The Strict/Tolerant Family Continued: Quantifiers and Modalities

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Abstract

This paper continues my work of [12] and [13], which showed there is a broad family of propositional many valued logics that have a strict/tolerant counterpart. Here we generalize those results from propositional to a range of both modal and quantified many valued logics, providing strict/tolerant counterparts for all. This paper is not self-contained; some results from [12] are called on, and are not reproved here. The key new machinery added to earlier work, allowing modalities and quantifiers to be handled in similar ways, is the central use of bilattices that are function spaces, and more generally lattices that are function spaces. Two versions of the central proofs are considered, one at length and the other in outline.

Keywords: strict/tolerant logic, bilattice, many-valued logic, modal logic

1 Introduction

I begin with a quick sketch of my work leading up to this paper, followed by a longer general discussion of just what it is that I'm talking about here. This paper is the fourth and probably last in a series. The first, [15], introduced a mechanism for constructing strict/tolerant propositional logics that correspond to a wide range of many-valued logics, generalizing the basic ideas of [3], [5] to an infinite family. The second and main paper, [12], presented this generalization in much more detail, and showed it extended to the hierarchy constructions of [2], [27]. The third, [13], examined to what extent the results extend to strict/tolerant logics based on natural generalizations of the weak Kleene three valued logic. Finally, the present paper further extends the work to encompass modal operators and quantifiers. In all cases, including this paper, the basic construction is largely the same. It is the scope of applicability of that construction that is being examined.

Let us represent a consequence relation using sequent notation, $\Gamma \Longrightarrow \Delta$, where Γ and Δ are finite sets of formulas built up using conjunction, disjunction, and negation. (Later modal and quantificational machinery will be considered as well.) Strict/Tolerant logic, ST, is a three-valued logic in which such a consequence relation holds premises to higher standards than it does the conclusions drawn from them. Say the truth values are $\{0, \frac{1}{2}, 1\}$, which also serve as truth values for both Kleene's strong three valued logic, $\mathsf{K_3}$, and for Priest's logic of paradox, LP. Truth tables for conjunction, disjunction, and negation in ST are the same as those for $\mathsf{K_3}$, which are the same as those for LP. The logics $\mathsf{K_3}$ and LP differ in their choice of *designated* truth values: $\mathsf{K_3}$ uses $\{1\}$, while LP adopts the more generous $\{\frac{1}{2}, 1\}$. ST mixes things by using the Kleene standard for premises of its consequence relation, but allowing the Priest standard for conclusions. This is the precise meaning attached to what was loosely described above as holding premises to higher standards than conclusions.

Remarkably, ST has the same set of valid sequents as classical logic, but differs from classical logic at the (local) meta-consequence level. This challenges a common view that logics are characterized by their consequence relations. Starting from this point, an entire hierarchy of strict/tolerant logics has been developed, whose members agree with classical logic on validities up to some meta-meta-meta-... level, but differ at the next. A dual version involving anti-validities has also been created. All this has given rise to much research centering around the questions of what constitutes a logic, and what makes two logics the same, or different. The primary ST developments are found in [2], [3], [5], [27], [29]. The present paper is the fourth in a sequence containing my contributions to the subject, all showing that the main features connecting classical logic and ST are more widespread than had first been realized: there is a broad range of many valued logics with strict/tolerant counterparts, and now that extends to many valued modal and quantificational logics as well. All of my papers are based on the same algebraic construction, which will be discussed informally after sketching the necessary background.

In the first of my papers on this subject, [15], the basic idea was to use the machinery of bilattices, associating them with many valued logics meeting certain natural conditions, then extracting appropriate strict/tolerant logics from the associated bilattices. My proofs were direct generalizations of the previous work of others for ST. Even the underlying ideas were the same, when properly looked at. In the three valued logic based on $\{0, \frac{1}{2}, 1\}$, one thinks of 0 as false, 1 as true, and one might think of $\frac{1}{2}$ as both true and false. When starting with a many valued logic extending 0 and 1, the role of bilattices is to provide a set of values, analogous to $\frac{1}{2}$, representing various degrees of inconsistency into which the original many values can be merged. Of course this is rather vague, but the similarities with the original ST are rather clear. What is surprising is the very broad range of propositional logics to which they can be applied.

The second of my papers on strict/tolerant logics, [12], carried this work further. First, the use of bilattice machinery was examined more carefully, and specific algebraic details about the resulting strict and tolerant designated sets were extracted. And second, the work on hierarchies of strict/tolerant logics, [2], [3], [5], [29], was extended to a broad range of many valued logics as were the results from [27] on anti-validity. The fundamentals were the same as in [15], but more of interest was extracted from them.

After [12] appeared, I realized that my conditions were too strong. Specifically, I had required that for the many valued logics I considered, the set of designated values should be a prime filter. It turned out that only being upward closed was needed, and in fact was all that I used. Details of the proofs need no changes. In [13] this was made explicit, and that usage is continued here. It was in [13], the third paper in my series, that I examined the general question of which of the many-valued logics from my earlier papers had a strict/tolerant counterpart in which the connectives obeyed conditions that generalized those of Kleene's *weak* three valued logic instead of his strong version. Briefly stated, it turned out to be those many-valued logics that were orthocomplemented. That is, for which $x \wedge \neg x = 0$ and $x \vee \neg x = 1$.

In the present paper I show that my results generally extend to a similarly broad range of many valued logics with quantifiers, and also with modal operators (which are close relatives of quantifiers). This should not be too surprising, and has been noted before for the original ST setting. In the modal case, for instance, one applies my methods world by world, verifying that these world by world results all fit together appropriately. Arguments are needed, of course, but they are rather natural. There is also a second approach, along somewhat different lines, that could have been taken, and this is discussed briefly in the concluding section of the paper. The modal logics considered here are all generalizations of normal modal logics. It is likely that the methods extend to generalizations of regular modal logics as well, but this has not been verified. How far things can be pushed is a subject for a different time, and perhaps a different investigator.

It must be emphasized that in the present paper *the terminology and results of [12] are assumed known, along with the weaker conditions presented in [13].* There is too much to repeat here in any detail, though there will be partial summaries as needed.

Here is a quick sketch of the basic idea for the modal case—the quantificational case is analogous—and details are given later in this paper. Semantically, modal frames can be used with many valued logics so that each world individually behaves in a many valued way; see [23] for instance. The basic idea here is that such a setup itself can be thought of as a single more complex many valued logic. Suppose L is the set of truth values of some many-valued logic, and $\mathcal{G}$ is a set of possible worlds. The function space $L^{\mathcal{G}}$ can be taken to be a many valued logic in its own right. Since we generally require a many valued logic L to be a lattice, the function space $L^{\mathcal{G}}$ will inherit a pointwise ordering and will, itself, be a lattice. Thus we have natural function space interpretations for $\wedge$ and $\vee$. We also require a truth value space L to be a Morgan algebra (a term introduced in [12] for a De Morgan algebra without a distributivity condition). This too is inherited pointwise by the function space $L^{\mathcal{G}}$, and provides a natural interpretation for negation. All this means that our earlier work can be applied, not to the underlying space L of truth values but to the function space $L^{\mathcal{G}}$. In a similar way the quantificational setting can be treated as a function space.

There do need to be some enhancements to our earlier work. The function space idea as sketched so far, using $L^{\mathcal{G}}$, provides no interpretation for the modal operators, or for quantifiers. This is quite straightforward, and exact details are deferred until later but essentially, in the same way that conjunction and disjunction can be modeled by lattice meet and join, quantifiers can be modeled algebraically by arbitrary meets and joins. However, we now must allow that such meets and joins may be infinitary. Likewise modal operators act like quantifiers relativized to accessible worlds. What this tells us is that a simple lattice structure for the underlying space L of truth values is not enough. We will need closure under *infinite* meets and joins as well, so we will need a *complete* lattice. There are no unexpected complications that result from a completeness condition, all earlier work extends easily, but details need to be checked.

Finally, there is actually more than one natural notion of consequence. 'Local' and 'global' are common pieces of terminology here; see Section 8 of [12] for instance. In modal logics there is a long and explicit history of such a distinction. In a Kripke model based on classical logic, for instance, we have *local* consequence: $\Gamma \Longrightarrow \Delta$ is validated by valuation v if, at *at each possible world* where every member of Γ is true under v, some member of Δ is true under v. But we also have *global* consequence: $\Gamma \Longrightarrow \Delta$ is validated by valuation v provided, if every member of Γ is valid under v in the model, then some member of Δ is valid under v in the model, where being *valid under v in a model* means that v assigns the value true at each possible world of the model. These are far from being the same. For instance, $\Box(X \wedge Y) \Longrightarrow \Box X$ is locally valid (no matter what the frame), and it easily follows that it is globally valid. But $X \Longrightarrow \Box X$ is globally valid, while it is not locally so. These examples were modal, but there are similar examples involving quantifiers. Using a function space $L^{\mathcal{G}}$, associated with a many valued Kripke model as briefly discussed above, both local and global consequence have their associated sets of designated truth values, and they are different.

But now it is time to begin introducing our formal material. Sections 2, 3, and 4 are background, to establish terminology and notation. Our new material begins in Section 5

2 Lattice Background

Lattices are fundamental to every aspect of our approach to the subject. In particular, truth value spaces will be lattices, generally denoted L to suggest 'lattice.' Here are standard definitions and terminology that we will use.

Definition 2.1 (Lattices) A *lattice* is a set with a partial ordering, $\langle L, \leq \rangle$, in which every pair of elements has a greatest lower bound or meet, and a least upper bound, or join. We generally abbreviate lattice $\langle L, \leq \rangle$ by L when no confusion will result. We denote meet of x and y by $x \wedge y$ and join by $x \vee y$. A lattice is *complete* if there are meets and joins for arbitrary subsets. For a subset S of complete lattice L its meet is denoted $\bigwedge S$ and its join by $\bigvee S$. A lattice is *bounded* if it has a least and a greatest element. A complete lattice is automatically bounded since $\bigwedge \emptyset$ is the largest member and $\bigvee \emptyset$ is the smallest.

Lattices will be our spaces of truth values. Generally some truth values are called *designated*. These can be thought of as generalized versions of *true* from classical logic. We typically denote the set of such values by D, to suggest 'designated.' It should be emphasized that we always make the general assumption that D is proper, that is, it is not empty and does not contain all truth values.

Definition 2.2 (Designated Set Conditions) Let $L = \langle L, \leq \rangle$ be a lattice, assumed complete for 4 and 5, and let D be a proper subset of L. D is:

1. *upward closed*, or *upclosed*, or an *upset*, if $x \in D$ and $x \leq y$ implies $y \in D$;
2. a *filter* if it is upward closed and $x, y \in D$ implies $x \wedge y \in D$;
3. a *prime filter* if it is a filter and $x \vee y \in D$ implies $x \in D$ or $y \in D$;
4. a *complete filter* if it is upward closed and, for every $E \subseteq D$, $\bigwedge E \in D$;
5. a *completely prime filter* it is a complete filter and for every $E \subseteq L$, if $\bigvee E \in D$ then $E \cap D \neq \emptyset$.

Trivially every complete filter is a filter, and every completely prime filter is a prime filter. Lattices generalize the usual truth value space, containing truth and falsehood. They serve in this role throughout, but a special kind is needed for semantically generalizing quantified and modal logics.

Definition 2.3 (Function Space Lattices) Suppose $L = \langle L, \leq \rangle$ is a lattice and S is a non-empty set. The function space L^S of all functions from S to L, is assigned a pointwise ordering: for $f, g \in L^S$, $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in S$.

It is straightforward to check that L^S is a lattice itself, with pointwise operations:

$$(f \wedge g)(x) = f(x) \wedge g(x),$$
$$(f \vee g)(x) = f(x) \vee g(x).$$

Further, if L is complete so is L^S, because for $E \subseteq L^S$ one can show

$$(\bigwedge E)(x) = \bigwedge \{f(x) \mid f \in E\},$$
$$(\bigvee E)(x) = \bigvee \{f(x) \mid f \in E\}.$$

Suppose L is a lattice and $D \subseteq L$. As a set, L^S has D^S as a subset. D^S inherits several properties from D, but this does not extend to being a prime filter. The following provides a way around this problem, if it actually is a problem. [I suspect this exists in the literature, and there is already terminology for it, but I can't find it.]

Definition 2.4 (Local and Global Designated Sets) Let L be a lattice, D be a subset of L, and S be a non-empty set. We call D^S a *global designated set*. And, for each $a \in S$ let $L^S_{a \to D}$ be the set of all functions from S to L that map a to some member of D. We call $L^S_{a \to D}$ a *local designated set*. The ordering of both sets of functions is pointwise, inherited from L^S.

Although we won't actually need most of the following for our main results, we state it all for the record.

Proposition 2.5 *For lattice L, $D \subseteq L$, and non-empty set S:*

1. *If D is upward closed in L then both D^S and $L^S_{a \to D}$ are upward closed on L^S, for every $a \in S$;*
2. *if D is a filter on L then both D^S and $L^S_{a \to D}$ are filters on L^S, for every $a \in S$;*

3. *if L is a complete lattice and D is a complete filter, both D^S and $L^S_{a\to D}$ are complete filters on L^S, for every $a \in S$;*
4. *if D is a prime filter on L then $L^S_{a\to D}$ is a prime filter on L^S, for every $a \in S$;*
5. *if L is a complete lattice and D is a completely prime filter then $L^S_{a\to D}$ is a completely prime filter on L^S, for every $a \in S$;*
6. *D^S is never a prime filter on L^S provided S has more than one member and L is a bounded lattice;*
7. *if L is a complete lattice then D^S is never a completely prime filter on L^S provided S has more than one member;*
8. *if L is a complete lattice then $D^S = \bigwedge_{a\in S} L^S_{a\to D}$ (here the meet operation is that of the function space D^S).*

Proof This is left to the reader, except that we supply details for 6. Suppose L is bounded, S has more than one member, and D is a prime filter in S. Since D is proper it cannot contain the least member of L, call it 0, because of upward closure, but likewise since D is non-empty it must contain the greatest member of L, call it 1. Suppose that S contains a and b, which are different. Let $f, g \in L^S$ be such that $f(a) = 0$ and otherwise $f(x) = 1$, and $g(b) = 0$ and otherwise $g(x) = 1$. Then $f \vee g \in D^S$ since it is the function that is identically 1, but neither f nor g is in D^S. ∎

We will see that these two kinds of filters, D^S and $L^S_{a\to D}$, are related to the local and global distinction mentioned in Section 1.

3 Morgan Algebra Background

Lattices provide machinery for interpreting conjunctions and disjunctions, but not negations. For this De Morgan algebras could be appropriate tools, but these require distributivity which we never need. In [12] we gave the name *Morgan algebra* to a system like a De Morgan algebra but without a requirement of distributivity, and we continue that practice here. (Dropping 'De' is to suggest dropping *d*istributivity.)

Definition 3.1 (Morgan Algebra) A *Morgan algebra* is a bounded distributive lattice with an order reversing involution, which we write as an overbar. Then, in a Morgan algebra, $a \le b$ implies $\overline{b} \le \overline{a}$, and also $\overline{\overline{a}} = a$. A *De Morgan algebra* is a Morgan algebra in which both distributive laws hold.

In a Morgan algebra $\overline{a \vee b} = \overline{a} \wedge \overline{b}$ and $\overline{a \wedge b} = \overline{a} \vee \overline{b}$. If a Morgan algebra is complete as a lattice, we also have infinitary versions, $\overline{\bigwedge E} = \bigvee \overline{E}$ and $\overline{\bigvee E} = \bigwedge \overline{E}$, where $\overline{E} = \{\overline{x} \mid x \in E\}$. The smallest and largest members of a Morgan algebra are mapped to each other by the involution. Morgan algebras provide us with natural truth value spaces and interpretations for $\wedge$, $\vee$, and $\neg$ as meet, join, and involution, and we assume this is how propositional formulas are generally evaluated from now on. Quantifiers will be interpreted using infinitary meets and joins, and modal operators similarly.

Function space lattices were characterized in Definition 2.3. If L is not just a lattice but a Morgan algebra and S is a non-empty set, then L^S, is also a Morgan algebra, where for $f \in L^S$ the involution is given pointwise by $\overline{f}(x) = \overline{f(x)}$.

Morgan algebras provide a family of truth value spaces but as noted earlier, to have logics we also need sets of *designated* truth values, and we want these to have some natural structural properties. Being a prime filter is common. In [12] we used the term *logical Morgan algebra* for the combination of Morgan algebra and a prime filter, but now we want primeness to be optional. We generally follow the terminology used in [13], but in addition we also need conditions of completeness to handle infinitary operations. This leads us to the following.

Definition 3.2 (Logical Morgan Algebras) Let L be a Morgan algebra and $D \subseteq L$. The structure $\langle L, D\rangle$ is:

1. an *upclosed logical Morgan algebra* if D is an upwardly closed proper subset of the lattice L;
2. an *upclosed complete logical Morgan algebra* if it is an upclosed logical Morgan algebra and L is complete lattice.

4 Many-Valued Logic Background

A semantically defined many valued propositional logic typically is characterized using some set of truth values, some operations on those truth values corresponding to the operation symbols of a propositional language, and a subset of truth values called designated. This give a very wide range of possibilities. Here things are considerably restricted within this general framework.

Our basic propositional language is built up from propositional letters using the operation symbols $\wedge$ (conjunction), $\vee$ (disjunction), and $\neg$ (negation). We use $\mathcal{L}$ for the formal language just described. Further on in the paper we will consider richer languages and different language names will be used, but all languages will include the connectives of $\mathcal{L}$. Note that we do not have an implication primitive. One can, of course, take implication as defined, $\neg X \vee Y$. Whether or not this is a good choice depends on the Morgan algebra used, and what features of implication are desired. A detailed investigation of implication in this context is called for, but is beyond the scope of the present paper.

A truth value space here will always be a Morgan algebra, with meet, join, and involution used to interpret conjunction, disjunction, and negation. Other logical operations will be added as appropriate. The subset of designated truth values will always be upwards closed, and thus the underlying many valued structure used propositionally is that of an upclosed logical Morgan algebra, Definition 3.2.

A *propositional valuation* in a weak logical Morgan algebra $\langle L, D\rangle$ is a mapping v from the atomic formulas of the language $\mathcal{L}$ to the Morgan algebra L. It is extended (uniquely) to all formulas using the recursive conditions

$$\begin{aligned} v(X \wedge Y) &= v(X) \wedge v(Y) \\ v(X \vee Y) &= v(X) \vee v(Y) \\ v(\neg X) &= \overline{v(X)} \end{aligned} \tag{1}$$

where, for instance, $\wedge$ on the left is an formal symbol of $\mathcal{L}$ and on the right is the meet operation of L. We use v for both the mapping on atomic formulas and for its extension to all formulas, and are generally casual about the distinction between them.

A *sequent* is an ordered pair of finite sets of formulas. Typically sequents are written with an arrow between the two sets. We will be using $\Longrightarrow$ as the arrow, and later it will have a 'decoration' to indicate properties more specialized than what is considered in this section. Then $\Gamma \Longrightarrow \Delta$ is a sequent, where Γ and Δ are finite sets of formulas. Validity for sequents, in our many valued setting, is defined as follows.

Definition 4.1 (Many Valued Validity) Let $\langle L, D\rangle$ be a logical Morgan algebra and let v be an arbitrary valuation in it.

$$\begin{aligned} \langle L, D\rangle \models_v \Gamma \Longrightarrow \Delta \;\text{ if }\; & v(X) \in D \text{ for every } X \in \Gamma \\ & \qquad \text{implies } v(Y) \in D \text{ for some } Y \in \Delta. \\ \langle L, D\rangle \models \Gamma \Longrightarrow \Delta \;\text{ if }\; & \langle L, D\rangle \models_v \Gamma \Longrightarrow \Delta \text{ for every valuation } v. \end{aligned}$$

Of course if L is the familiar Boolean structure $\{0, 1\}$, with $0 \leq 1$, and $D = \{1\}$, a prime filter in this structure, then $\langle L, D\rangle \models \Gamma \Longrightarrow \Delta$ is the usual classical consequence relation.

5 Many-Valued Modal Logics

We detail the many-valued modal logic machinery used here. To avoid any confusion it should be noted that in [8], [9] we investigated a class of many-valued modal logics in which the frame accessibility relations themselves could be many valued, but that is not the case now. Frames here are quite standard and accessibility is a conventional classical relation. Likewise we generally do not follow [23]; in particular our propositional connectives will be confined to $\wedge$, $\vee$, and $\neg$, without an implication, and are interpreted in a Morgan algebra as sketched in Section 4. This Morgan algebra provides us with the behavior of propositional connectives within worlds. In turn it, plus a frame, induces another Morgan algebra that takes the entire set of possible worlds into account, to allow the modeling of many valued modal operators. For this it is essential that our underlying space of truth values be a *complete* Morgan algebra since modal evaluation will involve meets and joins over the frame, which itself may be infinite. Here are the details.

Syntactically we expand the propositional language $\mathcal{L}$ from Section 4 by adding the familiar prefix symbols $\Box$ and $\Diamond$, and we use $\mathcal{M}$ to designate this modal language.

Semantically we work over a modal frame, $\mathcal{F} = \langle \mathcal{G}, \mathcal{R} \rangle$, where $\mathcal{G}$ is a non-empty set of possible worlds and $\mathcal{R}$ is an accessibility relation on $\mathcal{G}$. We assume we have a *complete* Morgan algebra L, which specifies the behavior of the propositional connectives. As usual, a modal valuation can behave differently at different possible worlds, so instead of a valuation simply assigning a value in L itself, it assigns a value that depends on a possible world. Consequently a modal valuation here maps formulas of the language $\mathcal{M}$ not to L but to $L^{\mathcal{G}}$, the set of all functions from $\mathcal{G}$ to L. $L^{\mathcal{G}}$ is given a pointwise ordering, Definition 2.3, and is itself a complete Morgan algebra. Then we have the usual Morgan algebra machinery in $L^{\mathcal{G}}$ to interpret propositional connectives. In addition we introduce two mappings to interpret the modal operators.

Definition 5.1 (Modal Mappings) $\boxdot$ and $\diamonddot$ are mappings from $L^{\mathcal{G}}$ to itself. For each $f \in L^{\mathcal{G}}$:

$$\boxdot f = \lambda a. \bigwedge \{f(b) \mid b \in \mathcal{G} \text{ and } a\mathcal{R}b\}$$
$$\diamonddot f = \lambda a. \bigvee \{f(b) \mid b \in \mathcal{G} \text{ and } a\mathcal{R}b\}$$

These operations are duals, and so the present generalization of modality is not broad enough to cover logics in which necessity and possibility are independent of each other.

Proposition 5.2 (Duality) *For the mappings* $\boxdot$ *and* $\diamonddot$ *from* $L^{\mathcal{G}}$ *to itself, and for each* $f \in L^{\mathcal{G}}$, $\boxdot f = \overline{\diamonddot \overline{f}}$ *and* $\diamonddot f = \overline{\boxdot \overline{f}}$.

Proof We show the first; the second is similar. For each $f \in L^{\mathcal{G}}$ and each $a \in \mathcal{G}$:

$$\begin{aligned} \diamonddot \overline{f}(a) &= \bigvee \{\overline{f}(b) \mid b \in \mathcal{G} \text{ and } a\mathcal{R}b\} && \text{Definition 5.1} \\ &= \bigvee \{\overline{f(b)} \mid b \in \mathcal{G} \text{ and } a\mathcal{R}b\} && \text{pointwise ordering} \\ &= \overline{\bigwedge \{f(b) \mid b \in \mathcal{G} \text{ and } a\mathcal{R}b\}} && \text{Morgan involution property} \\ &= \overline{\boxdot f(a)} && \text{Definition 5.1 again} \end{aligned}$$

so $\boxdot f(a) = \overline{\diamonddot \overline{f}(a)}$. ■

The operations also have a monotonicity property that plays a fundamental role when we come to strict/tolerant logics.

Proposition 5.3 (Monotonicity) *In the complete Morgan algebra* $L^{\mathcal{G}}$*, if* $f \leq g$ *then* $\boxdot f \leq \boxdot g$ *and* $\diamonddot f \leq \diamonddot g$.

Proof Suppose $f \le g$. Since this ordering is defined pointwise, for each $b \in \mathcal{G}$, $f(b) \le g(b)$. Using the monotonicity property of infinitary meets, for arbitrary $a \in \mathcal{G}$, $\bigwedge\{f(b) \mid b \in \mathcal{G}$ and $a\mathcal{R}b\} \le \bigwedge\{g(b) \mid b \in \mathcal{G}$ and $a\mathcal{R}b\}$, and hence $(\boxdot f)(a) \le (\boxdot g)(a)$. Since a was arbitrary, $\boxdot f \le \boxdot g$. The other is similar. ■

Definition 5.4 (Modal Valuation) A *modal valuation* v in the frame $\mathcal{F} = \langle \mathcal{G}, \mathcal{R} \rangle$, with complete Morgan algebra L as underlying truth value space, is a member of $(L^{\mathcal{G}})^{\mathcal{M}}$ that meets the propositional conditions (1) on $\wedge$, $\vee$, and $\neg$ for a valuation in the algebra $L^{\mathcal{G}}$, and interprets the modal operation symbols according to Definition 5.1. In detail, we have the following.

$$\begin{aligned} v(X \wedge Y) &= v(X) \wedge v(Y) \\ v(X \vee Y) &= v(X) \vee v(Y) \\ v(\neg X) &= \overline{v(X)} \\ v(\Box X) &= \boxdot v(X) \\ v(\Diamond X) &= \diamondsuit\!\!\cdot\, v(X). \end{aligned}$$

The two modal cases of the definition above, combined with Definition 5.1, give us the following. For each formula X of $\mathcal{M}$, and for each possible world $a \in \mathcal{G}$:

$$\begin{aligned} v(\Box X)(a) &= \bigwedge\{v(X)(b) \mid b \in \mathcal{G} \text{ and } a\mathcal{R}b\} \\ v(\Diamond X)(a) &= \bigvee\{v(X)(b) \mid b \in \mathcal{G} \text{ and } a\mathcal{R}b\} \end{aligned} \tag{2}$$

Similar world-by-world versions obtain for the propositional connectives as well. For instance, if the formula $X \wedge Y$ is in $\mathcal{M}$ then $v(X \wedge Y) \in L^{\mathcal{G}}$ and $v(X \wedge Y) = v(X) \wedge v(Y)$, where $\wedge$ on the right is the meet of $L^{\mathcal{G}}$. Since members of $L^{\mathcal{G}}$ are themselves mappings, for each $a \in \mathcal{G}$, $v(X \wedge Y)(a) = (v(X) \wedge v(Y))(a)$. And then since $\wedge$ is defined pointwise in $L^{\mathcal{G}}$, $(v(X) \wedge v(Y))(a) = v(X)(a) \wedge v(Y)(a)$. Similarly for the other propositional connectives. Then for each $a \in \mathcal{G}$ we have the following, where the operations on the right are in L instead of in $L^{\mathcal{G}}$.

$$\begin{aligned} v(X \wedge Y)(a) &= v(X)(a) \wedge v(Y)(a) \\ v(X \vee Y)(a) &= v(X)(a) \vee v(Y)(a) \\ v(\neg X)(a) &= \overline{v(X)(a)} \end{aligned} \tag{3}$$

Note that Proposition 5.2 has as an immediate consequence that $v(\Box X)(a) = v(\neg\Diamond\neg X)(a)$, and $v(\Diamond X)(a) = v(\neg\Box\neg X)(a)$.

Consequence, for modal logic, is complicated, and its nuances still have some ability to confuse people. Speaking semantically, there is not just a local/global distinction to be made, but a three level set of distinctions: frame, model, world. It might help to make use of some classical notation I introduced in [7, Ch 3], and again in [10, Sect 1.4]. This is for background only, and no actual use will be made of it in what follows. Suppose F is a set of frames and S, U, and V are sets of formulas. The notation $S \models_F U \to V$ means: for every (classical) Kripke model M based on a frame in F, if all members of S are true at every possible world of M, then at every world of M at which every member of U is true, some member of V is true. In what follows, a choice of frame is a parameter, so to speak, and what concerns us are the sets S, U, and V, of formulas. The members of S are *global premises*—they should hold throughout a model, that is, at every possible world. The members of U are *local premises*—we consider only those worlds where they hold. At those worlds we want some member of V to be true—this is our set of consequences.

In what follows we examine the many valued versions of model (or global) and possible world (or local) consequence. We do not consider them together, as the notation just discussed did. Instead we introduce two special purpose consequence relations, $\stackrel{m}{\Longrightarrow}$ for the model level version, and $\stackrel{p}{\Longrightarrow}$ for the possible world level version. We write $\mathsf{M}\langle L, D, \mathcal{F} \rangle$ to indicate we are considering *modal* validity based on truth values from the complete Morgan algebra L, where D, an upwardly closed set, is the set of designated values of L, and we are working over modal frame $\mathcal{F}$ (which is thus treated as a parameter). We remind the reader of Definition 2.4, for $L^S_{a\to D}$, which we will use with $S = \mathcal{G}$, the set of possible worlds of our frame.

Definition 5.5 (Modal Consequence) Let $\langle L, D\rangle$ be an upclosed complete logical Morgan algebra and let $\mathcal{F} = \langle \mathcal{G}, \mathcal{R}\rangle$ be a frame.

Model Validity:

For each modal valuation v in $L^{\mathcal{G}}$,
$\mathsf{M}\langle L, D, \mathcal{F}\rangle \models_v \Gamma \overset{m}{\Longrightarrow} \Delta$ at $a \in \mathcal{G}$ if
$v(X) \in D^{\mathcal{G}}$ for all $X \in \Gamma$ implies $v(Y) \in D^{\mathcal{G}}$ for some $Y \in \Delta$

$\mathsf{M}\langle L, D, \mathcal{F}\rangle \models \Gamma \overset{m}{\Longrightarrow} \Delta$ if
$\mathsf{M}\langle L, D, \mathcal{F}\rangle \models_v \Gamma \overset{m}{\Longrightarrow} \Delta$ for every modal valuation v in $L^{\mathcal{G}}$

Possible World Validity:

For each modal valuation v in $L^{\mathcal{G}}$,
$\mathsf{M}\langle L, D, \mathcal{F}\rangle \models_v \Gamma \overset{p}{\Longrightarrow} \Delta$ at $a \in \mathcal{G}$ if
$v(X) \in L^{\mathcal{G}}_{a\to D}$ for all $X \in \Gamma$ implies $v(Y) \in L^{\mathcal{G}}_{a\to D}$ for some $Y \in \Delta$

For each modal valuation v in $L^{\mathcal{G}}$,
$\mathsf{M}\langle L, D, \mathcal{F}\rangle \models_v \Gamma \overset{p}{\Longrightarrow} \Delta$ if
$\mathsf{M}\langle L, D, \mathcal{F}\rangle \models_v \Gamma \overset{p}{\Longrightarrow} \Delta$ at each $a \in \mathcal{G}$

$\mathsf{M}\langle L, D, \mathcal{F}\rangle \models \Gamma \overset{p}{\Longrightarrow} \Delta$ if
$\mathsf{M}\langle L, D, \mathcal{F}\rangle \models_v \Gamma \overset{p}{\Longrightarrow} \Delta$ for every modal valuation v in $L^{\mathcal{G}}$

We remark that for $\langle L, D\rangle$, if D is a prime filter then by Proposition 2.5 part 4, for each $a \in \mathcal{G}$, $L^{\mathcal{G}}_{a\to D}$ is also a prime filter, and then the condition for possible world validity is equivalent to the following.

$$\begin{aligned}&\mathsf{M}\langle L, D, \mathcal{F}\rangle \models_v \Gamma \overset{p}{\Longrightarrow} \Delta \text{ at } a \in \mathcal{G} \text{ if}\\ &\qquad \bigwedge\{v(X) \mid X \in \Gamma\} \in L^{\mathcal{G}}_{a\to D} \text{ implies} \bigvee\{v(Y) \mid Y \in \Delta\} \in L^{\mathcal{G}}_{a\to D}\end{aligned}$$

Since Γ and Δ are finite sets, primeness is enough—being completely prime is not needed for this, although completeness may come up in the evaluation of modal operators occurring in formulas of Γ or Δ. There is no similar equivalent for global validity.

Example 5.6 It is worth taking a moment to see what the two versions of modal consequence capture in the familiar classical setting. Let L be the familiar Morgan algebra of classical logic, $\{0, 1\}$ with $0 \leq 1$, with an involution operation that switches 0 and 1, and let $D = \{1\}$ be the usual set of designated truth values. Also let $\mathcal{F} = \langle \mathcal{G}, \mathcal{R}\rangle$ be any frame.

The function space filter $D^{\mathcal{G}}$ contains a single function, mapping every possible world of the frame to 1, so $v(X) \in D^{\mathcal{G}}$ simply says that v assigns truth to X at every possible world in $\mathcal{G}$. We can think of v as a modal model built on the frame $\mathcal{F}$ so, rephrased, $v(X) \in D^{\mathcal{G}}$ says X is valid in the model v; true at every possible world. Then $\mathsf{M}\langle L, D, \mathcal{F}\rangle \models_v \Gamma \overset{m}{\Longrightarrow} \Delta$ says that if every formula of Γ is valid in model v, then some formula of Δ is valid in model v.

The function space filter $L^{\mathcal{G}}_{a\to D}$ consists of functions that are arbitrary except at a, which must map to 1. So $v(X) \in L^{\mathcal{G}}_{a\to D}$ says that in the model v, X is true at possible world a. Then $\mathsf{M}\langle L, D, \mathcal{F}\rangle \models_v \Gamma \overset{p}{\Longrightarrow} \Delta$ says that in model v, at each possible world of $\mathcal{G}$ at which every formula of Γ is true, some formula of Δ is true.

Example 5.7 We are generally familiar with modal logics built on classical logic. This was the subject of the previous example. Now let $\langle L, D\rangle$ be any upclosed complete logical Morgan algebra, and let $\mathcal{F} = \langle \mathcal{G}, \mathcal{R} \rangle$ be any modal frame. We can still think of a valuation v as a modal model built on frame $\mathcal{F}$, but now it is a many valued model. We will say a valuation v validates a formula at a possible world if v maps the formula to a designated value at that world. We will say a valuation v simply validates a formula if it validates it at every possible world. Now the following is simple to check.

1. $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models_v \Gamma \overset{m}{\Longrightarrow} \Delta$ says that if valuation v validates every member of Γ then v validates some member of Δ.
2. $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models \Gamma \overset{m}{\Longrightarrow} \Delta$ says that every valuation that validates all members of Γ validates some member of Δ.
3. $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models_v \Gamma \overset{p}{\Longrightarrow} \Delta$ *at* a says if v validates every member of Γ at a, then v validates some member of Δ at a.
4. $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models_v \Gamma \overset{p}{\Longrightarrow} \Delta$ says that for each possible world, if v validates every member of Γ there, then v validates some member of Δ there.
5. $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models \Gamma \overset{p}{\Longrightarrow} \Delta$ says that for each valuation and for each possible world, if the valuation validates every member of Γ at the possible world, then the valuation validates some member of Δ there.

Here are some simple connections between the two modal consequence relations.

Proposition 5.8 *For an upclosed complete logical Morgan algebra $\langle L, D \rangle$ and a frame $\mathcal{F} = \langle \mathcal{G}, \mathcal{R} \rangle$ we have the following. (Note: being upward closed is not actually needed.)*

1. *For each valuation v in $L^{\mathcal{G}}$, if $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models_v \Gamma \overset{p}{\Longrightarrow} \{Y\}$ then $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models_v \Gamma \overset{m}{\Longrightarrow} \{Y\}$ (note that there is a singleton set on the right of the sequent arrows).*
2. *If $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models \Gamma \overset{p}{\Longrightarrow} \{Y\}$ then $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models \Gamma \overset{m}{\Longrightarrow} \{Y\}$.*
3. *Neither of the above extend generally to allow arbitrary sets on the right of the sequents.*

Proof Under the general assumptions we have the following.

1. Suppose $\mathsf{M}\langle L, D, \mathcal{F} \rangle \not\models_v \Gamma \overset{m}{\Longrightarrow} Y$. Then $v(X) \in D^{\mathcal{G}}$ for every $X \in \Gamma$ but $v(Y) \notin D^{\mathcal{G}}$. By the latter, there is some $a \in \mathcal{G}$ such that $v(Y)(a) \notin D$. Since $v(X) \in D^{\mathcal{G}}$ for every $X \in \Gamma$, then $v(X)(a) \in D$ for every $X \in \Gamma$. This tells us that $v(X) \in L^{\mathcal{G}}_{a \to D}$ for all $X \in \Gamma$ but $v(Y) \notin L^{\mathcal{G}}_{a \to D}$, so we do not have $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models_v \Gamma \overset{p}{\Longrightarrow} Y$ at a, and hence $\mathsf{M}\langle L, D, \mathcal{F} \rangle \not\models_v \Gamma \overset{p}{\Longrightarrow} Y$.
2. Item 2 follows immediately from item 1.
3. Let $\langle L, D \rangle$ be the classical frame of Example 5.6 (in which D is a prime filter). Suppose $\mathcal{G}$ has at least two elements, say a and b are in $\mathcal{G}$ where $a \neq b$. Let P, Q be atomic formulas of $\mathcal{M}$. It is easy to see that $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models_v P \vee Q \overset{p}{\Longrightarrow} P, Q$, for any valuation v. But suppose $v(P)(a) = v(Q)(b) = 1$ and $v(P)(b) = v(Q)(a) = 0$. Then $\mathsf{M}\langle L, D, \mathcal{F} \rangle \not\models_v P \vee Q \overset{m}{\Longrightarrow} P, Q$.

■

For the two modal consequence relations, the following are some general properties that are already very familiar in the usual classical setting. Note that by Proposition 5.8, items 1 – 4 transfer to the $\overset{m}{\Longrightarrow}$ consequence relation.

Proposition 5.9 *Let $\langle L, D \rangle$ be an upclosed complete logical Morgan algebra, and let $\mathcal{F} = \langle \mathcal{G}, \mathcal{R} \rangle$ be a frame.*

1. $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models_v \Box(X \wedge Y) \overset{p}{\Longrightarrow} \Box X$ *and* $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models_v \Box(X \wedge Y) \overset{p}{\Longrightarrow} \Box Y$.
2. $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models_v \Diamond X \overset{p}{\Longrightarrow} \Diamond(X \vee Y)$ *and* $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models_v \Diamond Y \overset{p}{\Longrightarrow} \Diamond(X \vee Y)$.
3. *If D is a complete filter,* $\mathsf{M}\langle L, D, \mathcal{F} \rangle \models_v \Box X, \Box Y \overset{p}{\Longrightarrow} \Box(X \wedge Y)$.

4. *If D is a completely prime filter,* $\mathsf{M}\langle L, D, \mathcal{F}\rangle \models_v \Diamond(X \vee Y) \overset{p}{\Longrightarrow} \Diamond X, \Diamond Y$.
5. *If D is a complete filter,* $\mathsf{M}\langle L, D, \mathcal{F}\rangle \models_v X \overset{m}{\Longrightarrow} \Box X$.
6. $\mathsf{M}\langle L, D, \mathcal{F}\rangle \models_v X \overset{p}{\Longrightarrow} \Box X$ *is not generally true.*

Proof Assume the hypotheses.

1. First a preliminary result. For any $b \in \mathcal{G}$, $v(X \wedge Y)(b) = [v(X) \wedge v(Y)](b) = v(X)(b) \wedge v(Y)(b) \le v(X)(b)$. So for any $a \in \mathcal{G}$, $\bigwedge\{v(X \wedge Y)(b) \mid a\mathcal{R}b\} \le \bigwedge\{v(X)(b) \mid a\mathcal{R}b\}$.
   Now let a be an arbitrary member of $\mathcal{G}$ and suppose $v(\Box(X \wedge Y)) \in L^{\mathcal{G}}_{a\to D}$. Then $v(\Box(X \wedge Y))(a) \in D$, so by definition, $\bigwedge\{v(X \wedge Y)(b) \mid a\mathcal{R}b\} \in D$. Then by the preliminary result and upward closure, $\bigwedge\{v(X)(b) \mid a\mathcal{R}b\} \in D$, and this says $v(\Box X)(a) \in D$, so $v(\Box X) \in L^{\mathcal{G}}_{a\to D}$.
2. Similar to 1.
3. Let $a \in \mathcal{G}$ be arbitrary and assume $v(\Box X), v(\Box Y) \in L^{\mathcal{G}}_{a\to D}$. Then $v(\Box X)(a) \in D$, and so $\bigwedge\{v(X)(b) \mid a\mathcal{R}b\} \in D$. Let b be any member of $\mathcal{G}$ such that $a\mathcal{R}b$. Then $\bigwedge\{v(X)(b) \mid a\mathcal{R}b\} \le v(X)(b)$ and, since D is upward closed, $v(X)(b) \in D$. Similarly $v(Y)(b) \in D$. Since D is closed under $\wedge$, $v(X \wedge Y)(b) = v(X)(b) \wedge v(Y)(b) \in D$. Since b was arbitrary, $\{v(X \wedge Y)(b) \mid a\mathcal{R}b\} \subseteq D$, and since D is a complete filter, $\bigwedge\{v(X \wedge Y)(b) \mid a\mathcal{R}b\} \in D$, that is $v(\Box(X \wedge Y))(a) \in D$, and so $v(\Box(X \wedge Y)) \in L^{\mathcal{G}}_{a\to D}$.
4. This is similar to item 3. Recall that since D is completely prime, so is $L^{\mathcal{G}}_{a\to D}$ by Proposition 2.5.
5. Assume $v(X) \in D^{\mathcal{G}}$ Then for every $b \in \mathcal{G}$ we have $v(X)(b) \in D$, and so for each $a \in \mathcal{G}$, $\{v(X)(b) \mid a\mathcal{R}b\} \subseteq D$. Then $v(\Box X)(a) = \bigwedge\{v(X)(b) \mid a\mathcal{R}b\} \in D$ because D is a complete filter.
6. A good exercise for the reader.

■

6 Many Valued Quantified Logics

Quantified many valued logic have an algebraic structure similar to that of many valued modal logic, Section 5, though of course there are substantial differences too. Here are the basic ideas.

We use $\mathcal{Q}$ for the set of formulas of a quantified language. There is an alphabet of free variables, for which we informally write x, y, etc., with or without subscripts. There is an infinite family of n-ary predicate letters, P, Q, R, etc., for each n. We do not have constant or function symbols. As punctuation there is a comma and a pair of parentheses. An atomic formula is of the form $P(x_1, \ldots, x_n)$, where P is n-ary. We have two quantifiers, $\forall$ and $\exists$. Formulas are built up from atomic formulas using $\wedge$, $\vee$, and $\neg$ in the usual way, and also $\forall xY$ and $\exists xY$ are formulas provided x is a variable and Y is a formula.

Semantically, quantifiers quantify over some domain which is required to be non-empty. The domain is a parameter of our work, much like a frame is for modal logics. Formulas can contain free variables, and a function which here is called an *assignment* maps these variables to members of the domain. Just as modal valuations behave differently at different possible worlds, a quantificational valuation will not simply assign a truth value to a formula, but to a formula *with respect to a choice of assignment*, thus assignments and possible worlds play analogous roles. We introduce some notation and terminology that helps make this analogy more obvious.

Definition 6.1 (Assignment) Let $\mathcal{V}$ be the set of variables of our language $\mathcal{Q}$, and let $\mathcal{D}$ be a domain, assumed to be non-empty. σ is an *assignment in $\mathcal{D}$* if σ is a mapping from the variables of language $\mathcal{Q}$ to $\mathcal{D}$, and thus $\sigma \in \mathcal{D}^{\mathcal{V}}$. We abbreviate $\mathcal{D}^{\mathcal{V}}$ by $\mathcal{A}$ (for 'assignment') leaving both $\mathcal{V}$ and $\mathcal{D}$ implicit.

Very much analogous to the modal case, Definition 5.1, a family of mappings is introduced to interpret quantification. These are functions from $L^{\mathcal{A}}$ to itself, where L is a many-valued space of truth values, a complete Morgan algebra. The mappings are denoted $\forall_x$ and $\exists_x$, for each variable x. Note that $L^{\mathcal{A}}$ is itself a complete Morgan algebra, with the ordering defined pointwise.

Definition 6.2 (Quantificational Mappings) Let L be a complete Morgan algebra, our space of truth values. Let $\mathcal{D}$ be a non-empty domain, and let $\mathcal{A} = \mathcal{D}^{\mathcal{V}}$ be our space of assignments in $\mathcal{D}$. For $\sigma, \tau \in \mathcal{A}$ we suggestively write $\sigma \mathcal{R}_x \tau$ to mean σ and τ agree on all variables except (possibly) for x. For each $f \in L^{\mathcal{A}}$:

$$\forall_x f = \lambda\sigma. \bigwedge\{f(\tau) \mid \tau \in \mathcal{A} \text{ and } \sigma\mathcal{R}_x\tau\}$$
$$\exists_x f = \lambda\tau. \bigvee\{f(\tau) \mid \tau \in \mathcal{A} \text{ and } \sigma\mathcal{R}_x\tau\}.$$

These operations are duals, that is, we have the following counterpart of Proposition 5.2. We omit the proof.

Proposition 6.3 (Duality) *For the mappings $\forall_x$ and $\exists_x$ from $L^{\mathcal{A}}$ to itself, and for each $f \in L^{\mathcal{A}}$, $\forall_x f = \overline{\exists_x \overline{f}}$ and $\exists_x f = \overline{\forall_x \overline{f}}$.*

Exactly as in the modal case, these operations have a monotonicity property, whose proof we omit.

Proposition 6.4 (Monotonicity) *In the complete Morgan algebra $L^{\mathcal{A}}$, if $f \leq g$ then $\forall_x f \leq \forall_x g$ and $\exists_x f \leq \exists_x g$.*

Now we come to quantificational valuations, which are much like modal valuations except that the role of individual variables needs to be taken into account.

Definition 6.5 (Quantificational Valuation) Let L be a complete Morgan algebra, our space of truth values. Let $\mathcal{D}$ be a non-empty domain, and let $\mathcal{A} = \mathcal{D}^{\mathcal{V}}$ be our space of assignments in $\mathcal{D}$. A *quantificational valuation* v is a member of $(L^{\mathcal{A}})^{\mathcal{Q}}$ that meets certain conditions. The first addresses the presence of individual variables.

A valuation v must meet the following *free variable condition.* For assignments $\sigma, \tau \in \mathcal{A}$, if σ and τ agree on the free variables of atomic formula A, then $v(A)(\sigma) = v(A)(\tau)$.

A valuation must meet the familiar conditions for $\wedge$, $\vee$, and $\neg$ from (1). In addition, it uses Definition 6.2 to interpret quantification. Specifically, we have the following.

$$\begin{aligned} v(X \wedge Y) &= v(X) \wedge v(Y) \\ v(X \vee Y) &= v(X) \vee v(Y) \\ v(\neg X) &= \overline{v(X)} \\ v(\forall x X) &= \forall_x v(X) \\ v(\exists x X) &= \exists_x v(X). \end{aligned}$$

Analogously to the modal case, the quantificational cases expand to the following.

$$\begin{aligned} v(\forall x Y)(\sigma) &= \bigwedge\{v(Y)(\tau) \mid \tau \in \mathcal{A} \text{ and } \sigma\mathcal{R}_x\tau\} \\ v(\exists x Y)(\sigma) &= \bigvee\{v(Y)(\tau) \mid \tau \in \mathcal{A} \text{ and } \sigma\mathcal{R}_x\tau\}. \end{aligned} \tag{4}$$

And of course we have the following by Proposition 6.3: $v(\forall x X)(\sigma) = v(\neg\exists x \neg X)(\sigma)$, and $v(\exists x X)(\sigma) = v(\neg\forall x \neg X)(\sigma)$.

Finally, the free variable condition from Definition 6.5 is at the atomic level, but it extends generally. The proof is by induction on formula degree, and is omitted.

Proposition 6.6 *For any formula X, if assignments σ and τ agree on the free variables of X then $v(X)(\sigma) = v(X)(\tau)$ for every valuation v.*

The terminology and general setup we use, when narrowed to two-valued logic, is not quite what one finds in treatments of classical first-order logic. A classical model specification typically has a domain $\mathcal{D}$ and an *interpretation*, assigning to each n-ary relation symbol P of the language $\mathcal{Q}$ some n-ary relation $\mathcal{I}(P)$ on $\mathcal{D}$. To mimic that with the present machinery, let us take L to be the classical two valued space, $\{0,1\}$, with the operations $\wedge$, $\vee$, $\neg$, and with Boolean negation as involution. The designated space of truth values is $D = \{1\}$. Suppose we have an interpretation $\mathcal{I}$ in the standard sense. Let us say a valuation v is $\mathcal{I}$ *compliant* if, for each n, each n-ary relation symbol P of the language, and each assignment σ,

$$v(P(x_1,\ldots,x_n))(\sigma) = 1 \text{ if } \langle\sigma(x_1),\ldots,\sigma(x_n)\rangle \in \mathcal{I}(P)$$
$$v(P(x_1,\ldots,x_n))(\sigma) = 0 \text{ otherwise.}$$

Then, using an $\mathcal{I}$ compliant valuation amounts to working in the classical model whose interpretation function is $\mathcal{I}$. Having noted this, we do not mention interpretations further.

We have a local, global split similar to the one that we had in the modal case, and so we introduce sequent notations analogous to those from Definition 5.5. Here we write $\mathsf{Q}\langle L, D, \mathcal{D}\rangle$ to indicate we are talking about *quantificational* validity using truth values from L, with D as the designated set of values, and with quantificational domain $\mathcal{D}$.

Definition 6.7 (Quantificational Consequence) Let $\langle L, D\rangle$ be an upclosed complete logical Morgan algebra and let $\mathcal{D}$ be a non-empty domain with $\mathcal{A} = \mathcal{D}^{\mathcal{V}}$ as the set of assignments.

Universal Validity:

For each quantificational valuation v in $L^{\mathcal{A}}$,
$$\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \Gamma \overset{u}{\Longrightarrow} \Delta \text{ if}$$
$$v(X) \in D^{\mathcal{A}} \text{ for all } X \in \Gamma \text{ implies } v(Y) \in D^{\mathcal{A}} \text{ for some } Y \in \Delta$$

$$\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models \Gamma \overset{u}{\Longrightarrow} \Delta \text{ if}$$
$$\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \Gamma \overset{u}{\Longrightarrow} \Delta \text{ for every quantificational valuation } v \text{ in } L^{\mathcal{A}}$$

Instance Validity:

For each quantificational valuation v in $L^{\mathcal{A}}$,
$$\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \Gamma \overset{i}{\Longrightarrow} \Delta \text{ at } \sigma \in \mathcal{A} \text{ if}$$
$$v(X) \in L^{\mathcal{A}}_{\sigma\to D} \text{ for all } X \in \Gamma \text{ implies } v(Y) \in L^{\mathcal{A}}_{\sigma\to D} \text{ for some } Y \in \Delta$$

For each quantificational valuation v in $L^{\mathcal{A}}$,
$$\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \Gamma \overset{i}{\Longrightarrow} \Delta \text{ if}$$
$$\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \Gamma \overset{i}{\Longrightarrow} \Delta \text{ at each } \sigma \in \mathcal{A}$$

$$\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models \Gamma \overset{i}{\Longrightarrow} \Delta \text{ if}$$
$$\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \Gamma \overset{i}{\Longrightarrow} \Delta \text{ for every quantificational valuation } v$$

Example 6.8 The modal Example 5.6 based on classical propositional logic has a quantificational analog. As before, let L be the classical structure $\{0,1\}$, with $D = \{1\}$ as the designated set of truth values, a complete, completely prime filter. Also let $\mathcal{D}$ be any non-empty domain of quantification. The space of valuations consists of those members of $(L^{\mathcal{A}})^{\mathcal{Q}}$ that meet the free variable condition and that respect the behavior of the propositional connectives and quantifiers (Definition 6.5), where $\mathcal{A}$ is the space of assignments.

As in Example 5.6, the function space filter $D^{\mathcal{A}}$ contains a single function, mapping every assignment in $\mathcal{A}$ to 1. Then for a valuation v, $v(X) \in D^{\mathcal{A}}$ says that v assigns truth to X under every assignment of values in $\mathcal{D}$ to variables. This is equivalent to saying that v maps the universal closure of X to 1, that is, to truth. Then $\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \Gamma \overset{u}{\Longrightarrow} \Delta$ says that if the universal closure of every formula of Γ is valid in model v, then the universal closure of some formula of Δ is valid in model v.

The function space filter $L^{\mathcal{D}}_{\sigma\to D}$ consists of functions that are arbitrary on assignments except for σ, which maps to 1. So $v(X) \in L^{\mathcal{D}}_{\sigma\to D}$ says that in the model v, X is true under assignment σ. And then $\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \Gamma \overset{i}{\Longrightarrow} \Delta$ says that in model v, each assignment that makes every member of Γ true makes some formula of Δ true.

Modal Propositions 5.8 and 5.9 have direct quantificational counterparts, with essentially the same proofs. We state them here, skipping the proofs.

Proposition 6.9 *Let $\langle L, D\rangle$ be an upclosed complete logical Morgan algebra, let $\mathcal{D}$ be a non-empty domain, and let $\mathcal{A} = \mathcal{D}^{\mathcal{V}}$ be the set of assignments. We have the following.*

1. *For each valuation v in $L^{\mathcal{A}}$, if $\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \Gamma \overset{i}{\Longrightarrow} Y$ then $\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \Gamma \overset{u}{\Longrightarrow} Y$. (As before, there is a singleton set on the right of the sequent arrows.)*
2. *If $\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models \Gamma \overset{i}{\Longrightarrow} Y$ then $\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models \Gamma \overset{u}{\Longrightarrow} Y$.*
3. *Neither of the above extend to arbitrary sets on the right of the sequents.*

Proposition 6.10 *Let $\langle L, D\rangle$ be an upclosed complete logical Morgan algebra, let $\mathcal{D}$ be a non-empty domain, and let $\mathcal{A} = \mathcal{D}^{\mathcal{V}}$ be the set of assignments.*

1. $\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \forall x(X \wedge Y) \overset{i}{\Longrightarrow} \forall x X$ *and* $\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \forall x(X \wedge Y) \overset{i}{\Longrightarrow} \forall x Y$.
2. $\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \exists x X \overset{i}{\Longrightarrow} \exists x(X \vee Y)$ *and* $\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \exists x Y \overset{i}{\Longrightarrow} \exists x(X \vee Y)$.
3. *If D is a complete filter, $\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \forall x X, \forall x Y \to \forall x(X \wedge Y)$.*
4. *If D is a completely prime filter, $\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v \exists x(X \vee Y) \overset{i}{\Longrightarrow} \exists x X, \exists x Y$.*
5. *If D is a complete filter, $\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v X \overset{u}{\Longrightarrow} \forall x X$.*
6. $\mathsf{Q}\langle L, D, \mathcal{D}\rangle \models_v X \overset{i}{\Longrightarrow} \forall x X$ *is not generally true, though it is if x does not occur free in X.*

7 Bilattices Briefly

Bilattices play a central role in what we do here. This section contains a summary of what we need, with few proofs. Details of bilattices can be found in a number of places. A recent survey is [11], [14]. Bilattices, are lattices with two orderings. Different conditions can be imposed, and so there are several versions of bilattices. To keep terminology at a minimum here, the convention used in [13] will be followed: we use *bilattice* for what would more fully be *bounded interlaced bilattice with negation and conflation that commute*. All this is spelled out below, but withoug much intuitive background.

Definition 7.1 (Bilattice) A *bilattice* (in this paper) is a structure $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k\rangle$ with two partial orderings, $\leq_t$ (the truth ordering) and $\leq_k$ (the information ordering). These meet the following conditions.

1. Each partial ordering is a lattice. Meet and join with respect to $\leq_t$ are denoted $\wedge$ and $\vee$. Meet and join with respect to $\leq_k$ are denoted $\otimes$ and $\oplus$.
2. Each partial ordering is bounded. The ordering $\leq_t$ has a least element, $\mathbf{f}$, and a greatest, $\mathbf{t}$. The ordering $\leq_k$ has a least element, $\perp$, and a greatest, $\top$. All four are distinct.
3. The two orderings are *interlaced*, the operations of each lattice are monotonic with respect to the other lattice:
   (a) $x \leq_t y$ implies $x \otimes z \leq_t y \otimes z$,
   (b) $x \leq_t y$ implies $x \oplus z \leq_t y \oplus z$,

(c) $x \leq_k y$ implies $x \wedge z \leq_k y \wedge z$,
(d) $x \leq_k y$ implies $x \vee z \leq_k y \vee z$.

4. There are two order reversing involutions. *Negation*, written as $\neg x$, reverses $\leq_t$ but does not change $\leq_k$. *Conflation*, written as $-x$, reverses $\leq_k$ but does not change $\leq_t$ The two operations commute, $\neg - x = -\neg x$.

The following items can be proved.

$$\begin{aligned}
\neg(a \wedge b) &= (\neg a \vee \neg b) & & \\
\neg(a \vee b) &= (\neg a \wedge \neg b) & & \\
-(a \wedge b) &= (-a \wedge -b) & \mathbf{f} \otimes \mathbf{t} &= \bot \\
-(a \vee b) &= (-a \vee -b) & \mathbf{f} \oplus \mathbf{t} &= \top \\
-(a \otimes b) &= (-a \oplus -b) & \bot \wedge \top &= \mathbf{f} \\
-(a \oplus b) &= (-a \otimes -b) & \bot \vee \top &= \mathbf{t} \\
\neg(a \otimes b) &= (\neg a \otimes \neg b) & & \\
\neg(a \oplus b) &= (\neg a \oplus \neg b) & &
\end{aligned}$$

Definition 7.2 (Complete Bilattice) A bilattice $\mathcal{B}$ is *complete* if arbitrary meets and joins exist with respect to both orderings. For a subset $\{a_i \mid i \in I\}$ of the bilattice, meet and join with respect to $\leq_t$ are written $\bigwedge_{i \in I} a_i$ and $\bigvee_{i \in i} a_i$; with respect to $\leq_k$, meet and join are $\prod_{i \in I} a_i$ and $\sum_{a \in I} a_i$. Bilattice $\mathcal{B}$ is *infinitarily interlaced* if the interlacing conditions above extend to arbitrary sets.

1. $a_i \leq_t b_i$ for all $i \in I$ implies $\prod_{i \in I} a_i \leq_t \prod_{i \in I} b_i$,
2. $a_i \leq_t b_i$ for all $i \in I$ implies $\sum_{i \in I} a_i \leq_t \sum_{i \in I} b_i$,
3. $a_i \leq_k b_i$ for all $i \in I$ implies $\bigwedge_{i \in I} a_i \leq_k \bigwedge_{i \in I} b_i$,
4. $a_i \leq_k b_i$ for all $i \in I$ implies $\bigvee_{i \in I} a_i \leq_k \bigvee_{i \in I} b_i$.

Definition 7.3 (Valuation) A *propositional valuation* in a bilattice $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k \rangle$ is a mapping v from atomic formula of $\mathcal{L}$ to members of $\mathcal{B}$. Valuations are extended to all formulas of $\mathcal{L}$ using the lattice operations of the *truth ordering*.

$$\begin{aligned}
v(X \wedge Y) &= v(X) \wedge v(Y) \\
v(X \vee Y) &= v(X) \vee v(Y) \\
v(\neg X) &= \neg v(X)
\end{aligned}$$

The following monotonicity result is now easy, using interlacing and the fact that negation does not change $\leq_k$.

Proposition 7.4 (Information Monotonicity) *Let v, w be propositional valuations in bilattice $\mathcal{B}$. If $v(P) \leq_k w(P)$ for every atomic formula P, then $v(X) \leq_k w(X)$ for every formula X.*

There is a simple way of constructing bilattices, and complete bilattices, starting with Morgan algebras.

Definition 7.5 (Bilattice Product) Let L be a Morgan algebra with $\leq$ as the partial ordering and $\overline{x}$ as the involution of x. The *bilattice product* of L with itself is $L \odot L = \langle L \times L, \leq_t, \leq_k \rangle$ where the structure is defined as follows.

1. $\langle a, b \rangle \leq_t \langle c, d \rangle$ iff $a \leq c$ and $d \leq b$
2. $\langle a, b \rangle \leq_k \langle c, d \rangle$ iff $a \leq c$ and $b \leq d$

3. $\neg\langle a, b\rangle = \langle b, a\rangle$
4. $-\langle a, b\rangle = \langle \overline{b}, \overline{a}\rangle$

It is always the case that a bilattice product, as defined above, is an interlaced bilattice with negation and conflation that commute. Indeed, every interlaced bilattice with commuting negation and conflation is isomorphic to such a product, though we will not need this. In addition, L is a complete lattice if and only if the product $L \odot L$ is a complete bilattice.

For motivation of the product construction, think of $\langle a, b\rangle$ as encoding that there is evidence a for some proposition and evidence b against. Evidences for and against are allowed to be incomplete or inconsistent, but we leave just what this means vague. Degree of truth goes up if evidence for goes up while evidence against goes down. (Note the reversal of the ordering of b and d in the definition of $\leq_t$.) Degree of information goes up if both the degree of evidence for and the degree of evidence against goes up. Negation switches the roles of for and against. Conflation is more complicated. If a is evidence for something, $\overline{a}$ represents that which is not evidence for that thing, and similarly for evidence against. Then $-\langle a, b\rangle$ arises from $\langle a, b\rangle$ by replacing evidence for by whatever was not evidence against, while evidence against is replaced by whatever was not evidence for.

The extreme elements of a bilattice product are $\bot = \langle 0, 0\rangle$, $\top = \langle 1, 1\rangle$, $\mathbf{f} = \langle 0, 1\rangle$, and $\mathbf{t} = \langle 1, 0\rangle$, where 0 is the least member of L and 1 is the greatest. So $\mathbf{f}$ represents no evidence for, but total evidence against, and similarly for the other cases. The bilattice operations can be shown to be the following, where on the right of the equality signs are the underlying lattice joins and meets of L. The right hand column assumes the lattice is complete.

$$\begin{array}{rclcrcl}
\langle a, b\rangle \wedge \langle c, d\rangle & = & \langle a \wedge c, b \vee d\rangle & \qquad & \bigwedge_{i\in I}\langle a_i, b_i\rangle & = & \langle \bigwedge_{i\in I} a_i, \bigvee_{i\in I} b_i\rangle \\
\langle a, b\rangle \vee \langle c, d\rangle & = & \langle a \vee c, b \wedge d\rangle & & \bigvee_{i\in I}\langle a_i, b_i\rangle & = & \langle \bigvee_{i\in I} a_i, \bigwedge_{i\in I} b_i\rangle \\
\langle a, b\rangle \otimes \langle c, d\rangle & = & \langle a \wedge c, b \wedge d\rangle & & \prod_{i\in I}\langle a_i, b_i\rangle & = & \langle \bigwedge_{i\in I} a_i, \bigwedge_{i\in I} b_i\rangle \\
\langle a, b\rangle \oplus \langle c, d\rangle & = & \langle a \vee c, b \vee d\rangle & & \sum_{i\in I}\langle a_i, b_i\rangle & = & \langle \bigvee_{i\in I} a_i, \bigvee_{i\in I} b_i\rangle
\end{array} \tag{5}$$

If members of bilattices are to play the role of generalized truth values, some notion of designated element is needed. In [1] the notions of a *prime bifilter* and a *logical bilattice* were introduced. We only need weaker versions here, and we introduce the term *upbiclosed*. It would be useful to recall the terminology from Definition 2.2.

Definition 7.6 Let $\mathcal{B} = \langle \mathcal{B}, \leq_t, \leq_k\rangle$ be a bilattice, and let D be a proper, non-empty subset of $\mathcal{B}$.

1. D is *upbiclosed* if it is upwardly closed with respect to both orderings $\leq_t$ and $\leq_k$.
2. $\langle \mathcal{B}, D\rangle$ is an *upbiclosed logical bilattice* if D is an upbiclosed set.

It is easy to check that for a Morgan algebra L, with $D \subseteq L$, D is upwardly closed in L if and only if $D \times L$ is upbiclosed in the bilattice $L \odot L$. Then, $\langle L, D\rangle$ is an upclosed logical Morgan algebra if and only if $\langle L \odot L, D \times L\rangle$ is an upbiclosed logical bilattice.

For our purposes we will not want the entire of a bilattice. It is what is called the *anticonsistent* part that is usefu to usl. Reasons for the terminology, and a general discussion of why it is of interest can be found in [11], [14].

Definition 7.7 (Anticonsistent, Exact) A member x of a bilattice is anticonsistent if $-x \leq_k x$, and is *exact* if $x = -x$. (There is also the notion of being *consistent*, $x \leq_k -x$, but it will not be needed here.)

In a product bilattice $L \odot L$, where L is a Morgan algebra, the definition above has a useful equivalent form: $\langle a, b\rangle$ is anticonsistent if $\overline{a} \leq b$ or equivalently $\overline{b} \leq a$, and $\langle a, b\rangle$ is exact if $\overline{a} = b$ or equivalently if $a = \overline{b}$.

Below are some important and useful properties of exactness and anticonsistency (and consistency because, why not).

Proposition 7.8 *In any bilattice $\mathcal{B}$ we have the following.*

1. *The sets of exact values, consistent values, and anticonsistent values are closed under $\wedge$, $\vee$, and $\neg$.*
2. *If $\mathcal{B}$ is a complete bilattice, the sets of exact values, consistent values, and anticonsistent values in $\mathcal{B}$ are closed under $\bigwedge$ and $\bigvee$.*
3. *The sets of exact values, consistent values, and anticonsistent values each contain* $\mathbf{f}$ *and* $\mathbf{t}$.
4. *$\perp$ is consistent and $\top$ is anticonsistent.*
5. *If x is anticonsistent and $x \le_k y$ then y is anticonsistent. Likewise if x is consistent and $y \le_k x$ then y is consistent.*
6. *If x and y are exact and $x \le_k y$, then $x = y$.*

Proof Assume we work in a bilattice $\mathcal{B}$.

1. Suppose x, y are anticonsistent, so $-x \le_k x$ and $-y \le_k y$. Then $-(x \wedge y) = -x \wedge -y \le_k x \wedge y$, using interlacing, so $x \wedge y$ is anticonsistent. The other cases are similar.
2. Similar to 1, using the infinitary interlacing conditions.
3. For every $x \in \mathcal{B}$, $-x \le_t \mathbf{t}$ since $\mathbf{t}$ is the truth ordering maximal element. Then $x \le_t -\mathbf{t}$ since conflation does not affect the truth ordering. Since x is arbitrary, $-\mathbf{t}$ is maximal, so $-\mathbf{t} = \mathbf{t}$, making $\mathbf{t}$ exact, anticonsistent, and consistent. The other cases are similar.
4. Similar to 3.
5. Suppose x is anticonsistent, so $-x \le_k x$. Also suppose $x \le_k y$. Then $-y \le_k -x \le_k x \le_k y$, so y is anticonsistent. The other case is similar.
6. Suppose x, y are exact, and $x \le_k y$. Then $y = -y \le_k -x = x$, so $x = y$.

■

We introduce some notation that will allow us to state things relatively easily. It is not the same as that of [12], but the relationship between the two notations is straightforward.

Definition 7.9 Let $\mathcal{B}$ be a bilattice. $\mathcal{A}(\mathcal{B})$ is the subset of *anticonsistent* members of $\mathcal{B}$. We will write $\mathcal{A}(\mathcal{B}) = \langle \mathcal{A}(\mathcal{B}), \le_t \rangle$ if we want to indicate we are interested in the anticonsistent members of $\mathcal{B}$ under the truth ordering of the bilattice, and similarly with $\le_k$. We use $\mathcal{E}(\mathcal{B})$ in a similar way for the subset of *exact* members of $\mathcal{B}$, again with or without orderings specified. Further, if S is a set of members of bilattice $\mathcal{B}$, by $\mathcal{A}(S)$ and $\mathcal{E}(S)$ we mean the set of anticonsistent (respectively exact) members of $\mathcal{B}$ that are in S, but with no ordering considered.

It follows from Proposition 7.8 that both $\langle \mathcal{A}(\mathcal{B}), \le_t \rangle$ and $\langle \mathcal{E}(\mathcal{B}), \le_t \rangle$ are Morgan algebras (with negation as De Morgan involution), and are complete Morgan algebras if $\mathcal{B}$ is a complete bilattice. It follows that if L is a complete Morgan algebra, $L \odot L$ is a complete bilattice, and so $\langle \mathcal{A}(L \odot L), \le_t \rangle$ and $\langle \mathcal{E}(L \odot L), \le_t \rangle$ are both complete Morgan algebras.

Finally, just as we had function space lattices, Definition 2.3, there are function space bilattices.

Definition 7.10 (Function Space Biattices) Suppose $\mathcal{B} = \langle \mathcal{B}, \le_t, \le_k \rangle$ is a bilattice and S is a non-empty set. The function space bilattice $\mathcal{B}^S = \langle \mathcal{B}^S, \le_t, \le_k \rangle$ has for its domain the set of all functions from S to $\mathcal{B}$, with orderings defined pointwise. That is, for $f, g \in \mathcal{B}^S$, $f \le_t g$ if and only if $f(x) \le_t g(x)$ in $\mathcal{B}$ for all $x \in S$; and likewise $f \le_k g$ if and only if $f(x) \le_k g(x)$ in $\mathcal{B}$ for all $x \in S$.

The function space $\mathcal{B}^S$ will be infinitarily interlaced if $\mathcal{B}$ is. The pointwise definitions of the operations imply that, in a function space bilattice, $f \wedge g$ is the function such that $(f \wedge g)(x) = f(x) \wedge g(x)$ for all $x \in \mathcal{B}$, and similarly for the other bilattice operations. Likewise if $\mathcal{B}$ is a complete bilattice so is the function space bilattice $\mathcal{B}^S$, and in it $\bigwedge_{i\in S} f_i$ is the function such that, for each $x \in S$, $[\bigwedge_{i\in S} f_i](x) = \bigwedge_{i\in S} f_i(x)$, and similarly for the other infinitary operators. Definition 7.9 is useful here for simplifying the statement of a couple of important items. For a bilattice $\mathcal{B}$ and a non-empty set S, $\mathcal{A}(\mathcal{B}^S) = [\mathcal{A}(\mathcal{B})]^S$, and likewise $\mathcal{E}(\mathcal{B}^S) = [\mathcal{E}(\mathcal{B})]^S$. All this is straightforward to verify, and is assumed in what follows.

8 Modal Strict/Tolerant Structures

We now reach the first payoff point of the paper. We show how to construct a strict/tolerant counterpart for each of our many valued modal structures. We begin with a very fast summary our earlier propositional construction from [12]. Then we lift this to the function space level, so we can bring in an interpretation for the modal operators.

Propositional Level: For the rest of this short propositional discussion we employ the propositional language $\mathcal{L}$, containing the connectives $\wedge$, $\vee$, and $\neg$, but no modal operators or quantifiers. In [12] the following was shown. The upclosed logical Morgan algebra $\langle L, D\rangle$ validates the same sequents in the many valued sense as the strict/tolerant structure $\langle \mathcal{A}(L \odot L), \mathcal{E}(D \times L), \mathcal{A}(D \times L)\rangle$ does in the strict/tolerant sense, where $\mathcal{A}(L \odot L) = \langle \mathcal{A}(L \odot L), \leq_t\rangle$, that is, we use the bilattice truth ordering restricted to the antioconsistent part. Unwinding the notation, the strict/tolerant structure has as its space of truth values the anticonsistent part of $L \odot L$, the strictly designated values are the exact part of $D \times L$, and the tolerantly designated values are the anticonsistent part of $D \times L$.

The proof from [12] actually made use of an intermediate structure, $\langle \mathcal{E}(L \odot L), \mathcal{E}(D \times L)\rangle$, where $\mathcal{E}(L \odot L) = \langle \mathcal{E}(L \odot L), \leq_t\rangle$. This is a many valued structure within the bilattice $L \odot L$, and was shown to be isomorphic to $\langle L, D\rangle$ on the one hand, and on the other hand to validate the same sequents as the strict/tolerant $\langle \mathcal{A}(L \odot L), \mathcal{E}(D \times L), \mathcal{A}(D \times L)\rangle$, thus establishing the connection between the strict/tolerant structure and $\langle L, D\rangle$.

It will be convenient to sketch why $\langle L, D\rangle$ and $\langle \mathcal{E}(L \odot L), \mathcal{E}(D \times L)\rangle$ are isomorphic. The proof is short, and we will need parts of it below. In remarks following Definition 7.7 we noted that in product bilattices, the exact members are those of the form $\langle x, \overline{x}\rangle$. It is easy to see that $\langle x, \overline{x}\rangle \leq_t \langle y, \overline{y}\rangle$ in $\mathcal{E}(L \odot L)$ if and only if $x \leq y$ in L, so it follows that $\langle \mathcal{E}(L \odot L), \leq_t\rangle$ is an isomorphic copy of the Morgan algebra $\langle L, \leq\rangle$.

Going further, we saw in Section 7 that D is upwardly closed in L if and only if $D \times L$ is upbiclosed in $L \odot L$. Further, $x \in D$ if and only if $\langle x, y\rangle \in D \times L$ for every y if and only if $\langle x, \overline{x}\rangle \in \mathcal{E}(D \times L)$. Putting all this together, $\langle L, D\rangle$ is an upclosed complete logical Morgan algebra if and only if $\langle \mathcal{E}(L \odot L), \mathcal{E}(D \times L)\rangle$ is, and the two are isomorphic.

Table 3 summarizes the details of what was just said. In it MV (for *many valued*) is an upclosed logical Morgan algebra. BMV (for *bilattice many valued*) is also an upclosed logical Morgan algebra that is isomorphic to MV but is a subsystem of a product bilattice. ST is a *strict/tolerant* structure for which S marks the strictly designated truth values and T marks the tolerant set of values.

Now we want to lift things from the underlying propositional truth value space to a function space to accomodate the modal operators.

Modal Level: For the rest of the discussion in this section the formal language is $\mathcal{M}$ which, in addition to the propositional connectives, also has the modal operators $\Box$ and $\Diamond$. We work with an upclosed complete Morgan algebra $\langle L, D\rangle$, and with a modal frame $\mathcal{F} = \langle \mathcal{G}, \mathcal{R}\rangle$. In Section 5 we saw that the complete Morgan algebra $L^{\mathcal{G}}$, a function space, could interpret the modal operators as well as the propositional connectives. But also we saw in Section 7 that $\mathcal{A}(L \odot L)$ is a complete Morgan algebra, so the function space $\mathcal{A}(L \odot L)^{\mathcal{G}}$ is yet another complete Morgan algebra, for which definitions of $\boxdot$ and $\diamonddot$ can be given. We thus have mappings of the

	Truth Value Set	Designated Set
MV	L	D
BMV	$\mathcal{E}(L \odot L)$	$\mathcal{E}(D \times L)$
ST	$\mathcal{A}(L \odot L)$	$S: \ \mathcal{E}(D \times L)$ $T: \ \mathcal{A}(D \times L)$

Table 3: Propositional Model Structures

following two types, both given by Definition 5.1. We use the same operator notation for both types since no confusion should result.

$$\boxdot : L^{\mathcal{G}} \to L^{\mathcal{G}} \qquad \boxdot : \mathcal{A}(L \odot L)^{\mathcal{G}} \to \mathcal{A}(L \odot L)^{\mathcal{G}}$$
$$\diamondsuit : L^{\mathcal{G}} \to L^{\mathcal{G}} \qquad \diamondsuit : \mathcal{A}(L \odot L)^{\mathcal{G}} \to \mathcal{A}(L \odot L)^{\mathcal{G}}$$

Though it is not used for our main results, it is of interest to note that the operations $\boxdot$ and $\diamondsuit$ on $L^{\mathcal{G}}$ and on $\mathcal{A}(L \odot L)^{\mathcal{G}}$ have direct connections with each other. In the statements below π_1 and π_2 are projection functions, mapping ordered pairs to their first and their second components respectively. Now, for $f \in \mathcal{A}(L \odot L)^{\mathcal{G}}$ and for $a \in \mathcal{G}$:

$$\boxdot f = \lambda a.\langle \boxdot(\pi_1 f)(a), \diamondsuit(\pi_2 f)(a)\rangle$$
$$\diamondsuit f = \lambda a.\langle \diamondsuit(\pi_1 f)(a), \boxdot(\pi_2 f)(a)\rangle.$$

We check the first of these.

$$\begin{aligned}
(\boxdot f)(a) &= \bigwedge\{f(b) \mid a\mathcal{R}b\} && \text{Def 5.1 for } \mathcal{A}(L \odot L)^{\mathcal{G}} \\
&= \bigwedge\{\langle(\pi_1 f)(b), (\pi_2 f)(b)\rangle \mid a\mathcal{R}b\} && \text{behavior of ordered pairs} \\
&= \langle \bigwedge\{(\pi_1 f)(b) \mid a\mathcal{R}b\}, \bigvee\{(\pi_2 f)(b) \mid a\mathcal{R}b\}\rangle && (5) \\
&= \langle \boxdot(\pi_1 f)(a), \diamondsuit(\pi_2 f)(a)\rangle && \text{Def 5.1 for } L^{\mathcal{G}}
\end{aligned}$$

We now have the modal language $\mathcal{M}$ interpreted in both $L^{\mathcal{G}}$ and $\mathcal{A}(L \odot L)^{\mathcal{G}}$, with propositional connectives interpreted using bilattice truth operations and modal operators interpreted as above. Before moving on, here are a few small but useful items.

Proposition 8.1 (Exactness Closure) *Suppose v is a valuation of modal language $\mathcal{M}$ in $[\mathcal{A}(L \odot L)]^{\mathcal{G}}$ (where L is complete). If $v(P)$ is exact for every propositional letter P then $v(X)$ is exact for every formula of $\mathcal{M}$. More briefly, if $v(P) \in [\mathcal{E}(L \odot L)]^{\mathcal{G}}$ for every atomic P, then $v(X) \in [\mathcal{E}(L \odot L)]^{\mathcal{G}}$ for every modal formula X.*

Proof By induction on the complexity of X. The atomic case is given. Conjunction, disjunction, and negation use item 1 of Proposition 7.8. The modal cases, using Definition 5.1, follow using item 2 of Proposition 7.8. ■

Proposition 8.2 (Information Order Monotonicity) *Let v, w be valuations, mapping the modal language $\mathcal{M}$ into $[\mathcal{A}(L \odot L)]^{\mathcal{G}}$. Using the knowledge ordering from the bilattice $(L \odot L)^{\mathcal{G}}$ we have the following. If $v(P) \leq_k w(P)$ for each propositional letter P, then $v(X) \leq_k w(X)$ for each modal formula X of $\mathcal{M}$.*

Proof The proof is by induction on formula complexity. Assume $v(P) \leq_k w(P)$ for each propositional letter P. We thus have the base case by assumption.

Suppose $v(X) \leq_k w(X)$. Then for each $a \in \mathcal{G}$, $v(X)(a) \leq_k w(X)(a)$ in the bilattice $L \odot L$. It follows by the properties of negation that $v(\neg X)(a) = \neg v(X)(a) \leq_k \neg w(X)(a) = w(\neg X)(a)$, for each $a \in \mathcal{G}$. Then $v(\neg X) \leq_k w(\neg X)$.

The argument is similar for the other propositional connectives using the interlacing conditions from Definition 7.1. The modal cases follow using Definition 5.1 and the infinitary interlacing conditions from Definition 7.2. ■

We next turn to designated sets of truth values, and hence to their corresponding consequence relations.

For L we assumed we have an upclosed set D as the set of designated truth values. Then Definition 2.4 gives us both global and local versions of designated set on the function space $L^{\mathcal{G}}$, denoted $D^{\mathcal{G}}$ and $L^{\mathcal{G}}_{a \to D}$ respectively. Since D is upclosed in L, both of these are upclosed in $L^{\mathcal{G}}$, Proposition 2.5, so each of $\langle L^{\mathcal{G}}, D^{\mathcal{G}} \rangle$ and $\langle L^{\mathcal{G}}, L^{\mathcal{G}}_{a \to D} \rangle$ are upclosed logical Morgan algebras. Each gives us a consequence relation, Definition 5.5: $\overset{m}{\Longrightarrow}$ for *model validity*, the global version using $D^{\mathcal{G}}$, and $\overset{p}{\Longrightarrow}$ for *possible world validity*, the local version using $L^{\mathcal{G}}_{a \to D}$. From now on, it is the choice of designated set that plays a direct role; the resulting consequence relation is secondary. The basic many valued setup is summarized in the first line of Table 4.

	Truth Value Set	Global Designated Set	Local Designated Set
MV	$L^{\mathcal{G}}$	$D^{\mathcal{G}}$	$L^{\mathcal{G}}_{a \to D}$
BMV	$[\mathcal{E}(L \odot L)]^{\mathcal{G}}$	$[\mathcal{E}(D \times L)]^{\mathcal{G}}$	$[\mathcal{E}(L \times L)]^{\mathcal{G}}_{a \to \mathcal{E}(D \times L)}$
ST	$[\mathcal{A}(L \odot L)]^{\mathcal{G}}$	S: $[\mathcal{E}(D \times L)]^{\mathcal{G}}$ T: $[\mathcal{A}(D \times L)]^{\mathcal{G}}$	S: $[\mathcal{A}(L \times L)]^{\mathcal{G}}_{a \to \mathcal{E}(D \times L)}$ T: $[\mathcal{A}(L \times L)]^{\mathcal{G}}_{a \to \mathcal{A}(D \times L)}$

Table 4: Modal Model Structures

Just as hapened in the propositional case, there is an isomorphic version of MV, occurring within the bilattice structure itself. This has the truth value structure $[\mathcal{E}(L \odot L)]^{\mathcal{G}} = \langle [\mathcal{E}(L \odot L)]^{\mathcal{G}}, \leq_t \rangle$ as shown on the BMV line. The isomorphism $\theta : L^{\mathcal{G}} \to [\mathcal{E}(L \odot L)]^{\mathcal{G}}$ given by

$$\theta(f) = \lambda a.\langle f(a), \overline{f(a)} \rangle$$

is one-one, onto, and preserves the ordering relation. Further, it maps $D^{\mathcal{G}}$ to $[\mathcal{E}(D \times L)]^{\mathcal{G}}$ and $L^{\mathcal{G}}_{a \to D}$ to $[\mathcal{E}(L \times L)]^{\mathcal{G}}_{a \to \mathcal{E}(D \times L)}$. All this establishes that the BMV line of the table does contain an isomorphic version of the MV line.

The ST line of Table 4 gives us two different versions of strict/tolerant modal structures, and consequently of logics, one of a global nature, the other local. It remains to show that validity for the two many valued BMV consequence relations is the same as for the two strict/tolerant ST consequence relations. At their core our proofs are the same as in our earlier papers. We begin with a formulation of some simple common features shared by both the global and the local columns of Table 4. These allow us to give a uniform proof covering both cases together. Essentially, all the unpleasant detail work is concentrated here.

Common Properties 8.3 *In Table 4 the following hold in both the local and the global case.*

1. *The* ST *strict and the* ST *tolerant designated sets have the same exact subset, and it is the* BMV *designated set.*
2. *Any exact function below a member of the* ST *strict designated set, in the $\leq_k$ ordering, is designated in* BMV*.*
3. *The* ST *tolerant designated set is upward closed under the $\leq_k$ ordering.*

4. *Let $\top$ be the largest member of the bilattice $(L \odot L)^{\mathcal{G}}$ in the $\leq_k$ ordering. $\top$ is in the* ST *tolerant designated set, but is not in the strict designated set.*

Proof There are four items to be proved, each of which has a global and a local case.

1. The global case is trivial. The local case is straightforward, and is left to the reader.
2. First the global case. Suppose g is in the strict designated set, that is, $g \in [\mathcal{E}(D \times L)]^{\mathcal{G}}$, $f \leq_k g$, and f is exact. Note that g is also exact. By Proposition 7.8 part 6, $f = g$, so $f \in [\mathcal{E}(D \times L)]^{\mathcal{G}}$ which is the BMV global designated set.

   Next the local case. Suppose $g \in [\mathcal{A}(L \times L)]^{\mathcal{G}}_{a \to \mathcal{E}(D \times L)}$, $f \leq_k g$, and f is exact. If $b \neq a$ then $f(b)$, being exact, must be in $\mathcal{E}(L \times L)$. And for the case of a, $f(a) \leq_k g(a) \in \mathcal{E}(D \times L)$. But then both $f(a)$ and $g(a)$ are exact, so by Proposition 7.8 part 6 again, $f(a) = g(a)$, so $f(a) \in \mathcal{E}(D \times L)$. We then have $f \in [\mathcal{E}(L \times L)]^{\mathcal{G}}_{a \to \mathcal{E}(D \times L)}$, the BVM local designated set.
3. We do the global case first. Suppose $f \in [\mathcal{A}(D \times L)]^{\mathcal{G}}$ and $f \leq_k g$. Let a be an arbitrary member of $\mathcal{G}$. Then $f(a) \in \mathcal{A}(D \times L)$, so $f(a)$ is anticonsistent and in $D \times L$. We have $f(a) \leq_k g(a)$, so $g(a)$ is anticonsistent by Proposition 7.8 part 5, and $g(a) \in D \times L$ because D is upward closed (as is L of course). Then $g(a) \in \mathcal{A}(D \times L)$ for every $a \in \mathcal{G}$, and so $g \in [\mathcal{A}(D \times L)]^{\mathcal{G}}$.

   For the local case, suppose $f \in [\mathcal{A}(L \times L)]^{\mathcal{G}}_{a \to \mathcal{A}(D \times L)}$ and $f \leq_k g$. If $b \neq a$, $f(b)$ is anticonsistent in $L \odot L$, and since $f \leq_k g$, the same is true for $g(b)$ by Proposition 7.8 part 5. Then $g(b) \in \mathcal{A}(L \times L)$. For the case of a, $f(a) \in \mathcal{A}(D \times L)$ and hence $g(a)$ is anticonsistent by Proposition 7.8 part 5 again, and $g(a) \in D \times L$ because D is upward closed. So $g(a) \in \mathcal{A}(D \times L)$.
4. $\top$ is the function that maps every member of $\mathcal{G}$ to the largest member of $L \odot L$ in the $\leq_k$ ordering. In turn this largest member is $\langle 1, 1 \rangle$, where 1 is the largest member of L.

   First the global case. Since D is upward closed and non-empty, $1 \in D$, so $\langle 1, 1 \rangle \in D \times L$. Also, using Proposition 7.8 item 4, $\langle 1, 1 \rangle$ is anticonsistent. Then $\langle 1, 1 \rangle \in \mathcal{A}(D \times L)$, so $\top \in [\mathcal{A}(D \times L)]^{\mathcal{G}}$, the global tolerant designated set. But $\top \notin [\mathcal{E}(D \times L)]^{\mathcal{G}}$ because $\langle 1, 1 \rangle \notin \mathcal{E}(D \times L)$, because $\langle 1, 1 \rangle$ is not exact.

   Finally the local case. For $\top$ to be in $[\mathcal{A}(L \times L)]^{\mathcal{G}}_{a \to \mathcal{A}(D \times L)}$ it is enough that $\top$ map a to $\mathcal{A}(D \times L)$. And we saw in the argument for the global case that the image of a, which is $\langle 1, 1 \rangle$, is in this set. On the other hand, for $\top$ to be in $[\mathcal{A}(L \times L)]^{\mathcal{G}}_{a \to \mathcal{E}(D \times L)}$, $\top$ should map a to $\mathcal{E}(D \times L)$, but we have seen that $\langle 1, 1 \rangle$ is not in this set.

■

In what follows we handle both the global and local cases of designated sets simultaneously, and we introduce some notation to make this easier. We use **D** for either the global or the local designated sets for BMV in Table 4, and **S** and **T** for the corresponding strict and tolerant designated sets in ST. Table 5 gives the definitions more precisely. And here is a restatement of the Common Properties, using this notation.

Case	**D**	**S**	**T**
Global	$[\mathcal{E}(D \times L)]^{\mathcal{G}}$	$[\mathcal{E}(D \times L)]^{\mathcal{G}}$	$[\mathcal{A}(D \times L)]^{\mathcal{G}}$
Local	$[\mathcal{E}(L \times L)]^{\mathcal{G}}_{a \to \mathcal{E}(D \times L)}$	$[\mathcal{A}(L \times L)]^{\mathcal{G}}_{a \to \mathcal{E}(D \times L)}$	$[\mathcal{A}(L \times L)]^{\mathcal{G}}_{a \to \mathcal{A}(D \times L)}$

Table 5: Uniform Notation

Common Properties 8.4 (Uniform Version) *Using notation from Table 5, the items from Common Properties 8.3 are the following.*

1. $\mathcal{E}(\mathbf{S}) = \mathcal{E}(\mathbf{T}) = \mathbf{D}$.
2. *If* $f \in \mathbf{S}$, $g \leq_k f$, *and* g *is exact, then* $g \in \mathbf{D}$.
3. *If* $f \in \mathbf{T}$ *and* $f \leq_k g$ *then* $g \in \mathbf{T}$.
4. $\top \in \mathbf{T}$ *but* $\top \notin \mathbf{S}$ *(where* $\top$ *is the largest member of* $(L \odot L)^{\mathcal{G}}$ *under the information ordering).*

With all this set up, we are finally in a position to prove the main results of this section. The first two Propositions below together show that for each modal model built on a complete logical Morgan algebra there is a corresponding strict/tolerant veersion that validates the same sequents, whether we use the local or the global notion of designated truth value. But the Morgan structure and the strict/tolerant structure differ on the local version of cut. In all of what follows in this section, L is a complete Morgan algebra (of truth values), and D is an upclosed subset (of designated truth values).

Proposition 8.5 *If the modal* $\mathcal{M}$ *sequent* $\Gamma \Longrightarrow \Delta$ *is not valid in the many valued structure* $\langle [\mathcal{E}(L \odot L)]^{\mathcal{G}}, \mathbf{D}\rangle$, *it is not valid in the strict/tolerant structure* $\langle [\mathcal{A}(L \odot L)]^{\mathcal{G}}, \mathbf{S}, \mathbf{T}\rangle$.

Proof Assume the $\mathcal{M}$ sequent $\Gamma \Longrightarrow \Delta$ is not valid in $\langle [\mathcal{E}(L \odot L)]^{\mathcal{G}}, \mathbf{D}\rangle$, and v is a valuation that invalidates it. That is, v maps formulas to $[\mathcal{E}(L \odot L)]^{\mathcal{G}}$, and $v(X) \in \mathbf{D}$ for all $X \in \Gamma$ but $v(Y) \notin \mathbf{D}$ for every $Y \in \Delta$.

Since exact values are also anticonsistent, v is a valuation in $[\mathcal{A}(L \odot L)]^{\mathcal{G}}$ as well. v maps all of Γ to $\mathbf{D}$, which is $\mathcal{E}(\mathbf{S})$ by **Common Properties** 8.4(1), a subset of $\mathbf{S}$, so v maps all of Γ to $\mathbf{S}$. If $v(Y) \in \mathbf{T}$ for some $Y \in \Delta$, then $v(Y)$ would have to be in the exact subset of it, since v maps formulas to $[\mathcal{E}(L \odot L)]^{\mathcal{G}}$. But this exact subset is $\mathcal{E}(\mathbf{T}) = \mathbf{D}$, by **Common Properties** 8.4(1) again, which does not contain $v(Y)$. Thus $v(Y) \notin \mathbf{T}$. Then v invalidates $\Gamma \Longrightarrow \Delta$ in the strict/tolerant sense in $\langle [\mathcal{A}(L \odot L)]^{\mathcal{G}}, \mathbf{S}, \mathbf{T}\rangle$ as well. ∎

Proposition 8.6 *If the modal* $\mathcal{M}$ *sequent* $\Gamma \Longrightarrow \Delta$ *is not valid in the strict/tolerant structure* $\langle [\mathcal{A}(L \odot L)]^{\mathcal{G}}, \mathbf{S}, \mathbf{T}\rangle$, *it is not valid in the many valued structure* $\langle [\mathcal{E}(L \odot L)]^{\mathcal{G}}, \mathbf{D}\rangle$.

Proof This time assume the sequent $\Gamma \Longrightarrow \Delta$ is not valid in $\langle [\mathcal{A}(L \odot L)]^{\mathcal{G}}, \mathbf{S}, \mathbf{T}\rangle$ in the strict/tolerant sense, and valuation v invalidates it. Then v is a valuation in $[\mathcal{A}(L \odot L)]^{\mathcal{G}}$ such that $v(X) \in \mathbf{S}$ for every $X \in \Gamma$, and for every $Y \in \Delta$, $v(Y) \notin \mathbf{T}$.

The problem is that for each formula Z, $v(Z)$ maps $\mathcal{G}$ to $\mathcal{A}(L \odot L)$, but for a counter-model in $\langle [\mathcal{E}(L \odot L)]^{\mathcal{G}}, \mathbf{D}\rangle$ we need a mapping from $\mathcal{G}$ to $\mathcal{E}(L \odot L)$. We define a new valuation v' as follows. For an atomic formula A, and $a \in \mathcal{G}$, we have that $v(A)(a) = \langle x, y\rangle \in \mathcal{A}(L \odot L)$; we set $v'(A)(a) = \langle x, \overline{x}\rangle \in \mathcal{E}(L \odot L)$. Since this is for any $a \in \mathcal{G}$, we have defined $v'(A)$ for atomic A. Extend v' to all formulas of $\mathcal{M}$ by recursion on formula complexity. Proposition 8.1 gives us that this is a valuation in $[\mathcal{E}(L \odot L)]^{\mathcal{G}}$.

Suppose A is atomic, and $a \in \mathcal{G}$. Then $v(A)(a) = \langle x, y\rangle$ for some $x, y \in L$ and then $v'(A)(a) = \langle x, \overline{x}\rangle$. Since $\langle x, y\rangle$ is anticonsistent $\overline{x} \leq y$, and so $\langle x, \overline{x}\rangle \leq_k \langle x, y\rangle$. This shows that $v'(A)(a) \leq_k v(A)(a)$. Since a was arbitrary, for every atomic A, $v'(A) \leq_k v(A)$ in $[\mathcal{E}(L \odot L)]^{\mathcal{G}}$. Then by Proposition 8.2, for every formula Z, $v'(Z) \leq_k v(Z)$ in $[\mathcal{E}(L \odot L)]^{\mathcal{G}}$.

Finally we show v' does not validate $\Gamma \Longrightarrow \Delta$ in $\langle [\mathcal{E}(L \odot L)]^{\mathcal{G}}, \mathbf{D}\rangle$. Suppose $X \in \Gamma$. We know that $v(X) \in \mathbf{S}$. We also know that $v'(X)$ is exact, and we know that $v'(X) \leq_k v(X)$, so by **Common Properties** 8.4(2), $v'(X) \in \mathbf{D}$. And this is for every $X \in \Gamma$.

Suppose $Y \in \Delta$. If we had $v'(Y) \in \mathbf{T}$, since $v'(Y) \leq_k v(Y)$, then by **Common Properties** 8.4(3) we would have $v(Y) \in \mathbf{T}$, which is not the case. So $v'(Y) \notin \mathbf{T}$. Then $v'(Y) \notin \mathcal{E}(\mathbf{T})$ and so by **Common Properties** 8.4(1), $v'(Y) \notin \mathbf{D}$. We thus have that v' does not validate $\Gamma \Longrightarrow \Delta$ in $\langle [\mathcal{E}(L \odot L)]^{\mathcal{G}}, \mathbf{D}\rangle$. ∎

Proposition 8.7 *For modal* $\mathcal{M}$ *formulas, the metaconsequence scheme*

$$\frac{\Gamma, A \Longrightarrow \Delta \qquad \Gamma \Longrightarrow \Delta, A}{\Gamma \Longrightarrow \Delta}$$

is locally valid in $\langle[\mathcal{E}(L\odot L)]^{\mathcal{G}},\mathbf{D}\rangle$*, but an instance is not locally valid in* $\langle[\mathcal{A}(L\odot L)]^{\mathcal{G}},\mathbf{S},\mathbf{T}\rangle$*.*

Proof *Validity* Suppose v is a valuation in $\langle[\mathcal{E}(L\odot L)]^{\mathcal{G}},\mathbf{D}\rangle$ that validates both premise sequents $\Gamma, A\Longrightarrow\Delta$ and $\Gamma\Longrightarrow\Delta, A$. To show v also validates $\Gamma\Longrightarrow\Delta$, we assume $v(X)\in\mathbf{D}$ for every $X\in\Gamma$, and we show $v(Y)\in\mathbf{D}$ for some $Y\in\Delta$.

Since v validates $\Gamma\Longrightarrow\Delta, A$ and it maps all of Γ to $\mathbf{D}$, it must map some member of Δ, A to $\mathbf{D}$. If it does this for a member of Δ, we are done. Otherwise $v(A)\in\mathbf{D}$. But then v maps all of Γ, A to $\mathbf{D}$ so, since it validates $\Gamma, A\Longrightarrow\Delta$, v must map some member of Δ to $\mathbf{D}$ and again we are done.

NonValidity We consider the following scheme instance, where P is an atomic formula.

$$\frac{\emptyset, P\Longrightarrow\emptyset \quad \emptyset\Longrightarrow\emptyset, P}{\emptyset\Longrightarrow\emptyset}$$

Let v be any valuation in $\langle[\mathcal{A}(L\odot L)]^{\mathcal{G}},\mathbf{S},\mathbf{T}\rangle$ that maps P to largest member, $\top$, of the bilattice $(L\odot L)^{\mathcal{G}}$. By **Common Properties** 8.4(4, $v(P)\in\mathbf{T}$ but $v(P)\notin\mathbf{S}$. Now, v validates $\emptyset, P\Longrightarrow\emptyset$ in the strict/tolerant structure $\langle[\mathcal{A}(L\odot L)]^{\mathcal{G}},\mathbf{S},\mathbf{T}\rangle$ because it does not map one of the premises, namely P, to $\mathbf{S}$. It validates $\emptyset\Longrightarrow\emptyset, P$ because it maps one of the conclusions, again P, to $\mathbf{T}$. Thus v validates both hypotheses. But v does not validate $\emptyset\Longrightarrow\emptyset$, because it maps every member of the antecedent to a strict designated value, but there is no member of the consequent that maps to a tolerant designated value. So v is a counterexample to the local validity of an instance of the cut scheme. ■

9 Modal Examples

Before getting to the main example we examine, we briefly discuss the simplest one, leaving the details of it to the reader. For it the underlying logical Morgan algebra is that of classical logic. The construction from Section 5 gives us an algebraic version of standard Kripke semantics. The corresponding strict/tolerant version that the construction in Section 8 produces is a modal structure with the standard ST propositional logic underneath. This is exactly what one would expect, and working through the details is a good exercise.

What we devote the rest of this section to is a somewhat more complex example. We do not propose an intuitive meaning for this. It is meant simply as a formal example, illustrating how the machinery presented in this paper works. Instead of the classical two valued space, we take for our underlying complete Morgan algebra the four valued one shown in Figure 1, ordered upward, with an involution operator given by $\overline{0}=1$, $\overline{1}=0$, $\overline{\alpha}=\alpha$, and $\overline{\beta}=\beta$. We call this L for the rest of the section. The subset of designated truth values D is $\{1\}$, shown circled in the diagram. This is trivially an upwardly closed set, so $\langle L, D\rangle$ is a complete upclosed logical Morgan algebra. (D is also a filter, but is not prime.) This was used in an earlier paper in our series on ST generalizations, as Example 8.2 of [13], where it provided an example of a strict/tolerant logic for which an interpretation of the connectives generalizing those of the *weak* Kleene three valued logic would not work. That plays no role here, but is noted for the record.

The propositional strict/tolerant truth value space given by our construction starts with a bilattice product, $L\odot L$. It has 16 members, but we only need the anticonsistent part, which narrows it to 9. These are shown in Figure 2. The ordering for this is the bilattice truth ordering, and in the diagram it is the vertical direction. Thus in the diagram $\langle x,y\rangle\leq\langle z,w\rangle$ just if $x\leq z$ and $y\geq w$ in L. The exact members are the four 'corners', $\langle 0,1\rangle, \langle\alpha,\alpha\rangle$, $\langle\beta,\beta\rangle$, and $\langle 1,0\rangle$, a structure easily seen to be isomorphic to the one in Figure 1.

The counterpart of D that we use for the bilattice is $D\times L$. This, of course, is $\{\langle 1,0\rangle$, $\langle 1,\alpha\rangle$, $\langle 1,\beta\rangle$, $\langle 1,1\rangle\}$. The tolerant values of $\mathcal{A}(L\odot L)$ are those in $D\times L$ that are anticonsistent, and the strict values are those that are exact. These are shown in Figure 2, exact in a small rectangle and tolerant in a larger tilted rectangle.

To interpret the modal operators we need function spaces mapping to the structure in Figure 2. Let $\mathcal{F}=\langle\mathcal{G},\mathcal{R}\rangle$ be an arbitrary modal frame. Following Section 5, $L^{\mathcal{G}}$ gives us an algebraic

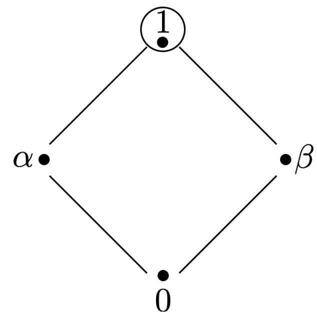


Figure 1: Morgan Algebra L

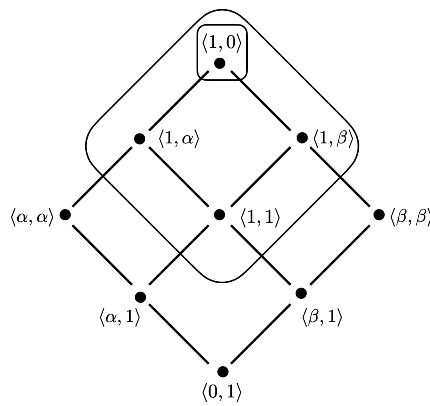
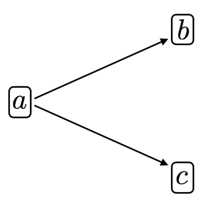


Figure 2: Anticonsistent Bilattice Values, $\mathcal{A}(L \odot L)$

version of many valued Kripke semantics, using the frame $\mathcal{F}$. The corresponding strict/tolerant modal structure has the truth value space $[\mathcal{A}(L \odot L)]^{\mathcal{G}}$. In both cases operations are interpreted as in Definition 5.4, or equivalently, using the conditions (2) and (3).

Now we narrow the example down further, and work with a particular frame. There is nothing especially significant about our choice of frame except that it is small, so one can easily see what is happening. It consists of three possible worlds, $\{a, b, c\}$, having the accessibility relation $a\mathcal{R}b$, $a\mathcal{R}c$, where these are the only cases in which accessibility holds. The frame appears in Figure 3a. We show the strict/tolerant non-validity of local sequent $\Diamond(P \vee Q) \overset{p}{\Longrightarrow} \Diamond P, \Diamond Q$ at a, where P, Q are atomic. We remind the reader of Definition 5.5 and Table 4. We must produce a valuation v in $[\mathcal{A}(L \odot L)]^{\mathcal{G}}$ such that $v(\Diamond(P \vee Q)) \in [\mathcal{A}(L \times L)]^{\mathcal{G}}_{a\to\mathcal{E}(D\times L)}$, a strict set of values, but $v(\Diamond P), v(\Diamond Q) \notin [\mathcal{A}(L \times L)]^{\mathcal{G}}_{a\to\mathcal{A}(D\times L)}$, a tolerant set of values. Unwinding this, we need a valuation such that $v(\Diamond(P \vee Q)(a) \in \mathcal{E}(D \times L)$ but $v(\Diamond P)(a), v(\Diamond Q)(a) \notin \mathcal{A}(D \times L)$.



(a) Frame $\mathcal{F}$

	P	Q	$P \vee Q$	$\Diamond(P \vee Q)$	$\Diamond P$	$\Diamond Q$
a	–	–	–	$\langle 1,0\rangle$	$\langle \alpha,\alpha\rangle$	$\langle \beta,\beta\rangle$
b	$\langle \alpha,\alpha\rangle$	$\langle \beta,1\rangle$	$\langle 1,\alpha\rangle$	–	–	–
c	$\langle \alpha,1\rangle$	$\langle \beta,\beta\rangle$	$\langle 1,\beta\rangle$	–	–	–

(b) Truth Value Calculations

Figure 3: Strict/Tolerant Local Non-Validity at Possible World a

The valuation v that we use is shown in Figure 3b. The columns labeled P and Q are what define v; for instance, $v(P)(b) = \langle\alpha, \alpha\rangle$, and so on. In the interests of simplicity, we did not show values for v at a since they will not enter into the calculation of what we need. We have similarly omitted other parts of the table which are not needed. The four columns to the right of the two defining v have calculated values. We check two cases from the table. The calculations can be done using (2) and (3), or read from the Hasse diagram in Figure 2. We use the first method here.

$$\begin{aligned} v(P \vee Q)(b) &= v(P)(b) \vee v(Q)(b) \\ &= \langle\alpha, \alpha\rangle \vee \langle\beta, 1\rangle \\ &= \langle\alpha \vee \beta, \alpha \wedge 1\rangle \\ &= \langle 1, \alpha\rangle \end{aligned} \qquad \begin{aligned} v(\Diamond(P \vee Q))(a) &= v(P \vee Q)(b) \vee v(P \vee Q)(c) \\ &= \langle 1, \alpha\rangle \vee \langle 1, \beta\rangle \\ &= \langle 1 \vee 1, \alpha \wedge \beta\rangle \\ &= \langle 1, 0\rangle \end{aligned}$$

Now, $v(\Diamond(P \vee Q))(a) = \langle 1, 0\rangle$ is in the strict set shown in Figure 2, but neither $v(\Diamond P)$ and $v(\Diamond Q)$ are in the tolerant set. Thus in the frame $\Diamond(P \vee Q) \overset{p}{\Longrightarrow} \Diamond P, \Diamond Q$ is not locally valid at a in the strict/tolerant sense.

By the results in Section 8, the sequent must $\Diamond(P \vee Q) \overset{p}{\Longrightarrow} \Diamond P, \Diamond Q$ must also be not valid at a in the many-valued sense, using the logical structure from Figure 1. The calculation shown in Figure 4 shows this directly.

	P	Q	$P \vee Q$	$\Diamond(P \vee Q)$	$\Diamond P$	$\Diamond Q$
a	–	–	–	1	α	β
b	α	β	1	–	–	–
c	α	β	1	–	–	–

Figure 4: Many-Valued Local Non-Validity at Possible World a

10 Quantified Strict/Tolerant Structures

A look at Sections 5 and 6 will quickly show that in our algebraic setting differences between modal and quantified many valued logics are minimal. Of course the languages, $\mathcal{M}$ and $\mathcal{Q}$ differ, but this not major. The primary difference is that modally we have two operators, $\boxdot$ and $\diamondsuit$, on the function space $L^{\mathcal{G}}$, Definition 5.1, while quantificationally we have an infinite family of operators, $\forall_x$, $\exists_x$, $\forall_y$, $\exists_y$, … on the function space $L^{\mathcal{A}}$, Definition 6.2, but it is easy to see that the algebraic behavior of both versions is really the same. Consequently we do not present a full treatment of a quantifier counterpart of Section 8, since it would simply amount to a translation from the one setting to the other. One could, of course, device an abstract version that had both the modal and the quantificational as instances. Some attempts were made in this direction, but finally it looked like the abstraction made things harder to follow without gaining much, primarily because it was a single abstract from just two concrete cases. The present approach seemed, and still seems, less cluttered and clearer.

11 Conclusion

It is worth noting that there is a second route to the strict/tolerant models whose construction has been discussed in this paper. We begin by sketching the important features of the version as presented, then those of an alternative but equivalent approach.

In this paper, given a complete logical Morgan algebra $\langle L, D\rangle$ we first constructed the bilattice $L \odot L$, narrowed it to the anticonsistent part, $\mathcal{A}(L \odot L)$, and finally constructed the function space $[\mathcal{A}(L \odot L)]^S$ where S is the set of possible worlds of a frame in the modal case, and is the set of assignments in the quantifier case. The restriction to anticonsistent values is flexible since, as we noted at the end of Section 7, $[\mathcal{A}(L \odot L)]^S$ and $\mathcal{A}[(L \odot L)^S]$ are isomorphic. Then ignoring the anticonsistency operator for now, the heart of our construction can be seen as first the formation of a product bilattice $L \odot L$, then of a function space bilattice $(L \odot L)^S$. One direct advantage of this construction is that the result has the structure of a function space, as did L^S, and so it relates easily to many-valued versions of Kripke models or quantificational structures.

The construction just sketched begins with methods that are like those of earlier papers in the present sequence, [12], [13], [15]. These are applied in the creation of $L \odot L$, after which a function space construction is applied. Instead, we could have proceeded as follows. Starting with the complete Morgan algebra L, construct the function space L^S first. It too will be a complete Morgan algebra. As such, methods from the earlier papers apply directly, and we can form the product bilattice $L^S \odot L^S$. The anticonsistent part of this provides us with a strict/tolerant structure, and this is simply by applying earlier results (extended to take completeness and modal/quantificational operators into account). The advantage of this is that it uses earlier results directly. The disadvantage is that it produces a product bilattice, not a function space one, and so relating the result to Kripke frames or quantificational structures is not immediate. However it is easy to show that for sets A, B, and C, $(A \times B)^C$ and $A^C \times B^C$ are isomorphic. The following are 1-1, onto, and inverses of each other.

$$\begin{aligned}
&\theta : A^C \times B^C \to (A \times B)^C \text{ where } \theta(\langle f, g\rangle) = \lambda x \in C.\langle f(x), g(x)\rangle \\
&\eta : (A \times B)^C \to A^C \times B^C \text{ where } \eta(f) = \langle \pi_1 f, \pi_2 f\rangle
\end{aligned} \tag{6}$$

These mappings provide an isomorphism not only at the set level, but at the full bilattice level. That is, $L^S \odot L^S$ and $(L \odot L)^S$ are isomorphic as bilattices. Both mappings θ and η preserve the bilattice orderings. As an example we show the information ordering is preserved by θ. By definition of bilattice product, $\langle f_1, g_1\rangle \leq_k \langle f_2, g_2\rangle$ in $L^S \odot L^S$ if both $f_1 \leq f_2$ and $g_1 \leq g_2$ in L^S. Since the ordering in L^S is defined pointwise, this means that $f_1(a) \leq f_2(a)$ and $g_1(a) \leq g_2(a)$ in L for all $a \in S$. But this is equivalent to saying that $\langle f_1(a), f_2(a)\rangle \leq_k \langle g_1(a), g_2(a)\rangle$ in the bilattice $L \odot L$ for all $a \in S$. And this in turn is equivalent to saying that $\theta(\langle f_1, f_2\rangle) \leq_k \theta(\langle g_1, g_2\rangle)$ in $(L \odot L)^S$.

Since θ and η preserve the two bilattice orderings, it follows that they preserve the various bilattice meets and joins defined using those orderings. It is also easy to check that bilattice negation is preserved. In our modal and quantificational settings it is not hard to extend all this to versions of $\boxdot$ and $\diamonddot$, or $\forall_x$, $\exists_x$, ...

The upshot is that we could have proceeded as follows. Starting with $\langle L, D\rangle$, and an appropriate indexing set S, possible worlds or assignments, form the bilattice $L^S \odot L^S$ using machinery from earlier papers directly, then apply the isomorphism just discussed to transition to $(L \odot L)^S$ (an isomorphism that also preserves the anticonsistent subsets). In short, earlier work (extended to incorporate infinitary bilattice operators) is already sufficient for what was done here. But it would have been somewhat more indirect and harder to follow. There is really only one construct behind my creation of strict/tolerant versions for a broad range of many valued logics: propositional, modal, quantified, based on weak Kleene operators generalized, and on strong Kleene operators generalized. The uniformity of treatment is striking.

Finally, the question is what is all this good for. Actually, I don't know. Here's a very brief sketch of the history of ST and TS investigations, with some of the primary references. Afterwards I can explain the reasons for my problem.

What is currently called TS was introduced by Grzegorz Malinowski starting back in 1990, in [21], [22], using the terminology of *Q-consequence*. The opening sentence of his first paper on the subject directly foreshadowed much of the subsequent work in the area: "The objective of the paper is a generalization of Tarski's concept of consequence operation related upon the idea that the rejection and acceptance need not be complementary." Malinowski's work was motivated by the work of Roman Suszko, [28], but it would be too far afield to discuss this here. While Malinowski's ideas correspond to what is now called TS, in 2004 Szymon Frankowski,

[16], [17], [18], introduced ST itself, under the name of *P consequence*. It was understood that Q consequence corresponded to something like classical logic without the reflexivity of consequence, while P consequence was similar but with transitivity removed instead. This suggested applications to various important paradoxes, which were investigated.

At some more recent point, ST and TS were given their current names, and there has been a notable sequence of papers, beginning with [3] and [5]. Applications specifically to issues of vagueness have been discussed in [4]. One of the newer topics involves a hierarchy of logics, and the basic ideas of its construction have proved to be of considerable interest, [2], [27]. And there have been further applications to paradoxes. In particular, [19] continues the work on nonreflexive logics, which originated with Malinowski, making use of TS. Also [20] should be mentioned in this paper, since it examines bilatices and their generalizations. [24] and [25] employ ST to handle languages with transparent truth while avoiding truth paradoxes, and [26] addresses paradoxes from set theory. Clearly current research in the area is very much ongoing.

Now, why do I ask what my work is good for? All of the fundamental work cited above took place in a three-valued setting, connecting with classical logic. My work shows that what I have called the "strict/tolerant phenomenon" has a very broad extent. But for the impact and significance of strict/tolerance, the single example based on classical logic is really enough. I have no idea what the role of the rich variety of examples coming from my work might be. Perhaps direct consequences of interest will emerge. Perhaps also, the machinery employed will itself have further significant uses. It all seems broad and versatile, but at the moment that opinion concerns potentiality and not actuality. To quote Michael Faraday, who was quoting Benjamin Franklin, [6], "What good is a newborn baby?"

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