Algebra-valued models for LP-Set Theory

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Abstract

In this paper, we explore the possibility of constructing algebra-valued models of set theory based on Priest's Logic of Paradox. We show that we can build a non-classical model of ZFC which has as internal logic Priest's Logic of Paradox and validates Leibniz's law of indiscernibility of identicals. This is achieved by modifying the interpretation map for $\in$ and $=$ in our algebra-valued model. We end by comparing our model constructions to Priest's model-theoretic strategy and point out that we have a trade-off between a classical notion of identity and the validity of ZF and its theorems.

Keywords: Priest's Logic of Paradox, Algebra-valued models, ZFC, Non-classical set theory

Introduction

Every formal theory is composed of two basic kinds of axioms; the logical axioms, which are determined by the choice of our underlying logic, and the proper theory axioms, which provide mathematical content. For example, Zermelo-Fraenkel Set Theory (ZF) is based on classical logic and the theory axioms of ZF. But it is also possible to think about non-classical counterparts of ZF, such as intuitionistic Zermelo-Fraenkel Set Theory or a paraconsistent Zermelo-Fraenkel Set Theory which are based on, respectively, intuitionistic logic or a paraconsistent logic and the theory axioms of ZF.

Moreover, given that the literature on intuitionistic Zermelo-Fraenkel Set Theory and related systems is more abundant, (e.g., see [5] and [13]), we will focus in this paper on the latter class of non-classical set theories, i.e., paraconsistent Zermelo-Fraenkel Set Theory. In particular, we would like to follow a recent approach which consists in using algebra-valued models to construct paraconsistent models of set theory. We will call this approach also the *Tarafder*-approach. The *Tarafder*-approach initiated in [12] where the first example of a paraconsistent model of the negation free fragment of ZF was provided. This result was then extended to a class of paraconsistent models of the negation free fragment of ZF in [10] and to a class of paraconsistent models of full ZF in [7]. Additionally, the possibility of doing cardinal arithmetic and forcing in these models was explored in [19] and [22].

The use of algebra-valued models within this approach is motivated by the fact that we can obtain non-classical models of set theory that bear a great resemblance to the cumulative hierarchy. This allows us to compare these models directly with their classical counterparts, i.e., Boolean-valued models, given that both Boolean and non-Boolean algebra-valued models have the same ontological status in the universe of sets $\mathbf{V}$. Furthermore, this paper aims to bring together the *Tarafder*-approach and the study of paraconsistent set theories based on Priest's logic of paradox (LP).

The possibility of embedding set theory within LP has been explored for many years. In particular, [18] has produced an LP-model which validates ZF minus Foundation, and in [17] we can find an LP-model that validates both ZF and naïve set theory. However, a considerable drawback for both these constructions is that they are model-theoretically poor. In the case of [18] we have a highly degenerated model (see Section 2) and in the case of [17] we lose fundamental properties of identity (see Section 4). Thus, we want to build algebra-valued models based on LP with the hope of finding a more refined LP-model which allows us to carry out a reasonable amount of mathematics. In particular, we want to explore the possibility of constructing an LP-model with a classical notion of identity.

The article is organized as follows. In Section 1 we review some preliminaries and introduce the set theories we will be using. In Section 2 we build the algebra-valued model $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ based on $\mathbb{LP}$ and the regular interpretation map (viz. $[\![\cdot]\!]$). We point out the limitations of this model. In particular, we show that Leibniz's law of indiscernibility of identicals fails and that all the inconsistent elements collapse to a single element. Thus, we argue that this approach is unfeasible. Section 3 is devoted to the construction of the algebra-valued model $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$, based on a modified interpretation map (viz. $[\![\cdot]\!]_{IN}$). We show that this model validates bounded quantification, the theory axioms of ZFC, and Leibniz's law of indiscernibility of identicals. Moreover, we prove that this model is indeed paraconsistent and that the propositional logic associated with $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ corresponds to LP. Finally, in Section 4, we introduce Priest's model-theoretic strategy and compare his model constructions to our algebra-valued models based on LP.

1 Technical Preliminaries

1.1 Set Theory

In this subsection, we introduce the axiom system ZFC and some abbreviations in the language of set theory which we will use. Notice that to avoid confusion we will denote with ZFC the usual Zermelo-Fraenkel Set Theory plus the axiom of choice, whereas with ZFC we refer in particular to the theory axioms of ZFC (such as $\mathsf{Extensionality}$, $\mathsf{Separation}$, etc.). Similarly, for ZF and ZF. We will list the relevant axioms below.

Let $\mathcal{L}_{\in}$ be the language of set theory, which contains apart from the logical language the binary predicate $\in$ denoting membership. In particular, we will use the following abbreviations:

(i) $z = \{x\} =_{df.} \exists y(y \in z) \wedge \forall y(y \in z \to y = x)$,

(ii) $z = \{x, y\} =_{df.} \exists s(z \in z \wedge s = x) \wedge \exists t(t \in z \wedge t = y) \wedge \forall w(w \in z \to w = x \vee w = y)$,

(iii) $\mathsf{Pair}(z;\ x, y) =_{df.} \exists s\big(s \in z \wedge (s = \{x\})\big) \wedge \exists t\big(t \in z \wedge (t = \{x, y\})\big)\wedge$
$\forall w\big(w \in z \to (w = \{x\}) \vee (w = \{x, y\})\big)$,

(iv) $\mathsf{Func}(f) =_{df.} \forall x\big(x \in f \to \exists s \exists t \mathsf{Pair}(x;\ s, t)\big) \wedge$
$\forall x \forall y \forall s \forall t \forall w \forall v\big((x \in f \wedge y \in f \wedge \mathsf{Pair}(x; w, s) \wedge \mathsf{Pair}(y;\ v, t) \wedge w = v) \to s = t\big)$,

(v) $\mathsf{Dom}(f;\ x) =_{df.} \forall y\big(y \in x \to \exists w \exists z\big(w \in f \wedge \mathsf{Pair}(w;\ y, z)\big)\big) \wedge$
$\forall w\big(w \in f \to \exists y \exists z \mathsf{Pair}(w;\ y, z) \wedge z \in x\big)$.

The ZFC axiom system, in the language $\mathcal{L}_\in$ is displayed in Figure 1. In the schemes, φ is a formula with $n + 2$ free variables. This formulation follows closely [2]. This definition of ZFC is classically equivalent and is chosen to simplify the task of checking the validity of the axioms in algebra-valued models. Moreover, we denote with ZF, ZFC minus Choice and with ZF^-, ZF minus $\mathsf{Regularity}_\varphi$.

Definition 1.1. *We denote with* $\mathsf{NLP}_=$*, the set theory that we obtain by combining the theory axioms of naïve set theory, i.e.,* $\mathsf{Extensionality}$ *and*

$$\exists y \forall x\big(x \in y \leftrightarrow \varphi(x)\big) \qquad (\mathsf{Comprehension}_\varphi)$$

with the logical axioms of $\mathrm{LP}_=$.

Here $\mathrm{LP}_=$ denotes the first-order version of LP with identity, which we obtain by adding a binary predicate $=$ to LP, where $x = y$ receives value $\mathbf{1}$ or $\frac{1}{2}$ just in case $x = y$. Furthermore, there exists an alternative presentation of naïve LP-set theory due to [18], where identity is not a primitive notion, but defined in the following way:

$$x = y =_{df.} \forall z(x \in z \leftrightarrow y \in z). \qquad (*)$$

We denote with NLP the naïve LP-set theory that we obtain by combining the logical axioms of LP and the theory axioms of naïve set theory and where identity is defined as (*).

1.2 Properties of the lattice $\mathbb{LP}$.

We briefly review some basic definitions.

$$\forall x \forall y \big(\forall z (z \in x \leftrightarrow z \in y) \to x = y\big) \qquad \text{(Extensionality)}$$
$$\forall x \forall y \exists z \forall w \big(w \in z \leftrightarrow (w = x \vee w = y)\big) \qquad \text{(Pairing)}$$
$$\exists x \big(\exists y (\forall z\, \neg(z \in y) \wedge y \in x) \wedge \forall w (w \in x \to \exists u (u \in x \wedge w \in u))\big) \qquad \text{(Infinity)}$$
$$\forall x \exists y \forall z \big(z \in y \leftrightarrow \exists w (w \in x \wedge z \in w)\big) \qquad \text{(Union)}$$
$$\forall x \exists y \forall z \big(z \in y \leftrightarrow \forall w (w \in z \to w \in x)\big) \qquad \text{(Power Set)}$$
$$\forall p_0 \cdots \forall p_n \forall x \exists y \forall z \big(z \in y \leftrightarrow (z \in x \wedge \varphi(z, p_0, \ldots, p_n))\big) \qquad (\mathsf{Separation}_\varphi)$$
$$\forall p_0 \cdots \forall p_{n-1} \forall x \big(\forall y (y \in x \to \exists z \varphi(y, z, p_0, \ldots, p_{n-1})) \to \exists w \forall v (v \in x \to \exists u (u \in w \wedge \varphi(v, u, p_0, \ldots, p_{n-1})))\big) \qquad (\mathsf{Replacement}_\varphi)$$
$$\forall p_0 \cdots \forall p_n \forall x \big((\forall y (y \in x \to \varphi(y, p_0, \ldots, p_n)) \to \varphi(x, p_0, \ldots, p_n)) \to \forall z \varphi(z, p_0, \ldots, p_n)\big) \qquad (\mathsf{Regularity}_\varphi)$$
$$\forall u \big(\neg(u = \varnothing) \to \exists f \big(\mathsf{Func}(f) \wedge \mathsf{Dom}(f; u) \wedge \forall x \big(x \in u \wedge \neg(x = \varnothing) \to \exists z \exists y (\mathsf{Pair}(z;\; x, y) \wedge z \in f \wedge y \in x)\big)\big)\big) \qquad \text{(Choice)}$$

Figure 1: ZFC axioms.

Definition 1.2. *We call a poset $(A; \leq)$ a* meet semilattice *if every pair $x, y \in A$ has an infimum, denoted by $x \wedge y$. If there also exists a supremum, $x \vee y$, for any two $x, y \in A$, then $(A; \leq)$ is a* lattice*. We say that $(A; \leq)$ is a* bounded lattice *if it is a lattice that has a greatest element 1_A and a least element 0_A. A lattice $(A; \leq)$ is* complete *if the supremum $\bigvee X$ and the infimum $\bigwedge X$ exist for every $X \subseteq A$. A lattice is called* distributive *if it satisfies the distributivity law, that is, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all x, y, z in its universe.*

Let us consider the complete bounded distributive lattice $\mathbb{LP} = \langle \mathbf{A}; \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0} \rangle$, where the algebraic operations of $\mathbb{LP}$ correspond extensionally to the truth tables of the logical connectives of LP as introduced in [14]. Furthermore, we take our universe to be $\mathbf{A} = \{1, \frac{1}{2}, 0\}$, where $0 < \frac{1}{2} < 1$ and $D_{\mathbb{LP}} = \{1, \frac{1}{2}\}$ acts as the set of designated values.

$\Rightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	1	1

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

x	x^*
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

Table 1. Algebraic operations of $\mathbb{LP}$.

The lattice $\mathbb{LP}$ satisfies the following properties which hold as well for the $\Rightarrow$ operation of a Heyting algebra;

Lemma 1.3. *For any a, b, c in the domain of $\mathbb{LP}$ we have:*

$$a \leq b \text{ implies } c \Rightarrow a \leq c \Rightarrow b, \tag{P2}$$

$$a \leq b \text{ implies } b \Rightarrow c \leq a \Rightarrow c, \tag{P3}$$

$$\big((a \wedge b) \Rightarrow c\big) = \big(a \Rightarrow (b \Rightarrow c)\big). \tag{P4}$$

Proof. See ([20], p. 176). □

Lemma 1.4. *For any subset $\{a_i : i \in I\} \cup \{b\}$ of the domain of $\mathbb{LP}$, where I is an index set,*

$$(\bigvee_{i \in I} a_i) \Rightarrow b = \bigwedge_{i \in I} (a_i \Rightarrow b). \tag{†}$$

Proof. It is known that, for any two elements p and q of $\mathbb{LP}$, $p \Rightarrow q$ is either $\mathbf{1}, \frac{1}{2}$ or $\mathbf{0}$.

$$\begin{aligned}(\bigvee_{i \in I} a_i) \Rightarrow b = \mathbf{0} &\text{ iff } \bigvee_{i \in I} a_i = \mathbf{1} \text{ and } b = \mathbf{0} \\ &\text{iff there exists } j \in I \text{ so that } a_j = \mathbf{1} \text{ and } b = \mathbf{0} \\ &\text{iff } \bigwedge_{i \in I} (a_i \Rightarrow b) = \mathbf{0}.\end{aligned}$$

$$\begin{aligned}(\bigvee_{i \in I} a_i) \Rightarrow b = \mathbf{1} &\text{ iff } \bigvee_{i \in I} a_i = \mathbf{0} \text{ or } b = \mathbf{1} \\ &\text{iff for every } j \in I \text{ so that } a_j = \mathbf{0} \text{ or } b = \mathbf{1} \\ &\text{iff } \bigwedge_{i \in I} (a_i \Rightarrow b) = \mathbf{1}.\end{aligned}$$

□

1.3 The logic corresponding to $(\mathbb{A}, D)$.

We go on with some basic definitions.

Definition 1.5. *Let $\mathbb{A}$ be a complete bounded distributive lattice. A set $D \subset \mathbb{A}$ is said to be a designated set if*

(i) $\mathbf{1} \in D$*, but* $\mathbf{0} \notin D$*,*

(ii) if $x \in D$ and $x \leq y$, then $y \in D$, and

(iii) for any $x, y \in D$, $x \wedge y \in D$.

Moreover, we define the *propositional logic of a complete bounded distributive lattice* $\mathbb{A}$ given a set of designated values D as follows.

Definition 1.6. *Given a complete bounded distributive lattice $\mathbb{A}$ and a set of designated values $D \subseteq A$, the* propositional logic $\mathbf{L}(\mathbb{A}, D)$ *is defined as*

$$\mathbf{L}(\mathbb{A}, D) = \{\varphi \in \mathcal{L}_{Prop} : \iota(\varphi) \in D \textit{ for all assignments } \iota\}.$$

1.4 $\mathbb{A}$-valued models of set theory

In this subsection, we introduce the notion of an $\mathbb{A}$-valued model. In particular, we follow the Boolean-valued model construction of [2]. Let $\mathbb{A} = \langle \mathbf{A}; \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0}\rangle$ be a complete bounded distributive lattice and $\mathbf{V}$ a model of set theory, which means that $\langle \mathbf{V}, \in\rangle \models \mathsf{ZFC}$, then we define an $\mathbb{A}$-valued universe as follows:

Definition 1.7. *We define by transfinite recursion the set-theoretic universe* $\mathbf{V}^{(\mathbb{A})}$.

$$\mathbf{V}^{(\mathbb{A})}_\alpha = \{x\,;\ \ x \textit{ is a function and } \operatorname{ran}(x) \subseteq A \textit{ and there is } \xi < \alpha \textit{ with } \operatorname{dom}(x) \subseteq \mathbf{V}^{(\mathbb{A})}_\xi)\} \textit{ and}$$
$$\mathbf{V}^{(\mathbb{A})} = \{x\,;\, \exists\alpha(x \in \mathbf{V}^{(\mathbb{A})}_\alpha)\}.$$

Let $\mathcal{L}_\mathbb{A}$ be the extended language of $\mathcal{L}_\in$, which we obtain by adding constant symbols for every element in $\mathbf{V}^{(\mathbb{A})}$. Moreover, to increase the readability, the name corresponding to each $u \in \mathbf{V}^{(\mathbb{A})}$ will be denoted by the symbol u in the extended language $\mathcal{L}_\mathbb{A}$. A mapping $[\![\cdot]\!]$ is recursively defined from the collection of all closed well-formed formulas in $\mathcal{L}_\mathbb{A}$ to the complete bounded distributive lattice $\mathbb{A}$ as follows (cf. [2]).

Definition 1.8. *For any pair of elements* $u, v \in \mathbf{V}^{(\mathbb{A})}$,

$$[\![u \in v]\!] = \bigvee_{x\in\operatorname{dom}(v)} \big(v(x) \wedge [\![x = u]\!]\big),$$
$$[\![u = v]\!] = \bigwedge_{x\in\operatorname{dom}(u)} \big(u(x) \Rightarrow [\![x \in v]\!]\big) \wedge \bigwedge_{y\in\operatorname{dom}(v)} \big(v(y) \Rightarrow [\![y \in u]\!]\big).$$

Then, we can extend the map $[\![\cdot]\!]$ to non-atomic formulas: for any two closed well-formed formulas φ and ψ,

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!],$$
$$[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \vee [\![\psi]\!],$$
$$[\![\varphi \to \psi]\!] = [\![\varphi]\!] \Rightarrow [\![\psi]\!],$$
$$[\![\neg\varphi]\!] = [\![\varphi]\!]^*,$$
$$[\![\forall x\varphi(x)]\!] = \bigwedge_{u\in\mathbf{V}^{(\mathbb{A})}} [\![\varphi(u)]\!], \textit{ and}$$
$$[\![\exists x\varphi(x)]\!] = \bigvee_{u\in\mathbf{V}^{(\mathbb{A})}} [\![\varphi(u)]\!].$$

Definition 1.9. *Let $\mathbf{V}^{(\mathbb{A})}$ be the universe of $\mathbb{A}$-valued functions. Then we denote with $\mathbf{V}^{(\mathbb{A},\, [\![\cdot]\!])}$ the $\mathbb{A}$-valued model that we obtain by using $[\![\cdot]\!]$ as interpretation map.*

Definition 1.10. *A formula $\varphi \in \mathcal{L}_\mathbb{A}$ is said to be valid in $\mathbf{V}^{(\mathbb{A},\, [\![\cdot]\!])}$ given a designated set D, whenever $[\![\varphi]\!] \in D$. We denote this fact by $\mathbf{V}^{(\mathbb{A},\, [\![\cdot]\!])} \models_D \varphi$.*

1.5 The logic corresponding to $(\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!])}, D)$ and some model-theoretic notions

In this subsection, we introduce the *propositional logic of the* $\mathbb{A}$*-valued model* $\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!])}$ given a set of designated values D, and some model-theoretic notions. Let $\mathsf{Sent}_\in$ be the class of sentences in the language $\mathcal{L}_\in$. By an $\in$-*translation* we mean an homomorphism $T : \mathcal{L}_{Prop} \to \mathsf{Sent}_\in$.

Definition 1.11. *Given an* $\mathbb{A}$*-valued model* $\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!])}$*, a set of designated values* $D \subset A$*, the* propositional logic $\mathbf{L}(\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!])}, D)$ *is defined as*

$$\mathbf{L}(\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!])}, D) = \{\varphi \in \mathcal{L}_{Prop} : [\![T(\varphi)]\!] \in D \text{ for all } \in\text{-translations } T\}.$$

We say that an $\mathbb{A}$-valued model $\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!])}$ is *loyal* to $(\mathbb{A}, D)$ if the propositional logic of $\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!])}$ is the propositional logic of the complete bounded distributive lattice $\mathbb{A}$ given a set of designated values D, whereas $\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!])}$ is *faithful* to $\mathbb{A}$ if every element $a \in A$ is the truth value of at least one sentence φ, where $\varphi \in \mathsf{Sent}_\in$. Formally:

Definition 1.12 ([11])**.** *The* $\mathbb{A}$*-valued model* $\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!])}$ *is called* loyal *to* $(\mathbb{A}, D)$ *if*

$$\mathbf{L}(\mathbb{A}, D) = \mathbf{L}(\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!])}, D)$$

and faithful *to* $\mathbb{A}$ *if for every* $a \in A$ *there exists a* $\varphi \in Sent_\in$ *such that* $[\![\varphi]\!] = a$*.*

The reader may find a short proof of the following fact in [11].

Lemma 1.13 ([11])**.** *Let* $\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!])}$ *be an* $\mathbb{A}$*-valued model. Then if* $\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!])}$ *is faithful to* $\mathbb{A}$*, then it is loyal to* $(\mathbb{A}, D)$ *for any set of designated values* D*.*

Finally, we use the following definition.

Definition 1.14. *Let* $\mathbb{A}$ *and* $\mathbb{C}$ *be any two complete bounded distributive lattices. We say that* $\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!]_1)}$ *and* $\mathbf{V}^{(\mathbb{C},\ [\![\cdot]\!]_2)}$ *are* $\in$-elementarily equivalent *with respect to* $G \subset A$ *and* $H \subset C$*, and write*

$$(\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!]_1)}, G) \equiv_\in (\mathbf{V}^{(\mathbb{C},\ [\![\cdot]\!]_2)}, H)$$

whenever $\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!]_1)} \vDash_G \varphi$ *if and only if* $\mathbf{V}^{(\mathbb{C},\ [\![\cdot]\!]_2)} \vDash_H \varphi$*, for any* $\varphi \in \mathsf{Sent}_\in$*.*

2 An $\mathbb{LP}$-valued model of set theory

In this section, we apply the Tarafder-approach to the construction of LP-models. In particular, we build an algebra-valued model based on $\mathbb{LP}$. So, let $\mathbf{V}$ be a model of set theory, then we define the $\mathbb{LP}$-valued universe as follows:

Definition 2.1. *We define by transfinite recursion the set-theoretic universe* $\mathbf{V}^{(\mathbb{LP})}$.

$$\mathbf{V}_\alpha^{(\mathbb{LP})} = \{x\,;\ x \text{ is a function and } \mathrm{ran}(x) \subseteq A \text{ and there is } \xi < \alpha \text{ with } \mathrm{dom}(x) \subseteq \mathbf{V}_\xi^{(\mathbb{LP})})\} \text{ and}$$
$$\mathbf{V}^{(\mathbb{LP})} = \{x\,;\, \exists\alpha(x \in \mathbf{V}_\alpha^{(\mathbb{LP})})\}.$$

Moreover, the mapping $[\![\cdot]\!]$ is recursively defined from the collection of all closed well-formed formulas in $\mathcal{L}_{\mathbb{LP}}$ to the complete bounded distributive lattice $\mathbb{LP}$ as follows.

Definition 2.2. *For any pair of elements* $u, v \in \mathbf{V}^{(\mathbb{LP})}$,

$$[\![u \in v]\!] = \bigvee_{x \in \mathrm{dom}(v)} \big(v(x) \wedge [\![x = u]\!]\big),$$
$$[\![u = v]\!] = \bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow [\![x \in v]\!]\big) \wedge \bigwedge_{y \in \mathrm{dom}(v)} \big(v(y) \Rightarrow [\![y \in u]\!]\big).$$

Then, we can extend the map $[\![\cdot]\!]$ *to non-atomic formulas as specified in Definition 1.8.*

We go on to define the model $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$.

Definition 2.3. *Let* $\mathbf{V}^{(\mathbb{LP})}$ *be the universe of* $\mathbb{LP}$*-valued functions. Then we denote with* $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ *the* $\mathbb{LP}$*-valued model that we obtain by using* $[\![\cdot]\!]$ *as interpretation map.*

We show in the following lemma, that due to the $\Rightarrow$ operation of $\mathbb{LP}$, the algebra-valued model $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ is too weak to validate certain set-theoretic properties which hold in the case of Boolean and Heyting-valued models. As a consequence, we lose some properties which would be helpful in further calculations and many standard arguments that we use generally in algebra-valued models break down. To worsen the situation many calculations are blocked in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$, given the failure of the transitivity of the conditional and the lack of modus ponens.

Lemma 2.4. *For any* $u, v, w \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ *the following claims do not hold in general:*

(i) $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])} \models_{D_{\mathbb{LP}}} u = v \wedge v = w$ *implies* $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])} \models_{D_{\mathbb{LP}}} u = w$,

(ii) $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])} \models_{D_{\mathbb{LP}}} u = v \wedge u \in w$ *implies* $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])} \models_{D_{\mathbb{LP}}} v \in w$,

(iii) $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])} \models_{D_{\mathbb{LP}}} u = v \wedge w \in u$ *implies* $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])} \models_{D_{\mathbb{LP}}} w \in v$.

Proof. (i) Consider the elements $u, v, w \in \mathbf{V}^{(\mathbb{LP})}$ defined as $p_{\mathbf{0}} = \{\langle\varnothing, \mathbf{0}\rangle\}$, $p_{\frac{1}{2}} = \{\langle\varnothing, \frac{1}{2}\rangle\}$, and $p_{\mathbf{1}} = \{\langle\varnothing, \mathbf{1}\rangle\}$. Then we calculate readily

$$[\![p_{\mathbf{0}} = p_{\frac{1}{2}}]\!] \wedge [\![p_{\frac{1}{2}} = p_{\mathbf{1}}]\!] \in D_{\mathbb{LP}},$$

but $[\![p_{\mathbf{0}} = p_{\mathbf{1}}]\!] = \mathbf{0}$.

(ii) Consider the $\mathbb{LP}$ name $z = \{\langle p_{\mathbf{1}}, \mathbf{1}\rangle\}$. Then we calculate

$$[\![p_{\mathbf{0}} = p_{\frac{1}{2}}]\!] \wedge [\![p_{\frac{1}{2}} \in z]\!] \in D_{\mathbb{LP}}$$

and $[\![p_{\mathbf{0}} \in z]\!] = \mathbf{0}$.

(iii) Consider the $\mathbb{LP}$-names $r = \{\langle p_{\mathbf{0}}, \frac{1}{2}\rangle\}$ and $q = \{\langle p_{\mathbf{0}}, \mathbf{0}\rangle\}$. Then we have

$$[\![r = q]\!] \wedge [\![p_{\mathbf{0}} \in r]\!] \in D_{\mathbb{LP}}$$

and $[\![p_{\mathbf{0}} \in q]\!] = \mathbf{0}$. □

In particular, we get:

Corollary 2.5. *For any $u, v \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ and any formula $\varphi(x)$ in $\mathcal{L}_{\mathbb{LP}}$ having one free variable x it is generally not the case that, if $[\![u = v]\!] \wedge [\![\varphi(u)]\!] \in D_{\mathbb{LP}}$ then $[\![\varphi(v)]\!] \in D_{\mathbb{LP}}$.* □

Therefore, we can raise the first line of criticism against the algebra-valued model $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$. In particular, Corollary 2.5 shows that the Leibniz's law of indiscernibility of identicals fails within $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$. On the one side, since we can not build equivalence classes we are unable to define natural numbers and other basic kinds of sets in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$. So we are also unable to quotient down our algebra-valued model and to build a model of set theory with a *proper* notion of identity. On the other side, we have a conceptual problem given that we are dealing with an uncontroversial and widely accepted property of equality (see [6, pp. 108–109]). It is not clear why we should abandon such an intuitive principle regarding equality within a paraconsistent set theory.

Thus, we believe that the failure of Leibniz's law of indiscernibility of identicals constitutes a serious challenge for the algebra-valued model $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$. Notice that [18] has shown that Leibniz's law of indiscernibility of identicals fails, as well, in NLP.

We go on to show that in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ we have non-well-founded sets.

Lemma 2.6. $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])} \models_{D_{\mathbb{LP}}} \exists x(x \in x)$.

Proof. Consider the $\mathbb{LP}$-name $p_{\frac{1}{2}}$ (as defined in Lemma 2.4). Then we can readily calculate that

$$\begin{aligned}
[\![p_{\frac{1}{2}} \in p_{\frac{1}{2}}]\!] &= \big(p_{\frac{1}{2}}(\varnothing) \wedge [\![\varnothing = p_{\frac{1}{2}}]\!]\big) \\
&= \big(\frac{1}{2} \wedge \frac{1}{2}\big) \\
&= \frac{1}{2} \in D_{\mathbb{LP}}.
\end{aligned}$$

□

Moreover, [18] has shown that the foundation axiom fails within NLP.

Theorem 2.7. ([18, Lemma 3 & 4]). $\mathsf{NLP} \models \mathsf{ZF}^{-}$. □

We go on to point out the second issue of $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$. We will need the definition of $\frac{1}{2}$-*like* elements (these names are constituted just as canonical names with the only difference that the range of these names is $\frac{1}{2}$ instead of $\mathbf{1}$): for any $x \in \mathbf{V}$ let

$$x^{\circ} = \{\langle y^{\circ}, \frac{1}{2}\rangle : y \in x\}.$$

It is easily observable that every $\frac{1}{2}$-*like* element u° is a non-well-founded set in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$, i.e., $[\![u^{\circ} \in u^{\circ}]\!] \neq \mathbf{0}$. We believe that the existence of these sets is not problematic by itself, however, it seems unsatisfactory that every $\frac{1}{2}$-*like* element is identical in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$. In other words, every $\frac{1}{2}$-*like* element collapses to a single element from the perspective of our algebra-valued model.

Lemma 2.8. *For any* $u^{\circ}, v^{\circ} \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ *we have* $[\![u^{\circ} = v^{\circ}]\!] \in D_{\mathbb{LP}}$.

Proof. Fix any two $\frac{1}{2}$-*like* $\mathbb{LP}$-names u° and v°. Then:

$$\begin{aligned} [\![u^{\circ} = v^{\circ}]\!] &= \big((\frac{1}{2} \Rightarrow \frac{1}{2}) \wedge (\frac{1}{2} \Rightarrow \frac{1}{2})\big) \\ &= \frac{1}{2} \in D_{\mathbb{LP}}. \end{aligned}$$

□

The situation is even worse since not only is every $\frac{1}{2}$-*like* element identical in our algebra-valued model, but every $\frac{1}{2}$-*like* element is, as well, identical to any **0**-*like* element. We call an $\mathbb{LP}$-name u a **0**-*like* element whenever $u = \varnothing$ or for any $x \in \mathrm{dom}(u)$ we have $u(x) = \mathbf{0}$, i.e. we can think of **0**-like elements as representatives of the empty set $\varnothing$ in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$. Thus every $\frac{1}{2}$-*like* and **0**-*like* element collapses to a single element from the perspective of our model, i.e., the empty set $\varnothing$. Hence, we believe that in the case that of $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ we have a case of an *excessive* duplication of $\mathbb{LP}$-names.

Moreover, it was observed by [24, pp. 393-395] that in the case of $\mathsf{NLP}_=$ we have also problems regarding identity. More specifically, there exists an $\mathsf{NLP}_=$-model where the formula $\exists x \exists y (x \neq y)$ does not hold. In the case of NLP, on the other hand, it is possible to find two sets x and y such that $x \neq y$ holds (see [18, Theorem 7]). However, NLP is still unable to prove that there exist two sets x and y such that $x = y$ does not hold. This is due to fact that in Restall's NLP-model every formula receives value $\frac{1}{2}$. The moral that we can draw from this, is that a non-classical notion of identity is problematic for $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$, (the models of) NLP and (the models of) $\mathsf{NLP}_=$.

Moreover, we have the following open question.

Open question: Is $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ a model of NLP or $\mathsf{NLP}_=$?

Finally, we conclude that the algebra-valued model $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ does not seem very fruitful. On one side, it is unclear how much set theory we can derive since various basic set-theoretic properties are blocked and due to the lack of basic inferential features of $\Rightarrow$. This carries over to $[\![\cdot = \cdot]\!]$, i.e., the interpretation of identity in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ since $[\![\cdot = \cdot]\!]$

is interpreted as the conjunction of conditional statements. As a consequence, Leibniz's law of indiscernibility of identicals fails in $\mathbf{V}^{(\mathbb{LP},\, [\![\cdot]\!])}$ and every $\frac{1}{2}$-*like* and $\mathbf{0}$-*like* element collapses.

3 Modifying the interpretation map

It has been claimed by [16] that the key problem of a set theory based on LP is the weak conditional. His solution consisted of supplying LP with a stronger conditional or modifying the consequence relation. This gave rise to many variations of LP such as *multiple conclusion* LP (originally introduced in [1]) or *minimally inconsistent* LP (originally introduced in [15]). In this paper, we want to explore another possibility of constructing a set theory based on LP without distorting the spirit of LP.

As in Section 2, we want to build an $\mathbb{LP}$-valued model where we retain the conditional, however, we modify the interpretation map of the algebra-valued model. In particular, we propose to define a new interpretation map, denoted by $[\![\cdot]\!]_{IN}$, which does not allow for glutty identity statements anymore.

Definition 3.1. *For any pair of elements $u, v \in \mathbf{V}^{(\mathbb{LP})}$;*

$$
\begin{aligned}
&[\![u \in v]\!]_{IN} = \bigvee_{x \in \mathrm{dom}(v)} \big(v(x) \wedge [\![x = u]\!]_{IN}\big),\\
&[\![u = v]\!]_{IN} = \mathbf{0} \textit{ iff}\\
&\qquad\qquad \textit{there exists a } x \in \mathrm{dom}(u) \textit{ such that } u(x) > [\![x \in v]\!]_{IN},\\
&\qquad\qquad \textit{or there exists a } y \in \mathrm{dom}(v) \textit{ such that } v(y) > [\![y \in u]\!]_{IN}.\\
&\qquad\qquad \textit{Otherwise; } [\![u = v]\!]_{IN} = \mathbf{1}.
\end{aligned}
$$

Then, we extend the map $[\![\cdot]\!]_{IN}$ to non-atomic formulas as in definition 1.8.

Definition 3.2. *Let $\mathbf{V}^{(\mathbb{LP})}$ be the universe of $\mathbb{LP}$-valued functions. Then we denote with $\mathbf{V}^{(\mathbb{LP},\, [\![\cdot]\!]_{IN})}$ the $\mathbb{LP}$-valued model that we obtain by using $[\![\cdot]\!]_{IN}$ as interpretation map.*

Definition 3.3. *A formula $\varphi \in \mathcal{L}_{\mathbb{LP}}$ is said to be valid in $\mathbf{V}^{(\mathbb{LP},\, [\![\cdot]\!]_{IN})}$ given a designated set D, whenever $[\![\varphi]\!]_{IN} \in D_{\mathbb{LP}}$. We denote this fact by $\mathbf{V}^{(\mathbb{LP},\, [\![\cdot]\!]_{IN})} \models_{D_{\mathbb{LP}}} \varphi$.*

Notice that now for any $u, v \in \mathbf{V}^{(\mathbb{LP})}$ we have either $[\![u = v]\!]_{IN} = \mathbf{1}$ or $[\![u = v]\!]_{IN} = \mathbf{0}$. In other words, the range of the modified interpretation map of identity is $\{\mathbf{0}, \mathbf{1}\}$, whereas the modified interpretation map of membership can range, as in the case of the usual interpretation map, over all the elements of the universe of $\mathbb{LP}$. For instance, if u is an $\mathbb{LP}$-name, then $v = \{\langle u, a\rangle\}$ (where $a \in \mathbb{LP}$) is also an $\mathbb{LP}$-name and $[\![u \in v]\!] = a$. Moreover, every time we want to prove that $[\![u = v]\!]_{IN} \in D_{\mathbb{LP}}$ it is enough to show that for any $x \in \mathrm{dom}(u)$ such that $u(x) \leq [\![x \in v]\!]_{IN}$ and similarly for the elements of the domain of v.

Theorem 3.4. *Consider any two elements $u, v \in \mathbf{V}^{(\mathbb{LP})}$. Then, $[\![u = v]\!]_{IN} \in D_{\mathbb{LP}}$ if and only if both of the following hold:*

(i) if $u(x) = \mathbf{1}$ then $[\![x \in v]\!]_{IN} = \mathbf{1}$, and if $v(y) = \mathbf{1}$ then $[\![y \in u]\!]_{IN} = \mathbf{1}$;

(ii) if $u(x) = \frac{1}{2}$ then $[\![x \in v]\!]_{IN} \in D_{\mathbb{LP}}$, and if $v(y) = \frac{1}{2}$ then $[\![y \in u]\!]_{IN} \in D_{\mathbb{LP}}$.

Proof. Let us consider two elements $u, v \in \mathbf{V}^{(\mathbb{LP})}$ such that $[\![u = v]\!]_{IN} \in D_{\mathbb{LP}}$.

For (i), suppose there exists an element $u(x) = \mathbf{1}$. We want to show that $[\![x \in v]\!]_{IN} = \mathbf{1}$. Suppose otherwise, so either $[\![x \in v]\!]_{IN} = \mathbf{0}$ or $[\![x \in v]\!]_{IN} = \frac{1}{2}$. In both cases we have a $x \in \mathrm{dom}(u)$ such that $u(x) > [\![x \in v]\!]_{IN}$, so by Definition 3.1 we get $[\![u = v]\!]_{IN} = \mathbf{0}$. Thus in both cases we are contradicting our initial assumption. Hence, we must have $[\![x \in v]\!]_{IN} = \mathbf{1}$, i.e., there exists a $y \in \mathrm{dom}(v)$ such that $v(y) = \mathbf{1}$ such that $[\![x = y]\!]_{IN} = \mathbf{1}$. Similarly, if there exists a $y \in \mathrm{dom}(v)$ such that $v(y) = \mathbf{1}$ then there also exists a $x \in \mathrm{dom}(u)$ such that $u(x) = \mathbf{1}$ and $[\![x = y]\!]_{IN} \in D_{\mathbb{LP}}$, otherwise $[\![u = v]\!]_{IN} = \mathbf{0}$, and hence our assumption fails.

For (ii), let there be a $x \in \mathrm{dom}(u)$ such that $u(x) = \frac{1}{2}$. If there is no $y \in \mathrm{dom}(v)$ such that $v(y) \in \{\mathbf{1}, \frac{1}{2}\}$ and $[\![x = y]\!]_{IN} \in D_{\mathbb{LP}}$ we must have $[\![x \in v]\!]_{IN} = \mathbf{0}$. So there exists a $x \in \mathrm{dom}(u)$ such that $u(x) > [\![x \in v]\!]_{IN}$. Then by Definition 3.1 we get $[\![u = v]\!]_{IN} = \mathbf{0}$, which contradicts our initial assumption. It follows immediately that there exists a $y \in \mathrm{dom}(v)$ such that $v(y) \in \{\mathbf{1}, \frac{1}{2}\}$ and $[\![x = y]\!]_{IN} \in D_{\mathbb{LP}}$. Similarly, if there exists $y \in \mathrm{dom}(v)$ such that $v(y) = \frac{1}{2}$ and there does not exist any $x \in \mathrm{dom}(u)$, then $[\![u = v]\!]_{IN} = \mathbf{0}$, leads to a contradiction.

Conversely, let (i) and (ii) hold. Suppose that $u(x) = \mathbf{1}$. By (i) we have $[\![x \in v]\!]_{IN} = \mathbf{1}$, so $u(x) \leq [\![x \in v]\!]_{IN}$. Similarly, if $u(x) = \frac{1}{2}$ we get by (ii) that $[\![x \in v]\!]_{IN} \in \{\mathbf{1}, \frac{1}{2}\}$, so again we have $u(x) \leq [\![x \in v]\!]_{IN}$. We proceed analogously for the elements of the domain of v. This leads to the fact that, $[\![u = v]\!]_{IN} \in D_{\mathbb{LP}}$. □

Lemma 3.5. *For any $u, v, w \in \mathbf{V}^{(\mathbb{LP})}$ the following hold:*

(i) $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_{D_{\mathbb{LP}}} u = u$,

(ii) for any $x \in \mathrm{dom}(u)$, $u(x) \in D_{\mathbb{LP}}$ implies $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_{D_{\mathbb{LP}}} x \in u$,

(iii) $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_{D_{\mathbb{LP}}} u = v \wedge v = w$ implies $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_{D_{\mathbb{LP}}} u = w$,

Proof. (i) Consider any $x \in \mathrm{dom}(u)$ such that $u(x) \in D_{\mathbb{LP}}$. Suppose $u(x) = \mathbf{1}$, then by (i) of theorem 3.4 we have $[\![x \in u]\!]_{IN} = \mathbf{1}$. Hence, $u(x) \leq [\![x \in u]\!]_{IN}$. Similarly, if $u(x) = \frac{1}{2}$ then by (ii) of theorem 3.4 we get $[\![x \in u]\!]_{IN} \in \{\mathbf{1}, \frac{1}{2}\}$. This means that for any $x \in \mathrm{dom}(u)$ we have $u(x) \leq [\![x \in u]\!]_{IN}$. We may conclude $[\![u = u]\!]_{IN} \in D_{\mathbb{LP}}$ for any $u \in \mathbf{V}^{(\mathbb{LP})}$.
(ii) Let $u(x) \in D_{\mathbb{LP}}$, so we have $[\![x \in u]\!]_{IN} \geq (u(x) \wedge [\![x = x]\!]_{IN}) \in D_{\mathbb{LP}}$, since $[\![x = x]\!]_{IN} \in D_{\mathbb{LP}}$ by item (i).
(iii) By induction on the domain of w. Assume that for all $z \in \mathrm{dom}(w)$ we have:

$$[\![u = v]\!]_{IN} \wedge [\![v = z]\!]_{IN} \in D_{\mathbb{LP}} \text{ implies } [\![u = z]\!]_{IN} \in D_{\mathbb{LP}}.$$

Take any $x \in \mathrm{dom}(u)$ such that $u(x) \in D_{\mathbb{LP}}$. We want to show that $u(x) \leq [\![x \in w]\!]_{IN}$. If $u(x) = \mathbf{1}$, then since $[\![u = v]\!]_{IN} \in D_{\mathbb{LP}}$ by item (i) of Theorem 3.4 we have $[\![x \in v]\!]_{IN} = \mathbf{1}$, i.e., there exists a $y \in \mathrm{dom}(v)$ such that $v(y) = \mathbf{1}$ and $[\![x = y]\!]_{IN} \in D_{\mathbb{LP}}$. Now, since $[\![v = w]\!]_{IN} \in D_{\mathbb{LP}}$ and $v(y) = \mathbf{1}$ we can apply the same argument again, so $[\![y \in w]\!]_{IN} = \mathbf{1}$, i.e., there exists a $z \in \mathrm{dom}(w)$ such that $w(z) = \mathbf{1}$ and $[\![z = y]\!]_{IN} \in D_{\mathbb{LP}}$. Then by induction hypothesis: $\big([\![x = y]\!]_{IN} \wedge [\![y = z]\!]_{IN}\big) \in D_{\mathbb{LP}}$ implies $[\![x = z]\!]_{IN} \in D_{\mathbb{LP}}$. Hence, there exists a $z \in \mathrm{dom}(w)$ such that $w(z) = \mathbf{1}$ and $[\![x = z]\!]_{IN} \in D_{\mathbb{LP}}$, i.e., $[\![x \in w]\!]_{IN} = \mathbf{1}$. Moreover, if $u(x) = \frac{1}{2}$ we simply apply (ii) of Theorem 3.4 instead of (i) and proceed similarly as in the previous case. We can proceed similar for any $z \in \mathrm{dom}(w)$ such that $w(z) = \mathbf{1}$. Likewise, for any $z \in \mathrm{dom}(w)$ such that $w(z) = \frac{1}{2}$ we can show that $w(z) \leq [\![z \in u]\!]_{IN}$. Hence, for any $x \in \mathrm{dom}(u)$ we have $u(x) \leq [\![x \in w]\!]_{IN}$ and for any $z \in \mathrm{dom}(w)$ we have $w(z) \leq [\![z \in u]\!]_{IN}$. Hence, we may conclude $[\![u = w]\!]_{IN} \in D_{\mathbb{LP}}$. □

Lemma 3.6. *For any $u, v \in \mathbf{V}^{(\mathbb{LP})}$ and any formula $\varphi(x) \in \mathcal{L}_{\mathbb{LP}}$, if $[\![u = v]\!]_{IN} \in D_{\mathbb{LP}}$ then the following hold:*

(i) if $[\![\varphi(u)]\!]_{IN} = \mathbf{1}$ then $[\![\varphi(v)]\!]_{IN} = \mathbf{1}$,

(ii) if $[\![\varphi(u)]\!]_{IN} = \frac{1}{2}$ then $[\![\varphi(v)]\!]_{IN} = \frac{1}{2}$.

Proof. By induction on the complexity of φ.

Base case (I). (i) Let $\varphi(x) := w = x$, where $w \in \mathbf{V}^{(\mathbb{LP})}$. If $[\![u = w]\!]_{IN} = \mathbf{1}$, then by Lemma 3.5$(iii)$ we have that $[\![v = w]\!]_{IN} = \mathbf{1}$. (ii) Follows vacuously, since we have either $[\![u = v]\!]_{IN} = \mathbf{1}$ or $[\![u = v]\!]_{IN} = \mathbf{0}$ for every $u, v \in \mathbf{V}^{(\mathbb{LP})}$.

Base case (II). (i) Let $\varphi(x) := w \in x$, where $w \in \mathbf{V}^{(\mathbb{LP})}$. Suppose $[\![\varphi(u)]\!]_{IN} = \mathbf{1}$. Then, there exists a $p \in \mathrm{dom}(u)$ such that $u(p) = \mathbf{1}$ and $[\![p = w]\!]_{IN} = \mathbf{1}$. Since we have $[\![u = v]\!]_{IN} \in D_{\mathbb{LP}}$, by item (i) of Theorem 3.4, there exists $q \in \mathrm{dom}(v)$ satisfying $v(q) = \mathbf{1}$ and $[\![p = q]\!]_{IN} \in D_{\mathbb{LP}}$. By Lemma 3.5$(iii)$, $[\![q = w]\!]_{IN} \in D_{\mathbb{LP}}$, i.e., $[\![q = w]\!]_{IN} = \mathbf{1}$. So there exists a $q \in \mathrm{dom}(v)$ such that $v(q) = \mathbf{1}$ and $[\![q = w]\!]_{IN} = \mathbf{1}$, i.e., $[\![w \in v]\!]_{IN} = \mathbf{1}$. Hence $[\![\varphi(v)]\!]_{IN} = \mathbf{1}$.
(ii) Now suppose $[\![\varphi(u)]\!]_{IN} = \frac{1}{2}$. Then, there exists $p \in \mathrm{dom}(u)$ such that $u(p) = \frac{1}{2}$ and $[\![p = w]\!]_{IN} \in D_{\mathbb{LP}}$. At the same time there does also not exist any $s \in \mathrm{dom}(u)$ such that $u(s) = \mathbf{1}$ and $[\![s = w]\!]_{IN} \in D_{\mathbb{LP}}$. Since, it is given that $[\![u = v]\!]_{IN} \in D_{\mathbb{LP}}$, Theorem 3.4 ensures the existence of $q \in \mathrm{dom}(v)$ satisfying $v(q) = \frac{1}{2}$ and $[\![p = q]\!]_{IN} \in D_{\mathbb{LP}}$, in addition, there does not exist any $t \in \mathrm{dom}(v)$ such that $v(t) = \mathbf{1}$ and $[\![t = w]\!]_{IN} \in D_{\mathbb{LP}}$. By Lemma 3.5 (iii), we have $[\![q = w]\!]_{IN} \in D_{\mathbb{LP}}$. Hence $[\![\varphi(v)]\!]_{IN} = \frac{1}{2}$.

Base case (III). Let $\varphi(x) := x \in w$, where $w \in \mathbf{V}^{(\mathbb{LP})}$. (i) Let $[\![\varphi(u)]\!]_{IN} = \mathbf{1}$, i.e.,

$$\bigvee_{z \in \mathrm{dom}(w)} (w(z) \wedge [\![u = z]\!]_{IN}) = \mathbf{1}$$

So there exists a $z_1 \in \mathrm{dom}(w)$ such that $w(z_1) = \mathbf{1} = [\![z_1 = u]\!]_{IN}$. Thus, we have that $[\![u = v]\!]_{IN} \wedge [\![z_1 = u]\!]_{IN} \in D_{\mathbb{LP}}$ and by Lemma 3.5(iii), $[\![z_1 = v]\!]_{IN} \in D_{\mathbb{LP}}$. So there exists a $z_1 \in \mathrm{dom}(w)$ such that $w(z_1) = \mathbf{1} = [\![z_1 = v]\!]_{IN}$, i.e., $[\![v \in w]\!]_{IN} = \mathbf{1}$. (ii) Now, suppose; $[\![\varphi(u)]\!]_{IN} = \frac{1}{2}$, i.e.,

$$\bigvee_{z \in \mathrm{dom}(w)} (w(z) \wedge [\![u = z]\!]_{IN}) = \frac{1}{2}$$

This can only be the case if;

1. There exists $z_1 \in \mathrm{dom}(w)$ such that $w(z_1) = \frac{1}{2}$ and $[\![z_1 = u]\!]_{IN} = \mathbf{1}$.
2. For any $z \in \mathrm{dom}(w)$, if $w(z) = \mathbf{1}$ then $[\![z = u]\!]_{IN} = \mathbf{0}$.

By Lemma 3.5(iii) we have $[\![z_1 = v]\!]_{IN} \in D_{\mathbb{LP}}$. So there exists a $z_1 \in \mathrm{dom}(w)$ such that $w(z_1) = \frac{1}{2}$ and $[\![z_1 = v]\!]_{IN} \in D_{\mathbb{LP}}$, i.e., $[\![v \in w]\!]_{IN} \in D_{\mathbb{LP}}$. We shall now prove that we have $[\![v \in w]\!]_{IN} < \mathbf{1}$. Suppose otherwise, so there exists a $z_2 \in \mathrm{dom}(w)$ such that $w(z_2) = \mathbf{1} = [\![z_2 = v]\!]_{IN}$. Since $[\![u = v]\!]_{IN} = \mathbf{1}$ by Lemma 3.5(iii), we have $[\![z_2 = u]\!]_{IN} = \mathbf{1}$. So there exists a $z \in \mathrm{dom}(w)$ such that $w(z) = \mathbf{1}$ and $[\![z = u]\!]_{IN} = \mathbf{1}$.This contradicts that $[\![u \in w]\!]_{IN} = \frac{1}{2}$. Hence we get, $[\![v \in w]\!]_{IN} = \frac{1}{2}$.

Induction step:

Case (I). Let $\varphi(x) := \psi(x) \wedge \gamma(x)$. (i) If $[\![\varphi(u)]\!]_{IN} = \mathbf{1}$ then both of $[\![\psi(u)]\!]_{IN}$ and $[\![\gamma(u)]\!]_{IN}$ get value $\mathbf{1}$. By the induction hypothesis, $[\![\psi(v)]\!]_{IN}$ and $[\![\gamma(v)]\!]_{IN}$ are $\mathbf{1}$, as well. Hence $[\![\varphi(v)]\!]_{IN} = \mathbf{1}$. (ii) Now, if $[\![\varphi(u)]\!]_{IN} = \frac{1}{2}$ holds, then we have $[\![\psi(u)]\!]_{IN} = \frac{1}{2}$ or $[\![\gamma(u)]\!]_{IN} = \frac{1}{2}$. Again, by the induction hypothesis it can be concluded that $[\![\varphi(v)]\!]_{IN} = \frac{1}{2}$.

Similarly, Case II, Case III and Case IV can also be proved.

Case (II). Let $\varphi(x) := \psi(x) \vee \gamma(x)$.

Case (III). Let $\varphi(x) := \psi(x) \to \gamma(x)$.

Case (IV). Let $\varphi(x) := \neg\, \psi(x)$.

Case (V). Let $\varphi(x) := \exists y\, \psi(y, x)$. (i)Suppose $[\![\varphi(u)]\!]_{IN} = \mathbf{1}$. So there exists $p \in \mathbf{V}^{(\mathbb{LP})}$ such that $[\![\psi(p, u)]\!]_{IN} = \mathbf{1}$. Therefore, $[\![\psi(p, v)]\!]_{IN} = \mathbf{1}$, by the induction hypothesis. Hence $[\![\varphi(v)]\!]_{IN} = \mathbf{1}$.

(ii) Let $[\![\varphi(u)]\!]_{IN} = \frac{1}{2}$. Then, there exists $p \in \mathbf{V}^{(\mathbb{LP})}$ such that $[\![\psi(p, u)]\!]_{IN} = \frac{1}{2}$ and there does not exist any $q \in \mathbf{V}^{(\mathbb{LP})}$ such that $[\![\psi(q, u)]\!]_{IN} = \mathbf{1}$. The induction hypothesis ensures that $[\![\psi(p, v)]\!]_{IN} = \frac{1}{2}$ and $[\![\psi(q, v)]\!]_{IN} \neq \mathbf{1}$, for all $q \in \mathbf{V}^{(\mathbb{LP})}$. Finally, $[\![\varphi(v)]\!]_{IN} = \frac{1}{2}$.

Case (VI). Let $\varphi(x) := \forall y\, \psi(y, x)$. By an immediate application of the induction hypothesis, both (i) and (ii) can be proved in this case also. □

Hence, we obtain as corollary the validity of Leibniz's law of indiscernibility of identicals in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$.

Corollary 3.7. *For any $u, v \in \mathbf{V}^{(\mathbb{LP})}$ and any formula $\varphi(x)$ in $\mathcal{L}_{\mathbb{LP}}$ having one free variable x, if $[\![u = v]\!]_{IN} \wedge [\![\varphi(u)]\!]_{IN} \in D_{\mathbb{LP}}$ then $[\![\varphi(v)]\!]_{IN} \in D_{\mathbb{LP}}$.*

Lemma 3.8. *For any $u \in \mathbf{V}^{(\mathbb{LP})}$, and a formula $\varphi(x)$, having one free variable x, in $\mathcal{L}_{\mathbb{LP}}$,*

$$[\![\forall x\big(x \in u \to \varphi(x)\big)]\!]_{IN} = \bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow [\![\varphi(x)]\!]_{IN}\big). \tag{BQ$_\varphi$}$$

Proof. By the definition of the assignment function $[\![\cdot]\!]_{IN}$,

$$\begin{aligned}
&[\![\forall x\big(x \in u \to \varphi(x)\big)]\!]_{IN} \\
&\qquad= \bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} [\![\big(y \in u \to \varphi(y)\big)]\!]_{IN} \\
&\qquad= \bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} \Big(\bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![y = x]\!]_{IN}) \Rightarrow [\![\varphi(y)]\!]_{IN}\Big) \\
&\qquad= \bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} \bigwedge_{x \in \mathrm{dom}(u)} \big((u(x) \wedge [\![x = y]\!]_{IN}) \Rightarrow [\![\varphi(y)]\!]_{IN}\big), \text{ by Lemma 1.4} \\
&\qquad= \bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} \bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow ([\![x = y]\!]_{IN} \Rightarrow [\![\varphi(y)]\!]_{IN})\big), \text{by (P4)} \\
&\qquad= \bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} \bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow ([\![x = y]\!]_{IN} \Rightarrow [\![\varphi(x)]\!]_{IN})\big), \text{ by Lemma 3.6} \\
&\qquad= \bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} \bigwedge_{x \in \mathrm{dom}(u)} \big((u(x) \wedge [\![x = y]\!]_{IN}) \Rightarrow [\![\varphi(x)]\!]_{IN}\big), \text{by (P4).}
\end{aligned}$$

Moreover, by (P3) we conclude that,

$$\begin{aligned}
\bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow [\![\varphi(x)]\!]_{IN}\big) &= \bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} \bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow [\![\varphi(x)]\!]_{IN}\big) \\
&\leq \bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} \bigwedge_{x \in \mathrm{dom}(u)} \big((u(x) \wedge [\![x = y]\!]_{IN}) \Rightarrow [\![\varphi(x)]\!]_{IN}\big).
\end{aligned}$$

On the other hand, for any $x \in \mathrm{dom}(u)$,

$$\begin{aligned}
\bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} \big((u(x) \wedge [\![x = y]\!]_{IN}) \Rightarrow [\![\varphi(x)]\!]_{IN}\big) &\leq (u(x) \wedge [\![x = x]\!]_{IN}) \Rightarrow [\![\varphi(x)]\!]_{IN} \\
&= u(x) \Rightarrow [\![\varphi(x)]\!], \text{ using Lemma 3.5}(i),
\end{aligned}$$

which implies,

$$\bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} \bigwedge_{x \in \mathrm{dom}(u)} \big((u(x) \wedge [\![x = y]\!]_{IN}) \Rightarrow [\![\varphi(x)]\!]_{IN}\big) \leq \bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow [\![\varphi(x)]\!]\big).$$

Hence,

$$\bigwedge_{y\in\mathbf{V}^{(\mathbb{LP})}}\ \bigwedge_{x\in\mathrm{dom}(u)}\big((u(x)\wedge[\![x=y]\!]_{IN})\Rightarrow[\![\varphi(x)]\!]_{IN}\big)=\bigwedge_{x\in\mathrm{dom}(u)}\big(u(x)\Rightarrow[\![\varphi(x)]\!]\big),$$

and as a conclusion,

$$[\![\forall x\big(x\in u\to\varphi(x)\big)]\!]_{IN}=\bigwedge_{x\in\mathrm{dom}(u)}\big(u(x)\Rightarrow[\![\varphi(x)]\!]\big).$$

□

We will use the following definitions to show the validity of Choice in our model:[1]

Definition 3.9. *Let* $u\in\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$*. Then we can define the subset* $\mathrm{dom}_{pos}(u)$ *of* $\mathrm{dom}(u)$ *as*

$$\mathrm{dom}_{pos}(u)=\{x\in\mathrm{dom}(u):u(x)\neq\mathbf{0}\}.$$

Definition 3.10. *We define* $\mathrm{dom}_{pos}(u)/\sim$ *as the partition of* $\mathrm{dom}_{pos}(u)$ *by* $\sim$ *where for any* $u,v\in\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$

$$u\sim v\ \text{iff}\ \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}\models_{D_{\mathbb{LP}}}u=v.$$

It is easy to check that $\sim$ is indeed an equivalence relation. Moreover, we denote with $[x]=\{v\in\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}:\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}\models_{D_{\mathbb{LP}}}x=v\}$ the elements of $\mathrm{dom}_{pos}(u)/\sim$ where $x\in\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$. Now, we are in a position to show that ZFC holds in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$. Moreover, notice that the following proof is a modification of the proof of Theorem 3.13 of [7]. Only that this time we are considering a different algebra and interpretation function.

Theorem 3.11. $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}\models_{D_{\mathbb{LP}}}$ ZFC*.*

Proof. Extensionality: We want to show that for any $u,v\in\mathbf{V}^{(\mathbb{LP})}$ we have

$$\begin{aligned}&[\![\forall w(w\in u\leftrightarrow w\in v)\to u=v]\!]_{IN}\\&=\big([\![\forall w(w\in u\leftrightarrow w\in v)]\!]_{IN}\Rightarrow[\![u=v]\!]_{IN}\big)\\&\in D_{\mathbb{LP}}.\end{aligned}$$

Suppose $[\![\forall w(w\in u\leftrightarrow w\in v)]\!]_{IN}\leq\frac{1}{2}$, then we get immediately $[\![\mathsf{Extensionality}]\!]_{IN}\in D_{\mathbb{LP}}$. So let $[\![\forall w(w\in u\leftrightarrow w\in v)]\!]_{IN}=\mathbf{1}$, i.e.,

$$\bigwedge_{w\in\mathrm{dom}(u)}[\![(u(w)\to w\in v)]\!]_{IN}\wedge\bigwedge_{w\in\mathrm{dom}(v)}[\![(v(w)\to w\in u)]\!]_{IN}=\mathbf{1}$$

This can only be the case if

[1] We would like to acknowledge Sourav Tarafder who was the first one to provide a complete proof of Choice in an algebra-valued model of a paraconsistent set theory in [19].

(1) for every $w \in \mathrm{dom}(u)$, $u(w) \leq [\![w \in v]\!]_{IN}$ and

(2) for every $w \in \mathrm{dom}(v)$, $v(w) \leq [\![w \in u]\!]_{IN}$.

Then by Definition 3.1 we have $[\![u = v]\!]_{IN} = \mathbf{1}$. Therefore, $[\![\mathsf{Extensionality}]\!]_{IN} \in D_{\mathbb{LP}}$.

Pairing: We show that for any two $u, v \in \mathbf{V}^{(\mathbb{LP})}$ there exists a $w \in \mathbf{V}^{(\mathbb{LP})}$ such that

$$[\![\forall z\big(z \in w \to (z = u \vee z = v)\big)]\!]_{IN} \wedge [\![\forall z\big((z = u \vee z = v) \to z \in w\big)]\!]_{IN} \in D_{\mathbb{LP}}.$$

We begin by showing that the first conjunct of Pairing holds. Consider two arbitrary $u, v \in \mathbf{V}^{(\mathbb{LP})}$. Let w be such that $\mathrm{dom}(w) = \{u, v\}$ and $\mathrm{ran}(w) = \{\mathbf{1}\}$. Then, applying BQ_φ it is enough to show that

$$\bigwedge_{z \in \mathrm{dom}(w)} \big(w(z) \Rightarrow [\![(z = u \vee z = v)]\!]_{IN}\big) \in D_{\mathbb{LP}}.$$

Now take any $z \in \mathrm{dom}(w)$ such that $w(z) \in D_{\mathbb{LP}}$, then due to the construction of w we have $[\![z = u \vee z = v]\!]_{IN} = \mathbf{1}$. Thus, it follows that the first conjunct of Pairing holds. Now, we show that also the second conjunct of Pairing holds. Take any $z \in \mathbf{V}^{(\mathbb{LP})}$ and suppose that $[\![z = u \vee z = v]\!]_{IN} \in D_{\mathbb{LP}}$. Then, by the construction of w, we get immediately that $[\![z \in w]\!]_{IN} \in D_{\mathbb{LP}}$. Therefore, it follows that the second conjunct of Pairing holds, as well. Hence, we get $[\![\mathsf{Pairing}]\!]_{IN} \in D_{\mathbb{LP}}$.

Infinity: We define for each $x \in \mathbf{V}$, where $\mathbf{V}$ is the ground model, $\check{x} = \{\langle \check{y}, 1\rangle : y \in x\}$. Then, we go on to show that

$$[\![\forall z \neg(z \in \varnothing)]\!]_{IN} \wedge [\![\varnothing \in \check{\omega}]\!]_{IN} \wedge [\![\forall w\big(w \in \check{\omega} \to \exists u(u \in \check{\omega} \wedge w \in u)\big)]\!]_{IN} \in D_{\mathbb{LP}},$$

where $\varnothing$ is the empty function in $\mathbf{V}^{(\mathbb{LP})}$ and ω is the collection of all natural numbers in $\mathbf{V}$. The first two conjuncts of Infinity do clearly hold. We go on to show that also the third conjunct of Infinity holds. By applying BQ_φ it is enough to show that

$$\bigwedge_{w \in \mathrm{dom}(\check{\omega})} \big(\check{\omega}(w) \Rightarrow [\![\exists u(u \in \check{\omega} \wedge w \in u)]\!]_{IN}\big) \in D_{\mathbb{LP}}.$$

Now take any $\check{w} \in \mathrm{dom}(\check{\omega})$. By the definition of $\check{\omega}$, we have $\check{\omega}(\check{w}) = \mathbf{1}$. Therefore, $\check{w} \in \omega$ holds in $\mathbf{V}$. Now, due to Infinity in $\mathbf{V}$ we know that there exists a $u \in \mathbf{V}$ (the successor of w) such that $u \in \omega$ and $w \in u$ holds in $\mathbf{V}$. Thus $\check{u} \in \mathbf{V}^{(\mathbb{LP})}$. It can be calculated readily that $[\![\check{u} \in \check{\omega}]\!]_{IN} = \mathbf{1}$ and $[\![\check{w} \in \check{u}]\!]_{IN} = \mathbf{1}$. Therefore, $[\![\exists u(u \in \check{\omega} \wedge \check{w} \in u)]\!]_{IN} \in D_{\mathbb{LP}}$. Furthermore, since the choice of $\check{w}$ was arbitrary we get immediately that the third conjunct of Infinity holds. Hence, $[\![\mathsf{Infinity}]\!]_{IN} \in D_{\mathbb{LP}}$.

Union: We have to prove, for any $u \in \mathbf{V}^{(\mathbb{LP})}$, there exists an element $v \in \mathbf{V}^{(\mathbb{LP})}$ such that

$$[\![\forall x\big(x \in v \to \exists y(y \in u \wedge x \in y)\big)]\!]_{IN} \wedge [\![\forall x\big(\exists y(y \in u \wedge x \in y) \to x \in v\big)]\!]_{IN} \in D_{\mathbb{LP}}.$$

Take any $u \in \mathbf{V}^{(\mathbb{LP})}$ and define $v \in \mathbf{V}^{(\mathbb{LP})}$, as follows:

$$\mathrm{dom}(v) = \bigcup\{\mathrm{dom}(y) \mid y \in \mathrm{dom}(u)\} \text{ and } v(x) = [\![\exists y(y \in u \wedge x \in y)]\!]_{IN}, \text{ for } x \in \mathrm{dom}(v).$$

We first prove that the first conjunct of Union holds. Applying BQ_φ we get:

$$\begin{aligned}
&[\![\forall x\big(x \in v \to \exists y(y \in u \wedge x \in y)\big)]\!]_{IN} \\
&= \bigwedge_{x \in \mathrm{dom}(v)} \big(v(x) \Rightarrow [\![\exists y(y \in u \wedge x \in y)]\!]_{IN}\big) \\
&= \bigwedge_{x \in \mathrm{dom}(v)} \big([\![\exists y(y \in u \wedge x \in y)]\!]_{IN} \Rightarrow [\![\exists y(y \in u \wedge x \in y)]\!]_{IN}\big) \\
&\in D_{\mathbb{LP}}, \text{ since } a \Rightarrow a \in D_{\mathbb{LP}}, \text{ for any element } a \in \mathbb{LP}.
\end{aligned}$$

We now show that also the second conjunct of Union holds. Fix a $x_0 \in \mathbf{V}^{(\mathbb{LP})}$ such that $[\![\exists y(y \in u \wedge x_0 \in y)]\!]_{IN} \in D_{\mathbb{LP}}$. By definition, $[\![\exists y(y \in u \wedge x_0 \in y)]\!]_{IN} \in D_{\mathbb{LP}}$ implies that, there exists $y_0 \in \mathbf{V}^{(\mathbb{LP})}$ such that $[\![y_0 \in u \ \wedge \ x_0 \in y_0]\!]_{IN} \in D_{\mathbb{LP}}$. Now, $[\![y_0 \in u]\!]_{IN} \in D_{\mathbb{LP}}$ guarantees the existence of an element $y_1 \in \mathrm{dom}(u)$ such that $u(y_1) \in D_{\mathbb{LP}}$ and $[\![y_1 = y_0]\!]_{IN} \in D_{\mathbb{LP}}$. So, we have, $[\![y_1 = y_0 \ \wedge \ x_0 \in y_0]\!]_{IN} \in D_{\mathbb{LP}}$ and by Lemma 3.5(v) we have $[\![x_0 \in y_1]\!]_{IN} \in D_{\mathbb{LP}}$. Hence, there exists $x_1 \in \mathrm{dom}(y_1)$ such that $y_1(x_1) \in D_{\mathbb{LP}}$ and $[\![x_0 = x_1]\!]_{IN} \in D_{\mathbb{LP}}$. Since, by our assumption, $u(y_1) \in D_{\mathbb{LP}}$ and $y_1(x_1) \in D_{\mathbb{LP}}$, by Lemma 3.5($ii$) we get $[\![y_1 \in u]\!]_{IN} \in D_{\mathbb{LP}}$ and $[\![x_1 \in y_1]\!]_{IN} \in D_{\mathbb{LP}}$ hold. Hence, $[\![y_1 \in u \ \wedge \ x_1 \in y_1]\!]_{IN} \in D_{\mathbb{LP}}$, which leads to the fact that $[\![\exists y(y \in u \ \wedge \ x_1 \in y)]\!]_{IN} \in D_{\mathbb{LP}}$, i.e., $v(x_1) \in D_{\mathbb{LP}}$. So, we have derived that, $[\![x_0 = x_1]\!]_{IN} \in D_{\mathbb{LP}}$ and $v(x_1) \in D_{\mathbb{LP}}$. Hence, $[\![x_0 \in v]\!]_{IN} \in D_{\mathbb{LP}}$. Therefore, it follows that the second conjunct of Union holds. Thus, $[\![\mathsf{Union}]\!]_{IN} \in D_{\mathbb{LP}}$.

Power Set: We have to prove that for any $u \in \mathbf{V}^{(\mathbb{LP})}$ there exists a $v \in \mathbf{V}^{(\mathbb{LP})}$ such that

$$[\![\forall z\big(z \in v \to \forall w(w \in z \to w \in u)\big)]\!]_{IN} \wedge [\![\forall z\big(\forall w(w \in z \to w \in u) \to z \in v\big)]\!]_{IN} \in D_{\mathbb{LP}}.$$

We begin by showing that the first conjunct of Power Set holds. So, take any $u \in \mathbf{V}^{(\mathbb{LP})}$ and define v such that

$$\mathrm{dom}(v) = \mathbb{LP}^{\mathrm{dom}(u)} \text{ and for any } z \in \mathrm{dom}(v), v(z) = [\![\forall w(w \in z \to w \in u)]\!]_{IN}.$$

Applying BQ_φ we get:

$$\begin{aligned}
&[\![\forall z(z \in v \to \forall w(w \in z \to w \in u))]\!]_{IN} \\
&= \bigwedge_{z \in \mathrm{dom}(v)} \big(v(z) \Rightarrow [\![\forall w(w \in z \to w \in u)]\!]_{IN}\big) \\
&= \bigwedge_{z \in \mathrm{dom}(v)} \big([\![\forall w(w \in z \to w \in u)]\!]_{IN} \Rightarrow [\![\forall w(w \in z \to w \in u)]\!]_{IN}\big) \\
&\in D_{\mathbb{LP}}, \text{ since } a \Rightarrow a \in D_{\mathbb{LP}}, \text{ for any element } a \in \mathbb{LP}.
\end{aligned}$$

For the second conjunct of Power Set, fix an arbitrary $z \in \mathbf{V}^{(\mathbb{LP})}$. Then,

$$\begin{aligned}
&[\![\forall w(w \in z \to w \in u)]\!]_{IN} \Rightarrow [\![z \in v]\!]_{IN} \\
&= \bigwedge_{w \in \mathrm{dom}(z)} \big(z(w) \Rightarrow [\![w \in u]\!]_{IN}\big) \Rightarrow \bigvee_{q \in \mathrm{dom}(v)} \big(v(q) \wedge [\![z = q]\!]_{IN}\big), \text{ by BQ}_{\varphi} \\
&= \bigwedge_{w \in \mathrm{dom}(z)} \big(z(w) \Rightarrow [\![w \in u]\!]_{IN}\big) \Rightarrow \bigvee_{q \in \mathrm{dom}(v)} \Big(\bigwedge_{p \in \mathrm{dom}(q)} \big(q(p) \Rightarrow [\![p \in u]\!]_{IN}\big) \wedge [\![z = q]\!]_{\mathrm{PA}}\Big).
\end{aligned}$$

Let us assume that,

$$\bigwedge_{w \in \mathrm{dom}(z)} \big(z(w) \Rightarrow [\![w \in u]\!]_{IN}\big) = \mathbf{1}.$$

Then it is enough to show that there exists a $q_0 \in \mathrm{dom}(v)$ for which,

$$\bigwedge_{p \in \mathrm{dom}(q_0)} \big(q_0(p) \Rightarrow [\![p \in u]\!]_{IN}\big) \wedge [\![z = q_0]\!]_{IN}\big) \in D_{\mathbb{LP}}. \tag{$\dagger$}$$

Notice that, for any $q \in \mathrm{dom}(v)$, we have $\mathrm{dom}(q) = \mathrm{dom}(u)$. Fix $q_0 \in \mathrm{dom}(v)$ such that $q_0(p) = (u(p) \wedge [\![p \in z]\!]_{IN})$, for any $p \in \mathrm{dom}(q_0)$. The first conjunct of (†) follows by Lemma 3.5(ii). Now, we show that $[\![z = q_0]\!]_{IN} \in D_{\mathbb{LP}}$. In the case that $z(w) = \mathbf{0}$ we are done. So for some $w \in \mathrm{dom}(z)$ let $z(w) \in D_{\mathbb{LP}}$. By our assumption, $z(w) \leq [\![w \in u]\!]_{IN}$.Hence, there exists $p_0 \in \mathrm{dom}(u)$ such that $z(w) \leq (u(p_0) \wedge [\![p_0 = w]\!]_{IN}) \in D_{\mathbb{LP}}$. Now, we notice that, $[\![p_0 \in z]\!]_{IN} \geq (z(w) \wedge [\![p_0 = w]\!]_{IN}) \in D_{\mathbb{LP}}$, by our assumptions. Hence, we get:

$$\begin{aligned}
[\![w \in q_0]\!]_{IN} &= \bigvee_{p \in \mathrm{dom}(q_0)} (q_0(p) \wedge [\![p = w]\!]_{IN}) \\
&\geq (q_0(p_0) \wedge [\![p_0 = w]\!]_{IN}), \text{ since } p_0 \in \mathrm{dom}(q_0) \text{ as well} \\
&= (u(p_0) \wedge [\![p_0 \in z]\!]_{IN} \wedge [\![p_0 = w]\!]_{IN}) \\
&\geq z(w).
\end{aligned}$$

Now, take any $p \in \mathrm{dom}(q_0)$ and notice that

$$q_0(p) = (u(p) \wedge [\![p \in z]\!]_{IN}) \leq [\![p \in z]\!]_{IN}.$$

Therefore, we get that the second conjunct of (†) holds. Thus, $[\![\text{Power Set}]\!]_{IN} \in D_{\mathbb{LP}}$.

Separation$_{\varphi}$: Let $\varphi(x)$ be any formula in $\mathcal{L}_{\mathbb{LP}}$, where x is the only free variable. We want to show that for any $u \in \mathbf{V}^{(\mathbb{LP})}$ there exists a $v \in \mathbf{V}^{(\mathbb{LP})}$ such that

$$[\![\forall z\big(z \in v \to (z \in u \wedge \varphi(z))\big)]\!]_{IN} \wedge [\![\forall z\big((z \in u \wedge \varphi(z)) \to z \in v\big)]\!]_{IN} \in D_{\mathbb{LP}}.$$

For any $u \in \mathbf{V}^{(\mathbb{LP})}$ define $v \in \mathbf{V}^{(\mathbb{LP})}$ as follows:

$$\mathrm{dom}(v) = \mathrm{dom}(u) \text{ and for any } z \in \mathrm{dom}(v) \text{ let } v(z) = (u(z) \wedge [\![\varphi(z)]\!]_{IN}).$$

We go on to prove that the first conjunct of $\mathsf{Separation}_\varphi$ holds. Applying BQ_φ, we have:

$$\begin{aligned}
&[\![\forall z\big(z \in v \rightarrow (z \in u \wedge \varphi(z))\big)]\!]_{IN} \\
&= \bigwedge_{z \in \mathrm{dom}(v)} \big(v(z) \Rightarrow [\![z \in u \wedge \varphi(z)]\!]_{IN}\big) \\
&= \bigwedge_{z \in \mathrm{dom}(v)} \big((u(z) \wedge [\![\varphi(z)]\!]_{IN}) \Rightarrow ([\![z \in u]\!]_{IN} \wedge [\![\varphi(z)]\!]_{IN})\big) \\
&\in D_{\mathbb{LP}}, \text{ by Lemma 3.5(ii).}
\end{aligned}$$

Now, we show that the second conjunct of $\mathsf{Separation}_\varphi$ holds as well. Since, **P4** holds for $\mathbb{LP}$, we have

$$\begin{aligned}
\bigwedge_{z \in \mathbf{V}^{(\mathbb{LP})}} [\![\big(z \in u \wedge \varphi(z)\big) \rightarrow z \in v]\!]_{IN} &= \bigwedge_{z \in \mathbf{V}^{(\mathbb{LP})}} \big([\![z \in u]\!]_{IN} \Rightarrow ([\![\varphi(z)]\!]_{IN} \Rightarrow [\![z \in v]\!]_{IN})\big) \\
&= [\![\forall z\big(z \in u \rightarrow (\varphi(z) \rightarrow z \in v)\big)]\!]_{IN} \\
&= \bigwedge_{z \in \mathrm{dom}(u)} \big(u(z) \Rightarrow [\![(\varphi(z) \Rightarrow z \in v)]\!]_{IN}\big).
\end{aligned}$$

Fix a $z_0 \in \mathrm{dom}(u)$ such that $u(z_0) \in D_{\mathbb{LP}}$ and $[\![\varphi(z_0)]\!]_{IN} \in D_{\mathbb{LP}}$. Then by construction of v we have $v(z_0) \in D_{\mathbb{LP}}$ and by Lemma 3.5(ii) we get $[\![z_0 \in v]\!]_{IN} \in D_{\mathbb{LP}}$. Therefore, we can conclude that for any $z_0 \in \mathrm{dom}(u)$, $\big(u(z_0) \Rightarrow [\![\varphi(z_0) \Rightarrow z_0 \in v]\!]_{IN}\big) \in D_{\mathbb{LP}}$. Hence, the second conjunct of $\mathsf{Separation}_\varphi$ holds. Thus, $[\![\mathsf{Separation}_\varphi]\!]_{IN} \in D_{\mathbb{LP}}$.

$\mathsf{Replacement}_\varphi$: Let $\varphi(x, y)$ be any formula in the language of set theory with two free variables. We want to proof that for every $u \in \mathbf{V}^{(\mathbb{LP})}$ we have

$$[\![\forall x\big(x \in u \rightarrow \exists y \varphi(x, y)\big) \rightarrow \exists v \forall x\big(x \in u \rightarrow \exists y(y \in v \wedge \varphi(x, y))\big)]\!]_{IN} \in D_{\mathbb{LP}}.$$

Take any $u \in \mathbf{V}^{(\mathbb{LP})}$ and assume the antecedent of $\mathsf{Replacement}_\varphi$ holds. Applying BQ_φ, we have;

$$\bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow \bigvee_{y \in \mathbf{V}^{(\mathbb{LP})}} [\![\varphi(x, y)]\!]_{IN}\big) \in D_{\mathbb{LP}}. \tag{1}$$

Now, we will show that the consequent of $\mathsf{Replacement}_\varphi$ holds. We know that $\mathbb{LP}$ is a set, so $\mathbb{LP} \in \mathbf{V}$. Thus, we may apply $\mathsf{Replacement}_\varphi$ in $\mathbf{V}$ so that for any $x \in \mathrm{dom}(u)$ we obtain an ordinal α_x such that

$$\bigvee_{y \in \mathbf{V}^{(\mathbb{LP})}} [\![\varphi(x, y)]\!]_{IN} = \bigvee_{y \in \mathbf{V}^{(\mathbb{LP})}_{\alpha_x}} [\![\varphi(x, y)]\!]_{IN}.$$

So we have

$$\bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow \bigvee_{y \in \mathbf{V}^{(\mathbb{LP})}_{\alpha_x}} [\![\varphi(x, y)]\!]_{IN}\big) \in D_{\mathbb{LP}}. \tag{2}$$

We apply Union in $\mathbf{V}$ to define $\alpha = \bigcup\{\alpha_x : x \in \mathrm{dom}(u)\}$. Then, we define the element $v \in \mathbf{V}^{(\mathbb{LP})}$ as $\mathrm{dom}(v) = \mathbf{V}^{(\mathbb{LP})}_{\alpha}$ and for every $y \in \mathrm{dom}(v), v(y) = \mathbf{1}$. We move on to show,

$$\bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow [\![\exists y\big(y \in v \wedge \varphi(x,y)\big)]\!]_{IN}\big) \in D_{\mathbb{LP}}.$$

Take any $x_0 \in \mathrm{dom}(u)$ such that $u(x_0) \in D_{\mathbb{LP}}$. By (2) we have $y_0 \in \mathbf{V}^{(\mathbb{LP})}_{\alpha_{x_0}}$ such that $[\![\varphi(x_0, y_0)]\!]_{IN} \in D_{\mathbb{LP}}$. By our construction $y_0 \in \mathrm{dom}(v)$ and $v(y_0) = \mathbf{1}$. Then, it follows by Lemma 3.5(ii) that $[\![y_0 \in v]\!]_{IN} \in D_{\mathbb{LP}}$. Therefore, $[\![y_0 \in v \wedge \varphi(x_0, y_0)]\!]_{IN} \in D_{\mathbb{LP}}$ and thus,

$$[\![\exists y\big(y \in v \wedge \varphi(x_0,y)\big)]\!]_{IN} \in D_{\mathbb{LP}}.$$

Since the choice of x_0 is arbitrary it follows that the consequent of $\mathsf{Replacement}_{\varphi}$ holds. Therefore, we conclude that $[\![\mathsf{Replacement}_{\varphi}]\!]_{IN} \in D_{\mathbb{LP}}$.

$\mathsf{Regularity}_{\varphi}$: We want to show that

$$[\![\forall x\big(\forall y(y \in x \to \varphi(y)) \to \varphi(x)\big) \to \forall x \varphi(x)]\!]_{IN} \in D_{\mathbb{LP}}.$$

Thus take any $x \in \mathbf{V}^{(\mathbb{LP})}$ and consider the following two cases:
(i) Let $[\![\varphi(x)]\!]_{IN} \in D_{\mathbb{LP}}$ for every $x \in \mathbf{V}^{(\mathbb{LP})}$, which implies $[\![\forall x \varphi(x)]\!]_{IN} \in D_{\mathbb{LP}}$. Therefore, we get immediately that $[\![\mathsf{Regularity}_{\varphi}]\!]_{IN} \in D_{\mathbb{LP}}$.
(ii) Let $[\![\varphi(x)]\!]_{IN} \notin D_{\mathbb{LP}}$ for some $x \in \mathbf{V}^{(\mathbb{LP})}$. Then, take a minimal $u \in \mathbf{V}^{(\mathbb{LP})}$ such that $[\![\varphi(u)]\!]_{IN} \notin D_{\mathbb{LP}}$ and for any $y \in \mathrm{dom}(u) \in D_{\mathbb{LP}}$ and $[\![\varphi(y)]\!]_{IN} \in D_{\mathbb{LP}}$. Using BQ_{φ} and our assumption, it is immediate that

$$[\![\forall y(y \in u \to \varphi(y))]\!]_{IN} \in D_{\mathbb{LP}}.$$

Now, we have two cases:

(1) $[\![\forall y(y \in u \to \varphi(y))]\!]_{IN} = \frac{1}{2}$ or

(2) $[\![\forall y(y \in u \to \varphi(y))]\!]_{IN} = \mathbf{1}$.

However, in both cases we get

$$[\![\forall x\big((\forall y(y \in x \to \varphi(y)) \to \varphi(x)\big)]\!]_{IN} \leq \frac{1}{2}.$$

Hence, the antecedent of $\mathsf{Regularity}_{\varphi}$ receives a value less or equal to $\frac{1}{2}$. Thus we have $[\![\mathsf{Regularity}_{\varphi}]\!]_{IN} \in D_{\mathbb{LP}}$.

Choice: Fix an arbitrary non-empty $u \in \mathbf{V}^{(\mathbb{LP},\, [\![\cdot]\!]_{IN})}$, i.e., $\mathbf{V}^{(\mathbb{LP},\, [\![\cdot]\!]_{IN})} \models \neg(u = \varnothing)$. In particular, this means $\mathrm{dom}_{pos}(u) \neq \varnothing$. Now, take any $[x] \in \mathrm{dom}_{pos}(u)/\sim$ and consider the following two cases.
Case (I): Suppose that $[x]$ does not contain any $\mathbf{0}$-like element. Fix an element $s_{[x]} \in [x]$.

By our assumption, we know that $\mathrm{dom}_{pos}(s_{[x]}) \neq \varnothing$. Moreover, choose a $t_{[x]} \in \mathrm{dom}_{pos}(s_{[x]})$. Then we define three elements $p_{[x]}, q_{[x]}, w_{[x]} \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ such that

$$p_{[x]} = \{\langle s_{[x]}, \mathbf{1}\rangle\}, q_{[x]} = \{\langle s_{[x]}, \mathbf{1}\rangle, \langle t_{[x]}, \mathbf{1}\rangle\} \text{ and } w_{[x]} = \{\langle p_{[x]}, \mathbf{1}\rangle, \langle q_{[x]}, \mathbf{1}\rangle\}.$$

Case (II): Suppose that $[x]$ is the class of $\mathbf{0}$-like elements in $\mathrm{dom}_{pos}(u)$. Let us arbitrarily fix any two $\mathbf{0}$-like elements $s, t \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$. Following the same construction as in Case I, we define three elements $p_{[x]}, q_{[x]}, w_{[x]} \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ such that

$$p_{[x]} = \{\langle s_{[x]}, \mathbf{1}\rangle\}, q_{[x]} = \{\langle s_{[x]}, \mathbf{1}\rangle, \langle t_{[x]}, \mathbf{1}\rangle\} \text{ and } w_{[x]} = \{\langle p_{[x]}, \mathbf{1}\rangle, \langle q_{[x]}, \mathbf{1}\rangle\}.$$

Then consider an element f such that

$$f = \{\langle w_{[x]}, \mathbf{1}\rangle : [x] \in \mathrm{dom}(u)/\sim\}.$$

The existence of f in $\mathbf{V}$ follows by the fact that Choice holds in $\mathbf{V}$. Then, by the construction $f \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$. Furthermore, it can be shown readily that

$$\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_{D_{\mathbb{LP}}} \mathsf{Func}(f) \wedge \mathsf{Dom}(f;\ u).$$

We are done if we prove that

$$\forall x\big(x \in u \wedge \neg(x = \varnothing) \to \exists z \exists y(\mathsf{Pair}(z;\ x, y) \wedge z \in f \wedge y \in x)\big).$$

Consider any $v \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ such that

$$\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_{D_{\mathbb{LP}}} v \in u \wedge \neg(v = \varnothing).$$

Then there exists an element $x \in \mathrm{dom}_{pos}(u)$ such that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_{D_{\mathbb{LP}}} v = x$, and x is not $\mathbf{0}$-like. Consider the equivalence class $[x]$ containing x in $\mathrm{dom}_{pos}(u)/\sim$. By the construction of f, there exists $w_{[x]} \in \mathrm{dom}(f)$ which is of the form $\{\langle p_{[x]}, \mathbf{1}\rangle, \langle q_{[x]}, \mathbf{1}\rangle\}$, where

$$p_{[x]} = \{\langle s_{[x]}, \mathbf{1}\rangle\}, q_{[x]} = \{\langle s_{[x]}, \mathbf{1}\rangle, \langle t_{[x]}, \mathbf{1}\rangle\}, s_{[x]} \in [x] \text{ and } t_{[x]} \in \mathrm{dom}_{pos}(s_{[x]}).$$

Since $s_{[x]} \in [x]$, we get $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_{D_{\mathbb{LP}}} s_{[x]} = x$, which implies $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_{D_{\mathbb{LP}}} s_{[x]} = v$. Hence, we can derive that

$$\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_{D_{\mathbb{LP}}} \mathsf{Pair}(w_{[x]};\ v, t_{[x]}) \wedge w_{[x]} \in f \wedge t_{[x]} \in v.$$

Thus, we can finally conclude that $[\![\mathsf{Choice}]\!] \in D_{\mathbb{LP}}$. □

Moreover, we show that the model $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ modulo the set of designated values $D_{\mathbb{LP}}$ is non-classical.

Lemma 3.12. *There exists a formula $\varphi \in \mathsf{Sent}_{\in}$ such that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_{D_{\mathbb{LP}}} \varphi \wedge \neg\varphi$.*

Proof. Consider the following sentence: $\varphi := \exists x \exists y(x \in y \wedge x \notin y)$. Now, simply consider the $\mathbb{LP}$-name $u = \{\langle v, \frac{1}{2}\rangle\}$ where v is an arbitrary $\mathbb{LP}$-name. We readily calculate that $[\![v \in u]\!]_{IN} = \frac{1}{2}$, as well as $[\![v \notin u]\!]_{IN} = [\![v \in u]\!]^*_{IN} = \frac{1}{2}^* = \frac{1}{2}$. Hence,

$$[\![\varphi \wedge \neg\varphi]\!]_{IN} = (\frac{1}{2} \wedge \frac{1}{2}^*) = \frac{1}{2} \in D_{\mathbb{LP}},$$

which completes the proof. □

It is interesting to notice that paraconsistent set theories have traditionally been explored with the purpose of accommodating $\mathsf{Comprehension}_\varphi$ and the contradictions that follow from it within a non-trivial set theory. We call these set theories, *naïve* paraconsistent set theories. Some of the advocates of this approach are authors such as [3], [4], [16], [18], and [23]. Moreover, the main argument of this approach goes along the following lines; the cost of removing contradictions of our universe and maintaining the underlying logic of our set theory is higher than the cost of weakening our underlying logic and accepting them. In particular, the LP-set theories that we can find in the existing literature follow this approach.

However, the LP-set theory that we have developed follows a different approach. We believe that a paraconsistent set theory does not necessarily have to validate the theory axioms of naïve set theory, rather it should have a model that resembles as closely as possible the cumulative hierarchy. We call these set theories, *iterative* paraconsistent set theories. Moreover, our LP-set theory is an iterative set theory as demonstrated by Theorem 3.11 (where we show that ZFC and, in particular, $\mathsf{Regularity}_\varphi$ holds, which is arguably the essential axiom behind the iterative conception for [6]) and Lemma 3.14 (where we show that $\mathsf{Comprehension}_\varphi$ fails).

The main motivation for developing an iterative paraconsistent set, rather than a naïve one, is that we can avoid the following criticism by [6, Chapter 4] against paraconsistent set theories. His arguments can be resumed by the following dilemma.

(1) Either a paraconsistent set theory is too *weak*,

(2) or a paraconsistent set theory is *unfaithful* to its underlying conception.

The first horn of the dilemma demands that a paraconsistent set theory should be both non-trivial and mathematically expressive. The second horn, on the other hand, dictates that a paraconsistent set theory should be coherent with respect to their basic (philosophical) assumptions. In the case of the naïve conception of set, this means for instance, that we can not argue on behalf of the intuitiveness of $\mathsf{Comprehension}_\varphi$ and at the same time give up on principles which seem equally intuitive (such as Leibniz's law of indiscernibility of identicals).

Moreover, to be precise, this criticism is only directed against naïve paraconsistent set theories that follow the material, relevant, or model-theoretic strategy, presented in [16]. However, it seems that this criticism can be extended to any naïve paraconsistent theory since the mentioned strategies are regarded as the most promising and well-known candidates of the naïve approach. In general, we believe that mathematical strength and faithfulness offer two solid maxims which any non-classical set theory should satisfy. Furthermore, in [8] it was shown that there exist iterative paraconsistent set theories which are both, strong enough to carry out a reasonable amount of standard set theory and which are faithful (to a reasonable degree) to the iterative conception of set. Thus, avoiding the mentioned criticism by [6].

We acknowledge that the iterative approach has also a certain drawback. By adopting an iterative paraconsisent set theory we abandon the intuitive appeal of $\mathsf{Comprehension}_\varphi$. However, we bite the bullet. We still believe that iterative paraconsistent set theories can be insightful for the study of classical and non-classical models of set theory. On the one hand, we have shown in this paper and in others, such as [10] and [7], that the class of models of ZFC and ZF exceeds the class of Boolean and Heyting-valued models of set theory. On the other hand, we can actually explore paraconsistent models of set theory which are mathematically very expressive as shown in [21]. Thus, the main imput of the models provided by the iterative approach, is that they allow us to do a reasonable amount of mathematics in a paraconsistent framework.

Despite this fact, we believe that both approaches to paraconsistent set theory are similar in spirit, they agree that the logical axioms of set theory should correspond to a paraconsistent logic, however, they disagree regarding which conception of set is the right one. So, in a certain way we can consider the iterative approach a radicalized version of the naïve approach. Because we are not only modifying the logical axioms of naïve set theory, but also the non-logical ones.

Lemma 3.13. *For any $u \in \mathbf{V}^{(\mathbb{LP})}$ we have $[\![u \in u]\!]_{IN} = \mathbf{0}$ and thus $[\![\exists y \forall x(x \in y)]\!]_{IN} = \mathbf{0}$.*

Proof. Suppose we have a minimal counterexample to the claim. So there exists a u such that $[\![u \in u]\!]_{IN} \neq \mathbf{0}$ (†) , however, for every $x \in \mathrm{dom}(u)$ we have $[\![x \in x]\!]_{IN} = \mathbf{0}$. Due to (†) we know that there exists a $x_0 \in \mathrm{dom}(u)$ such that $u(x_0) \neq \mathbf{0}$ and $[\![u = x_0]\!]_{IN} \neq \mathbf{0}$. In particular, $[\![u = x_0]\!]_{IN} = \mathbf{1}$, so for every $x \in \mathrm{dom}(u)$ we have $u(x) \leq [\![x \in x_0]\!]_{IN}$. Moreover, given that $u(x_0) \neq \mathbf{0}$ we have $[\![x_0 \in x_0]\!]_{IN} \neq \mathbf{0}$ which delivers us the desired contradiction. Hence, for any $u \in \mathbf{V}^{(\mathbb{LP})}$ we have $[\![u \in u]\!]_{IN} = \mathbf{0}$ and thus $[\![\exists y \forall x(x \in y)]\!]_{IN} = \mathbf{0}$. □

Now, we show that $\mathsf{Comprehension}$ fails in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$.

Lemma 3.14. $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \not\models_{D_{\mathbb{LP}}} \mathsf{Comprehension}_\varphi$.

Proof. Consider $\varphi =_{df.} y \notin y$. Due to Lemma 3.13, for every $u \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$, $[\![u \in u]\!]_{IN} = \mathbf{0}$ and thus $[\![u \notin u]\!]_{IN} = [\![u \in u]\!]^*_{IN} = \mathbf{1}$. Then we get:

$$[\![u \in u \leftrightarrow u \notin u]\!]_{IN} = \big((\mathbf{0} \Rightarrow \mathbf{1}) \wedge (\mathbf{1} \Rightarrow \mathbf{0})\big) = \mathbf{0}.$$

□

3.1 The logics of $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$

We go on to show that the model $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ is indeed faithful to the lattice $\mathbb{LP}$.

Theorem 3.15. *The model $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ is faithful to $\mathbb{LP}$ and hence loyal to $(\mathbb{LP}, D)$, for any set of designated values D.*

Proof. We know that the sentence $\forall x(x = x)$ receives value $\mathbf{1}$, i.e., $[\![\forall x(x = x)]\!]_{IN} = \mathbf{1}$ and hence we have $[\![\neg\forall x(x = x)]\!]_{IN} = \mathbf{0}$. So we are done in the case that we can find a sentence $\varphi \in \mathsf{Sent}_\in$ such that $[\![\varphi]\!]_{IN} = \frac{1}{2}$. Simply consider sentence φ of Lemma 3.12. □

Moreover, it is a well-known fact that the propositional logic associated to the lattice $\mathbb{LP}$ modulo $D_{\mathbb{LP}}$ is LP and that the propositional logic associated to the same lattice given the set of designated values that contains only the top element, i.e., $\{\mathbf{1}\}$, is Kleene's Logic K_3. Thus we get:

Corollary 3.16. $\mathbf{L}\big(\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}, D_{\mathbb{LP}}\big) = \mathrm{LP}$. □

Corollary 3.17. $\mathbf{L}\big(\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}, \{\mathbf{1}\}\big) = \mathrm{K}_3$. □

It is easy to notice that Theorem 3.4, Lemma 3.5, Lemma 3.6, Corollary 3.7 and Lemma 3.8 are still valid in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ given the set of designated values $\{\mathbf{1}\}$. Introspection of the relevant proofs shows that exactly the same calculations work for this case. The validity of ZFC, however, does not extend to this model due to the failure of $\mathsf{Extensionality}$.

Theorem 3.18. $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \not\models_{\{\mathbf{1}\}} \mathsf{Extensionality}$

Proof. Consider the $\mathbb{LP}$-names $p_{\frac{1}{2}}$ and $p_{\mathbf{1}}$. Then we calculate readily:

$$
\begin{aligned}
&[\![\mathsf{Extensionality}]\!]_{IN}\\
&= [\![\forall x \forall y\big(\forall z(z \in x \leftrightarrow z \in y) \to x = y\big)]\!]_{IN}.\\
&= \bigwedge_{u\in\mathbf{V}^{(\mathbb{LP})}} \bigwedge_{v\in\mathbf{V}^{(\mathbb{LP})}} \big([\![\forall z(z \in u \leftrightarrow z \in v)]\!]_{IN} \Rightarrow [\![u = v]\!]_{IN}\big)\\
&= \bigwedge_{u\in\mathbf{V}^{(\mathbb{LP})}} \bigwedge_{v\in\mathbf{V}^{(\mathbb{LP})}} \Big(\big(\bigwedge_{x\in \mathrm{dom}(u)} (u(x) \Rightarrow [\![x \in v]\!]_{IN}) \wedge \bigwedge_{y\in \mathrm{dom}(v)} (v(y) \Rightarrow [\![y \in u]\!]_{IN})\big) \Rightarrow [\![u = v]\!]_{IN}\Big)\\
&\leq \big((p_{\mathbf{1}}(\varnothing) \Rightarrow [\![\varnothing \in p_{\frac{1}{2}}]\!]_{IN}) \wedge (p_{\frac{1}{2}}(\varnothing) \Rightarrow [\![\varnothing \in p_{\mathbf{1}}]\!]_{IN})\big) \Rightarrow [\![u = v]\!]_{IN}\\
&= \big((\mathbf{1} \Rightarrow \frac{1}{2}) \wedge (\frac{1}{2} \Rightarrow \mathbf{1})\big) \Rightarrow \mathbf{0}\\
&= \frac{1}{2} \notin \{\mathbf{1}\}.
\end{aligned}
$$

□

The failure of $\mathsf{Extensionality}$ shows that if we choose a more classical set of designated values, i.e., $\{\mathbf{1}\}$, on $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ we end up with a model which is properly speaking not a model of set theory anymore. What do we learn from this? On the one hand, it shows that the choice of the set of designated values is relevant, unlike in the case of Boolean-valued models and the standard interpretation map $[\![\cdot]\!]$. Thus, even though the $[\![\cdot]\!]_{IN}$-interpretation map allowed us to build a non-classical model of ZFC based on LP, it comes with a price: we are bounded to one particular choice of designated values.

On the other hand, it is curious that in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ modulo $\{\mathbf{1}\}$ we get Leibniz's law of indiscernibility of identicals but not Extensionality since they are both conditions that describe identity. In particular, Leibniz's law is normally regarded as the stronger condition of both (see [8] for a more detailed discussion on these two conditions). Thus, $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ modulo $\{\mathbf{1}\}$ is the first algebra-valued model of set theory to our knowledge where Leibniz's law of indiscernibility of identicals is valid, but not Extensionality.

Moreover, given that Extensionality is a sentence in the language of set theory we can show that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ and $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ are non-$\in$-elementarily equivalent with each other.

Corollary 3.19. *We have*

$$(\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}, D_{\mathbb{LP}}) \not\equiv_{\in} (\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}, \{\mathbf{1}\}).$$

□

Moreover, we can show as well that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ is non-$\in$-elementarily equivalent with other algebra-valued models of paraconsistent set theory. In particular, we will compare $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ to the class of $(\mathbb{T}, \sim)$-valued models presented in [10].

Lemma 3.20. *Let* $D_{\mathbb{A}} = \{x : x \in A \wedge x \neq \mathbf{0}\}$*, then we have*

$$\big(\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}, D_{\mathbb{LP}}\big) \not\equiv_{\in} \big(\mathbf{V}^{((\mathbb{T},\sim),\ [\![\cdot]\!])}, D_{\mathbb{T}}\big).$$

Proof. Consider the following sentence

$$\psi =_{df.} \forall w\big(\exists xy(x \in y \wedge x \notin y) \rightarrow (w \neq w)\big).$$

We know by Lemma 3.5(i) that $[\![\forall w(w = w)]\!]^{\mathbb{LP}}_{IN} = \mathbf{1}$. Thus we get

$$[\![\neg\forall x(x = x)]\!]^{\mathbb{LP}}_{IN} = \big([\![\forall x(x = x)]\!]^{\mathbb{LP}}_{IN}\big)^* = 1^* = \mathbf{0}.$$

Moreover, due to Lemma 3.12 we get

$$[\![\psi]\!]^{\mathbb{LP}}_{IN} = (\frac{1}{2} \Rightarrow \mathbf{0}) = \frac{1}{2} \in D_{\mathbb{LP}}.$$

Then, we calculate readily $[\![\neg\forall w(w = w)]\!]^{(\mathbb{T},\sim)} = \mathbf{0}$ and $[\![\exists xy(x \in y \wedge x \notin y)]\!]^{(\mathbb{T},\sim)} = a$, where a is the co-atom of the universe of $\mathbb{T}$. We conclude

$$[\![\psi]\!]^{(\mathbb{T},\sim)}_{IN} = (a \Rightarrow \mathbf{0}) = \mathbf{0} \notin D_{\mathbb{T}}.$$

□

Similarly, if we choose $\{\mathbf{1}\}$ as set of designated values then we still have:

Lemma 3.21. *Let* $D = \{\mathbf{1}\}$, *then*

$$\left(\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}, \{\mathbf{1}\}\right) \not\equiv_{\in} \left(\mathbf{V}^{((\mathbb{T},\sim),\ [\![\cdot]\!])}, \{\mathbf{1}\}\right).$$

Proof. Consider the sentence $\sigma =_{df.} \psi \to \psi$, where ψ is the same sentence we were considering in the proof of Lemma 3.20. We calculate readily that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \not\models_{\{\mathbf{1}\}} \sigma$. However, we have $\mathbf{V}^{((\mathbb{T},\sim),\ [\![\cdot]\!])} \models_{\{\mathbf{1}\}} \sigma$. □

Notice that Lemma 3.20 and Lemma 3.21 are still valid in the case that we choose the same interpretation map $[\![\cdot]\!]$ on both $\mathbf{V}^{(\mathbb{LP})}$ and $\mathbf{V}^{(\mathbb{T},\sim)}$. Similarly, considerations hold for the choice of the interpretation map $[\![\cdot]\!]_{IN}$ for both models.

4 Comparision to Priest's model-theoretic strategy

In this section, we will briefly present Priest's set-theoretic models based on LP and compare them to the models developed in this paper. In particular, we presuppose familiarity with [17].

We have the following result (applying Priest's partition construction):

Theorem 4.1 ([17])**.** *Suppose that $\mathcal{M}$ is a classical model of* ZF *containing two inaccessible cardinals κ_1 and κ_2. Then there is an LP-model $\mathcal{M}^\sim = \langle D^\sim, I^\sim \rangle$ such that:*

(i) $\mathcal{M}^\sim$ *is a model of* ZF + $\mathsf{NLP}_=$ *and*

(ii) $\mathcal{M}^\sim$ *contains a model* $\mathcal{N}$ *where* $\mathcal{N}$ *is a classical model of* ZF.

This means, in particular, that we can construct a model of ZF which is compatible with LP. More importantly, in this model, we do not only have the validity of the theory axioms of ZF, but we get, as well, all the theorems of ZF. Nevertheless, we have two decisive drawbacks:

1. $\mathcal{M}^\sim$ is constructed by leaving alone *only* a proper fragment of the original model.

2. A single set a is the witness for all the instances of $\mathsf{Comprehension}_\varphi$.

To avoid these problems, Priest has used another construction which he calls the *Hamkins type-lift*. First of all, notice that Theorem 4.1 applies, as well, to the models produced by the type-lift. The type-lift construction is a big advance compared to the partition construction concerning problem (1) given that the partition construction is produced by leaving alone the *entire* original model. Furthermore, using the type-lift we obtain models where different sets witness different instances of $\mathsf{Comprehension}_\varphi$, thus providing more discriminating models. Thus, this construction also provides a solution to problem (2).

Priest concludes;

> Of the ways we have looked at for constructing models of ZF and naive set theory, the Hamkins type-lift produces perhaps the most natural candidate for a model of the universe of sets. The other constructions deliver a model of the cumulative hierarchy (as defined in ZF) with inconsistent sets inside it. The Hamkins construction delivers a consistent model of the cumulative hierarchy with extra inconsistent sets. ([17], p. 105)

Nevertheless, we have again a decisive drawback with this model construction which concerns the treatment of identity, i.e., Leibniz's law of indiscernibility of identicals fails. We, on the other hand, have shown that we can build an algebra-valued model that validates ZFC, which has as internal logic LP and which preserves all the intuitive properties we would like to attribute to identity. Moreover, we observe that we have two fundamental differences in our model construction.

(a) $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ is **not** a model of $\mathsf{Comprehension}_{\varphi}$.

(b) $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ does **not** allow us to derive all the theorems of ZF.

Notice that (a) is due to Lemma 3.14. Without going into technical details we believe that it might be possible to extend our algebra-valued model with class functions that might be used to interpret the universal set. Nevertheless, it is unclear how exactly our underlying model $\mathbf{V}$ has to look like. What we can say here is that $\mathbf{V}$ should be a model of a class theory, so we can talk about class-functions in our extended algebra-valued model and that the resulting model should avoid problem (2) of Priest's model-theoretic approach.[2]

Moreover, (b) is a drawback compared to Priest's model construction since it is unclear how many theorems of ZF we can derive in our model. At the same time, this is also a distinctive feature of our approach, i.e., the possibility of determining the set of valid theorems of ZF within a non-classical model of set theory. We leave this task for future work. However, we believe that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ is an excellent candidate for this enterprise due to the validity of Leibniz's law of indiscernibility of identicals. In particular, we can build a quotient model out of $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ following the strategy outlined in [9].

Finally, instead of arguing that one model construction is preferable over another one, we simply acknowledge that two different games are played. Whereas Priest is concerned in showing that we can make sense of the classical theorems of ZF from a paraconsistent perspective, we are concerned in finding out how much set theory we can obtain in a paraconsistent model of ZFC. It seems that a moral that we can draw from this is that we can not have the whole cake. There exists a trade-off between the validity of ZF (and its theorems) and a classical notion of identity. In the case of Priest we get the validity of ZF and all of its theorems, however, we have a non-classical notion of identity. In our case,

[2] We have tried to use a class theory à la NBG (i.e., a class theory that extends ZFC) and showed that the respective does not validate Comprehension. But we recognize that our result is not even near to be conclusive since other set theories that allow classes in their ontology, such as MK (Morse-Kelly set theory) or NF (New Foundation), might do the trick. We leave this as an open problem.

we get a model of the theory axioms of ZFC with a classical notion of identity, but we do not get all the theorems of ZFC. Thus, forcing identity to behave properly within our algebra-valued models is unfortunately not enough to obtain a model of all the theorems of ZFC.

5 Conclusion

In this paper, we offer new contributions to the study of LP-set theories. Before, pointing out these contributions we would like to situate this work within a more general project that explores the construction of algebra-valued models for paraconsistent set theories.

The technical main results of this project (that we have used for this paper) can be found in [9] and [19]. In particular, in [9], it was shown that there exists an interpretation map (viz. $[\![\cdot]\!]_{\mathrm{PA}}$) that allows us to validate an classically equivalent formulation of ZF for a class of algebra-valued models. Thus, providing the first paraconsistent models of full ZF. Additionally, in [19] it was shown that the same class of algebras together with the standard interpretation map (viz. $[\![\cdot]\!]$) gives rise to models of the negation-free fragment of ZFC.

However, non of these two interpretation maps applied to $\mathbb{LP}$ produce an algebra-valued model that validates ZFC (precisely because we have a non-classical notion of identity in these models). So, we loose Leibniz's law of indiscernibility of identicals and thus some axiom schemes such as $\mathsf{Separation}_\varphi$ fail. We could only overcome this difficulty by modifying the interpretation of identity and membership in our algebra-valued model. In particular, we had to tailor the interpretation map in such a way that the resulting notion of identity would be classical. This constitutes precisely the novelty of this paper: the $[\![\cdot]\!]_{IN}$-interpretation map. However, there remains an important open question which we leave for future work:

Open question: Which is the class of algebras that gives rise to models of ZFC under the $[\![\cdot]\!]_{IN}$-interpretation function ?

Let us now turn to the contributions towards LP-set theory. We have showed that we can build LP-models that come very close to the cumulative hierarchy and where identity behaves classically. Thus, from a classical perspective, these models are nicer than the LP-models that we find in the existing literature. A particular nice feature of these LP-models is that we can actually carry out a considerable amount of mathematics within these models as witnessed by Theorem 3.11. However, these models come at a big price. Namely, we do not get a model of all the theorems of ZFC and, even worse, if we choose the classical set of designated values then we do not even get a model of the theory axioms of ZFC.

Finally, notice that, we have used the interpretation map $[\![\cdot]\!]_{IN}$, in the case of LP, because we needed a classical notion of identity given that the glutty notion of identity was causing all the problems in previous model constructions. However, we can also try to apply a similar strategy for the construction of all kind of non-classical models of set theory. Thus, we conclude with the following open question.

Open Question: Given any non-classical logic $\mathbf{L}(\mathbb{A}, D)$, is it possible to define an interpretation function $[\![\cdot]\!]_X$ such that $\mathbf{V}^{(\mathbb{A},[\![\cdot]\!]_X)} \models_D \mathsf{ZFC}$? What are the minimal algebraic properties that the operations of $\mathbb{A}$ have to satisfy so we can find such an interpretation map ?

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