ and IR6$\prec$, IR3$\prec$ and IR4$\prec$. IR7$\prec$ imposes unrestricted improvement, whereas IR5$\prec$ and IR6$\prec$ impose a restricted improvement.

Darwiche and Pearl (1997, Theorem 14) showed that the unrestricted improvement operator $\kappa_{A \to \kappa(\overline{A})+1}$ satisfies IR1$\prec$–IR4$\prec$ for the threshold $z = 0$. The generalized improvement operator $\kappa_{A \to \kappa(\overline{A})+z+1}$ satisfies IR7$\prec$ (and the core postulates) for all thresholds $z \geq 0$ and violates IR0$\prec$. Violating IR0$\prec$ may actually be regarded as the point of unrestricted improvement. Thus it is possible to satisfy the core postulates without satisfying IR0$\prec$. It is also possible to satisfy the core postulates without satisfying some effective improvement postulate (IR5, IR6 or IR7), for example by moving only the minimal A-worlds to the bottom of the new order and keeping everything else fixed—which gives us something very close to Boutilier's (1996) 'natural revision'. We will later identify more compatibilities.

In our terminology, Darwiche and Pearl (1997, Theorem 13) proved the following in the total preorder context:

Theorem 30 (Darwiche and Pearl)**.** *Suppose that a belief state revision* $\langle \mathbb{D}, W, *, \mathrm{Bel} \rangle$ *is fully* AGM. *For* $i \in \{1, \ldots, 4\}$*, the revision operator satisfies* IRi *iff the operator and its corresponding total preorder (as in Theorem 13) satisfy* IR$i\prec$.

We obtain a similar correspondence, but in our much more general context of semiorder revisions:

[24] However, this makes no difference since Darwiche and Pearl's $\prec$ is a strict total preorder.

Theorem 31. *Suppose that a belief state revision* $\langle \mathbb{D}, W, *, \mathrm{Bel} \rangle$ *is semi-*AGM. *For* $i \in \{0, \dots, 7\}$, *the revision operator satisfies* IRi *iff the operator and its canonically corresponding semiorder (as in Theorem 16, see* (26) *below) satisfies* IR$i{\prec}$.

This shows that the Darwiche-Pearl correspondence still holds in the more general semiorder context.

From the above, we also get that the unrestricted improvement $\kappa_{A \to \kappa(\overline{A})+z+1}$ satisfies IR1–IR7 and violates IR0. We will now show that our ranking revisions are restricted improvements—they satisfy IR0–IR5 and violate IR6 and IR7. For this we need

Observation 32. *Let the threshold* $z \geq 0$ *be fixed. The map* $\langle \kappa, A \rangle \mapsto \kappa^*_A = \kappa^n_A$ *for some* $n > z$ *satisfies the following properties:*

($\kappa{*}0$) *If* $\mathrm{Bel}(\kappa) \subseteq A$, *then for all* w, $\kappa^*_A(w) = \kappa(w)$.

($\kappa{*}1$) *If* $v, w \in A$, *then* $\kappa^*_A(v) - \kappa^*_A(w) = \kappa(v) - \kappa(w)$.

($\kappa{*}2$) *If* $A \neq \emptyset$ *and* $v, w \in \overline{A}$, *then* $\kappa^*_A(v) - \kappa^*_A(w) = \kappa(v) - \kappa(w)$.

($\kappa{*}5$) *If* $\mathrm{Bel}(\kappa) \subseteq \overline{A}$ *and* $w \in A, v \in \overline{A}$, *then* $\kappa^*_A(w) + z < \kappa(w)$ *and* $\kappa(v) + z < \kappa^*_A(v)$.

($\kappa{*}6$) *If* $\mathrm{Bel}(\kappa) \not\subseteq A$ *and* $w \in A, v \in \overline{A}$, *then* $\kappa^*_A(w) \leq \kappa(w)$ *and* $\kappa^*_A(v) > \kappa(v)$.

($\kappa{*}0$) says that a revision by A triggers no change, when A is already believed—a vacuity condition. By ($\kappa{*}1$), a revision by A preserves distances within A, and by ($\kappa{*}2$), it also preserves distances within $\overline{A}$. In fact, these postulates, together with the success postulate, already determine that the revision must be (equivalent to) an n-revision for $A \neq \emptyset$. ($\kappa{*}5$) entails that when $\overline{A}$ is believed, a revision by A improves A relative to $\overline{A}$ by more than $2z$ ranks, since A is improved by more than z ranks and $\overline{A}$ is degraded by more than z ranks. ($\kappa{*}6$) says that when A is not believed then $\overline{A}$ gets degraded, whereas A does not get degraded. This last postulate does not tell us much if the threshold z is greater than 0, but it tells us something if $z = 0$, since one rank makes a difference in that context.

Theorem 33. *Let the threshold* $z \geq 0$ *be fixed and* $n, m > z$. *Let the semiorders* $\prec$ *and* $\prec_A$ *be associated with the ranking revisions of* κ *and* $\kappa * A = \kappa^n_A$, *with revision parameters* n *and* m *respectively.*

(a) The semiorders $\prec$ *and* $\prec_A$ *satisfy* IR0$\prec$–IR5$\prec$ *for all* $A \neq \emptyset$. IR6$\prec$ *is satisfied for* $z = 0$, *but in general fails for* $z > 0$.

(b) The iterated revision satisfies IR0–IR5 *for all* $A \neq \emptyset$ *and* B.

The most important consequences of our results are the following: (1) n-revisions satisfy IR0–IR5, but generally violate IR6 and IR7. IR6 is only satisfied if the threshold z equals 0, IR7 not even then. Thus n-revisions are restricted improvement operators. (2) All claims in (1) remain true if we alter the threshold in the second step and consider non-minimal revisions more generally (see the replacements in Theorem 33). (3) One can satisfy IR0–IR5 without satisfying IR6 or IR7. (4) IR0–IR2 and IR6 imply IR0–IR6. (5) IR0–IR2 and IR6 imply that $\prec$ is modular and thus a strict total preorder. (6) IR0–IR5 neither guarantee threshold stability nor revision parameter stability nor minimal revision.

In view of (5), we arrive at full AGM revision by assuming semi-AGM augmented by IR0–IR2 and IR6. In view of (6), one might ask whether there are further postulates which guarantee threshold stability or parameter stability or characterize minimal revision. We leave this as an open question.

8 Iterated contraction

We finally study iterated contraction. Consider the following postulates for iterated contraction:

(IC0) If $\mathrm{Bel}(\Psi) \not\subseteq A$, then $\mathrm{Bel}(\Psi \dot{-} B) = \mathrm{Bel}((\Psi \dot{-} A) \dot{-} B)$.

(IC1) If $\overline{A} \subseteq B$, then $\mathrm{Bel}(\Psi \dot{-} B) \cap \overline{B} = \mathrm{Bel}((\Psi \dot{-} A) \dot{-} B) \cap \overline{B}$.

(IC2) If $A \subseteq B$, then $\mathrm{Bel}(\Psi \dot{-} B) \cap \overline{B} = \mathrm{Bel}((\Psi \dot{-} A) \dot{-} B) \cap \overline{B}$.

(IC3) If $\mathrm{Bel}(\Psi \dot{-} B) \subseteq \overline{A} \cup B$, then $\mathrm{Bel}((\Psi \dot{-} A) \dot{-} B) \subseteq \overline{A} \cup B$.

(IC4) If $\mathrm{Bel}((\Psi \dot{-} A) \dot{-} B) \subseteq A \cup B$, then $\mathrm{Bel}(\Psi \dot{-} B) \subseteq A \cup B$.

Whereas the relation of iterated revision to order revision has been studied since the seminal paper of Darwiche and Pearl (1997), the case of contraction has only recently been investigated by Konieczny and Pino Pérez (2017) and Sauerwald, Kern-Isberner and Beierle (2020), and only in the total preorder setting (see Table 1). Here we consider the more general semiorder setting. Except for IC2, our postulates are new. The postulates most similar to ours are the postulates C8–C11 of Sauerwald, Kern-Isberner and Beierle (2020), which we rename SKB1–SKB4. SKB2 is our IC2. SKB1 is our IC1 with an intersection by A instead of an intersection by $\overline{B}$. Thus SKB1 and SKB2 don't have identical consequents, as we have in IC1 and IC2. However, SKB1 is equivalent to our IC1, due to the antecedent condition we have $\overline{B} \subseteq A$, so that larger intersection by A is irrelevant after contracting by B. SKB3 is a version of our IC3, but requires application to all supersets of $\overline{A}$ rather than just to $\overline{A} \cup B$. And SKB4 is a similar version of our IC4. It follows that SKB1–SKB4 imply IC1–IC4 one by one. But the reverse is not true for IC3 and IC4, at least not in a one-by-one fashion. We have added IC0 as a no-change postulate. Postulates KP1–4 of Konieczny and Pino Pérez (2017) are much more complicated than our postulates.

(KP1)	If $\overline{A} \subseteq B$, then $\mathrm{Bel}(\Psi \dot{-} (A \cup C)) \subseteq \mathrm{Bel}(\Psi \dot{-} A)$ iff $\mathrm{Bel}(\Psi \dot{-} B \dot{-} (A \cup C)) \subseteq \mathrm{Bel}(\Psi \dot{-} B \dot{-} A)$.
(KP2)	If $B \subseteq A$, then $\mathrm{Bel}(\Psi \dot{-} (A \cup C)) \subseteq \mathrm{Bel}(\Psi \dot{-} A)$ iff $\mathrm{Bel}(\Psi \dot{-} B \dot{-} (A \cup C)) \subseteq \mathrm{Bel}(\Psi \dot{-} B \dot{-} A)$.
(KP3)	If $\overline{C} \subseteq B$, then $\mathrm{Bel}(\Psi \dot{-} B \dot{-} (A \cup C)) \subseteq \mathrm{Bel}(\Psi \dot{-} B \dot{-} A)$ implies $\mathrm{Bel}(\Psi \dot{-} (A \cup C)) \subseteq \mathrm{Bel}(\Psi \dot{-} A)$.
(KP4)	If $B \subseteq C$, then $\mathrm{Bel}(\Psi \dot{-} B \dot{-} (A \cup C)) \subseteq \mathrm{Bel}(\Psi \dot{-} B \dot{-} A)$ implies $\mathrm{Bel}(\Psi \dot{-} (A \cup C)) \subseteq \mathrm{Bel}(\Psi \dot{-} A)$.
(SKB1)	If $\overline{A} \subseteq B$, then $\mathrm{Bel}(\Psi \dot{-} B) \cap A = \mathrm{Bel}((\Psi \dot{-} A) \dot{-} B) \cap A$.
(SKB2)	If $A \subseteq B$, then $\mathrm{Bel}(\Psi \dot{-} B) \cap \overline{B} = \mathrm{Bel}((\Psi \dot{-} A) \dot{-} B) \cap \overline{B}$.
(SKB3)	If $\overline{A} \subseteq C$, then $\mathrm{Bel}(\Psi \dot{-} B) \subseteq C$ implies $\mathrm{Bel}((\Psi \dot{-} A) \dot{-} B) \subseteq C$.
(SKB4)	If $A \subseteq C$, then $\mathrm{Bel}((\Psi \dot{-} A) \dot{-} B) \subseteq C$, then $\mathrm{Bel}(\Psi \dot{-} B) \subseteq C$.

Table 1: Iterated contraction postulates. KP1–KP4 are the postulates C8–C11 of Konieczny and Pino Pérez (2017). SKB1–SKB4 are the postulates C8–C11 of Sauerwald, Kern-Isberner and Beierle (2020). Our alternative postulates IC1–IC4 are in the main text.

Consider the following "semiorder contraction" postulates, in which Bel is short for $\mathrm{Bel}(\Psi)$, $\prec$ is short for $\prec_\Psi$ and $\prec_A^-$ is short for $\prec_{\Psi \dot{-} A}$:

(IC0$\prec$) If $\mathrm{Bel} \not\subseteq A$, then: $v \prec w$ iff $v \prec_A^- w$.

(IC1$\prec$) If $v, w \in A$, then: $v \prec w$ iff $v \prec_A^- w$.

(IC2$\prec$) If $v, w \in \overline{A}$, then: $v \prec w$ iff $v \prec_A^- w$.

(IC3$\prec$) If $v \in \overline{A}$ and $w \in A$, then $v \prec w$ implies $v \prec_A^- w$.

(IC4$\prec$) If $v \in A$ and $w \in \overline{A}$, then $v \prec_A^- w$ implies $v \prec w$.

IC1$\prec$ – IC4$\prec$ were used by Konieczny and Pino Pérez (2017) and by Sauerwald, Kern-Isberner and Beierle (2020) in the total preorder context, and IC1$\prec$ and IC2$\prec$ were formulated with non-strict rather than strict relations. IC0$\prec$ already implies one half of IC1$\prec$ – IC4$\prec$, namely for the case when A is not believed.[25]

In our terminology, Konieczny and Pino Pérez (2017) proved the fol-

[25] Spohn (2012, p. 96) gives a postulate resembling IC0$\prec$. We refrain from discussing in detail additional postulates that are similar to those of the revision case: (IC7$\prec$) If $v \in A$ and $w \in \overline{A}$, then $v \not\prec w$ implies $w \prec_A^- v$, as well as its weaker variants (IC5$\prec$) and (IC6$\prec$) which are restrictions of (IC7$\prec$) to the cases $\mathrm{Bel} \subseteq A$ and $\mathrm{Bel} \not\subseteq \overline{A}$, respectively. Only (IC5$\prec$) restricted to the standard case $z = 0$ is valid for ranking contractions. (Counterexample against (IC5$\prec$) for $z > 0$: Let $z = 1$, $n = 0$ and $W = \{w_0, w_2, w_3, w_4\}$ with the ranks of the worlds given by their indices, and $A = \{w_0, w_3\}$. Then we have $\mathrm{Bel}_1(\kappa) = \{w_0\} \subseteq A$ and $\kappa(w_4) - \kappa(w_3) = 4 - 3 = 1 \leq z - n$, so $w_3 \sim w_4$. But $\overline{\kappa}_A^0(w_3) - \overline{\kappa}_A^0(w_4) = 3 - 2 = 1 \leq z - n$, so still $w_3 \sim_A^- w_4$.)

lowing one-by-one correspondence in the total preorder context:

Theorem 34 (Konieczny and Pino Pérez)**.** *Suppose that a belief state contraction* $\langle\, \mathbb{D}, W, \dot{-}, \mathrm{Bel}\,\rangle$ *is* fully AGM*. For* $i \in \{1, \ldots, 4\}$*, the operator satisfies* KPi *iff the operator and its corresponding total preorder satisfy* IC$i\prec$*.*

In that same context, Sauerwald, Kern-Isberner and Beierle (2020) proved restricted one-by-one correspondences:

Theorem 35 (Sauerwald, Kern-Isberner and Beierle)**.** *Suppose that a belief state contraction* $\langle\, \mathbb{D}, W, \dot{-}, \mathrm{Bel}\,\rangle$ *is* fully AGM*. For* $i \in \{1, 2\}$*, the operator satisfies* SKBi *iff the operator and its corresponding total preorder satisfy* IC$i\prec$*; and given* SKB1–2*, the same holds for* SKBj *and* IC$j\prec$ *for* $j \in \{3, 4\}$*.*

We obtain one-by-one correspondences in the more general context of semiorder revisions:

Theorem 36. *Suppose that a belief state contraction* $\langle\, \mathbb{D}, W, \dot{-}, \mathrm{Bel}\,\rangle$ *is* semi-AGM*. For* $i \in \{0, \ldots, 4\}$*, the operator satisfies* ICi *iff the operator and its canonically corresponding semiorder (as in Corollary 26, see* (28) *below) satisfies* IC$i\prec$*.*

It is interesting that the one-one correspondence still holds in the more general semiorder context. Our result also improves on the axiomatizations of Konieczny and Pino Pérez (2017) and Sauerwald, Kern-Isberner and Beierle (2020).

The following result is remarkable:

Observation 37. *Suppose that a contraction operator* $\dot{-}$ *is semi-*AGM *and satisfies* AGM$\dot{-}$4 *in the second step. Then* IC2 *implies that* $\dot{-}$ *satisfies* AGM$\dot{-}$8 *in the first step.*

This observation shows that IC2 is not a harmless principle. For this reason, we think that one should add the principle IC0, and restrict all other postulates by the assumption $\mathrm{Bel} \dot{-} A \subseteq B$. For example, our substitute for IC2 is

(IC2′) If $\mathrm{Bel} \dot{-} A \subseteq B$ and $A \subseteq B$, then $\mathrm{Bel}(\Psi \dot{-} B) \cap \overline{B} = \mathrm{Bel}((\Psi \dot{-} A) \dot{-} B) \cap \overline{B}$.

Denote the other substitutes similarly by IC1′, IC3′, IC4′. Since IC0 takes care of the case when A is not initially believed, IC0 already implies the other three postulates for this case. Thus, in the context of IC0 (or AGM$\dot{-}$ 4) we may always consider IC1′–IC4′ as being additionally restricted to $\mathrm{Bel} \subseteq A$. The reason why only IC2 needs to be restricted by the above additional assumption is that the other principles don't create a conflict with AGM$\dot{-}$4. In fact, the stronger principles can be obtained from the weaker ones (except for IC2), provided we have AGM$\dot{-}$4.

From Theorem 24 and Observation 37, we can infer that our ranking contractions violate IC2 (remember that normal semi-AGM contractions do *not* in general satisfy AGM$\dot{-}$8). On the other hand, it is obvious from Definition 3 that ranking contractions satisfy IC2$\prec$ if the bridge principle (19) is used. This seems to violate our representation Theorem 36. However it does not. The semiorders obtained by (19) are the most straightforward ones, and as we have seen, they work for revisions. However, they are not suitable for contractions; here (23) is the right rule, as we showed in Corollary 25. And the semiorders thus obtained do not satisfy IC2$\prec$.[26]

This raises the question whether there is a similar result for the normal case. For reasons of space, we only answer one part of that question, and take it that it is natural to assume IC0, IC0$\prec$ for AGM$\dot{-}$4 contractions:

Theorem 38. *Suppose that a belief state contraction* $\langle\, \mathbb{D}, W, \dot{-}, \mathrm{Bel}\,\rangle$ *is normal semi-AGM.*

- *(a) If the contraction operator and its corresponding* normal *semiorder (as in Theorem24, see* (28) *below) satisfy* IC0$\prec$*, then the operator satisfies* IC0.
- *(b) If, furthermore, the operator and its corresponding* normal *semiorder also satisfy* IC$i\prec$*, then the operator satisfies* ICi'*, for* $i \in \{1, \dots, 4\}$.
- *(c)* IC1*,* IC3*,* IC4 *then follow from* IC1$'$*,* IC3$'$*,* IC4$'$.

We will need the following properties to investigate iterated ranking contractions. They are counterparts to the revision properties highlighted in Observation 32.

Observation 39. *Let the threshold* $z \geq 0$ *be fixed. The map* $\langle\, \kappa, A\,\rangle \mapsto \kappa^-_A := \overline{\kappa}^n_A$ *for some* $n \leq z$ *satisfies the following properties:*

($\kappa \dot{-} 0$) If $\mathrm{Bel} \nsubseteq A$, then $\kappa(w) = \kappa^-_A(w)$.

($\kappa \dot{-} 1$) If $v \in A$, then $\kappa(v) = \kappa^-_A(v)$.

($\kappa \dot{-} 2$) If $v, w \in \overline{A}$, then $\kappa(v) - \kappa(w) = \kappa^-_A(v) - \kappa^-_A(w)$.

($\kappa \dot{-} 5$) If $\mathrm{Bel} \subseteq A$ and $v \in \overline{A}$, then $\kappa^-_A(v) + (z - n) < \kappa(v)$.

($\kappa \dot{-} 0$) is a normal minimal change property for contraction: there is no change if A is initially not believed. By ($\kappa \dot{-} 1$), A-worlds are not moved at all in contractions by A. By ($\kappa \dot{-} 2$), the distances between $\overline{A}$-worlds are conserved. ($\kappa \dot{-} 5$) says that $\overline{A}$-worlds are moved downwards if A is initially believed. And the extent of the downwards move depends on how much the parameter n differs from threshold z. For maximal contraction, the downwards move is $\kappa(\overline{A}) > z$, for minimal contraction it is $\kappa(\overline{A}) - z > 0$. Overall,

[26]For a concrete counterexample against IC2 and IC2$\prec$, see the proof of Theorem 40 below.

A-worlds are uniformly degraded relative to $\overline{A}$-worlds, but only when A is initially believed. For this reason, we call our ranking contraction a *restricted degrading operator*. These properties characterize the ranking contractions completely. The only difference between different ranking contractions lies in how much they degrade A, i.e., how much they improve $\overline{A}$.

With the help of Observation 39, we obtain:

Theorem 40. *Let the threshold $z \geq 0$ be fixed and $n, m \leq z$. Let the semiorders $\prec$ and $\prec_A$ be associated with the ranking contractions of κ and $\kappa \dot{-} A = \overline{\kappa}^n_A$, with contraction parameters n and m respectively.*

(a) The semiorders $\prec$ and $\prec_A$ satisfy IC0$\prec$, IC1$\prec$, IC2$'$ $\prec$, IC3$\prec$, IC4$\prec$ *for all A.*

(b) The iterated contraction satisfies IC0, IC1, IC2$'$, IC3, IC4 *for all A and B.*

(c) IC2 *is satisfied for $n = z$, but in general fails for $n < z$.*

Whereas *minimal* ranking contraction satisfies all postulates IC0–IC4, non-minimal contraction only satisfies IC0, IC1, IC2$'$, IC3 and IC4 and violates IC2.

9 Conclusion

In this paper we have studied changes of beliefs in a ranking-theoretic setting using non-extremal implausibility thresholds for the fixation of beliefs. We have represented implausibilities as ranks and introduced natural rank changes subject to a minimal change criterion. We showed that many of the traditional AGM postulates for revision and contraction are preserved, but notably the postulate of Preservation is not valid any more. The diagnosis for belief contraction is similar, but not exactly the same. We have demonstrated that the one-shot versions of both revision and contraction can be represented as revisions based on semiorders, but in two subtly different ways. We have provided sets of postulates for contractions and revisions that are sound and complete in the sense that they allow us to prove representation theorems. We showed that, and explained why, the classical duality between revision and contraction, as exhibited by the Levi and Harper identities, is partly broken by threshold-based belief changes. We also studied the logic of iterated threshold-based revision and contraction. The traditional Darwiche-Pearl postulates for iterated revision continue to hold, as well as two additional postulates that characterize ranking-based revision as a restricted 'improvement' operator. We investigated the dual notion of iterated threshold-based belief contraction and provided a new set of postulates for it, characterizing contraction as a restricted 'degrading' operator. For a more detailed overview of the results of this paper, see Figure 1.

The attentive reader may be left with a nagging worry. On the one hand ranking revision violates Preservation, and on the other hand, ranking contraction satisfies Vacuity so that the revision induced by the Levi identity *does* satisfy Preservation. However, we have very consciously refrained from using the Levi identity, and instead argued that the Levi and Harper identities are rightly violated if the ideas of minimal change appropriate for the ranking-theoretic framework are heeded. The reason is this: there are two ways of generalizing the standard representation of revision and contraction from the total preorder case. Concerning revisions, for example, the standard representation is $\mathrm{Bel} = \min W$ and $\mathrm{Bel} * A = \min A$. For semiorders, we can also use this representation (we have called it the canonical representation), or we can introduce a case distinction and put $\mathrm{Bel} * A = \mathrm{Bel} \cap A$ if A is consistent with Bel, and else $\mathrm{Bel} * A = \min A$ (we have called this the normal representation, which we argue is *not* the standard for revisions based on ranking functions). For total preorders, canonical and normal representation are equivalent. For semiorders, however, they are not. The normal representation satisfies Preservation, the canonical representation does not. Similarly for contractions, except that what is standard and what is non-standard is reversed from the ranking-theoretic perspective. The (standard) normal representation satisfies Vacuity, the (non-standard) canonical one invalidates it. If we relate revision and contraction within each of the two schemes, then the Levi and Harper identities hold. But in the ranking framework the minimal-change criterion for revisions leads to minimally moving down A-worlds when A is compatible with the agent's beliefs (instead of simply intersecting beliefs with A)—hence the canonical representation, and hence the violation of Preservation. And the minimal-change criterion for contractions leads to not doing anything when A is a nonbelief from the start—hence the normal representation, and hence the satisfaction of Vacuity. Thus it is the minimal change idea in the setting of ranking theory which really creates what from the semiorder perspective might seem to be a double standard. But this difference only surfaces when we leave the total preorder case that has been studied in almost all of the literature. The lesson is that when it comes to rankings, contraction and revision are better understood as two cognitive actions *sui generis* rather than "two sides of the same coin" that are interdefinable by the Levi and Harper identities.

One last question that might be raised is whether belief changes should not be modelled by interval orders rather than the more special semiorders. While semiorders represent the situation in which the "just noticeable difference" between the plausibilities of worlds is constant across the board, interval orders allow that just noticeable differences generally depend on the particular worlds under consideration. More precisely, just noticeable differences might depend on the (ranks of the minimal worlds satisfying the) input A. Just noticeable differences are effective as threshold values for belief. We can, however, offer a nice interpretation that motivates our opting for semiorders: Threshold values don't change under normal processes of belief revision, and in particular, they are not dependent on what the input happens to be. Opting for interval orders would amount to giving up this assumption.

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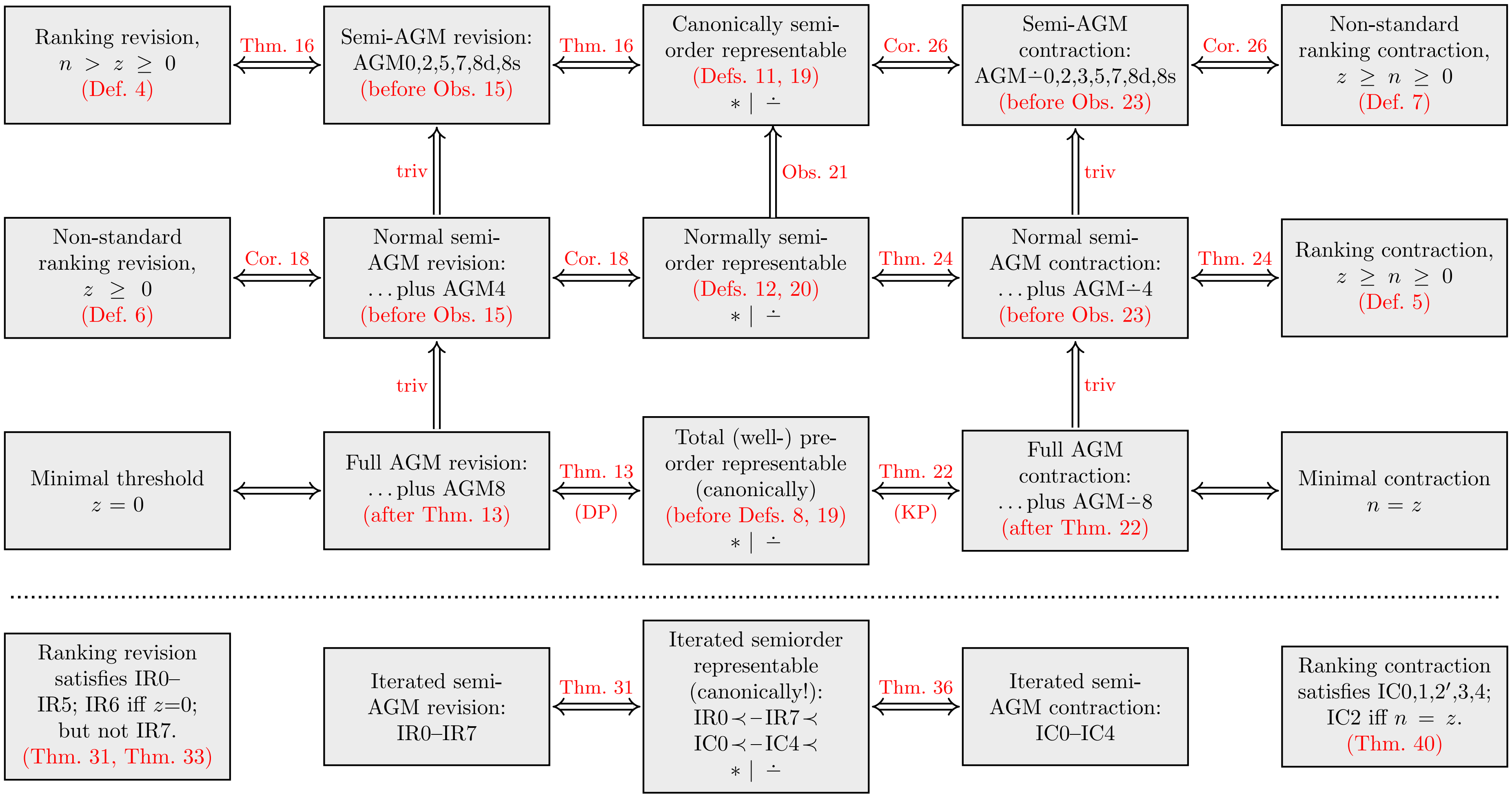

Figure 1: Overview of the results of this paper. The left half is about revisions, the right half about contractions. The upper three rows concern one-shot belief changes, the last row concerns iterated changes. Remarks: Thm. 16 mentions representability by a *minimal* ranking revision with $n = z + 1$. Thm. 24 mentions representability by a *maximal* ranking contraction with $n = 0$. For iterations, semiorder revisions and contractions need extra constraints, while ranking revisions and contractions are always iterable.

Appendix: Proofs

Proof of Theorem 9 (Scott-Suppes): Let W be finite. We define $v \sim w$ iff $v \nprec w$ and $w \nprec v$. $\sim$ is reflexive and symmetric. Also define $v \equiv w$ iff $\forall u(v \sim u \leftrightarrow w \sim u)$. The equivalence classes in $W_{\equiv}$ can be linearly ordered by the relation $\leq$ defined by $[v] \leq [w]$ iff for all $u \in W$, if $u \prec v$ then $u \prec w$, and if $w \prec u$ then $v \prec u$. $\leq$ is well-defined. Let the number of equivalence classes in $W_{\equiv}$ be $n+1$. We label the representatives w_i of them in such a way that $[w_0] < \cdots < [w_n]$. We then define a function f on the equivalence classes as follows:

$$f([w_i]) = \begin{cases} \frac{i}{i+1} & \text{if } w_0 \sim w_i, \\ \frac{1}{i+1} \cdot f([w_{j-1}]) + \frac{i}{i+1} \cdot f([w_j]) + 1 & \text{if } 0 < j, w_{j-1} \prec w_i, w_j \sim w_i. \end{cases} \tag{25}$$

This construction yields *rational* numbers $f([w_i])$ for all $i = 0, \ldots, n$. Notice that $f([w_0]) = 0$. Furthermore, f is injective, since $f([w_i]) < f([w_{i+1}])$ for all i, and $w_i \prec w_j$ if and only if $f([w_i]) + 1 < f([w_j])$ for all $i = 0, \ldots, n$. Intuitively, 1 is the constant value of a 'just noticeable difference'. Now we extend the domain of f to W by defining $f(w) = f([w_i])$ for all $w \in [w_i]$. We then have $v \prec w$ iff $f(v) + 1 < f(w)$. (For more details of the proof, see Suppes and Zinnes 1963, pp. 29–34.) □

Proof of Corollary 10: (a) Take the numerical representation $f(w)$ of the semiorder $\prec$ from Theorem 9. Any value $f(w_i)$ is a rational number k_i/m_i (where $k_i, m_i \in \mathbb{N}$). Multiply all these values uniformly with the least common multiple $z = \mathrm{lcm}(m_1, \ldots, m_n)$ of the denominators. Then we obtain natural numbers $\kappa(w_i) = z \cdot f(w_i)$, for all $i = 0, \ldots, n$. By construction, we have $\kappa(v) + z < \kappa(w)$ iff $f(v) + 1 < f(w)$. And κ is a world ranking.

(b) Suppose that $\kappa(v) \leq z$ and $z < \kappa(w)$, i.e., $f(v) \leq 1$ and $1 < f(w)$. By the construction of the function f, this means that $v \sim w_0$ and $w_0 \prec w$. But this is equivalent to $v \in \min_{\prec} W$ and $w \notin \min_{\prec} W$. So, since $\prec$ is bottom, $v \prec w$. By the construction of f, this implies $f(v) + 1 < f(w)$, and this is equivalent to $\kappa(v) + z < \kappa(w)$. □

Proof of Observation 14: Consider $W = \{x, y, z\}$ and the semiorder $x \prec z$ where no other relations hold. This is a semiorder. Consider the represented revision $*$. Let $A = \{y, z\}$. Note that $\mathrm{Bel} = \{x, y\}$. Thus the premiss $\mathrm{Bel} \cap A \neq \emptyset$ of AGM4 is satisfied. Also note that $\mathrm{Bel} * A = \{y, z\}$. But then $\mathrm{Bel} * A \nsubseteq \mathrm{Bel}$—a violation of AGM4. □

Proof of Observation 15: Suppose $\mathrm{Bel} * A \subseteq B$. Either (i) $\mathrm{Bel} * (A \cap \overline{B}) \subseteq \mathrm{Bel} * A$ or (ii) not. If (i), we get $\mathrm{Bel} * (A \cap \overline{B}) \subseteq B$, and thus $\mathrm{Bel} * (A \cap \overline{B}) = \emptyset$, by AGM2. Hence $A \cap \overline{B} = \emptyset$, by AGM5. Thus $A = A \cap B$.

Therefore $\mathrm{Bel} * (A \cap B) \subseteq \mathrm{Bel} * A$. If (ii), AGM8d delivers $\mathrm{Bel} * (A \cap B) \subseteq \mathrm{Bel} * ((A \cap B) \cup (A \cap \overline{B})) = \mathrm{Bel} * A$. $\square$

Proof of Theorem 16: We show that (a) implies (b), (b) implies (d), (d) implies (c), and (c) implies (a).

(a) implies (b). *Semi-AGM implies canonical semiorder representability.* Assume that $\langle *, \mathrm{Bel} \rangle$ satisfies AGM0, AGM2, AGM5, AGM7, AGM8d and AGM8s for all Ψ such that $\emptyset \neq \mathrm{Bel}(\Psi) \subseteq W$. For every such Ψ, we define

$$v \prec w \ \text{ iff } \ v \in \mathrm{Bel} * \{v, w\} \text{ and } w \notin \mathrm{Bel} * \{v, w\} \tag{26}$$

Part 1: We show that $\prec$ defined by equation (26) is a semiorder.

Irreflexivity. That $w \not\prec w$ is immediate by (26).

Interval condition. Suppose that $u \prec v$ and $x \prec y$. By (26) this means that $u \in, v \notin \mathrm{Bel} * \{u, v\}$ and $x \in, y \notin \mathrm{Bel} * \{x, y\}$. We need to show that $u \prec y$ or $x \prec v$, i.e., that $u \in, y \notin \mathrm{Bel} * \{u, y\}$ or $x \in, v \notin \mathrm{Bel} * \{x, v\}$. Consider the proposition $A = \{u, v, x, y\}$. By AGM7, we get $(\mathrm{Bel} * A) \cap \{u, v\} \subseteq \mathrm{Bel} * (A \cap \{u, v\}) = \mathrm{Bel} * \{u, v\}$. Since $v \notin \mathrm{Bel} * \{u, v\}$, we obtain $v \notin \mathrm{Bel} * A$. In the same way, we obtain $y \notin \mathrm{Bel} * A$. Now AGM8d implies that $\mathrm{Bel} * \{u, y\} \subseteq \mathrm{Bel} * A$ or $\mathrm{Bel} * \{v, x\} \subseteq \mathrm{Bel} * A$. If $\mathrm{Bel} * \{u, y\} \subseteq \mathrm{Bel} * A$, then $y \notin \mathrm{Bel} * \{u, y\}$. By AGM2 and AGM5, $\emptyset \neq \mathrm{Bel} * \{u, y\} \subseteq \{u, y\}$. So $u \in \mathrm{Bel} * \{u, y\}$. Together this means $u \prec y$, as desired. If $\mathrm{Bel} * \{v, x\} \subseteq \mathrm{Bel} * A$, an analogous argument establishes $x \prec v$.

Semitransitivity. Suppose that $u \prec v$ and $v \prec w$. By (26) this means that $u \in, v \notin \mathrm{Bel} * \{u, v\}$ and $v \in, w \notin \mathrm{Bel} * \{v, w\}$. So $(\mathrm{Bel} * \{u, v\}) \cap \{v\} = \emptyset$ and $(\mathrm{Bel} * \{v, w\}) \cap \{w\} = \emptyset$. Thus by AGM8s, $(\mathrm{Bel} * \{u, x\}) \cap \{x\} = \emptyset$ or $(\mathrm{Bel} * \{x, w\}) \cap \{w\} = \emptyset$. Then, by AGM2 and AGM5, $u \in, x \notin \mathrm{Bel} * \{u, x\}$ or $x \in, w \notin \mathrm{Bel} * \{x, w\}$. But this just means that $u \prec x$ or $x \prec w$, as desired.

Part 2: We show that $\prec$ represents $\langle *, \mathrm{Bel} \rangle$. Suppose that the revision $\star$ is obtained from $\prec$ by Definition 11. We show $\mathrm{Bel} * A = \mathrm{Bel} \star A$. Note that if $A = \emptyset$, then $\mathrm{Bel} * A = \emptyset$ by success, and similarly $\mathrm{Bel} \star A = \emptyset$ by (17). We have

$$\begin{aligned} w \in \mathrm{Bel} \star A \ \text{ iff } \ & w \in \min_{\prec} A \\ \text{ iff } \ & w \in A \text{ and } \neg \exists v \in A (v \prec w) \\ \text{ iff } \ & w \in A \text{ and } \neg \exists v \in A (v \in \mathrm{Bel} * \{v, w\} \text{ and } w \notin \mathrm{Bel} * \{v, w\}) \end{aligned}$$

Using the expression in the last line, we prove the following identity claim: for every proposition $A \neq \emptyset$ and every world w, $w \in \mathrm{Bel} * A$ iff $w \in \mathrm{Bel} \star A$.

$w \in \mathrm{Bel} * A$ implies the last line: Let $w \in \mathrm{Bel} * A$. By AGM2, $w \in A$. Let $v \in A$. By AGM7, we get $(\mathrm{Bel} * A) \cap \{v, w\} \subseteq \mathrm{Bel} * (A \cap \{v, w\}) = \mathrm{Bel} * \{v, w\}$. So $w \in \mathrm{Bel} * \{v, w\}$.

The last line implies $w \in \mathrm{Bel} * A$: Assume the last line, i.e., $w \in A$ and there is no $v \in A$ such that $v \in \mathrm{Bel} * \{v, w\}$ and $w \notin \mathrm{Bel} * \{v, w\}$. By AGM2, AGM5, and $A \neq \emptyset$, we have $\emptyset \neq \mathrm{Bel} * \{v, w\} \subseteq \{v, w\}$. Thus $w \in \mathrm{Bel} * \{v, w\}$. This holds for all $v \in A$. Since $A \subseteq W$ is finite and $A = \bigcup_{v \in A}\{v, w\}$, we can apply AGM8d repeatedly and get that there is a $v \in A$ such that $\mathrm{Bel} * \{v, w\} \subseteq \mathrm{Bel} * A$. Since $w \in \mathrm{Bel} * \{v, w\}$, we obtain $w \in \mathrm{Bel} * A$ as desired.

It remains to prove condition (16). From our identity claim, we know that $\mathrm{Bel} * W = \mathrm{Bel} \star W = \min_{\prec} W$, and from AGM0, we also have $\mathrm{Bel} = \mathrm{Bel} * W$. So $\mathrm{Bel} = \min_{\prec} W$, as desired.

(b) implies (d). *Semiorder representability implies ranking representability by a minimal ranking revision.* Suppose that $\prec$ can be used for the canonical representation of $*$, i.e., that (16) $\mathrm{Bel} = \min_{\prec} W$ and (17) $\mathrm{Bel} * A = \min_{\prec} A$. From the semiorder $\prec$, we construct a ranking κ and a threshold $z \geq 0$, using Corollary 10, such that $\kappa(v) + z < \kappa(w)$ iff $v \prec w$. But then $\min_{\prec} W = \min_{\kappa}^{z} W$ and $\min_{\prec} A = \min_{\kappa}^{z} A$, and so we get for arbitrary $n > z$:

$$
\begin{aligned}
\mathrm{Bel}_z(\kappa_A^n) &= \min_{\kappa}^{z} A \;=\; \{w \in A : \kappa(w) \leq \kappa(A) + z\} \\
&= \{w \in A : \neg\exists v \in A, \kappa(v) + z < \kappa(w)\} \\
&= \{w \in A : \neg\exists v \in A, v \prec w)\} \;=\; \min_{\prec} A \\
&= \mathrm{Bel} * A.
\end{aligned}
$$

Hence $\langle *, \mathrm{Bel} \rangle$ is ranking representable by a minimal revision: notice that here the revision parameter $n = z + 1$ is as suitable as any other $n > z$.

(d) implies (c). *Ranking representability by a minimal revision implies ranking representability.* This is trivial.

(c) implies (a). *Ranking representability implies semi-AGM.* Let $*$ be defined from κ and z by $\mathrm{Bel} = \mathrm{Bel}_z(\kappa) = \min_{\kappa}^{z} W$ and $\mathrm{Bel} * A = \mathrm{Bel}_z(\kappa_A^n) = \min_{\kappa}^{z} A$ for some $n > z$, by Definition 4 and equation (8). Since $\kappa^{-1}(0) \neq \emptyset$ (for $\kappa \neq \kappa_{\perp}$), we have $\mathrm{Bel} \neq \emptyset$.

AGM0. $\mathrm{Bel} * W = \min_{\kappa}^{z} W = \mathrm{Bel}$ is immediate.

AGM2. $\mathrm{Bel} * A = \min_{\kappa}^{z} A \subseteq A$ is also immediate.

AGM5. Assume $A \neq \emptyset$. Since $A \subseteq W$ is finite, $\mathrm{Bel} * A = \min_{\kappa}^{z} A$ is non-empty.

AGM7. Suppose that $w \in (\mathrm{Bel} * A) \cap B = (\min_{\kappa}^{z} A) \cap B$. Then $w \in A \cap B$ and $\kappa(w) \leq \kappa(A) + z$. But since $A \cap B \subseteq A$, $\kappa(A) \leq \kappa(A \cap B)$. So $\kappa(w) \leq \kappa(A \cap B) + z$, and we get $w \in \min_{\kappa}^{z}(A \cap B) = \mathrm{Bel} * (A \cap B)$.

AGM8d. Since A and B are subsets of $A \cup B$, it is clear that $\mathrm{Bel} * A = \min_{\kappa}^{z} A \subseteq \min_{\kappa}^{z}(A \cup B) = \mathrm{Bel} * (A \cup B)$ if $\kappa(A) \leq \kappa(B)$, and $\mathrm{Bel} * B = \min_{\kappa}^{z} B \subseteq \min_{\kappa}^{z}(A \cup B) = \mathrm{Bel} * (A \cup B)$ if $\kappa(B) \leq \kappa(A)$.

AGM8s. Let $\mathrm{Bel} * (A \cup B) \cap B = \emptyset$ and $\mathrm{Bel} * (B \cup C) \cap C = \emptyset$. Thus $\min^z_\kappa(A \cup B) \subseteq \overline{B}$ and $\min^z_\kappa(B \cup C) \subseteq \overline{C}$, i.e., $\kappa(A) + z < \kappa(B)$ and $\kappa(B) + z < \kappa(C)$. So $\kappa(A) + 2z < \kappa(C)$. It follows that for arbitrary D, $\kappa(A) + z < \kappa(D)$ or $\kappa(D) + z < \kappa(C)$. But this means that $\min^z_\kappa(A \cup D) \subseteq \overline{D}$ or $\min^z_\kappa(D \cup C) \subseteq \overline{C}$, i.e., $\mathrm{Bel} * (A \cup D) \cap D = \emptyset$ or $\mathrm{Bel} * (D \cup C) \cap C = \emptyset$. □

Proof of Corollary 18: We show that (a) implies (b), (b) implies (d), (d) implies (c), and (c) implies (a).

(a) implies (b). We take the semiorder constructed in the proof of Theorem 16. AGM4 forces this semiorder to be bottom: If $w \in \min W$ and $v \notin \min W$, then $w \prec v$. (This means that minimal elements have no ties or incomparabilities with non-minimal elements.) For bottom semiorders, canonical and normal representations coincide. To see this, one has to check that if $\mathrm{Bel} \not\subseteq \overline{A}$, then $\min_\prec A = \min_W \cap A$. For $\min_\prec A \subseteq \min_W \cap A$, let that $v \in \min_\prec A$ and suppose for reductio that $v \notin \min_\prec W$. Since $\mathrm{Bel} \not\subseteq \overline{A}$, there is a $w \in \min_A \prec W \cap A$. Since $\prec$ is bottom, $w \prec v$, contradicting the minimality of v in A. It follows from Theorem 16 that $*$ is normally representable by $\prec$.

(b) implies (d). If $\mathrm{Bel} \subseteq \overline{A}$, we take the ranking constructed in the proof of Theorem 16, with the minimal revision parameter $n = z + 1$. If $\mathrm{Bel} \not\subseteq \overline{A}$, normal representation puts $\mathrm{Bel} * A = \mathrm{Bel} \cap A$, and non-standard ranking representation does the same.

(d) implies (c). This is trivial.

(c) implies (a). It is easy to see that a non-standardly ranking representable revision can be standardly represented by its z-gappy transform κ', defined by $\kappa'(w) = \kappa(w)$ if $\kappa(w) \leq z$, and $\kappa'(w) = \kappa(w) + z$ if $\kappa(w) > z$. So by Theorem 16, the non-standard application of κ for $*$ satisfies the postulates for semi-AGM revisions. It remains to show that $*$ also satisfies AGM4. But it follows immediately from Definition 6 that if $\mathrm{Bel} \cap A \neq \emptyset$, then $\mathrm{Bel} * A = \mathrm{Bel} \cap A \subseteq \mathrm{Bel}$. □

Proof of Observation 21: (a) Suppose that $\langle \dot{-}, \mathrm{Bel} \rangle$ is normally represented by the semiorder $\prec$. We show that it is also canonically represented by the bottom semiorder $\prec'$ defined as follows:

$$w \prec' v \;\text{ iff }\; \text{(i) } w \prec v \text{ or (ii) } w \in \min_\prec W \text{ and } v \notin \min_\prec W. \tag{27}$$

First we show $\min_\prec W = \min_{\prec'} W$: Let $v \notin \min_\prec W$. Thus there is w such that $w \prec v$. But then $w \prec' v$, hence $v \notin \min_{\prec'} W$. Conversely, let $v \notin \min_{\prec'} W$. Thus there is w such that $w \prec' v$. Either (i) or (ii). When (i), we have $w \prec v$ and hence $v \notin \min_\prec W$. When (ii), we also have $v \notin \min_\prec W$. Thus $\min_\prec W = \min_{\prec'} W$, which we denote Bel.

Second, we show that $\prec'$ is a bottom semiorder.

Irreflexivity: $w \not\prec' w$: If we had $w \prec' w$, we would have (i) or (ii) for $v = w$. (i) is excluded by reflexivity, (ii) is excluded since $v = w$.

Interval condition: if $x \prec' y$, $w \prec' v$, then $x \prec' v$ or $w \prec' y$. Suppose $x \prec y$ and $w \prec v$, then $x \prec v$ or $w \prec y$ by the Interval condition. Suppose $x \prec y$ and $w \in \mathrm{Bel}$, $v \notin \mathrm{Bel}$. Thus $y \notin \mathrm{Bel}$ and therefore $w \prec' y$. Suppose $x \in \mathrm{Bel}, y \notin \mathrm{Bel}$ and $w \prec v$. Then $v \notin \mathrm{Bel}$ hence $x \prec' v$. Suppose $x \in \mathrm{Bel}, y \notin \mathrm{Bel}$, and $w \in \mathrm{Bel}, v \notin \mathrm{Bel}$. Then $x \prec' v$.

Semitransitivity: If $x \prec' y$, $y \prec' z$, then $x \prec' w$ or $w \prec' u$. There are four cases. Suppose first that $x \prec y$ and $y \prec z$: then $x \prec w$ or $w \prec z$ by Semitransitivity. Hence $x \prec' w$ or $w \prec' z$. The second case, $x \prec y$ and $y \in \mathrm{Bel}$, $z \notin \mathrm{Bel}$, is impossible. Suppose third that $x \in \mathrm{Bel}, y \notin \mathrm{Bel}$ and $y \prec z$. Either $w \in \mathrm{Bel}$, then $w \prec' y$. But $y \prec' z$, hence $w \prec' z$, by transitivity (wich follows from irreflexivity and the Interval condition). Or $w \notin \mathrm{Bel}$. Then $x \prec' w$. The fourth case, $x \in \mathrm{Bel}, y \notin \mathrm{Bel}$, and $y \in \mathrm{Bel}, z \notin \mathrm{Bel}$, is impossible.

Bottom: Let $v \in \min_{\prec'} W$, $w \notin \min_{\prec'} W$. Since $\min_{\prec'} W = \min_{\prec} W$, $v \prec' w$ by (ii).

For the identity of the two ways of obtaining $\mathrm{Bel}(\Psi \dot{-} A)$, it remains to show that (1) $\min_{\prec'} \overline{A} \subseteq \min_{\prec} W$ when $\mathrm{Bel} \nsubseteq A$ and (2) $\min_{\prec'} \overline{A} = \min_{\prec} \overline{A}$ otherwise.

(1) Let $\mathrm{Bel} \nsubseteq A$. Then there is a $w \in \mathrm{Bel}$, such that $w \in \overline{A}$. We already showed that $\mathrm{Bel} = \min_{\prec} W = \min_{\prec'} W$. Suppose for reductio that $v \in \min_{\prec'} \overline{A}$, but $v \notin \min_{\prec} W = \min_{\prec'} W$. Since $w \in \mathrm{Bel} = \min_{\prec'} W$ and $\prec'$ is bottom, we get $w \prec' v$, contradicting $v \in \min_{\prec'} \overline{A}$.

(2) Let $\mathrm{Bel} \subseteq A$. Suppose $v \notin \min_{\prec} \overline{A}$, thus either $v \in A$ and then $v \notin \min_{\prec'} \overline{A}$, or $v \in \overline{A}$ and there is $w \in \overline{A}$ such that $w \prec v$. But then $w \prec' v$. Hence $v \notin \min_{\prec'} \overline{A}$. Conversely, suppose $v \notin \min_{\prec'} \overline{A}$. Thus either $v \in A$ and then $v \notin \min_{\prec} \overline{A}$, or $v \in \overline{A}$ and there is $w \in \overline{A}$ such that $w \prec' v$. But then $w \prec v$ or $w \in \mathrm{Bel}$ and $v \notin \mathrm{Bel}$. The second case is excluded, since then $w \in A$. In the first case $v \notin \min_{\prec} \overline{A}$.

(b) Suppose that $\langle \dot{-}, \mathrm{Bel} \rangle$ is canonically represented by the bottom semiorder $\prec$. We want to show that it is also normally represented by $\prec$. For this we have to verify that if $\mathrm{Bel} \nsubseteq A$, i.e, if there is $w \in \mathrm{Bel}$ such that $w \in \overline{A}$, then $\min_{\prec} \overline{A} \subseteq \mathrm{Bel}$. But if this were not the case, there would be a $v \in \min_{\prec} \overline{A}$ such that $v \notin \mathrm{Bel}$. But then, since $\mathrm{Bel} = \min_{\prec} W$, $w \prec v$ by bottom, contradicting the minimality of v in $\overline{A}$. □

Proof of Observation 23: Suppose $\mathrm{Bel} \dot{-} (A \cap B) \subseteq B$. And $\mathrm{Bel} \dot{-} A \nsubseteq \mathrm{Bel} \dot{-} (A \cap B)$. Then by AGM$\dot{-}$8d, we obtain $\mathrm{Bel} \dot{-} B \subseteq \mathrm{Bel} \dot{-} (A \cap B)$. Therefore $\mathrm{Bel} \dot{-} B \subseteq B$. But if $B \neq W$, this violates AGM$\dot{-}$2. And if

$B = W$, the consequent of AGM∸8c is trivially true. □

Proof of Theorem 24: We show that (a) implies (b), (b) implies (d), (d) implies (c), and (c) implies (a).

(a) implies (b). (⇒). Assume that $\langle \dot{-}, \mathrm{Bel} \rangle$ is a normal semi-AGM contraction. We construct $\prec$ by

$$v \prec w \quad \text{iff} \quad v \in \mathrm{Bel} \dot{-} \overline{\{v,w\}} \text{ and } w \notin \mathrm{Bel} \dot{-} \overline{\{v,w\}} \tag{28}$$

We now show (Part 1) that $\prec$ is a bottom semiorder and (Part 2) that the belief contraction $\mathrm{Bel} \dot{-} A$ can be represented by it as in Equations (20) and (21).

Part 1: $\prec$ is a bottom semiorder.

Irreflexivity: $w \prec w$ is impossible, by (28).

Interval condition. Suppose that $u \prec v$ and $x \prec y$. By (28) this means that $u \in, v \notin \mathrm{Bel} \dot{-} \overline{\{u,v\}}$ and $x \in, y \notin \mathrm{Bel} \dot{-} \overline{\{x,y\}}$. We need to show that $u \prec y$ or $x \prec v$, i.e., that $u \in, y \notin \mathrm{Bel} \dot{-} \overline{\{u,y\}}$ or $x \in, v \notin \mathrm{Bel} \dot{-} \overline{\{x,v\}}$. Consider the proposition $A = \overline{\{u,v,x,y\}}$. By AGM∸7, we get $\mathrm{Bel} \dot{-} A = \mathrm{Bel} \dot{-}((A \cup \{u,v\}) \cap (A \cup \overline{\{u,v\}})) \subseteq \mathrm{Bel} \dot{-}(A \cup \{u,v\}) \cup \mathrm{Bel} \dot{-}(A \cup \overline{\{u,v\}})$. Thus, by set-theoretic reasoning, $(\mathrm{Bel} \dot{-} A) \cap \{u,v\} \subseteq (\mathrm{Bel} \dot{-}(A \cup \{u,v\}) \cap (A \cup \{u,v\})) \cup \mathrm{Bel} \dot{-} \overline{\{u,v\}} \subseteq$ (by AGM∸5) $\mathrm{Bel} \cup \mathrm{Bel} \dot{-} \overline{\{u,v\}} =$ (by AGM∸3) $\mathrm{Bel} \dot{-} \overline{\{u,v\}}$. Since $v \notin \mathrm{Bel} \dot{-} \overline{\{u,y\}}$, we get $v \notin \mathrm{Bel} \dot{-} A$. In the same way, we obtain $y \notin \mathrm{Bel} \dot{-} A$. Now AGM8∸d implies that $\mathrm{Bel} \dot{-} \overline{\{u,y\}} \subseteq \mathrm{Bel} \dot{-} A$ or $\mathrm{Bel} \dot{-} \overline{\{v,x\}} \subseteq \mathrm{Bel} \dot{-} A$. If $\mathrm{Bel} \dot{-} \overline{\{u,y\}} \subseteq \mathrm{Bel} \dot{-} A$, then $y \notin \mathrm{Bel} \dot{-} \overline{\{u,y\}}$. By $\overline{\{u,y\}} \neq W$ and AGM∸2, $\mathrm{Bel} \dot{-} \overline{\{u,y\}} \not\subseteq \overline{\{u,y\}}$. So $u \in \mathrm{Bel} \dot{-} \overline{\{u,y\}}$. Together this means $u \prec y$, as desired. If $\mathrm{Bel} \dot{-} \overline{\{v,x\}} \subseteq \mathrm{Bel} \dot{-} A$, an analogous argument establishes $x \prec v$.

Semitransitivity. Suppose that $u \prec v$ and $v \prec w$. By (28) this means that $u \in, v \notin \mathrm{Bel} \dot{-} \overline{\{u,v\}}$ and $v \in, w \notin \mathrm{Bel} \dot{-} \overline{\{v,w\}}$. So $\mathrm{Bel} \dot{-} \overline{\{u,v\}} \subseteq \overline{\{v\}}$ and $\mathrm{Bel} \dot{-} \overline{\{v,w\}} \subseteq \overline{\{w\}}$. Thus by AGM∸8s, $\mathrm{Bel} \dot{-} \overline{\{u,x\}} \subseteq \overline{\{x\}}$ or $\mathrm{Bel} \dot{-} \overline{\{x,w\}} \subseteq \overline{\{w\}}$. If $\mathrm{Bel} \dot{-} \overline{\{u,x\}} \subseteq \overline{\{x\}}$, then $x \notin \mathrm{Bel} \dot{-} \overline{\{u,x\}}$. By $\overline{\{u,x\}} \neq W$ and AGM∸2, $\mathrm{Bel} \dot{-} \overline{\{u,x\}} \not\subseteq \overline{\{u,x\}}$. So $u \in \mathrm{Bel} \dot{-} \overline{\{u,x\}}$. Together this means $u \prec x$, as desired. If $\mathrm{Bel} \dot{-} \overline{\{x,w\}} \subseteq \overline{w}$, an analogous argument establishes $x \prec w$.

Bottom. We first show that $\min_{\prec} W = \mathrm{Bel}$. Suppose that $v \in \min_{\prec} W$. Then there is no $u \in W$ such that $u \in, v \notin \mathrm{Bel} \dot{-} \overline{\{u,v\}}$, or equivelently, for all $u \in W$ such that $u \in \mathrm{Bel} \dot{-} \overline{\{u,v\}}$, we have $v \in \mathrm{Bel} \dot{-} \overline{\{u,v\}}$, too. But also, by $\overline{\{u,v\}} \neq W$ and AGM∸2, we have $\mathrm{Bel} \dot{-} \overline{\{u,v\}} \not\subseteq \overline{\{u,v\}}$, so for all $u \in W$ such that $u \notin \mathrm{Bel} \dot{-} \overline{\{u,v\}}$, we have $v \in \mathrm{Bel} \dot{-} \overline{\{u,v\}}$. Taken together, this implies that for all $u \in W$, $v \in \mathrm{Bel} \dot{-} \overline{\{u,v\}}$. So by AGM∸8d, $v \in \mathrm{Bel} \dot{-} \bigcap_{u \in W} \overline{\{u,v\}} = \mathrm{Bel} \dot{-} \emptyset = \mathrm{Bel}$, by AGM∸0. For the converse suppose that $v \in \mathrm{Bel}$. Then by AGM$\dot{-}$3, $v \in \mathrm{Bel} \dot{-} \overline{\{u, v\}}$ and hence $u \not\prec v$. for all $u \in W$. This means that $v \in \min_\prec W$.

So in order to show that $\prec$ is bottom we have to show that if $v \in, w \notin \mathrm{Bel}$, we get that $v \prec w$, i.e., $v \in, w \notin \mathrm{Bel} \dot{-} \overline{\{v, w\}}$. But by AGM$\dot{-}$3, $\mathrm{Bel} \subseteq \mathrm{Bel} \dot{-} \overline{\{v, w\}}$, and since $v \in \mathrm{Bel} \nsubseteq \overline{\{v, w\}}$, we get $\mathrm{Bel} \dot{-} \overline{\{v, w\}} \subseteq \mathrm{Bel}$ from AGM$\dot{-}$4.

Part 2: We show that $\prec$ canonically and thus normally, by Observation 21 (a) (since $\prec$ is bottom), represents $\langle \dot{-}, \mathrm{Bel} \rangle$.

Equation (21): $\mathrm{Bel} \dot{-} A = \mathrm{Bel} \cup \min \overline{A}$.

Suppose that the contraction $\ddot{-}$ is obtained from $\prec$ by Definition 20. Note that if $A = W$ or $\mathrm{Bel} \nsubseteq A$, then $\mathrm{Bel} \dot{-} A = \mathrm{Bel}$, by AGM$\dot{-}$3 to AGM$\dot{-}$5, and similarly $\mathrm{Bel} \ddot{-} A = \mathrm{Bel}$ by equation (22). If $\mathrm{Bel} \subseteq A \neq W$, then we have, by equations (22) and (28)

$$\begin{aligned}
w \in \mathrm{Bel} \ddot{-} A \;\text{ iff }\; & w \in \mathrm{Bel} \cup \min\nolimits_\prec \overline{A} \\
\text{iff }\; & w \in \mathrm{Bel} \text{ or } (w \in \overline{A} \text{ and } \neg\exists v \in \overline{A}(v \prec w)) \\
\text{iff }\; & w \in \mathrm{Bel} \text{ or } (w \in \overline{A} \text{ and } \neg\exists v \in \overline{A}(v \in \mathrm{Bel} \dot{-} \overline{\{v, w\}} \text{ and} \\
& \qquad\qquad\qquad\qquad\qquad\qquad w \notin \mathrm{Bel} \dot{-} \overline{\{v, w\}}))
\end{aligned}$$

Using the expression in the last line, we prove the following identity claim: for every proposition A such that $\mathrm{Bel} \subseteq A \neq W$ and every world w, $w \in \mathrm{Bel} \dot{-} A$ iff $w \in \mathrm{Bel} \ddot{-} A$.

$w \in \mathrm{Bel} \dot{-} A$ implies the last line: Let $w \in \mathrm{Bel} \dot{-} A$, and suppose that $w \notin \mathrm{Bel}$. Then $w \in \overline{A}$, by AGM$\dot{-}$5. Suppose for reductio that there is a $v \in \overline{A}$ such that $v \in, w \notin \mathrm{Bel} \dot{-} \overline{\{v, w\}}$. From $w \in \mathrm{Bel} \dot{-} A$, AGM$\dot{-}$7 gives us that either $w \in \mathrm{Bel} \dot{-} (A \cup \{v, w\})$ or $w \in \mathrm{Bel} \dot{-} (A \cup \overline{\{v, w\}}) = \mathrm{Bel} \dot{-} \overline{\{v, w\}}$. But the latter is false by supposition. So $w \in \mathrm{Bel} \dot{-} (A \cup \{v, w\})$. But also $w \in A \cup \{v, w\}$. Thus, by AGM$\dot{-}$5, $w \in \mathrm{Bel}$, which gives us a contradiction.

The last line implies $w \in \mathrm{Bel} \dot{-} A$: Assume the last line, i.e., $w \in \mathrm{Bel}$, or $w \in \overline{A}$ and there is no $v \in \overline{A}$ such that $v \in, w \notin \mathrm{Bel} \dot{-} \overline{\{v, w\}}$. If $w \in \mathrm{Bel}$, then $w \in \mathrm{Bel} \dot{-} A$ by AGM$\dot{-}$3. So assume that $w \in \overline{A}$ and for every $v \in \overline{A}$ such that $v \in \mathrm{Bel} \dot{-} \overline{\{v, w\}}$, it also holds that $w \in \mathrm{Bel} \dot{-} \overline{\{v, w\}}$. Since $\overline{\{v, w\}} \neq W$, AGM$\dot{-}$2 gives us $(\mathrm{Bel} \dot{-} \overline{\{v, w\}}) \cap \{v, w\} \neq \emptyset$. Hence $w \in \mathrm{Bel} \dot{-} \overline{\{v, w\}}$ for every $v \in \overline{A}$. Since $\overline{A} \subseteq W$ is finite and $A = \bigcap_{v \in \overline{A}} \overline{\{v, w\}}$, we can apply AGM8d repeatedly and get by AGM$\dot{-}$8d that $w \in \mathrm{Bel} \dot{-} (\bigcap_{v \in \overline{A}} \overline{\{v, w\}}) = \mathrm{Bel} \dot{-} A$, as desired.

It remains to prove condition (20). From our identity claim, we know that $\mathrm{Bel} \dot{-} \emptyset = \mathrm{Bel} \ddot{-} \emptyset = \mathrm{Bel} \cup \min_\prec W$, and from AGM$\dot{-}$0, we also have $\mathrm{Bel} \dot{-} \emptyset = \mathrm{Bel}$. So $\min_\prec W \subseteq \mathrm{Bel}$, We prove the converse, $\mathrm{Bel} \subseteq \min_\prec W$, by contraposition. Let $w \notin \min_\prec W$. Then there is a $v \in W$ such that $v \prec w$, i.e., $v \in, w \notin \mathrm{Bel} \dot{-} \overline{\{v, w\}}$. By AGM$\dot{-}$3, it follows from $w \notin \mathrm{Bel} \dot{-} \overline{\{v, w\}}$ that $w \notin \mathrm{Bel}$, as desired.

(b) implies (d). *Normal semi-representability implies ranking representability by a maximal contraction.* Let $\langle \dot{-}, \mathrm{Bel} \rangle$ be given. Suppose that $\prec$ can be used for the normal representation with (20) $\mathrm{Bel} = \min_{\prec} W$ and (22) $\mathrm{Bel} \dot{-} A = \mathrm{Bel}$ if $\mathrm{Bel} \not\subseteq A$, and $\mathrm{Bel} \dot{-} A = \mathrm{Bel} \cup \min_{\prec} \overline{A}$ otherwise. From the semiorder $\prec$, we construct a ranking κ and a threshold $z \geq 0$, using Corollary 10, such that $\kappa(v) + z < \kappa(w)$ iff $v \prec w$. But then $\min_{\prec} W = \min_{\kappa}^{z} W$ and $\min_{\prec} \overline{A} = \min_{\kappa}^{z} \overline{A}$, and so we get

$$\begin{aligned}
\mathrm{Bel}(\kappa \dot{-} A) &= \mathrm{Bel}_z(\overline{\kappa}_A^0) \\
&= \begin{cases} \min_{\kappa}^{z} W & \text{if Bel} \not\subseteq A, \\ \min_{\kappa}^{z} W \cup \min_{\kappa}^{z} \overline{A} & \text{if Bel} \subseteq A \end{cases} \\
&= \begin{cases} \min_{\prec}(W) & \text{if Bel} \not\subseteq A, \\ \min_{\prec}(W) \cup \min_{\prec} \overline{A} & \text{if Bel} \subseteq A \end{cases} \\
&= \mathrm{Bel} \dot{-} A.
\end{aligned}$$

Hence $\langle \dot{-}, \mathrm{Bel} \rangle$ is ranking representable by a maximal contraction: notice that here we have chosen the contraction parameter $n = 0$.

Note: In contrast to the revision case, the contraction parameter $n = 0$ is forced here. If the semiorder $\prec$ is bottom, the recipes for normal and canonical representation coincide. If $\prec$ is bottom, the constructed ranking κ is z-gappy in the sense of Corollary 10.

(d) implies (c). *Ranking representability by a maximal contraction implies ranking representability.* This is trivial.

(c) implies (a). *Ranking representability implies normal semi-AGM.* Let $\dot{-}$ be defined from κ and z by $\mathrm{Bel} = \mathrm{Bel}_z(\kappa) = \min_{\kappa}^{z} W$ and $\mathrm{Bel} \dot{-} A = \mathrm{Bel}_z(\overline{\kappa}_A^n)$ for some $n \leq z$. The latter is equal to Bel if $\mathrm{Bel} \not\subseteq A$, and to $\mathrm{Bel} \cup \min_{\kappa}^{z-n} \overline{A}$ if $\mathrm{Bel} \subseteq A$, by Definition 5 and equation (9). Since $\kappa^{-1}(0) \neq \emptyset$ (for $\kappa \neq \kappa_{\perp}$), we have $\mathrm{Bel} \neq \emptyset$.

AGM$\dot{-}$0. $\mathrm{Bel} \dot{-} \emptyset = \mathrm{Bel}$ by definition, since $\mathrm{Bel} \not\subseteq \emptyset$.

AGM$\dot{-}$2. Let $A \neq W$. If $\mathrm{Bel} \not\subseteq A$, then $\mathrm{Bel} \dot{-} A = \mathrm{Bel} \not\subseteq A$. If $\mathrm{Bel} \subseteq A$, then $\min_{\kappa}^{z-n} \overline{A} \subseteq \mathrm{Bel} \dot{-} A$ and $\emptyset \neq \min_{\kappa}^{z-n} \overline{A} \subseteq \overline{A} \not\subseteq A$.

AGM$\dot{-}$3. $\mathrm{Bel} \subseteq \mathrm{Bel} \dot{-} A$ is immediate from the definition.

AGM$\dot{-}$4. Let $\mathrm{Bel} \not\subseteq A$. Then $\mathrm{Bel} \dot{-} A \subseteq \mathrm{Bel}$ is also immediate from the definition. (Note that AGM$\dot{-}$4 is the postulate that makes a semi-AGM contraction normal.)

AGM$\dot{-}$5. $(\mathrm{Bel} \dot{-} A) \cap A \subseteq (\mathrm{Bel} \cup \min_{\kappa}^{z-n} \overline{A}) \cap A \subseteq \mathrm{Bel}$.

AGM$\dot{-}$7. If $\mathrm{Bel} \not\subseteq A$, then $\mathrm{Bel} \dot{-}(A \cap B) = \mathrm{Bel} = \mathrm{Bel} \dot{-} A$, and the claim is immediate. Similarly if $\mathrm{Bel} \not\subseteq B$. So let $\mathrm{Bel} \subseteq A \cap B$. Then $\mathrm{Bel} \dot{-}(A \cap B) \subseteq (\mathrm{Bel} \dot{-} A) \cup (\mathrm{Bel} \dot{-} B)$ follows from the fact that $\min_{\kappa}^{z-n}(\overline{A \cap B}) \subseteq$

$\min_\kappa^{z-n} \overline{A} \cup \min_\kappa^{z-n} \overline{B}$.

AGM$\dot{-}$8d. The cases Bel $\not\subseteq A$ and Bel $\not\subseteq B$ are the same as for AGM$\dot{-}$7. So let Bel $\subseteq A \cap B$. Then Bel $\dot{-}\, A \subseteq$ Bel $\dot{-}(A \cap B)$ or Bel $\dot{-}\, B \subseteq$ Bel $\dot{-}(A \cap B)$ follows from the facts that $\min_\kappa^{z-n} \overline{A} \subseteq \min_\kappa^{z-n}(\overline{A \cap B})$ if $\kappa(\overline{A}) \leq \kappa(\overline{B})$, and $\min_\kappa^{z-n} \overline{B} \subseteq \min_\kappa^{z-n}(\overline{A \cap B})$ if $\kappa(\overline{B}) \leq \kappa(\overline{A})$.

AGM$\dot{-}$8s. Let Bel $\dot{-}(A \cap B) \subseteq B$ and Bel $\dot{-}(B \cap C) \subseteq C$. Note that this implies that Bel $\subseteq B$ and Bel $\subseteq C$, and thus Bel $\dot{-}(B \cap C) \subseteq C$ implies that $\min_\kappa^{z-n}(\overline{B \cap C}) \subseteq C$, i.e., $\kappa(\overline{B}) + (z-n) < \kappa(\overline{C})$. Bel $\dot{-}(A \cap B) \subseteq B$ implies that either Bel $\not\subseteq A$ or $\kappa(\overline{A}) + (z-n) < \kappa(\overline{B})$. Suppose first that Bel $\not\subseteq A$. Then if Bel $\dot{-}(A \cap D) \not\subseteq D$, this means that Bel $\not\subseteq D$ and thus Bel $\dot{-}(D \cap C) =$ Bel $\subseteq C$. Suppose second that Bel $\subseteq A$ and $\kappa(\overline{A}) + (z-n) < \kappa(\overline{B})$. It follows that $\kappa(\overline{A}) + 2(z-n) < \kappa(\overline{C})$. But then either $\kappa(\overline{A}) + (z-n) < \kappa(\overline{D})$ or $\kappa(\overline{D}) + (z-n) < \kappa(\overline{C})$, and thus either Bel $\dot{-}(A \cap D) \subseteq D$ or Bel $\dot{-}(D \cap C) \subseteq C$. □

Proof of Corollary 26: We show that (a) implies (b), (b) implies (c) and (c) implies (a).

(a) implies (b). Suppose that $\langle \dot{-}, \mathrm{Bel} \rangle$ is semi-AGM. We can use the construction (28) of the proof of Theorem 24, and know that the relation $\prec$ thus defined is a semiorder (AGM$\dot{-}$4 was only used for the bottom condition). We have also shown in this proof (without the help of AGM$\dot{-}$4) that $\prec$ canonically represents $\dot{-}$.

(b) implies (d). *Canonical semi-representability implies non-standard ranking representability by a maximal contraction.* Let $\langle \dot{-}, \mathrm{Bel} \rangle$ be given. Suppose that $\prec$ can be used for the canonical representation with (20) Bel $= \min_\prec W$ and (21) Bel $\dot{-}\, A =$ Bel $\cup \min_\prec \overline{A}$. From the semiorder $\prec$, we construct a ranking κ and a threshold $z \geq 0$, using Corollary 10, such that $\kappa(v) + z < \kappa(w)$ iff $v \prec w$. But then $\min_\prec W = \min_\kappa^z W$ and $\min_\prec \overline{A} = \min_\kappa^z \overline{A}$. If we choose the non-standard ranking contraction with the contraction parameter $n = 0$, we get

$$\begin{aligned} \mathrm{Bel}(\kappa \dot{-} A) &= \mathrm{Bel}_\kappa^z W \cup \min\nolimits_\kappa^{z-0} \overline{A} \\ &= \min\nolimits_\prec(W) \cup \min\nolimits_\prec \overline{A} \\ &= \mathrm{Bel} \dot{-} A. \end{aligned}$$

Hence $\langle \dot{-}, \mathrm{Bel} \rangle$ is non-standardly ranking representable by a maximal contraction: notice that the contraction parameter is $n = 0$ here.

(d) implies (c). *Non-standard ranking representability by a maximal contraction implies non-standard ranking representability.* This is trivial.

(c) implies (a). *Non-standard ranking representability implies semi-AGM.* Let $\dot{-}$ be non-standardly defined from κ and z, i.e., Bel $= \mathrm{Bel}_z(\kappa) = \min_\kappa^z W$

and $\mathrm{Bel} \dot{-} A = \mathrm{Bel} \cup \min_\kappa^{z-n} \overline{A}$ for some $n \leq z$ (Definition 7, equation (15)). Since $\kappa^{-1}(0) \neq \emptyset$ (for $\kappa \neq \kappa_\perp$), we have $\mathrm{Bel} \neq \emptyset$.

AGM$\dot{-}$0. $\mathrm{Bel} \dot{-} \emptyset = \mathrm{Bel}$ since $\min_\kappa^{z-n} \overline{\emptyset} \subseteq \mathrm{Bel}$.

AGM$\dot{-}$2. Let $A \neq W$. Then $\emptyset \neq \min_\kappa^{z-n} \overline{A} \subseteq \overline{A} \nsubseteq A$ and, since $\min_\kappa^{z-n} \overline{A} \subseteq \mathrm{Bel} \dot{-} A$, $\mathrm{Bel} \dot{-} A \nsubseteq A$.

AGM$\dot{-}$3. $\mathrm{Bel} \subseteq \mathrm{Bel} \dot{-} A$ is trivial.

AGM$\dot{-}$5. $(\mathrm{Bel} \dot{-} A) \cap A \subseteq (\mathrm{Bel} \cup \min_\kappa^{z-n} \overline{A}) \cap A \subseteq \mathrm{Bel}$.

AGM$\dot{-}$7. $\mathrm{Bel} \dot{-} (A \cap B) \subseteq (\mathrm{Bel} \dot{-} A) \cup (\mathrm{Bel} \dot{-} B)$ follows from the fact that $\min_\kappa^{z-n}(\overline{A \cap B}) \subseteq \min_\kappa^{z-n} \overline{A} \cup \min_\kappa^{z-n} \overline{B}$.

AGM$\dot{-}$8d. $\mathrm{Bel} \dot{-} A \subseteq \mathrm{Bel} \dot{-} (A \cap B)$ or $\mathrm{Bel} \dot{-} B \subseteq \mathrm{Bel} \dot{-} (A \cap B)$ follows from the facts that $\min_\kappa^{z-n} \overline{A} \subseteq \min_\kappa^{z-n}(\overline{A \cap B})$ if $\kappa(\overline{A}) \leq \kappa(\overline{B})$, and $\min_\kappa^{z-n} \overline{B} \subseteq \min_\kappa^{z-n}(\overline{A \cap B})$ if $\kappa(\overline{B}) \leq \kappa(\overline{A})$.

AGM$\dot{-}$8s. Let $\mathrm{Bel} \dot{-} (A \cap B) \subseteq B$ and $\mathrm{Bel} \dot{-} (B \cap C) \subseteq C$. This implies that $\mathrm{Bel} \subseteq B$ and $\mathrm{Bel} \subseteq C$, i.e., $\kappa(\overline{B}) > z$ and $\kappa(\overline{C}) > z$. It also s that $\min_\kappa^{z-n}(\overline{A \cap B}) \subseteq B$ and $\min_\kappa^{z-n}(\overline{B \cap C}) \subseteq C$, i.e., that $\kappa(\overline{A}) + (z-n) < \kappa(\overline{B})$ and $\kappa(\overline{B}) + (z-n) < \kappa(\overline{C})$. It follows that $\kappa(\overline{A}) + 2(z-n) < \kappa(\overline{C})$. But then either $\kappa(\overline{A}) + (z-n) < \kappa(\overline{D})$ or $\kappa(\overline{D}) + (z-n) < \kappa(\overline{C})$, and thus either $\min_\kappa^{z-n}(\overline{A \cap D}) \subseteq D$ or $\min_\kappa^{z-n}(\overline{D \cap C}) \subseteq C$. If the latter, we have $\mathrm{Bel} \dot{-} (D \cap C) \subseteq C$. If not the latter, we have $\kappa(\overline{D}) + (z-n) \geq \kappa(\overline{C}) > \kappa(\overline{B}) + (z-n) > 2z-n$, so $\kappa(\overline{D}) > z$. Thus $\mathrm{Bel} \subseteq D$, and we get $\mathrm{Bel} \dot{-} (A \cap D) \subseteq D$. □

Proof of Observation 27: Let $\prec$ be a semiorder.

(a) Let $\mathrm{Bel} * A$ and $\mathrm{Bel} \dot{-} A$ be *canonically* defined from $\prec$ (Definitions 11 and 19).

LI. We have

$$\mathrm{Bel} \cap A = (\min W) \cap A \subseteq \min A. \tag{+}$$

Thus we obtain $\mathrm{Bel} * A \overset{\text{Def. 11}}{=} \min A \overset{(+)}{=} (\mathrm{Bel} \cap A) \cup \min A = (\mathrm{Bel} \cup \min A) \cap A \overset{\text{Def. 19}}{=} (\mathrm{Bel} \dot{-} \overline{A}) \cap A$.

HI. $\mathrm{Bel} \dot{-} A \overset{\text{Def. 19}}{=} \mathrm{Bel} \cup \min \overline{A} \overset{\text{Def. 12}}{=} \mathrm{Bel} \cup (\mathrm{Bel} * \overline{A})$.

(b) Now let $\mathrm{Bel} * A$ and $\mathrm{Bel} \dot{-} A$ be *normally* defined from $\prec$ (Definitions 12 and 20).

LI. We obtain the following equations, where the upper line applies when $\mathrm{Bel} \nsubseteq \overline{A}$, the lower line when $\mathrm{Bel} \subseteq \overline{A}$:

$$\mathrm{Bel} * A \overset{\text{Def. 12}}{=} \left\{ \begin{array}{lcl} \mathrm{Bel} \cap A & & \\ \min A & \overset{(+)}{=} & (\mathrm{Bel} \cup \min A) \cap A \end{array} \right\} \overset{\text{Def. 20}}{=} (\mathrm{Bel} \dot{-} \overline{A}) \cap A$$

HI. We obtain the following equations, where the upper line applies when

Bel $\not\subseteq A$, the lower line when Bel $\subseteq A$:

$$\mathrm{Bel} \dot{-} A \overset{\text{Def. 20}}{=} \left\{ \begin{array}{ll} \mathrm{Bel} & = \quad \mathrm{Bel} \cup (\mathrm{Bel} \cap \overline{A}) \\ \mathrm{Bel} \cup \min \overline{A} & \end{array} \right\} \overset{\text{Def. 12}}{=} \mathrm{Bel} \cup (\mathrm{Bel} * \overline{A})$$

□

Proof of Observation 28: RLI. We get

$$\mathrm{Bel} * A = \begin{cases} \min A & \text{if Bel} \not\subseteq \overline{A}, \\ \min A \overset{(+)}{=} (\mathrm{Bel} \cup \min A) \cap A = (\mathrm{Bel} \dot{-} \overline{A}) \cap A & \text{otherwise.} \end{cases}$$

RHI. We get

$$\mathrm{Bel} \dot{-} A = \begin{cases} \mathrm{Bel} & \text{if Bel} \not\subseteq A, \\ \mathrm{Bel} \cup \min \overline{A} = \mathrm{Bel} \cup (\mathrm{Bel} * \overline{A}) & \text{otherwise.} \end{cases}$$

Counterexample against RCI: Let $W = \{w_0, w_1, w_2\}$ and suppose that only w_0 and w_2 are related by $\prec$: $w_0 \prec w_2$. The relation $\prec$ is a semiorder over W. Let $A = \{w_1, w_2\}$. Then $\mathrm{Bel}(\kappa * A) = \{w_1, w_2\}$ (canonical representation), but $\mathrm{Bel}(\kappa \dot{-} \overline{A}) = \mathrm{Bel}(\kappa) = \{w_0, w_1\}$, since $\mathrm{Bel}(\kappa) \not\subseteq \overline{A}$ (normal representation). Hence $\mathrm{Bel}(\kappa * A) \not\subseteq \mathrm{Bel}(\kappa \dot{-} \overline{A})$, and RCI fails. Since each of LI and HI implies RCI, they must fail as well. □

Proof of Theorem 29: (a) Counterexample against RCI with $z > 0$: Let $z = 1$ and $W = \{w_0, w_1, w_2\}$, and suppose that the κ-ranks of the worlds are given by their indices. Let $A = \{w_1, w_2\}$. Writing $\kappa * A$ for κ_A^n with $n > 1$ and $\kappa \dot{-} \overline{A}$ for $\overline{\kappa}_{\overline{A}}^m$ with $m \leq 1$, we get $\mathrm{Bel}(\kappa * A) = \{w_1, w_2\}$, but $\mathrm{Bel}(\kappa \dot{-} \overline{A}) = \mathrm{Bel}(\kappa) = \{w_0, w_1\}$, since $\mathrm{Bel}(\kappa) \not\subseteq \overline{A}$. Hence $\mathrm{Bel}(\kappa * A) \not\subseteq \mathrm{Bel}(\kappa \dot{-} \overline{A})$, and RCI fails. Since each of LI and HI implies RCI, they must fail as well.

(b) If $z = 0$, ranking revision and contraction are fully AGM, and hence the normal and canonical representation are equivalent. □

Proof of Theorem 31: It is easily seen that IR0 is equivalent to IR0$\prec$. IR1–IR4 are analogous to the cases proven by Darwiche and Pearl (1997, pp. 24–25) for the total preorder case.

IR5 is equivalent to IR5$\prec$.

($\Leftarrow$) Suppose IR5$\prec$ holds. Assume $\mathrm{Bel}(\Psi) \subseteq \overline{A}$ and $\mathrm{Bel}(\Psi * B) \not\subseteq \overline{A}$. Suppose for reductio that $\mathrm{Bel}((\Psi * A) * B) \not\subseteq A$. Thus $\min_{\prec_A} B \not\subseteq A$. Hence there is an $\overline{A}$-world $v \in \min_{\prec_A} B$, i.e., such that for no $b \in B$, $b \prec_A v$. But since $\mathrm{Bel}(\Psi) \subseteq \overline{A}$ and $\mathrm{Bel}(\Psi * B) \not\subseteq \overline{A}$, we have $\min_{\prec} B \not\subseteq \overline{A}$ and thus there is an A-world $w \in \min_{\prec} B$, i.e., such that for no $b \in B$, $b \prec w$. Thus we must have $v \not\prec w$. Since $\mathrm{Bel}(\Psi) \subseteq \overline{A}$, $w \in A$ and $v \in \overline{A}$, $v \not\prec w$ implies $w \prec_A v$ by IR5$\prec$. This contradicts $v \in \min_{\prec_A} B$.

($\Rightarrow$) Suppose IR5 holds. Assume $\mathrm{Bel}(\Psi) \subseteq \overline{A}$, $w \in A$ and $v \in \overline{A}$. Now suppose that $w \not\prec_A v$. Consider $B = \{w, v\}$. Then, using AGM2 and AGM5, $v \in \min_{\prec_A} B$, since $w \not\prec_A v$. Thus $\mathrm{Bel}((\Psi * A) * B) \nsubseteq A$. Hence $\mathrm{Bel}(\Psi * B) \subseteq \overline{A}$ by IR5. Thus $\min_{\prec} B \subseteq \overline{A}$, and by AMG5 and AGM2, $\min_{\prec} B = \{v\}$. Since $w \neq v$, there is $b \in B$ such that $b \prec w$. By construction $b = v$, so $v \prec w$. Thus $w \not\prec v$, by the asymmetry of $\prec$.

The reasoning is similar for IR6 and IR7. □

Proof of Observation 32: Let κ be a ranking, $n > z \geq 0$ and $A \neq \emptyset$. Consider $\kappa^*_A = \kappa^n_A$. (κ∗0)–(κ∗2) hold by Definition 3.

(κ∗5): Let $\mathrm{Bel}(\kappa) \subseteq \overline{A}$. Thus $\kappa(\overline{A}) = 0$ and $\kappa(A) > z$. Therefore $\kappa^*_A = \kappa_{A \to n}$ (Definition 3). Hence for $w \in A$, $\kappa^*_A(w) + z = \kappa(w) - \kappa(A) + z < \kappa(w)$. And for $v \in \overline{A}$, $\kappa^*_A(v) = \kappa(v) - \kappa(\overline{A}) + n = \kappa(v) + n > \kappa(v) + z$ (since $z < n$).

(κ∗6). Let $\mathrm{Bel}(\kappa) \nsubseteq A$. Thus $\kappa(\overline{A}) \leq z$. Therefore $\kappa^*_A = \kappa_{A \to n}$ (by Definition 3). For $w \in A$, $\kappa^*_A(w) = \kappa(w) - \kappa(A) \leq \kappa(w)$. And for $v \in \overline{A}$, $\kappa^*_A(v) = \kappa(v) - \kappa(\overline{A}) + n \geq \kappa(v) - z + n > \kappa(v)$ (since $\kappa(\overline{A}) \leq z < n$). □

Proof of Theorem 33: (a) Let $n, m > z \geq 0$ and $A \neq \emptyset$. Define the semiorder $\prec$ by $w \prec v$ iff $\kappa(w) + z < \kappa(v)$, and the semiorder $\prec_A$ by $\kappa^n_A(w) + z < \kappa^n_A(v)$.

IR0$\prec$–IR4$\prec$ follow respectively from (κ ∗0)–(κ ∗4).

IR5$\prec$. Assume $\mathrm{Bel}(\kappa) \subseteq \overline{A}$, $w \in A, v \in \overline{A}$, and also $v \not\prec w$. But the latter implies that $\kappa(w) \leq \kappa(v) + z$. Thus $\kappa^n_A(w) + z < \kappa(w) \leq \kappa(v) + z < \kappa^n_A(v)$, by ($\kappa$ ∗5) applied twice.

IR6$\prec$. Suppose $z = 0$. Assume $\mathrm{Bel}(\kappa) \nsubseteq A$, $w \in A$, $v \in \overline{A}$ and $v \not\prec w$. We have $\kappa^n_A(w) \leq \kappa(w) \leq \kappa(v) + 0 < \kappa^n_A(v)$ by (κ ∗6) applied twice. Since $z = 0$, this suffices to prove $w \prec_A v$.

Counterexample against IR6$\prec$ with $z > 0$: Let $z = 1$, $n = 2$ and $W = \{w_0, w_1, w_2, w_3\}$, and suppose the κ-ranks of the worlds are given by their indices. Let $A = \{w_0, w_3\}$. We have $\mathrm{Bel}_z(\kappa) = \{w_0, w_1\} \nsubseteq A$, $w_3 \in A$ and $w_2 \in \overline{A}$. Since $\kappa(w_2) + z \geq \kappa(w_3)$, we have $w_2 \not\prec w_3$. So the assumptions of IR6$\prec$ are satisfied. But $\kappa^n_A(w_2) = \kappa^n_A(w_3) = 3$, so that $w_3 \not\prec_A w_2$, contrary to the conclusion of IR6$\prec$.

Counterexample against IR7$\prec$ with $z = 0$: Let $z = 0$, $n = 1$ and $W = \{w_0, w_1, v_1\}$, and suppose the κ-ranks of the worlds are given by their indices. Let $A = \{w_0, w_1\}$. We have $w_1 \in A$ and $v_1 \in \overline{A}$ and $\kappa(w_1) = \kappa(v_1)$, so $v_1 \not\prec w_1$. So the assumptions of IR7$\prec$ are satisfied. But $\kappa^n_A = \kappa$, so that

$w_1 \not\prec_A v_1$, contrary to the conclusion of IR7$\prec$.

(b) By Theorem 31, it follows from (a) that the induced canonical revision satisfies IR0–IR5 (for all $A \neq \emptyset$). It also satisfies IR6 for $z = 0$, but it does not in general satisfy IR6 for $z > 0$, and it does not in general satisfy IR7. See the above counterexamples against IR6$\prec$ and IR7$\prec$. $\square$

Proof of Theorem 36: Abbreviations: We write $\prec$ for $\prec_\Psi$, min for $\min_\prec$, $\prec_A^-$ for $\prec_{\Psi \dot{-} A}$ and $\min_A$ for $\min_{\prec_A^-}$, etc. Assume that $\dot{-}$ is semi-AGM. Thus, by Corollary 26, there is a semiorder representing it canonically, i.e., such that $\mathrm{Bel}(\Psi) = \min W$ and $\mathrm{Bel} \dot{-} A = \mathrm{Bel} \cup \min \overline{A}$. In particular

$$\mathrm{Bel} \dot{-} B = \mathrm{Bel} \cup \min \overline{B} \quad \text{and} \quad (\mathrm{Bel} \dot{-} A) \dot{-} B = \mathrm{Bel} \cup \min \overline{A} \cup \min_A \overline{B}. \quad (29)$$

($\Leftarrow$) IC0$\prec$ implies IC0: Suppose $\mathrm{Bel} \not\subseteq A$. By IC0, $\prec_A^- = \prec$. So $\min_A W = \min W$ and $\min_A \overline{B} = \min \overline{B}$, which by (29) suffices to prove $\mathrm{Bel} \dot{-} B \subseteq (\mathrm{Bel} \dot{-} A) \dot{-} B$. For the converse, we use the fact that $\mathrm{Bel} \dot{-} A = \min_A W$.

IC1$\prec$ implies IC1: Suppose $\overline{A} \subseteq B$. Thus $\overline{B} \subseteq A$. By (29), we have $(\mathrm{Bel} \dot{-} B) \cap \overline{B} = (\mathrm{Bel} \cap \overline{B}) \cup \min \overline{B}$ and $(\mathrm{Bel} \dot{-} A \dot{-} B) \cap \overline{B} = (\mathrm{Bel} \cap \overline{B}) \cup ((\min \overline{A}) \cap \overline{B}) \cup \min_A \overline{B} = (\mathrm{Bel} \cap \overline{B}) \cup \min_A \overline{B}$. It thus suffices to prove $\min \overline{B} = \min_A \overline{B}$. This follows by IC1$\prec$, since $\overline{B} \subseteq A$.

IC2$\prec$ implies IC2: Suppose $A \subseteq B$. Thus $\overline{B} \subseteq \overline{A}$. Then, by (29), $(\mathrm{Bel} \dot{-} B) \cap \overline{B} = (\mathrm{Bel} \cap \overline{B}) \cup \min \overline{B}$ and $(\mathrm{Bel} \dot{-} A \dot{-} B) \cap \overline{B} = (\mathrm{Bel} \cap \overline{B}) \cup ((\min \overline{A}) \cap \overline{B}) \cup \min_A \overline{B}$. Thus it suffices to prove $\min \overline{B} = ((\min \overline{A}) \cap \overline{B}) \cup \min_A \overline{B}$. First suppose that $w \in \min \overline{B}$. Thus $w \in \overline{A}$, since $\overline{B} \subseteq \overline{A}$. Thus, by IC2$\prec$, $w \in \min_A \overline{B}$. For the converse, we note that $(\min \overline{A}) \cap \overline{B} \subseteq \min \overline{B}$, and apply the same reasoning.

IC3$\prec$ implies IC3: Suppose $\mathrm{Bel} \dot{-} B \subseteq \overline{A} \cup B$. Thus $\mathrm{Bel} \cup \min \overline{B} \subseteq \overline{A} \cup B$. Therefore $\mathrm{Bel} \subseteq \overline{A} \cup B$, and since $\min \overline{A} \subseteq \overline{A}$, it remains to prove $\min_A \overline{B} \subseteq \overline{A}$. Suppose for reductio that $\min_A \overline{B} \not\subseteq \overline{A}$. Thus there is $w \in \min_A \overline{B}$ such that $w \in A$. Hence $w \notin \min \overline{B}$, because $\min \overline{B} \subseteq \overline{A}$. Hence there is $v \in \overline{B}$, such that $v \prec w$, and we can chose $v \in \min \overline{B}$ (finiteness of W and transitivity of $\prec$). But since $v \in \overline{A}$, $w \in A$, IC3$\prec$, yields $v \prec_A^- w$. Yet $v \in \overline{B}$, contradicting $w \in \min_A \overline{B}$.

IC4$\prec$ implies IC4: Suppose $(\mathrm{Bel} \dot{-} A) \dot{-} B \subseteq A \cup B$. Thus, by (29), $\mathrm{Bel} \subseteq A \cup B$, $\min \overline{A} \subseteq B$ and $\min_A \overline{B} \subseteq A$. We prove $\min \overline{B} \subseteq A$. Suppose for reductio that $\min \overline{B} \not\subseteq A$. Thus there is $v \in \min \overline{B}$ such that $v \in \overline{A}$. But then $v \notin \min_A \overline{B}$, since $\min_A \overline{B} \subseteq A$. Hence there is $w \in \overline{B}$ such that $w \prec_A^- v$, and we can choose w such that $w \in \min_A \overline{B}$ (finiteness). Thus $w \in A$, $v \in \overline{A}$ and $w \prec_A^- v$. Hence by IC4$\prec$, $w \prec v$. But $w \in \overline{B}$ contradicts minimality of v in $\overline{B}$.

($\Rightarrow$) We use the same construction of the order $\prec = \prec_\Psi$ as in Theorem 24, viz., $v \prec w$ iff $v \in \mathrm{Bel} \dot{-} \overline{\{v, w\}}$ and $w \notin \mathrm{Bel} \dot{-} \overline{\{v, w\}}$ (equation (28)).

All claims IC$i \prec$ are trivial when $v = w$. So we assume $v \neq w$, and always form the set $B = \overline{\{v, w\}}$.

IC0 implies IC0$\prec$: Let Bel $\not\subseteq A$. Suppose that $v \prec w$. By (28), $v \in \mathrm{Bel}(\Psi \dot{-} B)$ but not w. Thus, by IC0, $v \in \mathrm{Bel}((\Psi \dot{-} A) \dot{-} B)$ but not w. By (28), $v \prec_A^- w$. Similarly in the other direction.

IC1 implies IC1$\prec$: Let $v, w \in A$. So $\overline{A} \subseteq B$. Suppose that $v \prec w$. By (28), $v \in \mathrm{Bel}(\Psi \dot{-} B)$ but not w. This is true, for $v, w \in \overline{B} \subseteq A$ and by IC1, just in case $v \in \mathrm{Bel}((\Psi \dot{-} A) \dot{-} B)$, but not w. Thus, by (28), $v \prec_A^- w$. Similarly in the other direction.

IC2 implies IC2$\prec$: Let $v, w \in \overline{A}$. So $A \subseteq B$. Suppose that $v \prec w$. By (28), $v \in \mathrm{Bel}(\Psi \dot{-} B)$, but not w. This is true, by $v, w \in \overline{B} \subseteq \overline{A}$ and IC2, just in case $v \in \mathrm{Bel}((\Psi \dot{-} A) \dot{-} B)$, but not w. Thus, by (28) $v \prec_A^- w$. Similarly in the other direction.

IC3 implies IC3$\prec$: Let $v \in \overline{A}$, $w \in A$. Note that $\overline{A} \cap \overline{B} = \{v\}$ and $\overline{A} \cup B = W \setminus \{w\}$. Suppose that $v \prec w$. This means, by (28), $v \in \mathrm{Bel}(\Psi \dot{-} B)$ and $w \notin \mathrm{Bel}(\Psi \dot{-} B)$. Hence $\mathrm{Bel}(\Psi \dot{-} B) \subseteq W \setminus \{w\} = \overline{A} \cup B$. Thus, by IC3, $\mathrm{Bel}((\Psi \dot{-} A) \dot{-} B) \subseteq \overline{A} \cup B = W \setminus \{w\}$. But then $w \notin \mathrm{Bel}((\Psi \dot{-} A) \dot{-} B)$. By AGM$\dot{-}$2, $v \in \mathrm{Bel}((\Psi \dot{-} A) \dot{-} B)$. This means, by (28), that $v \prec_A^- w$.

IC4 implies IC4$\prec$: Let $v \in \overline{A}$, $w \in A$. Note $A \cup B = W \setminus \{v\}$. Suppose that $w \prec_A^- v$. This implies that $v \notin \mathrm{Bel}((\Psi \dot{-} A) \dot{-} B) \subseteq A \cup B = W \setminus \{v\}$. So by (IC4), $\mathrm{Bel}(\Psi \dot{-} B) \subseteq A \cup B$. So $v \notin \mathrm{Bel}(\Psi \dot{-} B)$. But then, by AGM$\dot{-}$2, $w \in \mathrm{Bel}(\Psi \dot{-} B)$. But this means that $v \prec w$. $\square$

Proof of Observation 37: Assume that $\dot{-}$ is semi-AGM. By Corollary 26, it has a canonical representation. Suppose $\mathrm{Bel} \dot{-} (A \cap B) \not\subseteq B$. For AGM$\dot{-}$8, we show $\mathrm{Bel} \dot{-} B \subseteq \mathrm{Bel} \dot{-} (A \cap B)$. Since $A \cap B \subseteq B$, IC2 yields $(\mathrm{Bel} \dot{-} B) \cap \overline{B} = ((\mathrm{Bel} \dot{-} (A \cap B)) \dot{-} B) \cap \overline{B}$. From our assumption $\mathrm{Bel} \dot{-} (A \cap B) \not\subseteq B$, AGM$\dot{-}$4 on the second step implies $(\mathrm{Bel} \dot{-} (A \cap B)) \dot{-} B \subseteq \mathrm{Bel} \dot{-} (A \cap B)$. Thus $(\mathrm{Bel} \dot{-} B) \cap \overline{B} \subseteq ((\mathrm{Bel} \dot{-} (A \cap B)) \cap \overline{B}$. By canonicity, this implies $(\mathrm{Bel} \cap \overline{B}) \cup \min \overline{B} \subseteq (\mathrm{Bel} \cap \overline{B}) \cup (\overline{B} \cap \min(\overline{A} \cup \overline{B}))$. Hence $\mathrm{Bel} \cup \min \overline{B} \subseteq \mathrm{Bel} \cup \min(\overline{A} \cup \overline{B})$. By canonicity, $\mathrm{Bel} \dot{-} B \subseteq \mathrm{Bel} \dot{-} (A \cap B)$. $\square$

Proof of Theorem 38: Assume that $\dot{-}$ is normal semi-AGM. Thus it satisfies AGM$\dot{-}$4 and is normally representable by $\prec$ (Theorem 24).

(a) IC0$\prec$ implies IC0: Suppose Bel $\not\subseteq A$. Thus $\mathrm{Bel} \dot{-} A = \mathrm{Bel}$ by AGM$\dot{-}$4. Hence Bel $\not\subseteq B$ iff $\mathrm{Bel} \dot{-} A \not\subseteq B$. Thus when Bel $\not\subseteq B$, IC0 is trivial by AGM$\dot{-}$4. When Bel $\subseteq B$, it suffices to prove $\min \overline{B} = \min_A \overline{B}$. This holds by IC0$\prec$.

From now on, we assume IC0.

(b) IC2$\prec$ implies IC2$'$: Suppose $\mathrm{Bel} \dot{-} A \subseteq B$, and $A \subseteq B$. From the first, we obtain Bel $\subseteq B$ by AGM$\dot{-}$3. Bel $\subseteq A$ is treated as in the canonical case, and Bel $\not\subseteq A$ is covered by IC0.

(c) We now prove directly IC1, IC3, IC4. The reasoning is as in the canonical case, when Bel $\subseteq A$ and Bel $\dot{-} A \subseteq B$ (*Case 0*). When Bel $\not\subseteq A$ (*Case 1*), IC0 guarantees all other postulates. Thus, assuming IC0, we concentrate on Bel $\subseteq A$ and Bel $\dot{-} A \not\subseteq B$. It suffices to check (*Case 2*) Bel $\subseteq B$, and (*Case 3*) Bel $\not\subseteq B$. On the normal representation, this means:

Case 2: $\text{Bel} \dot{-} B = \text{Bel} \cup \min \overline{B}$ and $\text{Bel} \dot{-} A \dot{-} B = \text{Bel} \cup \min \overline{A}$.

Case 3: $\text{Bel} \dot{-} B = \text{Bel}$ and $\text{Bel} \dot{-} A \dot{-} B = \text{Bel} \cup \min \overline{A}$.

IC1$\prec$ implies IC1: Suppose $\overline{A} \subseteq B$. Case 2 is impossible: $\text{Bel} \dot{-} A \not\subseteq B$ implies $\text{Bel} \cup \min \overline{A} \not\subseteq B$, contradicting our assumptions Bel $\subseteq B$ and $\overline{A} \subseteq B$. Case 3: $\text{Bel} \cap \overline{B} = (\text{Bel} \cup \min \overline{A}) \cap \overline{B}$, due to $\overline{A} \subseteq B$.

IC3$\prec$ implies IC3: Cases 2 and 3: When $\text{Bel} \dot{-} A \not\subseteq B$, then $\text{Bel} \dot{-} B \subseteq \overline{A} \cup B$ implies $\text{Bel} \dot{-} A \dot{-} B = \text{Bel} \dot{-} A \subseteq \overline{A} \cup B$ by AGM$\dot{-}$4.

IC4$\prec$ implies IC4: Case 2: Assume $((\text{Bel} \dot{-} A) \dot{-} B) \subseteq A \cup B$. By AGM$\dot{-}$4, $\text{Bel} \cup \min \overline{A} \subseteq A \cup B$, since $\text{Bel} \dot{-} A \not\subseteq B$. Thus $\min \overline{A} \subseteq B$. This contradicts the case assumptions $\text{Bel} \dot{-} A \not\subseteq B$ and Bel $\subseteq B$. Case 3: $\text{Bel} \dot{-} B = \text{Bel} \subseteq A \cup B$ is trivially satisfied, due to Bel $\subseteq B$. $\square$

Proof of Observation 39: Let $\kappa^-_A = \overline{\kappa}^n_A$ where $n \leq z$. $(\kappa \dot{-} 0)$ and $(\kappa \dot{-} 1)$ by Definition 3. $(\kappa \dot{-} 2)$ follows from distance conservation of $\kappa_{A \to n}$ and Definition 3.

$(\kappa \dot{-} 5)$: Let Bel $\subseteq A$. Therefore $\kappa^-_A = \kappa_{A \to n}$ by Definition 3. By the ranking axioms $\kappa(A) = 0$. Thus $\kappa^-_A(v) = \kappa(v) - \kappa(\overline{A}) + n$ for $v \in \overline{A}$. But since $\kappa(\overline{A}) > z$, we have $\kappa(\overline{A}) - n > z - n$. Therefore $\kappa^-_A(v) + (z - n) < \kappa(v)$. $\square$

Proof of Theorem 40: (a) Let $\prec^-_A$ be derived by (19) from the ranking contraction $\kappa^-_A = \overline{\kappa}^n_A$ for some $n \leq z$. By Definition 3, IC0$\prec$–IC4$\prec$ are trivially true for $A = W$, so we assume $A \neq W$ in the following.

Then $v \prec^-_A w$

$$\text{iff} \quad \kappa^n_A(w) - \kappa^n_A(v) > z$$

$$\text{iff} \quad \begin{cases} \kappa(w) - \kappa(v) > z & \text{if Bel} \not\subseteq A \\ \kappa_{A \to n}(w) - \kappa_{A \to n}(v) > z & \text{if Bel} \subseteq A \end{cases}$$

$$\text{iff} \quad \begin{cases} \kappa(w) - \kappa(v) > z & \text{if Bel} \not\subseteq A \\ \kappa(w) - \kappa(v) > z & \text{if Bel} \subseteq A \text{ and } (w, v \in A \text{ or } w, v \notin A) \\ \kappa(w) - \kappa(\overline{A}) + n - (\kappa(v) - \kappa(A)) > z & \text{if Bel} \subseteq A \text{ and } v \in A, w \notin A \\ \kappa(w) - \kappa(A) - (\kappa(v) - \kappa(\overline{A}) + n) > z & \text{if Bel} \subseteq A \text{ and } v \notin A, w \in A \end{cases}$$

by Definitions 3 and 2. We call the last four rows "the case distinction".

Now IC0-$\prec$ follows immediately from the first row of the case distinction, and both IC1-$\prec$ and IC2-$\prec$ follow immediately from the second row of the case distinction.

For IC3-$\prec$, suppose that $v \notin A$, $w \in A$ and $v \prec w$, i.e., $\kappa(w) - \kappa(v) > z$. If Bel $\not\subseteq A$, the claim is immediate. If Bel $\subseteq A$, then the fourth row of the case distinction applies, and we need to show that $\kappa(w) - \kappa(A) - (\kappa(v) - \kappa(\overline{A}) + n) > z$. But this follows from $\kappa(w) - \kappa(v) > z$, $\kappa(A) = 0$, $\kappa(\overline{A}) > z$ (since Bel $\subseteq A$) and $n \le z$.

For IC4-$\prec$, suppose that $v \in A$, $w \notin A$ and $v \prec_A^- w$. If Bel $\not\subseteq A$, the claim is immediate. If Bel $\subseteq A$, then the third row of the case distinction applies, and $v \prec_A^- w$ means that $\kappa(w) - \kappa(\overline{A}) + n - (\kappa(v) - \kappa(A)) > z$. Since $\kappa(A) = 0$, $\kappa(\overline{A}) > z$ and $n \le z$, we get $\kappa(w) - \kappa(v) > \kappa(w) - \kappa(\overline{A}) + n - \kappa(v) + \kappa(A) > z$, which means $v \prec w$, as desired.

(b) By Theorem 38, it follows from (a) that the induced normal contraction satisfies IC0, IC1, IC2$'$, IC3, IC4. However, it does not in general satisfy IC2.

Counterexample against IC2 with $n < z$: Let $z = 1$, $n = 0$ and $W = \{w_0, w_2, w_3, w_4\}$, and suppose the κ-ranks of the worlds are indicated by their indices. Let $A = \{w_0\}$ and $B = \{w_0, w_2\}$. Clearly Bel $= A \subseteq B$. Writing κ_A^- for $\overline{\kappa}_A^0$, we have

$$\kappa_A^-(w_0) = 0, \kappa_A^-(w_2) = 0, \kappa_A^-(w_3) = 1, \kappa_A^-(w_4) = 2.$$

Since $\mathrm{Bel}(\kappa_A^-) = \{w_0, w_2, w_3\}$, we have $\mathrm{Bel}(\kappa_A^-) \not\subseteq B$. Thus $(\kappa_A^-)_B^- = \kappa_A^-$ and $\mathrm{Bel}((\kappa_A^-)_B^-) = \{w_0, w_2, w_3\}$. We also have

$$\kappa_B^-(w_0) = 0, \kappa_B^-(w_2) = 2, \kappa_B^-(w_3) = 0, \kappa_B^-(w_4) = 1.$$

So $\mathrm{Bel}(\kappa_B^-) = \{w_0, w_3, w_4\}$. Thus $w_4 \in \mathrm{Bel}(\kappa_B^-) \cap \overline{B}$ and $w_4 \notin \mathrm{Bel}((\kappa_A^-)_B^-) \cap \overline{B}$, violating IC2.

(Note that IC2-$\prec$ is also violated: Consider $w_3, w_4 \in \overline{A}$. In the semiorder $\prec$ derived from the original κ by (23), we have $w_3 \not\prec w_4$, since $z < \kappa(w_3) = 3 \not< 4 - 1 = \kappa(w_4) - (z - n)$. But this changes to $w_3 \prec_A^- w_4$ in the semiorder $\prec_A^-$ derived from $\kappa \dot{-} A = \overline{\kappa}_A^0$ by (23), since $\overline{\kappa}_A^0(w_3) = 1 \le z < 2 = \overline{\kappa}_A^0(w_4)$. The fact that the κ-conditionalization is distance preserving within $\overline{A}$ does not help here.) $\square$

References

Alchourrón, C. E., Gärdenfors, P., & Makinson, D. (1985). On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic*, *50*(2), 510–530.

Booth, R., & Meyer, T. (2006). Admissible and restrained revision. *Journal of Artificial Intelligence Research*, *26*, 127–151.

Boutilier, C. (1996). Iterated revision and minimal change of conditional beliefs. *Journal of Philosophical Logic*, *25*(3), 263–305.

Darwiche, A., & Pearl, J. (1997). On the logic of iterated belief revision. *Artificial Intelligence*, *89*(1–2), 1–29.

Fishburn, P. C. (1975). Semiorders and choice functions. *Econometrica*, *43*(5/6), 975–977.

Jamison, D. T., & Lau, L. J. (1973). Semiorders and the theory of choice. *Econometrica*, *41*(5), 901–912.

Jin, Y., & Thielscher, M. (2007). Iterated belief revision, revised. *Artificial Intelligence*, *171*(1), 1–18.

Katsuno, H., & Mendelzon, A. O. (1991). Propositional knowledge base revision and minimal change. *Artificial Intelligence*, *52*(3), 263–294.

Konieczny, S., Medina Grespan, M., & Pino Pérez, R. (2010). Taxonomy of improvement operators and the problem of minimal change. In F. Lin, U. Sattler, & M. Truszczynski (eds.), *Principles of knowledge representation and reasoning: Proceedings of the twelfth international conference (KR 2010)* (pp. 161–170). Menlo Park, CA: AAAI Press.

Konieczny, S., & Pino Pérez, R. (2017). On iterated contraction: Syntactic characterization, representation theorem and limitations of the Levi identity. In S. Moral, O. Pivert, D. Sánchez, & N. Marín (eds.), *Scalable uncertainty management: Proceedings of the 11th international conference (SUM 2017)* (pp. 348–362). Cham: Springer.

Luce, R. D. (1956). Semiorders and a theory of utility discrimination. *Econometrica*, *24*(2), 178–191.

Peppas, P., & Williams, M.-A. (2014). Belief change and semiorders. In C. Baral, L. Kovács, G. De Giacomo, & T. Eiter (eds.), *Proceedings of the fourteenth international conference on principles of knowledge representation and reasoning* (pp. 160–167). Palo Alto, CA: AAAI Press.

Rott, H. (2001). *Change, choice and inference: A study in belief revision and nonmonotonic reasoning.* Oxford: Oxford University Press.

Rott, H. (2009). Degrees all the way down: Beliefs, non-beliefs and disbeliefs. In F. Huber & C. Schmidt-Petri (eds.), *Degrees of belief* (pp. 301–339). Dordrecht: Springer.

Rott, H. (2014). Four floors for the theory of theory change: The case of imperfect discrimination. In E. Fermé & J. Leite (eds.), *Logics* *in artificial intelligence. Proceedings of the 14th European conference (JELIA 2014)* (pp. 368–382). Cham: Springer.

Sauerwald, K., Kern-Isberner, G., & Beierle, C. (2020). A conditional perspective for iterated belief contraction. In G. De Giacomo et al. (eds.), *ECAI 2020 – 24nd European conference on artificial intelligence* (pp. 889–896). Amsterdam: IOS Press.

Scott, D., & Suppes, P. (1958). Foundational aspects of theories of measurement. *Journal of Symbolic Logic*, *23*(2), 113–128.

Spohn, W. (2012). *The laws of belief.* Oxford: Oxford University Press.

Spohn, W. (2014). AGM, ranking theory, and the many ways to cope with examples. In S. Hansson (ed.), *David Makinson on classical methods for non-classical problems* (Vol. 3, pp. 95–118). Dordrecht: Springer.

Suppes, P., & Zinnes, J. L. (1963). Basic measurement theory. In R. D. Luce, R. R. Bush, & E. Galanter (eds.), *Handbook of mathematical psychology* (Vol. 1, pp. 1–76). New York: Wiley.