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STRONG DEPTH RELEVANCE

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Abstract. Relevant logics infamously have the variable sharing property: they only validate conditionals that share some propositional variable between antecedent and consequent. This property has been strengthened in a variety of ways over the last half-century. Two of the more famous of these strengthenings are the *strong variable sharing property* and the *depth relevance property*. In this paper I demonstrate that an appropriate class of relevant logics has a property that might naturally be characterized as the supremum of these two properties. I also show how to use this fact to demonstrate that these logics seem to be *constructive* in previously unknown ways.

1. Some Setup

Recall that Routley and Meyer's basic logic B is given by the following axioms and rules:[1]

A1: $A \to A$

A2: $(A \wedge B) \to A/B$

A3: $A/B \to (A \vee B)$

A4: $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

A5: $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

A6: $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$

A7: $\neg\neg A \to A$

R1: $\dfrac{A \qquad A \to B}{B}$

R2: $\dfrac{A \qquad B}{A \wedge B}$

R3: $\dfrac{A \to \neg B}{B \to \neg A}$

R4: $\dfrac{A \to B \qquad C \to D}{(B \to C) \to (A \to D)}$

Following [3], call any set of formulas containing these axioms and closed under these rules a *logic*. For the purposes of this paper, there are two particularly important logics to single out.

[1] See [13, 11, 12] and [6].

The first, DR^-, is a fragment of the logic called DR in [2].[2] It adds the following two axioms and one rule to B:

A8: $((A \to B) \wedge (B \to C)) \to (A \to C)$
A9: $(A \to \neg B) \to (B \to \neg A)$
R5: $\dfrac{A}{\neg(A \to \neg A)}$

The second, R, is the logic that adds the following four axioms to B:

A10: $(A \to \neg B) \to (B \to \neg A)$
A11: $(A \to B) \to ((B \to C) \to (A \to C))$
A12: $A \to ((A \to B) \to B)$
A13: $(A \to (A \to B)) \to (A \to B)$

We note that this is a redundant axiomatization of R; in particular R3 and R4 are no longer independent of the remaining system.

R is famously a *relevant* logic. DR^- is a sublogic of R. So DR^- is relevant as well. What exactly relevance amounts to is, of course, a matter of some debate. But one natural way of cashing out what makes relevant logics *relevant* is that they prove $A \to B$ only if the 'content' of A is somehow related to the content of B. So a natural-enough place to look for formal symptoms of relevance is in the propositional variables that occur in A and B, respectively.

Of course, any propositional-variable-related condition can *at best* give a necessary but not sufficient condition for a logic's being relevant. Be that as it may, there's enough intuitive oomph to the connection between variable sharing and relevance that it's worthwhile to be precise about various types of variable sharing that might be of interest. To that end, we will borrow the following vocabulary from [1, p. 240]:

Definition 1. We specify the *antecedent parts* (aps) and *consequent parts* (cps) of the formula A as follows:

- A is a cp of A.
- If $B \wedge C$ is a cp (ap) of A, then B is a cp (ap) of A and C is a cp (ap) of A.
- If $B \vee C$ is a cp (ap) of A, then B is a cp (ap) of A and C is a cp (ap) of A.
- If $B \to C$ is a cp (ap) of A, then B is an ap (cp) of A and C is a cp (ap) of A.
- If $\neg B$ is a cp (ap) of A, then B is an ap (cp) of A.

And, borrowing now from [2, p. 64], we also introduce the following vocabulary:

[2] The reason for choosing only a fragment of DR rather than all of DR is somewhat embarrassing: the proof technique used in this paper requires a great deal of bookkeeping and I quite simply could not keep track of all the details when it came to the disjunctive rules in DR.

Definition 2. Each occurrence of C as a subformula of a given formula is assigned a number called its *depth* as follows:

- C occurs at depth 0 in its unique occurrence in the formula C.
- If C occurs at depth n in A, then the corresponding occurrence of C in $\neg A$, in $A \wedge B$, in $B \wedge A$, in $A \vee B$, and in $B \vee A$ are all depth n occurrences of C as well.
- If C occurs at depth n in A, then the corresponding occurrence of C in $A \to B$ and in $B \to A$ is a depth $n + 1$ occurrence of C.

With these definitions in hand, we will single out three 'flavors' of variable sharing for formulas:

- $A \to B$ has *the variable sharing property* iff some variable that occurs in A also occurs in B.
- $A \to B$ has *the strong variable sharing property* iff either some variable occurs as an ap of both A and B or some variable occurs as a cp of both A and B.
- $A \to B$ has *the depth relevance property* iff for some natural number d and variable p, p occurs at depth d in A and in B.

The definition of the variable sharing property is due to Anderson and Belnap. While they never explicitly define what I've here called the strong variable sharing property, they do in fact prove (see below) theorems concerning it. The term 'strong variable sharing property', however, is due (as far as the author knows) to [8]. The depth relevance property was defined by Brady in [2].

We say a logic L has one of these properties when $\vdash_L A \to B$ just if $A \to B$ has the property. We are now in position to state some fairly well-known results:

Theorem 1 (Anderson and Belnap, 1975)**.** *Sublogics of R have the strong variable sharing property. It follows that sublogics of R have the (weak) variable sharing property as well.*

Theorem 2 (Brady, 1984)**.** *Sublogics of DR$^-$ have the depth relevance property.*

Brady in fact proved a stronger result, namely that sublogics of DR have the depth relevance property. DR, in turn, is a logic that extends DR$^-$ by including several additional disjunctive rules and axioms.

Strong variable sharing and depth relevance can be fairly naturally 'hybridized' as it were. More to the point, say that $A \to B$ has *the strong depth relevance property* just if either

- For some natural number d and variable p, p occurs at depth d as an ap of A and p occurs at depth d as an ap of B or
- For some natural number d and variable p, p occurs at depth d as a cp of A and p occurs at depth d as a cp of B.

As before, say that L has the strong depth relevance property when $\vdash_L A \to B$ only if $A \to B$ has the strong depth relevance property. The main purpose of this paper is to prove the following theorem:

The Strong Depth Relevance Theorem Sublogics of DR^- have the strong depth relevance property.

In outline, here is how the remainder of the paper will go: first, before actually proving the Strong Depth Relevance Theorem, we will give an application of it. The point is to show that the theorem is of more than 'merely technical' interest.[3] The application we provide shows that sublogics of DR^- are constructive in ways that, to my knowledge, had not previously been known. Only after demonstrating that the Strong Depth Relevance Theorem has this application will we then turn to proving it.

2. An Application

Before we turn to the application itself, a bit of discussion is in order. To begin, recall that a logic L *has the disjunction property* when $\vdash_L A \vee B$ just if $\vdash_L A$ or $\vdash_L B$. Just as variable sharing in its various forms is a formal symptom of *relevance*, the disjunction property is a formal symptom of *constructiveness*. Given this, it's unsurprising that we have the following:

Theorem 3 (Gödel, 1932)**.** *Intuitionistic logic has the disjunction property.*[4]

Perhaps more surprising is the following:

Theorem 4 (Slaney, 1984)**.** *The usual contractionless relevant logics have the disjunction property.*[5]

A natural question to ask, then, is this: of the various formal symptoms of constructiveness exhibited by intuitionistic logic, which of them are also exhibited by (perhaps weak members of) the family of contractionless relevance logics?[6]

There's reason to think we won't get much from this sort of an investigation. After all, apart from the disjunction property, the most famous formal symptom of constructivity is probably the rejection of double negation elimination, and every logic we're looking at here takes double negation elimination *as an axiom*!

Be that as it may, the fact that the contractionless relevant logics seem to exhibit at least the disjunction-related symptoms of constructiveness gives us reason to look at other such symptoms. That is, it gives us reason to look

[3] Other forms of variable sharing have also been shown to have interesting applications; see e.g. [9] and [10].

[4] See [4].

[5] See [14].

[6] There are other reasons to be interested in constructive members of the family of relevant logics. See [15] for more discussion of this point.

into what are known as *generalized disjunction properties*; the most famous of which is Harrop's rule:

Definition 3. We say that L *obeys Harrop's rule* when for all formulas A that lack disjunctions as consequent parts, $\vdash_L A \to (B \vee C)$ just if $\vdash_L A \to B$ or $\vdash_L A \to C$.

Theorem 5 (Harrop, 1960)**.** *Intuitionistic logic obeys Harrop's rule.*[7]

Our goal in this section is to prove that sublogics of DR$^-$ obey a certain weak form of Harrop's rule, demonstrating at least one other symptom of constructiveness that they exhibit. As we will see, the proof of this result relies crucially on the Strong Depth Relevance Theorem.

Before getting more precise about the theorem we'll prove, let's first set up all the rest of the machinery we'll need in the proof.

Definition 4. An *intensional elementary conjunction* (iec) is a finite conjunction each of whose conjuncts is either a propositional variable, the negation of a propositional variable, a formula of any complexity whose main connective is an arrow, or the negation of a formula of any complexity whose main connective is an arrow.

Definition 5. An *intensional disjunctive normal form* (idnf) is a finite disjunction of iecs.

Lemma 1 (See, e.g. §26.3 of [1])**.** *In any logic, every formula is equivalent to an idnf.*[8]

Before the next lemma, we note the following abbreviations: we will write '(d)ap' for 'depth d ap', and similarly for (d)cp. We will also write (d)p for 'depth d part' (though this won't come up for a while).

Lemma 2. *Suppose L is a sublogic of DR$^-$. Then*

- *If p is a (0)cp of A, then for some formula A' either*
  - (i) $\vdash_L A \leftrightarrow p$, *or*
  - (ii) $\vdash_L A \leftrightarrow (p \wedge A')$, *or*
  - (iii) $\vdash_L A \leftrightarrow (p \vee A')$*; also*
- *If p is a (0)ap of A, then for some formula A' either*
  - (i) $\vdash_L A \leftrightarrow \neg p$, *or*
  - (ii) $\vdash_L A \leftrightarrow (\neg p \wedge A')$, *or*
  - (iii) $\vdash_L A \leftrightarrow (\neg p \vee A')$*; also*

The proof, which is rather straightforward and left to the reader, is by simultaneous induction over the complexity of A on both parts of the result. And now at last, we turn to the promised pseudo-Harrop's rule:

Theorem 6. *If L is a sublogic of DR$^-$ and P is a propositional variable or the negation of a propositional variable, then $\vdash_L P \to (A \vee B)$ only if $\vdash_L P \to A$ or $\vdash_L P \to B$.*

[7] See [5].

[8] Note here that we are using 'logic' in the technical sense introduced above.

Proof. We give only the case where $P = p$ is a propositional variable; the other case being essentially the same. By Lemma 1 we can safely assume both A and B are idnfs. The proof is by induction on the total complexity of $A \vee B$. The base case is trivial. By the Strong Depth Relevance Theorem, since $\vdash_L p \to (A \vee B)$, p is a (0)cp of $A \vee B$. Without loss of generality, assume that p is a (0)cp of A. So by Lemma 2, either

(i) $\vdash_L A \leftrightarrow p$, or
(ii) $\vdash_L A \leftrightarrow (p \vee A')$, or
(iii) $\vdash_L A \leftrightarrow (p \wedge A')$.

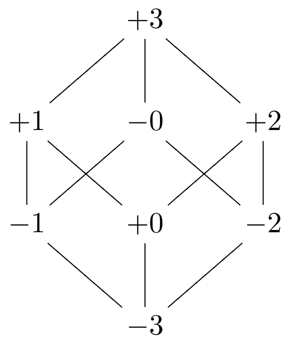
In the first two cases, we are done. In the third case, we clearly have $\vdash_L A \to (p \wedge A')$, and thus $\vdash_L A \to p$ as well. Since A is an idnf, there are iec's $C_1, \ldots, C_m$ so that $A = C_1 \vee \cdots \vee C_m$. Since $\vdash_L A \to p$, it follows that $\vdash_L C_i \to p$ for $1 \leq i \leq m$. So by the Strong Depth Relevance Theorem again, p is a (0)cp of each C_i. Since each C_i is an iec, it follows that each C_i in fact contains p as a conjunct. So each C_i either *is* p or (without loss of generality) has the form $p \wedge C'_i$ for some iec C'_i that is strictly less complex that C_i. Note that if for some i, $C_i = p$, then clearly $\vdash_L p \to A$ and we are done. So we assume that we can write A in the form $(p \wedge C'_1) \vee \cdots \vee (p \wedge C'_m)$ with each C'_i strictly less complex than C_i. Thus, letting $C^* = C'_1 \vee \cdots \vee C'_m$, we see that (a) C^* is strictly less complex than A and (b) $\vdash_L A \leftrightarrow (p \wedge C^*)$. So since $\vdash_L p \to (A \vee B)$, we also have that $\vdash_L p \to ((p \wedge C^*) \vee B)$. So $\vdash_L p \to ((p \vee B) \wedge (C^* \vee B))$. Thus $\vdash_L p \to (C^* \vee B)$. Since C^* is strictly less complex than A, induction then gives that either $\vdash_L p \to C^*$ or $\vdash_L p \to B$. In the latter case we are done. In the former, we observe that since $\vdash_L p \to p$ as well, we quickly have that $\vdash_L p \to A$. $\square$

3. Proving the Strong Depth Relevance Theorem

The basic idea of my proof of the Strong Depth Relevance Theorem is this: I adapt the proof of what is called the antecedent-parts and consequent-parts theorem in [1, §22.1.3] using the 'supervaluational' trick Brady used in the proof of Theorem 2 given in [2].[9] More to the point, we follow [1, §18.4] in

[9] I suspect that the tricks and tools in [8] and [7] would allow the proof I give to be simplified and possibly extended, but I haven't actually checked to see whether this is true.

calling the lattice with the following Hasse diagram M_0:



Still following [1], we recognize four functions $m_\neg$, $m_\wedge$, $m_\vee$, and $m_\rightarrow$ defined as follows:

- $m_\neg : \pm i \mapsto \mp i$.
- $m_\wedge : (x, y) \mapsto \inf(x, y)$.
- $m_\vee : (x, y) \mapsto \sup(x, y)$.

$m_\rightarrow$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$
-3	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$
-2	-3	$+2$	-3	$+2$	-3	-3	$+2$	$+3$
-1	-3	-3	$+1$	$+1$	-3	$+1$	-3	$+3$
-0	-3	-3	-3	$+0$	-3	-3	-3	$+3$
$+0$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$
$+1$	-3	-3	-1	-1	-3	$+1$	-3	$+3$
$+2$	-3	-2	-3	-2	-3	-3	$+2$	$+3$
$+3$	-3	-3	-3	-3	-3	-3	-3	$+3$

We take an M_0 assignment to be a function mapping propositional variables into M_0 and an M_0 valuation to be an infinite sequence of M_0 assignments.[10] Corresponding to an M_0 valuation $\alpha = v_0, v_1, \ldots$, we construct an infinite family $v_0^+, v_1^+, \ldots$ of functions from arbitrary formulas to M_0 by using the following clauses:

(1) $v_i^+(p) = v_i(p)$ for any i and any propositional variable p.
(2) $v_i^+(\neg A) = m_\neg(v_i^+(A))$.
(3) $v_i^+(A \wedge B) = m_\wedge(v_i^+(A), v_i^+(B))$.
(4) $v_i^+(A \vee B) = m_\vee(v_i^+(A), v_i^+(B))$.
(5) $v_0^+(A \rightarrow B) = +2$.
(6) $v_{j+1}^+(A \rightarrow B) = m_\rightarrow(v_j^+(A), v_j^+(B))$

We then say that $\alpha(A) = \inf_{i<\omega}(v_i^+(A))$. We say that A is true on α when $\alpha(A)$ is positive. We say that A is valid when A is true on all valuations.

Theorem 7. *Every theorem of DR^- is valid.*

[10] Taking assignments to be *mere* ω-sequences of valuations rather than ω+1-sequences of such is a significant but easy-to-overlook difference between the proof I give and the proof Brady gives.

Proof. By induction on the complexity of the proof. For the base case, look, for example at A8. For $i = 0$, $v_i^+(((A \to B) \wedge (B \to C)) \to (A \to C)) = +2$, for $i = 1$, $v_i^+(((A \to B) \wedge (B \to C)) \to (A \to C)) = m_\to(m_\wedge(+2, +2), +2) = +2$ and for $i > 2$ we instead get

$$m_\to(m_\wedge(m_\to(v_{i-2}^+(A), v_{i-2}^+(B)), m_\to(v_{i-2}^+(B), v_{i-2}^+(C))), m_\to(v_{i-2}^+(A), v_{i-2}^+(C))).$$

Observe that we are evaluating each of A, B, and C at the same 'level'; namely, $i - 2$. But Belnap's proof already showed that every theorem of R is true on every M_0 assignment, which then gives us what we need. By inspection, we see that in every other axiom, the same thing will be true. Thus every axiom of DR$^-$ is valid.

That the rules preserve validity is straightforward though tedious to check; we show only R4. To that end, suppose $A \to B$ and $C \to D$ are valid. The discussion will be aided by adopting the following abbreviations:

- $a_i := v_i^+(A)$, and similarly for b_i, c_i, and d_i.
- $w_i := v_i^+((B \to C) \to (A \to D))$.

Our goal is to show that $\inf_{i<\omega} w_i$ is positive. We do this by showing that for all i, w_i is positive. To begin, note that $w_0 = v_0^+((B \to C) \to (A \to D)) = +2$ by definition. For w_1 we compute as follows:

$$\begin{aligned} w_1 = v_1^+((B \to C) \to (A \to D)) &= m_\to(v_0^+(B \to C), v_0^+(A \to D)) \\ &= m_\to(+2, +2) \\ &= +2 \end{aligned}$$

For $i \geq 2$ we have that

$$w_i = v_i^+((B \to C) \to (A \to D)) = m_\to(m_\to(b_{i-2}, c_{i-2}), m_\to(a_{i-2}, d_{i-2}))$$

Since by assumption $i \geq 2$, $v_{i-1}^+(A \to B) = m_\to(a_{i-2}, b_{i-2})$ and $v_{i-1}^+(C \to D) = m_\to(c_{i-2}, d_{i-2})$. Thus it suffices to verify that whenever $m_\to(a_{i-2}, b_{i-2})$ and $m_\to(c_{i-2}, d_{i-2})$ are both positive, so is $w_i = m_\to(m_\to(b_{i-2}, c_{i-2}), m_\to(a_{i-2}, d_{i-2}))$. This, in turn, we prove by cases.

To begin, suppose $m_\to(a_{i-2}, b_{i-2}) = +3$. Then by inspection we see that either $a_{i-2} = -3$ or $b_{i-2} = +3$. In the former case, $m_\to(a_{i-2}, d_{i-2}) = +3$, so $w_i = +3$ as well. In the latter case, we have two subcases: either $c_{i-2} = +3$ or $m_\to(b_{i-2}, c_{i-2}) = -3$. In the second subcase, it is immediate that $w_i = +3$. In the first subcase, note that since $m_\to(a_{i-2}, b_{i-2})$ and $m_\to(c_{i-2}, d_{i-2})$ are both positive, it follows that $a_{i-2} = b_{i-2} = c_{i-2} = d_{i-2} = +3$. But then again $w_i = +3$. If $m_\to(c_{i-2}, d_{i-2}) = +3$, a very similar argument gives the same result.

Next suppose $m_\to(c_{i-2}, d_{i-2}) = +0$. Then either $c_{i-2} = d_{i-2} = -0$ or $c_{i-2} = d_{i-2} = +0$. If $c_{i-2} = d_{i-2} = +0$, inspecting the $m_\to$-matrix reveals that the following subcases are exhaustive:

- $b_{i-2} = -3$ and $m_\to(b_{i-2}, c_{i-2}) = +3$.
- $b_{i-2} = c_{i-2} = d_{i-2} = +0$ and $m_\to(b_{i-2}, c_{i-2}) = +0$.
- $m_\to(b_{i-2}, c_{i-2}) = -3$.

In the third subcase we immediately see that $w_i = +3$. In the first subcase, since $m_\to(a_{i-2}, b_{i-2})$ is positive, $m_\to(a_{i-2}, b_{i-2}) = +3$ so we reduce to a previous case. In the second subcase, since $m_\to(a_{i-2}, b_{i-2})$ is positive, either $a_{i-2} = b_{i-2} = c_{i-2} = d_{i-2} = +0$ and thus $w_i = +0$ or $a_{i-2} = -3$ and thus $m_\to(a_{i-2}, b_{i-2}) = +3$ and we reduce to a previous case. If instead $c_{i-2} = d_{i-2} = -0$, then $m_\to(b_{i-2}, v^+_{i-2}C) = -1 \cdot b_{i-2}$ and $m_\to(a_{i-2}, v^+_{i-2}D) = -1 \cdot a_{i-2}$. Thus since the $m_\to$-matrix is symmetric about the bottom-left to top-right diagonal and $m_\to(a_{i-2}, b_{i-2})$ is positive, so is w_i. If $m_\to(a_{i-2}, b_{i-2}) = +0$, a very similar argument gives the result.

This leaves four cases to examine: $m_\to(a_{i-2}, b_{i-2}) = m_\to(c_{i-2}, d_{i-2}) = +2$; $m_\to(a_{i-2}, b_{i-2}) = m_\to(c_{i-2}, d_{i-2}) = +1$; $m_\to(a_{i-2}, b_{i-2}) = +2$ and $m_\to(c_{i-2}, d_{i-2}) = +1$; and $m_\to(a_{i-2}, b_{i-2}) = +1$ and $m_\to(c_{i-2}, d_{i-2}) = +2$. As is probably expected from the symmetries of M_0, the first two cases are essentially the same, as are the last two cases. So we do only the first and the third. For the first, we resort to simply examining each of the 25 available options directly:

For $m_\to(a_{i-2}, b_{i-2}) = m_\to(c_{i-2}, d_{i-2}) = +2$, we compute as follows:

a_{i-2}	b_{i-2}	c_{i-2}	d_{i-2}	$m_\to(b_{i-2}, c_{i-2})$	$m_\to(a_{i-2}, d_{i-2})$	w_i
-2	-2	-2	-2	$+2$	$+2$	$+2$
-2	-2	-2	-0	$+2$	$+2$	$+2$
-2	-2	-2	$+2$	$+2$	$+2$	$+2$
-2	-2	$+0$	$+2$	-3	$+2$	$+3$
-2	-2	$+2$	$+2$	$+2$	$+2$	$+2$
-2	-0	-2	-2	-3	$+2$	$+3$
-2	-0	-2	-0	-3	$+2$	$+3$
-2	-0	-2	$+2$	-3	$+2$	$+3$
-2	-0	$+0$	$+2$	-3	$+2$	$+3$
-2	-0	$+2$	$+2$	-3	$+2$	$+3$
-2	$+2$	-2	-2	-2	$+2$	$+2$
-2	$+2$	-2	-0	-2	$+2$	$+2$
-2	$+2$	-2	$+2$	-2	$+2$	$+2$
-2	$+2$	$+0$	$+2$	-3	$+2$	$+3$
-2	$+2$	$+2$	$+2$	$+2$	$+2$	$+2$
$+0$	$+2$	-2	-2	-2	-2	$+2$
$+0$	$+2$	-2	-0	-2	-0	$+2$
$+0$	$+2$	-2	$+2$	-2	$+2$	$+2$
$+0$	$+2$	$+0$	$+2$	-3	$+2$	$+3$
$+0$	$+2$	$+2$	$+2$	$+2$	$+2$	$+2$
$+2$	$+2$	-2	-2	-2	-2	$+2$
$+2$	$+2$	-2	-0	-2	-2	$+2$
$+2$	$+2$	-2	$+2$	-2	$+2$	$+2$
$+2$	$+2$	$+0$	$+2$	-3	$+2$	$+3$
$+2$	$+2$	$+2$	$+2$	$+2$	$+2$	$+2$

For $m_\to(a_{i-2}, b_{i-2}) = +2, m_\to(c_{i-2}, d_{i-2}) = +1$, the situation is easier. Since $m_\to(a_{i-2}, b_{i-2}) = +2$, $b_{i-2} \in \{-2, -0, +2\}$. Since $m_\to(c_{i-2}, d_{i-2}) = +1$, $c_{i-2} \in \{-1, +0, +1\}$. As it turns out, in all nine options this leaves us with, $m_\to(b_{i-2}, c_{i-2}) = -3$, from which $w_i = +3$ immediately follows:

b_{i-2}	c_{i-2}	$m_\to(b_{i-2}, c_{i-2})$
-2	-1	-3
-2	$+0$	-3
-2	$+1$	-3
-0	-1	-3
-0	$+0$	-3
-0	$+1$	-3
$+2$	-1	-3
$+2$	$+0$	-3
$+2$	$+1$	-3

□

Suppose $A \to B$ lacks the strong depth relevance property. Then for all natural numbers d and variables p, if p occurs as a (d)ap of A, then p does not occur as a (d)ap of B and if p occurs as a (d)cp of A, then p does not occur as a (d)cp of B. Our goal will be to show that $A \to B$ is invalid from which, in conjunction with Theorem 7, it follows that $A \to B$ is not a theorem of DR$^-$. Contraposing will then give the Strong Depth Relevance Theorem as desired.

Taking our lead from Brady's proof in [2], let M be the maximum depth at which any variable occurs in $A \to B$. We define what amounts to a depth-sensitive version of the assignment given in [1, §22.1.3]. In particular we use the assignment $\alpha = v_0, v_1, \ldots$ defined as follows:

- If p is a (d)cp of B, then

$$v_{M-d-1}(p) = \begin{cases} -3 & \text{if } p \text{ is a (d)ap of } A \\ +2 & \text{otherwise} \end{cases}$$

- If p is not a (d)p of B, then

$$v_{M-d-1}(p) = \begin{cases} +1 & \text{if } p \text{ is a (d)ap of } A \\ -1 & \text{otherwise} \end{cases}$$

- Otherwise,

$$v_{M-d-1}(p) = \begin{cases} +3 & \text{if } p \text{ is a (d)cp of } A \\ -2 & \text{otherwise} \end{cases}$$

Lemma 3. *If C is a (d)cp of B, then $v^+_{M-d-1}(C) \in \{\pm 2, -3\}$, and if C is a (d)ap of B, then $v^+_{M-d-1}(C) \in \{\pm 2, +3\}$.*

Proof. By induction on the complexity of C. Suppose $C = p$ is a variable and a (d)cp of B. Then by the definitions, $v_{M-d-1}(p) \in \{-3, +2\} \subseteq \{\pm 2, -3\}$.

Suppose instead that $C = p$ is a variable and a (d)ap of B. We consider two cases:

(1) p also a (d)cp of B.
(2) p is not a (d)cp of B.

In case (1), note that since p is a (d)ap of B, p is *not* a (d)ap of A. Thus $v_{M-d-1}(p) = +2$, while in case (2), since p is a (d)p of B but is not a (d)cp of B, we see that $v_{M-d-1}(C) \in \{-2, +3\} \subseteq \{\pm 2, +3\}$.

The negation, conjunction, and disjunction cases are straightforward and left to the reader.

For conditionals, note that if $C = D_1 \to D_2$ is a (d)cp of B, then D_1 is a $(d+1)$ap of B and D_2 is a $(d+1)$cp of B. Note that it follows from B having $(d+1)$p's *at all* that $d+1 \leq M-1$ since $M-1$ is the maximum depth at which any formula can occur in B. Thus $1 \leq M-d-1$.

By the inductive hypothesis $v^+_{M-(d+1)-1}(D_1) = v^+_{M-d-2}(D_1) \in \{\pm 2, +3\}$. Similarly, $v^+_{M-d-2}(D_2) \in \{\pm 2, -3\}$. Finally, notice that since $M-d-1 > 0$, $v^+_{M-d-1}(D_1 \to D_2) = m_\to(v^+_{M-d-2}(D_1), v^+_{M-d-2}(D_2))$. By inspecting the $m_\to$-matrix, we see it follows that $v^+_{M-d-1}(C) \in \{\pm 2, -3\}$. Mutatis mutandis, the same argument works when $D_1 \to D_2$ is a (d)ap of B instead. □

Lemma 4. *If C is a (d)cp of A, then $v^+_{M-d-1}(C) \in \{\pm 1, +3\}$, and if C is a (d)ap of A, then $v^+_{M-d-1}(C) \in \{\pm 1, -3\}$.*

Proof. By induction on the complexity of C. Only the base cases differ from the previous proof in any interesting way. For the base case, suppose $C = p$ is a variable and a (d)cp of A. Then p is not a (d)cp of B. We consider two cases:

(1) p not a (d)p of B at all.
(2) p is a (d)ap of B.

In case (1), we immediately have that $v^+_{M-d-1}(p) \in \{\pm 1\} \subseteq \{\pm 1, +3\}$. In case (2), $v^+_{M-d-1}(p) = +3 \in \{\pm 1, +3\}$.

Now suppose instead that $C = p$ is a variable and a (d)ap of A. Then p is not a (d)ap of B. We consider two cases:

(1) p not a (d)p of B at all.
(2) p is a (d)cp of B.

In case (1), we immediately have that $v^+_{M-d-1}(p) = +1 \in \{\pm 1, -3\}$. In case (2), $v^+_{M-d-1}(p) = -3 \in \{\pm 1, -3\}$. □

Lemma 5. *$A \to B$ is not valid.*

Proof. A is a (0)cp of A. So $v^+_{M-1}(A) \in \{\pm 1 + 3\}$. B is a (0)cp of B. So $v^+_{M-1}(B) \in \{\pm 2, -3\}$. Inspecting the $m_\to$-matrix finishes the job. □

By contraposing, the Strong Depth Relevance Theorem is proved.

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