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The Formalization of Arithmetic in a Logic of Meaning Containment

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Abstract

We assess Meyer's formalization of arithmetic in his [21], based on the strong relevant logic R and compare this with arithmetic based on a suitable logic of meaning containment, which was developed in Brady [7]. We argue in favour of the latter as it better captures the key logical concepts of meaning and truth in arithmetic. We also contrast the two approaches to classical recapture, again favouring our approach in [7]. We then consider our previous development of Peano arithmetic including primitive recursive functions, finally extending this work to that of general recursion.

1 Introduction

Bob Meyer, together with Richard Routley/Sylvan, was largely responsible for the outstanding international reputation of Australian logic in the 1970s right up to his untimely demise in 2009. He has been a huge influence on the Australian logic community, having had close relationships with many logicians including myself. It was indeed an honour to have had him as a close friend, and to contribute to this journal issue, publishing his incomplete works on relevant arithmetic. Although my paper is generally critical in nature, I wish to put on record my admiration of his sheer inspiration, determination and diligence in his carrying out the work in Meyer [21] and [22].

The most important requirement of a logic is its application, though there is much good technical work in setting up such a logic in the first place. And, the application to arithmetic is one of two key testing grounds for a logic, differing somewhat from the other testing ground, viz. set theory. Meyer's great effort on this in [21] takes the application of the logic R a very long way whilst showing up the complexities of the enterprise. Further, most logicians would expect the familiar recursive parts of arithmetic to be simply consistent and there has always been a keenness to try and show the consistency of arithmetic as best as is possible, despite Gödel's second theorem, which states that the full classical arithmetic cannot be shown to be simply consistent by using finitary methods. Meyer, over a number of years, had tried to prove the admissibility of Ackermann's γ rule, $A, {\sim}A \vee B \Rightarrow B$ for relevant arithmetic,[1] from which the simple consistency of this arithmetic would then follow, as was proved on p. 72 of Meyer [22]. This result assumes the non-triviality of relevant arithmetic, which was proved by Meyer in [24]. Alas, Friedman provided a counter-example to this admissibility result in Friedman and Meyer [16]. Nevertheless, we appreciate all of Meyer's work, making an effort in this very interesting direction.

2 The Central Role of Meaning in Logic

We start with the roles of meaning in logic and subsequently truth in logic, though much has been said about this in Brady's earlier works. (See Brady [8], as a starting point.) Our brief account just serves to clarify some points in the area, in preparation for a critique of Meyer's approach to arithmetic. Logical formalization has always attempted to capture the meanings of sentences and the words in them, as best as it can. This is what we have taught our students in our introductory logic classes. Meaning must play a key role in formalization and this should naturally include the meanings of the logical words: 'and', 'or', 'not', 'implies' and/or 'entails', 'for all', 'for some'. In particular, 'A entails B' would need meaning analysis in proceeding from antecedent A to consequent B as it is used when B follows from A as a matter of necessity or certainty. 'A implies B', on the other hand, is commonly used for truth-preservation, of a purely classical sort or subject to some condition such as relevance, say for the logic R of relevant implication. We will formally introduce the '$\rightarrow$' of entailment in §4 and we will consider the issue of whether 'implies' should be a connective or not in §5.

[1] See Ackermann [1] for the first full relevant logic, which includes the γ rule as primitive.

Before proceeding in further detail, we briefly consider some of the major alternatives to our focus on meanings. First, relevance in the form of the Relevance Condition (if $A \to B$ is a theorem then A and B share a common sentential variable) is not a suitable concept to base a logic on. It acts as a necessary condition for a good logic, but we need both necessary and sufficient conditions to capture a suitable concept. Taken alone, relevance is not even transitive. It is not even a clear concept by itself, as relevant logics such as R have to adhere to its logical rules as well as the Relevance Condition to make a somewhat sensible logic. And, a clear logical concept is needed in order to make accurate applications of the logic, such applications not only conforming to concepts of the logic but also to the non-logical concepts introduced.

The standard relevant logic R is justified in practice by its being a strong system satisfying the Relevance Condition with a neatly presented natural deduction system. This neatness is superficial since its intrinsic complexity is shown up by its undecidability.[2] We will have more to say on the application of R in §6, when examining Meyer's arithmetic. In the meantime, let us consider the following example from Brady [4], p. 158, which is indeed due to Meyer: $m = n \to l = l$, for natural numbers l, m and n. This implication is clearly irrelevant in an extended arithmetical sense, but it can be proved using transitivity and symmetry of '=', together with axioms $m = n \to m' = n'$ and $m' = n' \to m = n$ in Meyer's relevant arithmetic $\mathsf{R}^\sharp$.

Second, information has been used to motivate relevant logics by Mares in [18] and [19]. However, information is a derivative concept, such as true content, where content is taken to be an analytic closure of a sentence or set of sentences, as argued for in Brady [9]. That is, information would be dependent on both truth and meaning. However, one should point out that Dunn does not agree with Brady on the inclusion of truth and would prefer information to be just contents or ranges. (See Dunn [15] and, for contents and ranges, see Brady [5].) However, both of these concepts are derivatives of meaning.

Further, Meyer, in his later years, considered the positive basic logic B_+, based on a comparison with combinators, but this was given a technical rather than a conceptual justification.

So, after considering the above alternatives, we are left with the key logical concepts of meaning and truth, the latter being considered in §3 below as it

[2]See Urquhart [31] for the proof of the undecidability of the logic R.

applies to a logic of meaning containment. However, we need to separate out two central uses of meaning in logic in more detail, thus enhancing its general use in logical formalization, as discussed above. First, we use meaning to obtain the concept of entailment represented by the connective '$\rightarrow$' of relevant logic. This is in the form of meaning containment, which provides the natural necessary and sufficient condition for a good relevant logic which satisfies transitivity, as it should. Indeed, logic is about capturing concepts rather than an exploration of what might be technically interesting combinations of axioms. Moreover, the concept of meaning containment enables one to capture intensional reasoning, as opposed to extensional reasoning which is more suited to classical logic and the use of rules of inference, the intertwining of which can be seen in the axiomatization and development of arithmetic in Brady [7] and in §7 below. This leads us to the next usage of meaning.

The second use of meaning is in valid deductive arguments, which require the certainty of the conclusion, given the premises. Such certainty is ensured by meaning analysis, usually from the premises, but it can also just occur from within the conclusion, making it an analytic truth. (There is more on this point in §3 to follow.) Valid deductive arguments are represented by rules of inference in an axiomatic system and so these rules also require meaning analysis to derive their conclusions from their premises. However, this needs to be distinguished from meaning containment, which is represented by a connective. There are two key differences here. The obvious one is that rules require their premises to be derivable before the rule can be activated and the conclusion drawn. If a premise is not derived the rule cannot be applied. There is no such requirement for a connective, which must take into account the prospect of its antecedent being false, as well as true. So, the meaning containment must allow for the antecedent to be either true or false. (See the discussion regarding the axiom-form $A \mathbin{\&} (A \rightarrow B) \rightarrow B$ versus the Modus Ponens rule, $A, A \rightarrow B \Rightarrow B$ in Brady [4] and [5].) The second difference is that analytic truths can be added as extra premises to a rule which can then enhance its deductive base, this addition being made possible by our allowance of analytic truths to be the conclusions of valid deductive arguments. (This is discussed further in §3 below.) This is of particular interest for our axiomatization of arithmetic in Brady [7] and in §7 below, where analytic truths involving numerical structure need to be taken into account. We note, by way of comparison, that analytic truths cannot always be added as extra antecedents of entailments in relevant logics, as their removal can induce invalidity.

3 The Role of Truth in a Logic of Meaning Containment

We start by briefly giving some clarification of the concepts of truth, proof, certainty, necessary truth and analytic truth, and their inter-relations, as applied to a logic of meaning containment. For truth, we start by referring to the argument in Brady [10] and elsewhere that the proof-theoretic concepts of disjunction and existential quantification are not precisely captured in truth-theoretic semantics, which is based on formula induction. The disjunction $A \vee B$ requires a witness A or a witness B for each world of a truth-theoretic semantics, in order to perform the induction. However, this is not required in the proof theory, where the induction is on proof-steps, as can especially be seen in a Fitch-style natural deduction setting by the general shape of the $\vee E$ rule: if $A \vee B_a$, $A \to C_b$ and $B \to C_b$ then $C_{a \cup b}$, with possible conditions on the indices a and b if needed. (See Brady [3] for Fitch-style natural deduction systems for relevant logics.) Note that in applied settings such as arithmetic, it is not Cut-free Gentzen systems that are used but Hilbert-style formal deduction, with natural deduction easily worked into it. (See Mendelson [20].) Also, note that worlds of truth-theoretic semantics mostly contain non-theorems and thus readily correspond to sub-proofs rather than the main proof of a natural deduction system. The similar addition of existential witnesses is required in the canonical model of the Henkin-style completeness proof of the classical predicate calculus. (See Henkin [17].) Further, the logic MC of meaning containment that follows in §4 should not contain the distribution axiom, $A \,\&\, (B \vee C) \to (A \,\&\, B) \vee (A \,\&\, C)$, which is needed for the relevant logics which are given a Routley-Meyer semantics. (See Routley, Meyer, Plumwood and Brady [28] for Routley-Meyer semantics and see Brady and Meinander [14] for the removal of the distribution axiom.) So, our truth cannot be the truth of this truth-theoretic semantics. However, we can and do use the content semantics of Brady [4] and [5], which is algebraic in style and thus relates closely to the proof theory, and it is proof theory that logic is about. (See Brady [10] and [12] on this last point.) Content semantics is ideal for our purposes, as it has a closer relationship to the proof theory than other algebraic-style semantics which generally have a closer relation to algebraic practice in mathematics.

Necessary truth is usually taken to be truth in all possible worlds but, as argued above, worlds do not fit into our semantical framework. However, analytic truths do fit into the framework of content semantics, as contents are appropriately interpreted as analytic closures in its canonical modelling. So, we take necessary truths to be analytic and modalities can be added to our logic to capture the various additional concepts that require a single-place operator.

The key use of certainty in logic is the certainty of the conclusions of valid deductive arguments, in contrast to strong inductive arguments which have high probability. Such certainty is guaranteed by meaning analysis, as any other analysis would fall short of certainty, given our view on the main contender, necessity.

Truth is used in our content semantics to determine logical truths and the validity of arguments, in the standard manner of an algebraic-style semantics. Nevertheless, meaning analysis is still used in deriving conclusions of arguments from premises, using the meanings of logical words and any non-logical words appearing in non-logical axioms or background information. Further, any sentence can still imply an analytic truth as meaning analysis is used in determining such a truth. This is borne out in the arithmetic of Brady [7], where theorem suppression is quite reasonably allowed in a set of premises, i.e. if $A, T \Rightarrow B$ then $A \Rightarrow B$, where T is a theorem, this being a deductive equivalent of $A \Rightarrow T$. Note that theorem suppression is in keeping with the definition of a valid deductive argument as discussed in §2. Note too that this is not classical logic, where any sentence can imply a contingent truth and a contingent falsehood can imply any sentence. So, whilst every such valid derivation of a conclusion from premises (in sentential logic, anyway) is truth-preserving, not every truth-preservation would yield a valid argument, as meaning analysis must be used to establish a derivation from truths to another truth. (We will see an example of such meaning analysis in §6, viz. for the rule, $s = t \Rightarrow s' = t'$.) As we will see in §4, all the rules of the logic MC of meaning containment preserve truth in content semantics. It should be noted, however, that the rule QR1 below of the quantified logic MCQ preserves validity rather than truth.

4 The Logic MCQ of Meaning Containment

In accordance with earlier works, we set out our sentential logic MC of meaning containment and its quantificational extension MCQ, with standard bracketing conventions from Anderson and Belnap [2]. This is our standard logic for meaning containment, presented and argued for in Brady [5] in the form of the logic $\mathsf{DJ^d}$ and then modified to MC in Brady and Meinander [14], replacing the distribution axiom by the distribution rule through a strengthening of the meta-rule (see below). Note also that the logic $\mathsf{MCQ^-}$ used in Brady [7] for arithmetic is weaker in regard to its quantificational part, but the sentential part remains the same.

Definition 1 (MC)**.**

Primitives:

- $\sim, \&, \vee, \to$.

Axioms.

1. $A \to A$.
2. $A \mathbin{\&} B \to A$.
3. $A \mathbin{\&} B \to B$.
4. $(A \to B) \mathbin{\&} (A \to C) \to . A \to B \mathbin{\&} C$.
5. $A \to A \vee B$.
6. $B \to A \vee B$.
7. $(A \to C) \mathbin{\&} (B \to C) \to . A \vee B \to C$.
8. $\sim\sim A \to A$.
9. $A \to \sim B \to . B \to \sim A$.
10. $(A \to B) \mathbin{\&} (B \to C) \to . A \to C$.

Rules.

1. A, $A \to B \Rightarrow B$.
2. A, $B \Rightarrow A \mathbin{\&} B$.
3. $A \to B$, $C \to D \Rightarrow B \to C \to . A \to D$.

Meta-Rule.

1. If $A, B \Rightarrow C$ then $D \vee A, D \vee B \Rightarrow D \vee C$.

Note that the rule-form of distribution, $A \mathbin{\&} (B \vee C) \Rightarrow (A \mathbin{\&} B) \vee (A \mathbin{\&} C)$ is derivable in MC, whilst its axiom-form is omitted, on account of its lack of meaning containment. (See Brady and Meinander [14], where this point is argued.)

The preservation of truth for the rules can be seen from the content semantics of MC in Brady [4] and on p. 63 in Brady [5]. Note that the logic of Brady [5] is DJ^{d} and that MC can be obtained from it by just removing the distribution axiom, $A \mathbin{\&} (B \vee C) \to (A \mathbin{\&} B) \vee (A \mathbin{\&} C)$, and by strengthening MR1 to apply to 2-premise rules, as given above, instead of applying to 1-premise rules as for DJ^{d}.

We now add the quantifiers to yield MCQ. As in previous presentations, we separate free and bound variables to simplify the conditions on the axioms.

Definition 2 (MCQ)**.**

Primitives:

- $\forall, \exists$.
- $a, b, c, ...$ (free variables)
- $x, y, z, ...$ (bound variables)

Axioms.

1. $\forall x A \to A^a/x$.
2. $\forall x(A \to B) \to . \ A \to \forall x B$.
3. $A^a/x \to \exists x A$.
4. $\forall x(A \to B) \to . \ \exists x A \to B$.

Rule.

1. $A^a/x \Rightarrow \forall x A$, where a does not occur in A.

Meta-Rule.

1. If $A, B^a/x \Rightarrow C^a/x$ then $A, \exists xB \Rightarrow \exists xC$, where a does not occur in A, B, or C, and where QR1 does not generalize on any free variable in the premises A and B^a/x in the derivation $A, B^a/x \Rightarrow C^a/x$. The same condition then applies to the premises A and B of the meta-rule MR1 of MC.

Note that the existential distribution rule, $A \mathbin{\&} \exists xB \Rightarrow \exists x(A \mathbin{\&} B)$, is clearly derivable, but the universal distribution rule, $\forall x(A \vee B) \Rightarrow A \vee \forall xB$, fails as it does for intuitionist logic and for the same reasons. (Intuitionist logic and MCQ differ only with respect to negation and implication/entailment. Discussion of this point can be found in Brady [13].)

As can be seen on p. 74 in Brady [5], QMR1 preserves truth in the content semantics of MCQ, but QR1 preserves validity rather than truth. (We make the above sentential adjustments to obtain the logic MCQ from $\mathsf{DJ^dQ}$, but also dropping the quantificational distribution axioms and using the 2-premise rule in QMR1 of MCQ.) Nevertheless, we can say that QR1 preserves metavaluational truth. As can be seen on p. 160 of Brady [5], the truth of A^a/x for a given variable a, which then extends to that for all terms, enables the metavaluational truth of $\forall xA$ to be obtained.

5 Classical Recapture

There are "cut and dried" situations that do not allow for under-determination or over-determination of concepts, which would then be just true or just false, thus being suitable for classical logic to apply. However, these situations are special, usually contrived to be such, and certainly do not include the whole gamut of sentences which express everyday concepts that may indeed be under- or over-determined in practice. More generally, these classical situations are a part of an overall logical theory containing non-classically determined sentences as well, which arise from this under- or over-determination. (For discussion of under- and over-determination of concepts, see Brady [11] and [13].)

The question then is: how do we best capture such classical systems or subsystems within the context of a meaning-based logic such as MCQ? The key classical properties which would apply to such systems or sub-systems would appropriately be the LEM and the DS. Indeed, if a set of sentences A are classical in that the LEM, of the form $A \vee {\sim}A$, and the DS, of the form

$\sim A, A \vee B \Rightarrow B$, both hold for A then any formula built up from such a set, using only $\sim$, &, and $\vee$, also satisfies the LEM and the DS (in the same way), and are thus classical, using simple properties from a meaning-based logic.[3] So, we can build up a sentential classical system or subsystem from such sentences. However, we do not have the universal distribution rule over disjunction, $\forall x(A \vee B) \Rightarrow A \vee \forall xB$, which would then prevent us from carrying through the argument extending the LEM to the two quantifiers, given the LEM in the form $Fa \vee \sim Fa$, for a free variable a. In applying this to arithmetic, this will involve us in extra work, as will be indicated in §8. Nevertheless, the argument for the DS does extend to the two quantifiers, given the DS in the form $\sim Fa, Fa \vee B \Rightarrow B$, where a does not occur in B, but as discussed below this does not matter very much as it turns out.[4] Nevertheless, this does mean that any formula built using the classical connectives and quantifiers from a set of formulae satisfying the DS will continue to do so and also satisfy Ex Falso Quodlibet. (See note 3 for EFQ.)

Before going further, we note the difference between the proof-based classical system, as above, and the formal system as a whole being classical, which is meta-theoretic in nature. The theorem-scheme LEM does not guarantee that the whole system is negation-complete nor does the rule DS guarantee that the whole system is simply consistent. However, with the priming property (if $A \vee B$ is a theorem, then either A is a theorem or B is a theorem), negation-completeness does follow from the LEM and, with non-triviality, simple consistency does follow from the DS, with the help of some very simple inclusions in the logic. Nevertheless, as argued in Brady [13], a disjunction should not be a theorem in the case where neither disjunct is provable, as applied to theorem instances. This would mean rejecting $G \vee \sim G$, where G is Gödel's sentence in classical arithmetic, since $G \vee \sim G$ would otherwise hold without either disjunct holding, assuming simple consistency as for Gödel's First Theorem. So, in the case of conceptual under-determination, which is

[3]The case for the DS can be shown by noting that the DS, $\sim A, A \vee C \Rightarrow C$, and Ex Falso Quodlibet, $A, \sim A \Rightarrow C$, are inter-derivable, using MR1, and that if $A, \sim A \Rightarrow C$ and $B, \sim B \Rightarrow C$ then $A \,\&\, B, \sim(A \,\&\, B) \Rightarrow C$ and $A \vee B, \sim(A \vee B) \Rightarrow C$ follow. Here, we use the meta-rule, $(A \Rightarrow C, B \Rightarrow C) \Rightarrow (A \vee B \Rightarrow C)$, established using repeated applications of MR1.

[4]We prove these results for the two quantifiers as follows. Let $\sim Fa, Fa \vee B \Rightarrow B$. Then $\exists x(Fx \,\&\, \sim Fx) \Rightarrow \exists xB$ and $\exists x(\forall yFy \,\&\, \sim Fx) \Rightarrow B$, which then yields $\forall yFy \,\&\, \exists x \sim Fx \Rightarrow B$ and $\forall xFx, \sim\forall xFx \Rightarrow B$. (Note that the free x cannot occur in B.) A similar proof can then be given for $\exists xFx$ by switching conjuncts. Note that these results use existential distribution over conjunction, rather than universal distribution over disjunction.

indeed ubiquitous in practice, the LEM would be restricted in application to those cases where one of its disjuncts has been shown. Indeed, having the LEM as a theorem of the system is hard to justify. (See Brady [13] on this.) Further, it is hard to achieve both negation-completeness and simple consistency for the whole system, as can also be seen in Brady [12]. There, it is stated that if one restricts the sentential logic to a recursive set of sentential constants and restricts quantification to a finite domain then one can achieve this whole-of-system classicality. (See Brady [6], for the proof of this.) So, in determining a suitable classical recapture, we must distinguish the proof-based approach from the whole-system-based approach. Moreover, we proceed with the proof-based classical recapture, as we wish that the classicality be proved within a logical system, despite its short-comings for the system as a whole.

To place this proof-based classicality within a logic of meaning containment such as MC, the addition of the LEM and the DS are vital in relating the rule '$\Rightarrow$' and the classical '$\supset$'. As can easily be shown using MR1, the rule $A \Rightarrow B$ is deductively equivalent to the theorem $A \supset B$, provided the LEM and the DS are both provable for the formula A.[5] (Also, see Brady [7], p. 374 on this.) This result raises the question of how implication rather than entailment can be expressed in the logic. Obviously, $A \supset B$ is an implication, as it is classical and hence preserves truth with help from its Boolean negation. When the LEM for A is not available, as it often would be, we could reasonably extend the use of the term 'implication' to include the rule $A \Rightarrow B$, which is more general and available, despite '$\Rightarrow$' not being a connective. And, it does not need to be a connective as the classical relationship between implication and entailment, as a necessitated (or modalized) implication, has already been broken down by the arguments in §3 and §4.

This classicality can ideally be restricted to a subsystem as it allows one to focus on particular sentences A for which the LEM and DS hold. However, because of its relationship with non-triviality and simple consistency, the DS is best introduced as an admissible rule for the whole system when simple consistency is proved, hoping that this can be done. The proof of the LEM is then best left to those instances where one of its disjuncts is provable, as carried out in the development of arithmetic in Brady [7] and, indeed, as occurs for constructive logics such as MC and intuitionist logic.

[5]Given the DS in the form $\sim A, A \vee B \Rightarrow B$ then one can detour through $\sim A, A \Rightarrow B$ and $A, \sim A \Rightarrow B$ to establish $A, A \supset B \Rightarrow B$, using MR1.

6 A Critique of Meyer's Arithmetic

We now compare our approach to arithmetic with that of Meyer's in his two papers [21] and [22], in the light of what we have said above. We have two major concerns. First, Meyer uses the logic R of relevant implication, which does have problems in its application. (See Brady [4] and [5] for some examples of what can go wrong.) This is brought about by its not having a concept such as meaning containment which can be used to judge each application to see that it fits the concept. The key cases in point are the inferential Peano axioms such as $m = n \rightarrow m' = n'$. The argument that would be made by people such as Meyer is that the '$\rightarrow$' is justified on the basis of relevance, due to the common variables m and n in both antecedent and consequent. This would extend the use of the commonality of sentential variables of the Relevance Condition. However, this same condition applies to a whole raft of relevant logics, which can have quite different interpretations, some of which could reasonably be construed as meaning containments as opposed to a kind of truth-preservation as applies to the logic R. As stated in §2, the variable-sharing property is not even transitive, a key property that the '$\rightarrow$' would be expected to satisfy.

We now move on to the relationship between $m = n$ and $m' = n'$. $m = n$ would hold because m and n represent the same natural number and similarly with m' and n'. How can we then move from $m = n$ to $m' = n'$? We have just added 1 to both sides. However, this introduces new numbers and so the relevance (and meaning) is lost in that we are dealing with different numbers. These different numbers would be represented by two disjoint sets of sets, i.e. the set of sets of m objects has no overlap with the set of sets of m' objects. More specifically, a 4-membered set is not a 5-membered set, though they may nevertheless have common members themselves. Moreover, the two numbers m and m' can be pushed further apart by repeated application of the above "axiom" together with transitivity. So, in moving from $m = n$ to $m' = n'$ is to say that $m' = n'$ holds because $m = n$ does. That is, it is a recursive move, which generates one theorem from another, expanding the set of theorems of arithmetic. As argued above, there is a loss of meaning containment between $m = n$ and $m' = n'$. However, this inference is not just a truth-preservation as there has been meaning analysis used in yielding the truth of $m' = n'$ from the truth of $m = n$, as required for a valid deductive argument. This meaning analysis depends not just on the identity $m = n$ as it would if it was a meaning containment but also on the structure of the numerical system which takes one from a number to its unique successor. Hence, this meaning analysis uses an additional analytic truth concerning the uniqueness of successors. Such an additional analytic truth can be used as part of the justification for a rule, but not for an entailment, as discussed in §2 and §3 above. Thus, the inference from $m = n$ to $m' = n'$ would be represented by a rule '$\Rightarrow$' rather than a meaning containment '$\rightarrow$'. Indeed, this type of justification applies to all the numerical Peano inferences that are represented as rules in the axiomatization of arithmetic in Brady [7].

Our second major concern is Meyer's classical recapture. The logic R of his arithmetic $\mathsf{R}^\sharp$ has, of course, the LEM as a theorem, leaving the main focus on the DS to achieve classicality. Meyer does this by trying to prove the admissibility of Ackermann's γ rule, $A, {\sim}A \vee B \Rightarrow B$, which is deductively equivalent to the DS. Unlike Ackermann, who included γ as a primitive rule of his logic in [1], Anderson and Belnap in [2] decided to exclude it on the grounds that the Deduction Theorem ought to be maintained as best as possible and, of course, the '$\rightarrow$' form of γ, $A \,\&\, ({\sim}A \vee B) \rightarrow B$ is not a theorem of R as $A \,\&\, {\sim}A \rightarrow B$ would follow. (See p. 13 of Meyer [21], where a reference is made to Belnap's requirement "that the primitive rules *for* a system should correspond to theses *of* the system".) It is a more recent idea that γ should be left unproven to maintain paraconsistency. The proof of admissibility of γ would then serve as a consistency proof, given the non-triviality of relevant arithmetic, which was subsequently proved by Meyer in [24]. (See also Meyer and Mortensen [25] on this.) Alas, this was not to be for Meyer's relevant arithmetic, as Friedman found a counter-example to the admissibility of γ in their paper [16].

The overall difficulty we would see here in Meyer's approach is that his (proof-based) classicality would be established for the whole system and there is no mechanism to apply it to a proper subsystem such as the one advocated above. This is because the LEM is part of the logic R whilst its partial failure would provide a better mechanism, with the DS or γ better established for the whole system through their admissibility or through simple consistency, given the non-triviality of $\mathsf{R}^\sharp$. In any case, the LEM is not justified as a theorem of arithmetic, given the failure of the priming property for theorem instances, as we argued in Brady [13]. For, as mentioned in §5 above, neither the Gödel sentence G nor its negation are provable in classical arithmetic, given its consistency, and this would extend to relevant arithmetic (with $G \vee {\sim}G$), and also to a logic of meaning containment (without $G \vee {\sim}G$), both being weaker logics than classical logic. Thus, we can see here the failure of the priming property for a theorem instance of $\mathsf{R}^\sharp$: if $G \vee \sim G$ then G or $\sim G$. As argued in Brady [13], such a case of the priming property ought to hold for theorem instances such as $G \vee \sim G$.

The same general argument can be given for other approaches that add or incorporate classical logic into the relevant logic R, as occurs in Meyer and Routley's classical relevant logics CR and CR^* in [26] and [27], respectively, and Meyer and Abraham's superclassical system KR to be found in Routley, Meyer, Plumwood and Brady [28], pp. 378–9.

Further, Meyer, on p. 127 of [21], proved that there is an exact translation of the theorems of classical arithmetic $\mathsf{C}^\sharp$ to R^*, which is $\mathsf{R}^\sharp$ restricted to the language of classical formulae, this result assuming the admissibility of γ for $\mathsf{R}^\sharp$. Note that Meyer's idiot's translation and his direct translation are the same here, this being the simple identity translation. Thus, the theorems of $\mathsf{C}^\sharp$ and R^* are identical, providing a classical recapture for the classical subformulae of $\mathsf{R}^\sharp$, assuming that the γ rule (and the DS) are admissible for $\mathsf{R}^\sharp$. This would provide a restricted classical recapture for $\mathsf{R}^\sharp$, with the LEM as a theorem of $\mathsf{R}^\sharp$, but with admissibility of the DS just applying to the classical formulae of $\mathsf{C}^\sharp$, this being in contrast to our restriction of the LEM but with the DS holding overall. However, this still leaves open the question of the admissibility of γ or the full relevant arithmetic $\mathsf{R}^\sharp$, which would then provide a full classical recapture. To this end, Meyer has attempted to prove the admissibility of γ for $\mathsf{R}^\sharp$. However, had he succeeded with a finitary proof of this, given its non-triviality, Meyer would have proved the simple consistency of $\mathsf{R}^\sharp$, as can be seen on p. 72 of his [22]. Given the above translation, the simple consistency of $\mathsf{C}^\sharp$ would then follow, contradicting Gödel's Second Theorem.

7 The Consistency of Arithmetic

We follow the procedure in Brady [7], by first setting up our arithmetic $\mathsf{MC}^\sharp$ and then proving its simple consistency by using metavaluations. We leave till §8 the formal development of $\mathsf{MC}^\sharp$ with its focus on recursion, both primitive and general. As on pp. 357–8 of [7], we set up the logic MCQ^- as follows: [The sentential logic MC is set up as in §4.]

Definition 3 (MCQ^-)**.**

Quantificational Primitives:

- ∘ $\forall$ (universal quantifier)
- ∘ $a, b, c, ...$ (free individual variables)
- ∘ $x, y, z, ...$ (bound individual variables)
- ∘ $f, g, h, ...$ (predicate variables)
- ∘ [$k, l, m, n...$ (individual constant schemes)]
- ∘ [$r, s, t...$ (schemes for terms, which are variable or constant)]

Definition.

- ∘ $\exists xA =_{df} {\sim}\forall x{\sim}A$.

Quantificational Axioms.

1. $\forall xA \to A^t/x$, for any term t.

2′. $A \to \forall xA$. [Note that x cannot occur in A.]

Quantificational Rule.

1. $A^a/x \Rightarrow \forall xA$, where a does not occur in A.

Quantificational Meta-Rule.

1. If $A, B^m/x \Rightarrow C^m/x$ then $A, \exists xB \Rightarrow \exists xC$, where m does not occur in A, B, or C, and QR1 does not generalize on any free variable in A or in B^m/x in the derivation $A, B^m/x \Rightarrow C^m/x$. The same condition applies to the premises A and B of the meta-rule MR1 of MC.

As explained on p. 358 of [7], we need to reduce the logic MCQ to MCQ$^-$ because the method of proof of consistency uses metavaluations which cannot sufficiently distinguish the rule $\forall x(A \to B) \Rightarrow A \to \forall xB$ from the rule $\forall x(A \lor B) \Rightarrow A \lor \forall xB$, and since this latter one must be excluded, so must the former. This is then replaced by the considerably weaker $A \to \forall xA$. And, $\exists xA$ needs to be defined in terms of $\forall xA$ as its corresponding equivalence is proved using $\forall x(A \to B) \Rightarrow A \to \forall xB$ and $\forall x(A \to B) \Rightarrow \exists xA \to B$, neither of which are available. Nevertheless, this weakening does not impact much on the development of arithmetic, because we will be focussing on recapturing as much of the classical arithmetic as possible.

Based on MCQ^-, we set up the numerical axioms and rules below to yield what we call $\mathsf{MC}^\sharp$. As discussed in §6, we will replace the '$\to$'s by '$\Rightarrow$'s in each of the inferential Peano axioms.

Definition 4 ($\mathsf{MC}^\sharp$)**.**

Identity Axioms.

1. $a = a$.
2. $a = b \to b = a$.
3. $a = b \,\&\, b = c \to a = c$.

Identity Rule.

1. $s = t, A(s) \Rightarrow A(t)$, where t is substituted for s in a single argument place.

Note that the rule $s = t \Rightarrow A(s) \to A(t)$ is derivable.

Number-Theoretic Axioms.

1. $\sim a' = 0$.
2. $a + 0 = a$.
3. $a + b' = (a + b)'$.
4. $a \times 0 = 0$.
5. $a \times b' = (a \times b) + a$

Number-Theoretic Rules.

1. $s = t \Rightarrow s' = t'$.
2. $s' = t' \Rightarrow s = t$.
3. $\sim s = t \Rightarrow \sim s' = t'$.
4. $\sim s' = t' \Rightarrow \sim s = t$.

Number-Theoretic Meta-Rule.

1. If $A(m) \Rightarrow A(m')$ then $A(0) \Rightarrow A(t)$, where t is an arbitrary numerical constant or variable. [Mathematical Induction.]

Note that the more familiar form with conclusion $\forall x A(x)$ is then derivable.

Classicality Axiom.

1. $a = b \vee {\sim} a = b$. [The LEM for identities.]

Note that the Classicality Rule (CR1), ${\sim} m = n, m = n \vee B \Rightarrow B$, is not added here, but we wait until simple consistency is proved and then add it as an admissible rule to the system. Moreover, CR1 cannot be added at this point as inconsistent arithmetic is used to show that $0 = m'$ is unprovable in $\mathsf{MC}^\sharp$, for any numerical constant m. Such classicality of the numerical identity statements holds as there is no room for under- or over-determination of the concept of identity in this context. Moreover, CR1 is used to kick-start the proofs of the LEM in the development of arithmetic.

To prove the simple consistency of $\mathsf{MC}^\sharp$ we introduce the following metavaluations v and v^* for the formulae of $\mathsf{MC}^\sharp$, bearing in mind that exactly one of the values $\mathbf{T}$ and $\mathbf{F}$ are assigned by v and v^*: (For background on metavaluations, see Meyer [23], Slaney [29] and [30].)

(i). ∘ $v(s = t) = \mathbf{T}$ iff $s = t$ is a theorem of $\mathsf{MC}^\sharp$, for *constant terms* s and t.
  ∘ $v^*(s = t) = v(s = t)$, for *constant terms* s and t.

Let A and B be *sentences*.

(ii). ∘ $v(A \,\&\, B) = \mathbf{T}$ iff $v(A) = \mathbf{T}$ and $v(B) = \mathbf{T}$.
  ∘ $v^*(A \,\&\, B) = \mathbf{T}$ iff $v^*(A) = \mathbf{T}$ and $v^*(B) = \mathbf{T}$.

(iii). ∘ $v(A \vee B) = \mathbf{T}$ iff $v(A) = \mathbf{T}$ or $v(B) = \mathbf{T}$.
  ∘ $v^*(A \vee B) = \mathbf{T}$ iff $v^*(A) = \mathbf{T}$ or $v^*(B) = \mathbf{T}$.

(iv). ∘ $v({\sim} A) = \mathbf{T}$ iff $v^*(A) = \mathbf{F}$.
  ∘ $v^*({\sim} A) = \mathbf{T}$ iff $v(A) = \mathbf{F}$.

(v). ∘ $v(A \to B) = \mathbf{T}$ iff $A \to B$ is a theorem of $\mathsf{MC}^\sharp$, if $v(A) = \mathbf{T}$ then $v(B) = \mathbf{T}$, and if $v^*(A) = \mathbf{T}$ then $v^*(B) = \mathbf{T}$.

- $v^*(A \to B) = \mathbf{T}$ (This is for an M1-metavaluation. See Slaney [30] for this.)

(vi). ∘ $v(\forall xA) = \mathbf{T}$ iff $v(A^n/x) = \mathbf{T}$, for all numerical constants n, recursively generated.

- $v^*(\forall xA) = \mathbf{T}$ iff $v^*(A^n/x) = \mathbf{T}$, for some numerical constant n, finitely determined.

To take into account *free variables*, we add the following:

(vii). ∘ $v(A) = \mathbf{T}$ iff $v(A_i) = \mathbf{T}$, for all constant instances A_i of A, recursively generated.

- $v^*(A) = \mathbf{F}$ iff $v^*(A_i) = \mathbf{F}$, for all constant instances A_i of A, recursively generated.

As explained on p. 366 of Brady [7], the above recursive generation for numerical constants n in $v(A^n/x) = \mathbf{T}$ is the conjunction: $v(A^0/x) = \mathbf{T}$ and, for all m, if $v(A^m/x) = \mathbf{T}$ then $v(A^{m'}/x) = \mathbf{T}$. (The other two recursive generations are similar.) Given the metacompleteness result below, this means that that all universal statements are provable using mathematical induction. Further, finite determination for v^* in (vi) is existential and just requires a finite process to find a witness for such an existential. It is finite in that our logic is both disjunctively and existentially constructive and so such a numerical constant n in $v^*(A^n/x) = \mathbf{F}$ is predetermined due to $\sim A^n/x$ being provable prior to that of $\sim\forall xA$ in this inductive metavaluational proof process. (See the metacompleteness result for $\mathsf{MC}^\sharp$ below, which relates metavaluations to proof. Also, see metavaluations (iii) and (vi) above for the disjunctive and existential constructivity.) Note that we have replaced the 'recursive determination' of (vi) in Brady [7], pp. 365–6, by 'finite determination', since recursion, as a concept that applies to an infinite set, is not needed in a finite setting and, of course, finiteness is preferable to recursion when it is appropriate.

The simple consistency proof can now proceed as in pp. 364–371 of Brady [7]. We start by first showing metacompleteness for $\mathsf{MC}^\sharp$, i.e. $v(A) = \mathbf{T}$ iff A is provable in $\mathsf{MC}^\sharp$ and also $v^*(A) = \mathbf{F}$ iff $\sim A$ is provable in $\mathsf{MC}^\sharp$. Given metacompleteness, the simple consistency of $\mathsf{MC}^\sharp$ can then be stated as: if $v(A) = \mathbf{T}$ then $v^*(A) = \mathbf{T}$, which is easily shown by induction on formulae in a finitary process. As in [7], we can then go on to show that CR1 and the

DS are admissible rules of $\mathsf{MC}^\sharp$. On this basis, we add the DS to the rules of $\mathsf{MC}^\sharp$, but we realize that it would need to be checked in any extension of the system. Further, we can add Ex Falso Quodlibet, $A, \sim A \Rightarrow B$, and both this rule and the DS are used in developing arithmetic in §8 below.

8 Formal Development of Arithmetic with Primitive and General Recursion

As in Brady [7], pp. 374–380, following Mendelson [20], classical arithmetic can be largely rebuilt. The LEM plays a key role here as it can convert any rule-form $A \Rightarrow B$ to the material implication form $A \supset B$, by applying MR1. This material form then allows us to insert it, as a formula, into the Mathematical Induction Scheme, NTMR1, which applies to formulae, but not to rules. So, the proofs of the LEM for formulae built up from the identities $a = b$ are vital to this process. Note that the bar $\bar{\cdot}$ over a natural number constant scheme is introduced to indicate a numerical constant.

We start with the Peano axioms and rules, including those for $+$ and $\times$, and add definitions for the familiar arithmetic relations: $<$ (less than), $>$ (greater than), $\leq$ (less than or equal to), $\geq$ (greater than or equal to), and $|$ (divides). These additional definitions are all defined using an existential quantifier together with positive and negative identities. The LEM holds for the identities and negated identities, but does not immediately extend to the existential quantifier, as indicated in §5 above. Nevertheless, as in pp. 376–8 of Brady [7], this can be circumvented by proving the LEM for $t < s$, i.e. $\exists w(\sim w = 0 \;\&\; t + w = s)$, by applying rule-forms of some $\supset$-theorems in Mendelson [20] and using the LEM for some identity statements. The detailed proof of the LEM for $t|s$, i.e. $\exists z(s = t \times z)$, is given in pp. 379–80 of Brady [7], and again the existential quantifier is circumvented.[6] The LEM for any further definitions involving quantification, existential or universal, would need to be proved on an individual basis. (See note 6 on this.) The functions ! (factorial) and a^b (power) are recursively introduced, these just expanding the range of terms representing natural numbers, and thus the

[6]Nevertheless, there are inductive definitions for these existentially defined concepts. $t < s$ can be inductively introduced as follows with base case $t < t'$ and inductive step, $t < s \Rightarrow t < s'$, whilst $t|s$ can be introduced with base case $t|t$ and induction step, $t|s \Rightarrow t|s \times n'$. This, we believe, provides a process for dealing with the problem of classically defined existential concepts.

LEM is unaffected. Similarly, any further primitive recursive definitions can be easily dealt with.

There is some complexity required for general recursion, however. On pp. 380–1 of Brady [7], there was some doubt expressed about whether general recursion could be accommodated in $\mathsf{MC}^\sharp$. This was because the LEM was proved for each formula at a time and the process was not able to deal with formula schemes such as A. There was also a problem with the classical Least Number Principle, $A(x) \supset \exists y(A(y) \,\&\, \forall z(z < y \supset {\sim}A(z))$, in that it did not seem to be provable, except in its contraposed rule-form. We will put these concerns aside and we will proceed to deal with general recursion using rules, in what turns out to be a relatively straightforward manner.

Here, the term $\mu x A(x)$ is introduced representing the least x such that $A(x)$ and added to the numerical terms that can be included in the usual inductive process. The introduction of this term depends on the existence of a number x such that $A(x)$ holds, i.e. on $\exists x A(x)$. We first raise the question of whether $A(x)$ needs to be classical or not. Given the inclusion of the DS in $\mathsf{MC}^\sharp$, this comes down to whether $A(x)$ satisfies the LEM or not, i.e. whether $\forall x(A(x) \vee {\sim}A(x))$ is provable in $\mathsf{MC}^\sharp$ or not. At this level of generality, it can only be assumed to be classical, if indeed it should for a particular $A(x)$, because our processes of proof of the LEM in $\mathsf{MC}^\sharp$ are based on particular formulae rather than schemes such as $A(x)$.

We next set out the general axiomatic principles for the introduction of $\mu x A(x)$, in a way that is appropriate for the system $\mathsf{MC}^\sharp$ and in accordance with the above arguments. Thus, we set out the following principles for the formula A to follow:

Definition 5 (*Least Number Principles*)**.**

1. $A(a) \vee {\sim}A(a)$.
2. $\exists x A(x) \Rightarrow A(\mu x A(x))$.
3. $\exists x A(x), m < \mu x A(x) \Rightarrow {\sim}A(m)$.

To justify these principles, we start with the following argument as to why the LEM for such $A(a)$ should be assumed to be the case, as in LNP1. Nevertheless, we have to assume $\exists x A(x)$ in any case, as in LNP2 and LNP3. Going back to its metavaluation, $v(\exists x A(x)) = \mathbf{T}$ iff $v({\sim}\forall x{\sim}A(x)) = \mathbf{T}$ iff $v^*(\forall x{\sim}A(x)) = \mathbf{F}$ iff $v^*({\sim}A(n)) = \mathbf{F}$ for some numerical constant n, finitely determined. [Recall that A is provable in $\mathsf{MC}^\sharp$ iff $v(A) = \mathbf{T}$ and $\sim A$ is provable iff $v^*(A) = \mathbf{F}$.] Such finite determination is a proof-theoretic process leading to the witness n, done by examining the proof of $\exists x A(x)$ for a prior proof of $A(n)$, this being due to the constructivity of $\exists x A(x)$ in the metavaluational proof process. We then need to look for the finite set, $A(0), A(1), ..., A(n)$, to find the least such number, $\mu x A(x)$. But, where do we look for these?

The following question first needs to be asked: what happens to $A(m)$ between $A(0)$ and $A(\mu x A(x))$? Here, for $0 \leq m < \mu x A(x)$, $A(m)$ must fail to hold, as $\mu x A(x)$ is the least number for which $A(m)$ holds. The issue now is: Is $A(m)$ unprovable or is $\sim A(m)$ provable, over this range? If $A(m)$ is unprovable, this is a metatheoretic result depending on proof within the whole system, this taking us outside the proof system of $\mathsf{MC}^\sharp$. This would not be in accordance with what a finitary process is, especially in the absence of a decidability result for $\mathsf{MC}^\sharp$, and a proof-theoretic finitary process is preferable to one that takes one outside the proof system of $\mathsf{MC}^\sharp$. Thus, $\sim A(m)$ would need to be provable for each such m, and this is accordance with the standard classical requirement for the introduction of $\mu x A(x)$, as for LNP3. Hence the LEM, $A(m) \vee \sim A(m)$, would be provable through one of its disjuncts, for $0 \leq m < \mu x A(x)$, and indeed for $m = \mu x A(x)$, as for LNP2. The LEM may well apply further, not only for $m \leq n$, where n is the witness for the existential $\exists x A(x)$, but also for all natural numbers, since the proof of the LEM normally focusses on the structure of the formula A and its application over all natural numbers, rather than applying just to some specific natural numbers. This has certainly been the case so far in the Hilbert-style development of arithmetic and it is also in accordance with a finitary metavaluational proof process for the general LEM. Thus, we would prove $A(a) \vee \sim A(a)$, as for LNP1. Since the logic is disjunctively constructive, there must be prior proofs of $A(m)$ or $\sim A(m)$, applying to each of the natural numbers m, introduced as part of a mathematical induction up to m. So, all one needs to do is to examine the proof of the LEM, for natural numbers m such that $0 \leq m \leq n$, to find $\mu x A(x)$ such that $A(\mu x A(x))$ and $\sim A(m)$ for each $m < \mu x A(x)$. This is the advantage of proving the LEM generally for all numbers, as it gives a finite piece of proof to look for the finite set $A(0), ..., A(n)$, from which $\mu x A(x)$ can be determined on a par with other numerical constants. So, $\mu x A(x)$ can take its place in general recursive procedures along with primitive recursion, once these conditions for its use are satisfied. So, the consistency of arithmetic of §7 would continue to apply to general recursion, subject to the proof of the above three principles for such formulae A occurring in $\mu x A(x)$, given $\exists x A(x)$.

Lastly, we raise the further question of the uniqueness for $\mu x A(x)$. Normally, one assumes there are two such terms $\mu_1 x A(x)$ and $\mu_2 x A(x)$ and goes on to prove their identity. The problem here is that there is nothing in LNP2 and LNP3 which says anything about whether either of $A(m)$ or $\sim A(m)$ is provable, for $m > \mu x A(x)$. What this means is that the identity of $\mu_1 x A(x)$ and $\mu_2 x A(x)$ has to be established by examining each $A(m)$ in the respective ranges $0 \leq m \leq \mu_1 x A(x)$ and $0 \leq m \leq \mu_2 x A(x)$. But, since exactly one of $A(m)$ and $\sim A(m)$ is provable for each m, given consistency, there can be no variation over the respective ranges, both ending at the same point. Thus, the identity is clear, in which case there is a unique least number $\mu x A(x)$ such that LNP1 and LNP2 hold, with respect to the only formula that matters, viz. A, and over the only range that matters, $0 \leq m \leq \mu x A(x)$.

9 In Conclusion

We have critiqued Meyer's account of relevant arithmetic on two counts. First, we argued against his use of the relevant logic R on its lack of a clear inferential concept. Second, we argued against his approach to classical recapture, which relies on the inclusion of the LEM in the logic R and the proof of the admissibility of the DS, shown to be impossible. Indeed, these are the key differences between Meyer's arithmetic and our own in Brady [7], where we use the logic of meaning containment MC, which better captures the key concept of meaning in logic and subsequently truth as it relates to MC. As for classical recapture, MC does not include the LEM, and we instead prove the simple consistency of the arithmetic $\mathsf{MC}^\sharp$ and, as a result, show that the DS is admissible and finally add it to the logic. This leaves us to prove the LEM for each formula, as required, through one of its disjuncts in a constructive fashion, re-creating classical arithmetic up to primitive and general recursion, as we go along.

The second aim was to extend our simple consistency result from the primitive recursion established in Brady [7] to include general recursion. Whilst the primitive recursion itself was straightforward, it was pointed out that there was a need to circumvent the existential definitions for '$<$' and '$|$', in order to prove that the LEM holds for $t < s$ and $t|s$. There was concern about the difficulty of including general recursion, expressed in Brady

[7]. However, we captured the classical least number principle for the least number $\mu x A(x)$ by rephrasing it as two rules, without quantifiers. In the end, it was quite clear that it could be done once the LEM was assumed for the formula $A(a)$. This produces a finite procedure for determining the least number $\mu x A(x)$ by examining a finite piece of constructive proof within that of $A(a) \vee {\sim}A(a)$. Given $\exists x A(x)$, the constructive determination of a natural number n satisfying $A(n)$ provides an upper bound on the natural numbers to be considered in this process. This then gives us a very sensible result that arithmetic, set up using the above metavaluational proof process and incorporating primitive and general recursion, is simply consistent. This proof is finitary, but that does not mean that Gödel's Second Theorem is contradicted, as the classical component of the logic is restricted by not including the rule, $\forall x(A \vee B) \Rightarrow A \vee \forall x B$, which then prevents the LEM from automatically extending to the two quantifiers. However, there is quite some prospect that existential definitions can be replaced by inductive definitions using rules without quantifiers. (On this, see note 6.)

Further, there is no usage of '$\rightarrow$' in the development of arithmetic occurring in Brady [7] as we used rules rather than '$\rightarrow$'s for inferences, followed by proofs of the LEM for the antecedents of the rules, which then allowed us to replace the rule '$\Rightarrow$' by the classical '$\supset$', creating a formula that can then fit into the mathematical induction scheme. This then allowed us to prove (almost all) the theorems of classical arithmetic from Mendelson [20], including primitive recursion and now general recursion. So, the metavaluational proof process that is used for this arithmetic need not include proofs of formulae with '$\rightarrow$'. This simplifies the proof process to a formula-inductive one involving just $\sim$, &, $\vee$, $\forall$, $\exists$, based on the atomic arithmetic identities and non-identities. The key difference between this formula-induction and that of classical arithmetic is the metavaluation for the universal quantifier $\forall$, this ensuring that each universal must be established by mathematical induction. This enables the proof of consistency to be finitary, removing the need for the infinitary rule: $A(0), A(1), A(2), ..., A(n), ... \Rightarrow \forall x A(x)$, which can be used to complete classical arithmetic. It is this constructivity of the metavaluational process that causes the rule $\forall x(A \vee B) \Rightarrow A \vee \forall x B$ to be rejected, as with intuitionist logic.[7]

[7]Interestingly, the rule $A \vee {\sim}A, \forall x(A \vee B) \Rightarrow A \vee \forall x B$ does hold in MCQ, but this does not help in the process of proving the LEM for the existential and universal quantifiers from their unquantified forms.

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