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SOME INTERRELATIONS BETWEEN GEOMETRY AND MODAL LOGIC

Doctoral dissertation

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Note. Axioms, theorems, definitions and figures
are labelled in a single numerical sequence.

INTRODUCTION

Modal logicians have studied many properties of binary relations, in exploring the Kripke semantics of modal systems. It seems timely to ask where such techniques can be applied in mathematics. Two of the well-known plane geometries, projective and elliptic, can be axiomatized using a single binary relation. The usual two-sorted theory of projective planes is unsuitable for normal modal semantics, but Chapters I and VI show how it can be made one-sorted (by keeping careful account of whether the incidence relation is iterated an even or odd number of times). A one-sorted theory of elliptic planes is already available from the work of Dr. M. Kordos.

These ideas lead to a range of new modal systems (Chapters II and III) with clear geometrical interpretations (Chapter V). But unfortunately Chapter VII shows that the modal language is not powerful enough to describe the geometry adequately. Nevertheless, the theory of Chapters I, IV and VI may have some value in its own right, and a few possible ways forward are outlined in Chapter VIII.

It is a pleasure to acknowledge the support and encouragement of Dr. L.W. Szczerba. In particular, he urged me to replace an earlier non-triviality axiom

$$(\forall a,b)(\exists c) \sim (a = c \vee b = c \vee aIc \vee bIc)$$

by a purely existential axiom such as (4). It was while working on the resulting new independence proofs for the projective axioms that I realized how appropriate it would be to introduce elliptic planes as well.

I. SOME PROPERTIES COMMON TO PROJECTIVE AND ELLIPTIC PLANES

The following sentences, (1) to (4), are in a one-sorted language.

I stands for the (symmetric) incidence relation in any projective or elliptic plane, and each variable may be interpreted as either a point or a line in the plane. (If aIb , then a is a point and b is a line, or vice versa.)

- (1) $(\forall a,b)(aI^4b \Rightarrow aI^2b)$
- (2) $(\forall a,b)(aI^2b \vee aI^3b)$
- (3) $(\forall a,b,c,d)((aIbIcIdIa) \Rightarrow (a=c \vee b=d))$
- (4) $(\exists a,b,c,d,e,f)(aIbIcId \wedge eIf \wedge \sim(aId \vee aIe \vee aIf \vee bIe \vee bIf \vee cIe \vee cIf \vee dIe \vee dIf))$

The validity of these may be seen as follows.

Suppose $aI^n b$, i.e. $(\exists x_1, x_2, \dots, x_{n-1})(aIx_1Ix_2I\dots Ix_{n-1}Ib)$. It will be seen that if n is even then a and b are both points or both lines, whereas if n is odd then one of them is a point and the other a line. Now if a and b are both $\begin{Bmatrix} \text{points} \\ \text{lines} \end{Bmatrix}$, some $\begin{Bmatrix} \text{line} \\ \text{point} \end{Bmatrix} z$ is incident with them both, i.e. $aIzIb$, so aI^2b . (1) follows from this.

Also if a is a point and b is a line, then a line x and point y may be chosen as in Fig. 5 so that $aIxIyIb$, whence aI^3b . Likewise if a is a line and b is a point. But each of a,b is either a point or a line, so (2) follows.

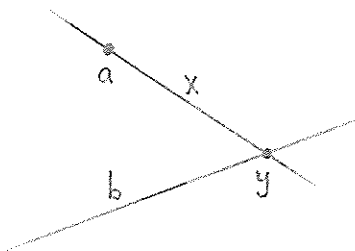


Figure 5.

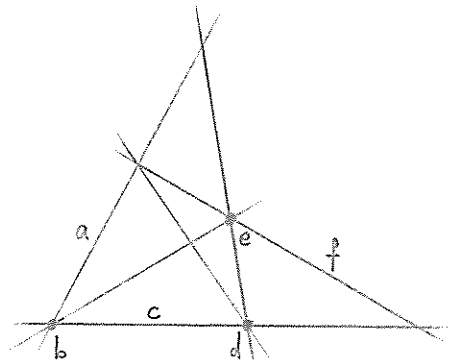


Figure 6.

(3) states the basic uniqueness property that two distinct points cannot both lie on two distinct lines. (4) is a non-triviality condition, and Fig. 6 shows how it follows from the existence of a quadrangle.

Theorems

(1) to (4) will now be treated as axioms, and Theorems (7) to (13) and (15) to (18) derived from them. In (7) to (15) Axioms (1) and (2) only are used.

Axiom (1) means that I^2 is transitive. Hence if $aI^n b$ with $n \geq 4$, it follows that $aI^{n-2} b$, $aI^{n-4} b$, ..., so either $aI^2 b$ or $aI^3 b$. (Thus when Axiom (2) states that $I^2 \cup I^3$ is universal, it asks for a connected model, i.e. a single plane rather than several disjoint planes.) This extended version of (1) will be assumed in proofs.

However the first theorem is a simple consequence of (2) alone.

(7) I is serial, i.e. $(\forall a)(\exists x)(aIx)$.

Proof. $aI^2 a \vee aI^3 a$ by (2).

$\therefore (\exists x)(aIxIa) \vee (\exists x,y)(aIxIyIa)$.

In either case, $(\exists x)aIx$.

(8) $(\forall a,b,c)(aI^2 b \vee bI^2 c \vee aI^2 c)$.

Proof. $(\sim aI^2 b \wedge \sim bI^2 c) \Rightarrow aI^3 b I^3 c$ by (2)

$\Rightarrow aI^6 c$

$\Rightarrow aI^2 c$ by (1).

(9) I^2 is reflexive.

Proof. Substitute a for b and c in (8).

(10) I^2 is symmetric.

Proof. Let $aI^2 b$. Then

$bI^3 a \Rightarrow bI^3 a I^2 b I^3 a \Rightarrow bI^8 a$

$\Rightarrow bI^2 a$ by (1).

But (2) gives $bI^2 a \vee bI^3 a$. $\therefore bI^2 a$.

(11) I^2 is an equivalence relation.

Proof. (9), (10), (1) .

(12) There are at most two equivalence classes for I^2 .

Proof. (8).

At this stage a projective plane could be defined as a model having two equivalence classes, and an elliptic plane as a model having only one. But the rest of this chapter will continue to treat the two cases together.

The next theorem is an expansion of (12), giving more information about the equivalence classes.

(13) For any element a , $\{x : aI^2x\}$ and $\{x : aI^3x\}$ are the only equivalence classes for I^2 .

Proof. By (9), aI^2a . $\therefore (\exists b)(aIbIa)$.

Also $aI^3x \Rightarrow bIaI^3x \Rightarrow bI^4x$
 $\Rightarrow bI^2x$ by (1) ,

and $bI^2x \Rightarrow aIbI^2x \Rightarrow aI^3x$.

$\therefore \{x : aI^3x\} = \{x : bI^2x\} =$ the equivalence class containing b .

But $(\forall x)(aI^2x \vee aI^3x)$ by (2),

so the only equivalence classes are those containing a and b .

b exists if a does, and both classes are then non-empty.

(14) Definitions. In any model having two I^2 -classes, the elements of one class (either) are lines; of the other, points. In a model having only one I^2 -class, every element is both a line and a point. I may be read is incident with or lies on.

(15) $aIb \Rightarrow aI^3b \Leftrightarrow$ (one of a, b is a point and the other is a line).

Proof. $aIb \Rightarrow aIbI^2b$ by (9)

$\Rightarrow aI^3b$.

If there are two I^2 -classes, then

$aI^3b \Leftrightarrow$ (a and b are in different classes) by (13).

The result follows from (14).

The symmetry of the incidence relation has not been postulated, as it can be proved from Axioms (1), (2) and (3).

(16) I is symmetric.

Proof. Let aIb .

Then (15) shows that one of a, b is a point and the other a line, and hence that bI^3a .

$\therefore (\exists x, y)(aIbIxIyIa)$.

$\therefore a = x \vee b = y$ by (3).

But bIx and yIa , so in either case bIa .

Earlier proof of (16) would have simplified (10), but that would have used (3) unnecessarily early. Anyway the obvious intention of (3) concerns the following theorem.

(17) Any two points lie on a unique line.

Any two lines lie on a unique point.

Proof. If a, b are distinct $\begin{Bmatrix} \text{points} \\ \text{lines} \end{Bmatrix}$ then aI^2b by (14).

Thus $(\exists x)(aIxIb)$, and x is a $\begin{Bmatrix} \text{line} \\ \text{point} \end{Bmatrix}$ by (15).

(16) then gives bIx , so any two $\begin{Bmatrix} \text{points} \\ \text{lines} \end{Bmatrix}$ lie on at least

one $\begin{Bmatrix} \text{line} \\ \text{point} \end{Bmatrix}$.

If a and b also lie on y , then by (16) $a \times b \in y$,
 so either $a = b$ or $x = y$ by (3).

But $a \neq b$. Hence any two $\begin{Bmatrix} \text{points} \\ \text{lines} \end{Bmatrix}$ lie on only one $\begin{Bmatrix} \text{line} \\ \text{point} \end{Bmatrix}$.

It will sometimes be convenient to argue even less formally, and to assume the usual definitions of collinear, concurrent and other standard geometrical terms.

The following proof derives from (4) one of the usual non-triviality conditions for projective (and elliptic) planes - cf. [4, p.231].

(18) There exists a quadrangle.

Proof. In the situation described by (4), $a|b|c|d$ may be represented as in Fig. 19 or 20, and $e|f$ may be represented as in Fig. 21 or 22.

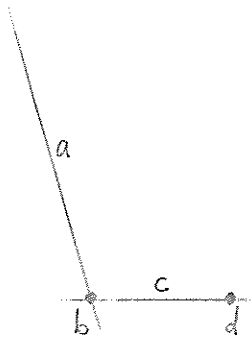


Figure 19.

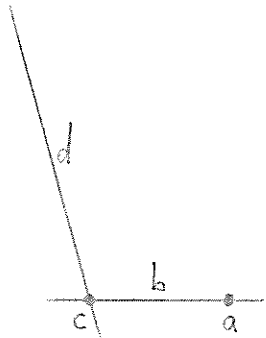


Figure 20.



Figure 21.

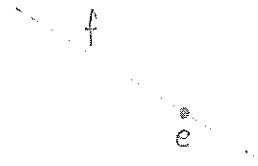


Figure 22.

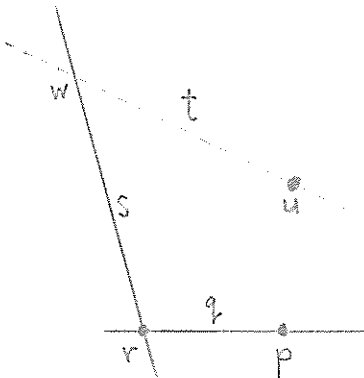


Figure 23.

Combining these gives Fig. 23, in whose notation p, q, r, s are either a, b, c, d respectively, or d, c, b, a respectively; and t, u are either e, f respectively, or f, e respectively.

s and t lie on a point w , by (17) (or by (7) if $s = t$).

It remains to show that $pruw$ is a quadrangle.

$(p,r,u \text{ colline}) \vee (p,r,w \text{ colline}) \vee (p,u,w \text{ colline}) \vee (r,u,w \text{ colline})$

$\Rightarrow (qIu \vee p=r) \vee (pIs \vee r=w) \vee (pIt \vee u=w) \vee (rIt \vee sIu)$

$\Rightarrow pIs \vee pIt \vee qIu \vee rIt \vee sIu$

$\Rightarrow aId \vee aIe \vee aIf \vee bIe \vee bIf \vee cIe \vee cIf \vee dIe \vee dIf$

which contradicts (4).

This incidentally shows that a, b, c, d, e, f are all distinct, as in the figures.

II. THE MODAL SYSTEM 12g

Each model for the theory of the previous chapter is a set with one binary relation. Many such models occur in the Kripke semantics of modal propositional logics ([7], developed by Lemmon and Scott [8] and others). Of the foregoing axioms and theorems which fairly obviously correspond to modal formulae, (9), (10), (11) and the first half of (15) all follow easily from (1), (7) and (16). Some of the less obvious cases will be considered later (Chapter VII).

Systems of modal logic are commonly based on ordinary propositional calculus (PC) with the extra formation rule that if α is a well-formed formula then so is $L\alpha$. (The notation of [5] is used here.) $M\alpha$ is defined as $\sim L\sim\alpha$.

Definition. 12g is the modal system based on all theses of PC, the rules of modus ponens and uniform substitution for propositional variables, and (24) to (28):

- (24) $L(p \supset q) \supset (Lp \supset Lq)$
- (25) $Lp \supset Mp$
- (26) $MLp \supset p$
- (27) $LLp \supset LLLLp$
- (28) Necessitation rule: if $\vdash \alpha$ then $\vdash L\alpha$.

Every normal modal system has (24) and (28). (25) corresponds to the semantic condition (7) that I is serial: cf. [8, p.55, 4.7]. (26) corresponds to the condition (16) that I is symmetric: cf. [8, p.54, 4.5]. (27) corresponds to the condition (1): cf. [8, p.59, A(i)].

A syntactic property of the system $l2g$ is that it has only finitely many inequivalent modalities. To establish this, some technicalities of modal logic are now needed.

An affirmative modality is a finite word in the letters L, M . We use A, B, C as metalogical variables ranging over such modalities, writing $A \rightarrow B$ for $\vdash (Ap \supset Bp)$, and $A = B$ for $\vdash (Ap \equiv Bp)$. If $\vdash (Ap \equiv p)$ then we write $A = 1$ (the improper affirmative modality - others are called proper).

From (28) and (24) come the standard rules [5, pp.33, 37]: if $\vdash (\alpha \supset \beta)$ then both $\vdash (L\alpha \supset L\beta)$
and $\vdash (M\alpha \supset M\beta)$.

From these, (29) follows by induction.

- (29) For any affirmative modality C , if $\vdash (\alpha \supset \beta)$ then $\vdash (C\alpha \supset C\beta)$.

The case of (29) where α is Ap and β is Bp gives part of (30). The other part uses the substitution p/Cp . (31) follows.

- (30) If $A \rightarrow B$, then $CA \rightarrow CB$ and $AC \rightarrow BC$.

- (31) If $A = B$, then $CA = CB$ and $AC = BC$.

From $\vdash (Ap \supset Bp)$ easily follows $\vdash (\sim B\sim p \supset \sim A\sim p)$. But $\vdash (\sim A\sim p \equiv \bar{A}p)$, where \bar{A} is the modality obtained from A by replacing each L by M and vice versa (the modality dual to A). Hence (32) and (33).

(32) If $A \rightarrow B$ then $\bar{B} \rightarrow \bar{A}$ (where the bars denote duals).

(33) If $A = B$ then $\bar{A} = \bar{B}$.

These principles will now be applied to the modalities of 12g, first to obtain a well-known property (34) of the system B° which has (26) as proper axiom.

(34) $LML = L$, $MLM = M$.

Proof. $ML \rightarrow 1$ by (26), so $1 \rightarrow LM$ by (32).

$\therefore LML \rightarrow L$ and also $L \rightarrow LML$ by (30).

Thus $LML = L$, and dually (33) $MLM = M$.

(35) $L^4 = ML^3 = L^2$, $M^4 = LM^3 = M^2$.

Proof. $L \rightarrow M$ by (25), so $L^4 \rightarrow ML^3$ by (30).

$ML \rightarrow 1$ by (26), so $ML^3 \rightarrow L^2$ by (30).

But $L^2 \rightarrow L^4$ by (27). $\therefore L^4 = ML^3 = L^2$.

Dually (33), $M^4 = LM^3 = M^2$.

(36) $ML^2 = L^3$, $LM^2 = M^3$.

Proof. $M.L^2 = M.L^4$ (35), (31)

$= ML^3.L = L^2.L$ (35), (31)

$= L^3$. Dually (33), $LM^2 = M^3$.

The proper affirmative modalities of 12g form the semigroup generated by L and M subject to (34), (35), (36). It is not yet clear why there are no defining relations independent of these, but that question will be taken up after (37).

To prove that the 12 modalities (37), together with 1, are all distinct, it will be enough to show why there are no implications connecting them except those that can be seen from Fig. 39. Chapter III will settle that question, by showing that any extra implication gives rise to a system stronger than 12g.

It is perhaps worth mentioning here the geometrical interpretations of the basic modalities L and M. Suppose a formula α is assigned a set X of points and lines. Then by the usual semantic rules $M\alpha$ is assigned $\{y : (\exists x \in X)(yIx)\}$ and $L\alpha$ is assigned $\{y : (\forall x \in X)(yIx)\}$. At this stage it is not obvious whether these set constructions are enough to express, say, (1) to (4) in the modal language; or whether any extra implications between modalities in Fig. 39 are geometrically valid.

III. MODALITIES OF SYSTEMS CONTAINING 12g

We next consider all systems obtained from 12g solely by adding extra implications between modalities. Every extension of 12g has the same pattern of modalities as one of these, which is the weakest normal system having that pattern.

First we dispose of negative modalities (still using A,B to stand for affirmative modalities). $(\sim Ap \supset \sim Bp)$ is merely the contrapositive of $(Bp \supset Ap)$. Also the substitution $p/\sim p$ interchanges $(Ap \supset \sim Bp)$ and $(\sim \bar{A}p \supset \bar{B}p)$, where bars denote duals. Hence negative modalities can be ignored after the following theorem.

(40) Every formula $(\sim Ap \supset Bp)$ is inconsistent with 12g.

Proof. Suppose $\sim Ap \supset Bp$.

.'. $Ap \vee Bp$ by PC.
 .'. $MMp \vee MMMp$ from Fig. 39.
 .'. $\sim LL(p \supset p) \vee \sim LLL(p \supset p)$ $p/\sim(p \supset p)$, $\sim L$ for $M\sim$.
 But $LL(p \supset p)$ PC, (28), (28)
 .'. $LLL(p \supset p)$ (28).

The last three lines are contradictory.

Implications between affirmative modalities give rise to nine different systems (including 12g itself). Each of these is named by the number of its proper affirmative modalities and then a distinguishing letter (e, f or g) . If the number is n , there are $2(n+1)$ distinct modalities altogether.

Figure 39 shows no implication linking modalities of even degree with those of odd degree. The weakest extra implications which preserve this property are $ML \rightarrow L^2$, $M^3L \rightarrow 1$, $L \rightarrow L^3$, $M^2L \rightarrow L$, their duals (32) $M^2 \rightarrow LM$, $1 \rightarrow L^3M$, $M^3 \rightarrow M$, $M \rightarrow L^2M$, and $L^3M \rightarrow M^3L$, $L^2M \rightarrow M^2L$.

The last two are equivalent. For $L^3M \rightarrow M^3L$ by (30) and (35) entails $L^2M = L^4M \rightarrow LM^3L = M^2L$, and $L^2M \rightarrow M^2L$ by (30) and (36) entails $L^3M \rightarrow LM^2L = M^3L$. The resulting system is called 10g (cf. Fig. 48).

The remaining eight are equivalent. For from $ML \rightarrow L^2$, (30) and (34) give $L \rightarrow L^3$, whence (30) and Fig. 39 give $1 \rightarrow L^3M$, whence (30) and (35) give $M \rightarrow L^2M$, whence (30) and (34) give back $ML \rightarrow L^2$; and (32) finishes the proof. The resulting system is 4g (cf. Fig. 44).

Proofs of this kind will be so numerous that most of their details will be omitted from here on.

Still keeping even and odd modalities separate, the weakest extra implications to add to Fig. 48 are $ML \rightarrow L^2$, $M^3L \rightarrow 1$, $L \rightarrow L^3$, $M^2L \rightarrow L$ and their duals, each giving 10g + 4g. This system 1g (cf. Fig. 42) has $L = M^2L = L^2M = M$. The weakest additions to Fig. 44 are the equivalents $1 \rightarrow L^2$, $M^2 \rightarrow 1$, $M \rightarrow L$, also giving 1g.

Now, the weakest way of linking the even and odd modalities in Fig. 39 is by $L^2 \rightarrow M^3$ or its dual $L^3 \rightarrow M^2$. The resulting system 8f has $L^2 = L^4 \rightarrow M^3L^2 = L^3$, whence Fig. 47.

One of the weakest additions to Fig. 47 is $L^2M \rightarrow M^2L$, which gives 7f (cf. Fig. 45). The others are $ML \rightarrow L$, $L \rightarrow 1$ and their duals; but $ML \rightarrow L$ gives $L = LML \rightarrow L^2 \rightarrow 1$, so at this stage we merely add $L \rightarrow 1 \rightarrow M$ to 8f, obtaining 8e (cf. Fig. 46).

The weakest additions to Fig. 45 also are $ML \rightarrow L$, $L \rightarrow 1$ and their duals; so again we consider only $L \rightarrow 1$ which gives 7f + 8e. This system also arises from adding $L^2M \rightarrow M^2L$ to 8e. Now in fact this addition gives the trivial system 0e (in which every affirmative modality collapses to 1), but the proof is very long and will not be given here.

The other weakest additions to Fig. 46 are the equivalents $L \rightarrow L^2$, $ML \rightarrow L$, $M^2L \rightarrow 1$ and their duals, giving 2e (cf. Fig. 43). Further additions to this give 0e (cf. Fig. 41).

For each of these eight extensions of 12g, the following list gives modality reduction laws to supplement (34), (35), (36), and then the appropriately reduced version of Fig. 39.

0e (the trivial system). $L = 1$
 $M = 1$

1

Figure 41. Affirmative modalities of 0e.

1g. $L^2 = 1$
 $M^2 = 1$

1 L

Figure 42. Affirmative modalities of 1g.

2e (Lewis's S5). $L^2 = L$
 $M^2 = M$

$$L \rightarrow 1 \rightarrow M$$

Figure 43. Affirmative modalities of 2e.

4g. $L^3 = L$
 $M^3 = M$

$$L^2 \rightarrow 1 \rightarrow M^2 \qquad L \rightarrow M$$

Figure 44. Affirmative modalities of 4g.

7f. $L^3 = L^2$, $M^2L = L^2M$,
 $M^3 = M^2$

$$\begin{array}{ccccccc} L^2 & \rightarrow & ML & \rightarrow & 1 & \rightarrow & LM & \rightarrow & M^2 \\ & \searrow & & \searrow & & \nearrow & & \nearrow & \\ & & L & \rightarrow & M^2L & \rightarrow & M & & \end{array}$$

Figure 45. Affirmative modalities of 7f.

8e, 8f. $L^3 = L^2$
 $M^3 = M^2$

8e and 8f have the same modalities, but different implications between them. 8e is the system T_2^+ of Thomas [10], cf. [5, p.260].

$$\begin{array}{ccccccccccc} L^2 & \rightarrow & L & \rightarrow & ML & \rightarrow & 1 & \rightarrow & LM & \rightarrow & M & \rightarrow & M^2 \\ & & & & & & \searrow & & & & \nearrow & & \\ & & & & & & M^2L & \rightarrow & L^2M & & & & \end{array}$$

Figure 46. Affirmative modalities of 8e.

$$\begin{array}{ccccccc} L^2 & \rightarrow & ML & \rightarrow & 1 & \rightarrow & LM & \rightarrow & M^2 \\ & \searrow & & \searrow & & \nearrow & & \nearrow & \\ & & L & \rightarrow & M^2L & \rightarrow & L^2M & \rightarrow & M \end{array}$$

Figure 47. Affirmative modalities of 8f.

10g. $M^2L = L^2M$

$$\begin{array}{c} 1 \\ \swarrow \quad \searrow \\ L^2 \rightarrow ML \quad LM \rightarrow M^2 \\ \searrow \quad \swarrow \\ M^3L \end{array} \qquad L^3 \rightarrow L \rightarrow M^2L \rightarrow M \rightarrow M^3$$

Figure 48. Affirmative modalities of 10g.

Figure 49 is a lattice diagram showing the inclusions between the nine systems. Since all are contained in O_e , the systems are consistent: cf. [5, pp.41-42].

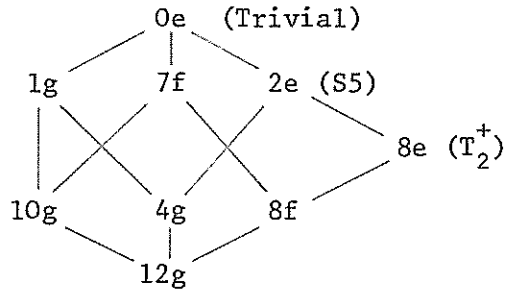


Figure 49. The nine systems.

The 13 affirmative modalities of 12g give 169 formulae $A_p \supset B_p$ which may be added to 12g as axioms. Finding what system results in each case is a tedious continuation of the foregoing argument, so the details will be omitted. The conclusions are tabulated in Table 50 (whose blank spaces are for theses of 12g itself).

		B												
		L ²	ML	M ³ L	1	L ³ M	LM	M ²	L ³	L	M ² L	L ² M	M	M ³
A	L ²	-	-	-	-	-	-	-	8f	8f	8f	8f	8f	8f
	ML	4g	-	-	-	-	-	-	2e	2e	8f	8f	8f	8f
	M ³ L	4g	4g	-	4g	-	-	-	2e	2e	8f	8f	8f	8f
	1	1g	1g	1g	-	4g	-	-	0e	0e	0e	2e	8e	8f
	L ³ M	1g	1g	10g	1g	-	-	-	0e	0e	7f	8f	8f	8f
	LM	1g	1g	1g	1g	4g	-	-	0e	0e	0e	2e	8e	8f
	M ²	1g	1g	1g	1g	4g	4g	-	0e	0e	0e	2e	2e	8f
	L ³	8f	8f	8f	8f	8f	8f	8f	-	-	-	-	-	-
	L	2e	8e	8f	8e	8f	8f	8f	4g	-	-	-	-	-
	M ² L	2e	2e	8f	2e	8f	8f	8f	4g	4g	-	-	-	-
	L ² M	0e	0e	7f	0e	8f	8f	8f	1g	1g	10g	-	-	-
	M	0e	0e	0e	0e	2e	2e	8f	1g	1g	1g	4g	-	-
	M ³	0e	0e	0e	0e	2e	2e	8f	1g	1g	1g	4g	4g	-

Table 50. Systems 12g + (A_p ⊃ B_p).

The distinctness of the nine systems in Fig. 49 will be proved using four models (undirected graphs). In the diagrams for these, each element is represented by a small square, and links between squares show the relation I. (25), (26) and (27) are valid on all of these models (as they satisfy (7), (16) and (1)), so all four are models for 12g.

(51) Every system contained in 1g is distinct from every system containing 8f.

Proof. Consider the 12g-model of Fig. 52, i.e. the set $\{w_1, w_2\}$ with relation $I = \{(w_1, w_2), (w_2, w_1)\}$. For each subset X, the following table gives $\underline{MX} = \{y : (\exists x \in X)(yIx)\}$.

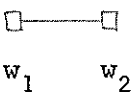


Figure 52.

	X	\emptyset	$\{w_1\}$	$\{w_2\}$	$\{w_1, w_2\}$
	\underline{MX}	\emptyset	$\{w_2\}$	$\{w_1\}$	$\{w_1, w_2\}$
Hence also	\underline{MMX}	\emptyset	$\{w_1\}$	$\{w_2\}$	$\{w_1, w_2\}$

In each case $\underline{\underline{MMX}} \cup X = \{w_1, w_2\}$, the whole set, so (by the usual evaluation rules) $(\underline{MMp} \supset p)$ is valid. But $\underline{\underline{M}}\{w_1\} \cup \underline{\underline{MM}}\{w_1\} = \{w_1\}$, not the whole set, so $(Mp \supset \underline{MMp})$ is invalid.

Hence (cf. Table 50) Fig. 52 gives a model for 1g but not for 8f.

(53) Every system contained in 7f is distinct from every system containing 4g or 8e.

Proof. Consider Fig. 54, in which the small loop indicates a reflexive element.



Figure 54.

Here $(\underline{LLMp} \supset \underline{MMMLp})$ is valid, but both $(\underline{MMMp} \supset Mp)$ and $(p \supset Mp)$ are false when p is assigned $\{w_1\}$.

Hence we have a model for 7f but not for 4g or 8e.

(55) Every system contained in 2e is distinct from every system containing 10g.

Proof. In Fig. 56, $(\underline{MMp} \supset Mp)$ is valid, but $(\underline{LLMp} \supset \underline{MMLp})$ is false when p is assigned $\{w_1\}$.

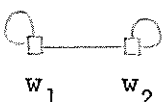


Figure 56.

Hence we have a model for 2e but not for 10g.

(57) 8e is distinct from 2e. (This is well-known.)

Proof. In Fig. 58, $(p \supset Mp)$ is valid, but $(MMp \supset Mp)$ is false

when p is assigned $\{w_1\}$.

Hence we have a model for 8e but not for 2e.



Figure 58.

(51), (53), (55) and (57) prove that all the systems in Fig. 49 are distinct from one another. It then follows that any implication between 12g modalities not already apparent from Fig. 39 gives (cf. Table 50) a system stronger than 12g; so Fig. 39 already shows all such implications in 12g itself. It then also follows that the 13 modalities in Fig. 39 are distinct, with no modality reduction laws independent of (34), (35), (36). Similar remarks apply to the other eight systems, with Figs. 41 to 48.

IV. ELLIPTIC PLANES

Chapter II set up the system 12g so that its semantics would include all models of the theory of Chapter I. This gives a geometrical significance to modal formulae, including those in Table 50. In particular, the 8f axiom $(LLp \supset LLLp)$ corresponds to the semantic condition:

$$(59) \quad (\forall a,b)(aI^3b \Rightarrow aI^2b) . \quad [8, p.59, A(i).]$$

But from (13), this is just the condition that there be only one equivalence class for I^2 , i.e. (cf. Def. (14)) that every element be both a point and a line. Thus geometrical models of 8f are (possibly degenerate) elliptic planes as treated by Kordos [6].

Any variable in any formula of Kordos's theory may be interpreted either as a line or as its pole. The corresponding four interpretations of aIb (cf. [1, p.48]) are: the point a lies on the line b ,
the point a is maximally distant from the point b ,
the line a lies on the point b ,
the line a is perpendicular to the line b .

(2) and (59) entail:

$$(60) \quad (\forall a,b)(aI^2b) .$$

In turn, (60) entails (2), (59) and also (1). So in specializing the theory of Chapter I to elliptic planes we may replace (1) and (2) by (60). Since (7) to (17) rest on (1), (2) and (3) only, they follow equally well from (60) and (3) as axioms. In particular, (17) now gives:

$$(61) \quad (\forall a,b)((a \neq b) \Rightarrow (\exists !c)(aIcIb))$$

(any two distinct elements have a unique common neighbour).

The theory based on (60) and (3) can be regarded as elliptic plane geometry in a general sense which permits degenerate planes. A little of this theory is developed in Theorems (62) to (74).

(62) No elements can be related as in Fig. 56, 63 or 64.

Proof. Figure 56 shows $w_1 I w_1 I w_2 I w_2 I w_1$ but $w_1 \neq w_2$, which violates (3).

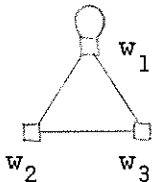


Figure 63.

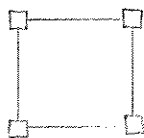


Figure 64.

Figure 63 shows $w_1 I w_1 I w_2 I w_3 I w_1$ but

$w_1 \neq w_2$ and $w_1 \neq w_3$, violating (3).

Figure 64 obviously violates (3).

(65) $(\sim aIb) \Rightarrow$ (the sets of neighbours of a and of b have the same cardinal number).

Proof. Let $A = \{x : aIx\}$, $B = \{y : bIy\}$. It is required to prove $|A| = |B|$.

Assuming $\sim aIb$, we have $b \notin A$.

$\therefore (\forall x \in A)(\exists ! y)(xIyIb)$ by (61)

But $yIb \Leftrightarrow y \in B$.

$\therefore (\forall x \in A)(\exists ! y \in B)(xIy)$.

Similarly $(\forall y \in B)(\exists ! x \in A)(xIy)$.

Hence the restriction of I is a 1-1 correspondence between A and B .

(66) Either (i) every element has the same number of neighbours; or
 (ii) some element a is related to all the others, and every element except a has the same number of neighbours.

Proof. First, suppose $(\forall b,c)(b=c \vee bIc)$.

If there are at least four elements, this violates (3).

If there are at most three elements, (62) permits only the

four models shown in Fig. 67. The first

three of these satisfy (i), and the last

three satisfy (ii).

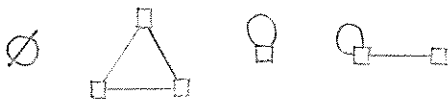


Figure 67. Some degenerate elliptic planes.

Henceforth, assume $(\exists b,c) \sim (b=c \vee bIc)$.

Such elements b and c have the same number of neighbours,

by (65). Call this number $n+1$ (a finite or infinite cardinal).

Let $A = \{x : x \text{ has } n+1 \text{ neighbours}\}$, so $b,c \in A$.

Then $a \notin A \Rightarrow$ (number of neighbours of a) $\neq n+1$

$\Rightarrow (\forall x \in A)(aIx)$ by the contrapositive of (65).

So if distinct elements $a_1, a_2 \notin A$, (3) is violated (Fig. 68).

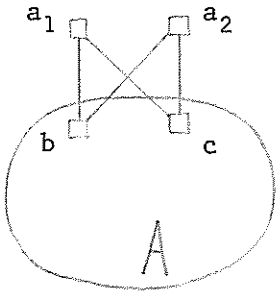


Figure 68.

Hence either every element is in A , giving case (i), or just one element $a \notin A$, and $(\forall x \in A)(aIx)$, giving case (ii).

(69) Any model of (66)(ii) must be as in Fig. 71 or 72.

Proof. With the notation of the previous proof, assume case (ii) with $n \geq 2$.

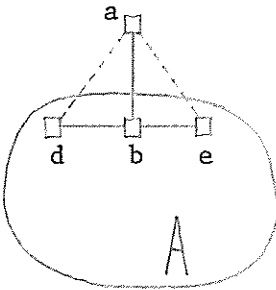


Figure 70.

Then any element b of A has at least three neighbours a, d, e as in Fig. 70. (d or e may coincide with b .) But $(\forall x \in A)(aIx)$, whence the dotted links in the figure lead to a violation of (3). $\therefore n \leq 1$.

($n = 0$) gives the situation of Fig. 71, where a is reflexive because of (60).

($n = 1$) gives the situation of Fig. 72, where a is irreflexive because of (62).

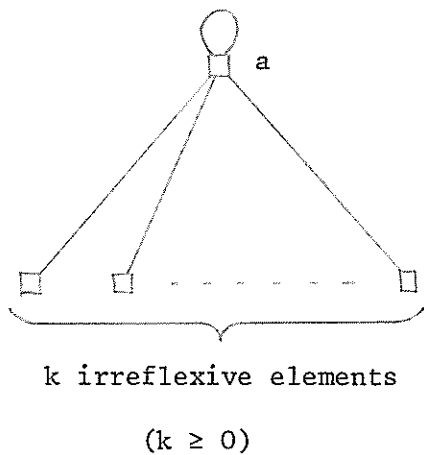


Figure 71.

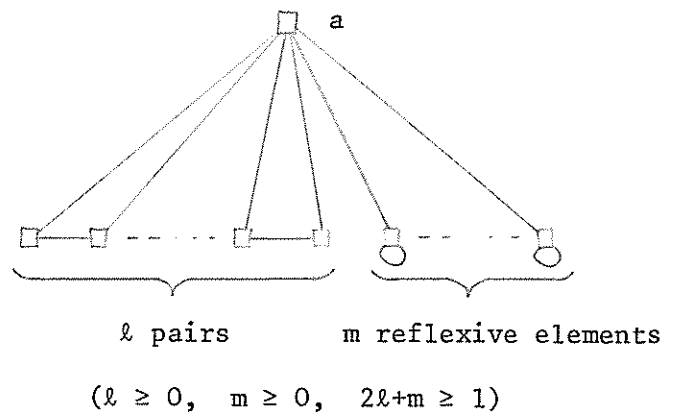






Figure 72.

Note. From Fig. 71, the cases $k = 0, 1$ give  and .
 From Fig. 72, the cases $(\ell=1, m=0)$, $(\ell=0, m=1)$ give  and .

But these make up all of Fig. 67 except \emptyset , so (69) embraces all non-empty degenerate elliptic planes considered so far.

(73) Definition. A non-empty model of (66)(i) in which each element has $n+1$ neighbours, is called an elliptic plane of order n .

(74) An elliptic plane of order n has $n^2 + n + 1$ elements.

Proof. Being non-empty, the model is generated by an element a .

First suppose $n = 0$, so a has just one neighbour b .

If $b \neq a$, we obtain Fig. 52, violating (60).

Hence $b = a$, so we have , i.e. $0^2 + 0 + 1$ elements.

Now assume $n \geq 1$, so a has at least two neighbours.

Hence there are at least two elements.

Since Fig. 56 violates (62), there is at least one irreflexive element, say a itself.

Let its neighbours be a_0, a_1, \dots, a_n (distinct from one another and from a).

Then $(\forall a_j)(\exists! a_i)(aIa_iIa_j)$ by (61).

This gives the situation of Fig. 72, with $2\ell+m = n+1$.

By (62) there are no other links amongst the elements

a, a_0, a_1, \dots, a_n .

For any other element x , $(\exists! a_i)(aIa_iIx)$.

But each a_i has $n+1$ neighbours including a and some a_j , so $n-1$ other neighbours (which may be imagined as added to the bottom of Fig. 72).

$$\begin{aligned} \therefore \text{total number of elements} &= 1 + (n+1) + (n+1)(n-1) \\ &= n^2 + n + 1. \end{aligned}$$

Of course this is strongly reminiscent of a projective plane of order n , which has $2(n^2+n+1)$ elements (points and lines). In fact every elliptic plane E fairly obviously corresponds to a projective plane P whose points and lines may each be labelled in the same way as the elements of E . Whenever $a \perp b$ in E , then in P the point a lies on the line b and also the line a lies on the point b . Thus P is self-dual. If P is a Pappus plane then the elements of E may be given homogeneous coordinates over a field, with:

$$(75) \quad (x_1, y_1, z_1) \perp (x_2, y_2, z_2) \iff x_1 x_2 + y_1 y_2 + z_1 z_2 = 0.$$

Cf. [3, pp.217-8, Th.1].

(74) and even (69) could have been obtained indirectly through standard projective results, instead of directly from (60) and (3).

All elliptic planes having orders up to 3 are illustrated in Fig. 76. The plane of order 0 is also included in Fig. 71 (case $k = 0$), and those of order 1 are included in Fig. 72 (case $\ell = 0, m = 2$, and case $\ell = 1, m = 0$). Those of orders 2 and 3 correspond to (75) over the fields $GF(2)$ and $GF(3)$.

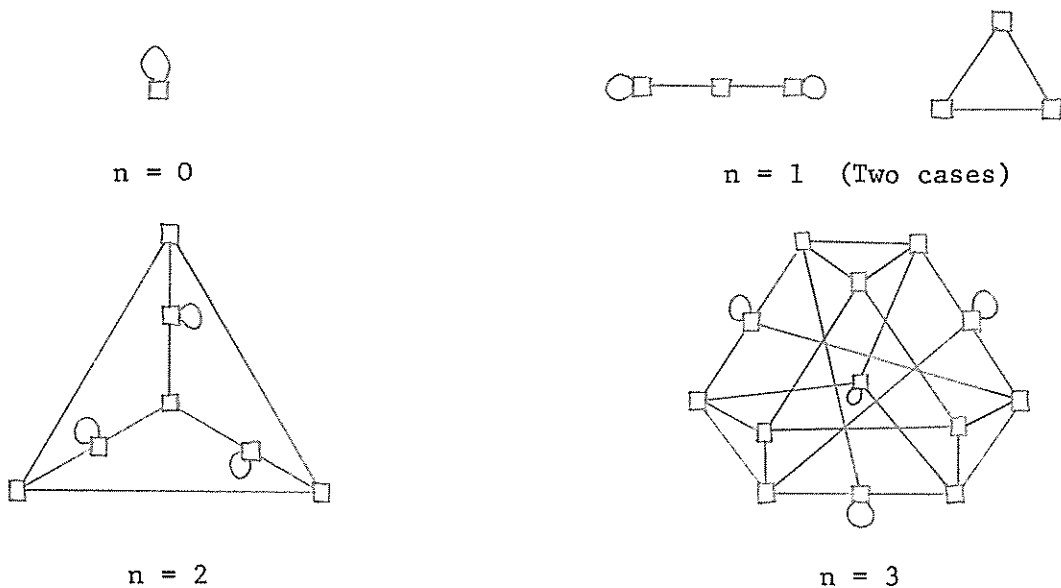


Figure 76. Elliptic planes of small order n .

In Chapter I, the axiom to exclude degenerate planes was (4). This requires a model to contain at least one of the six graphs in Fig. 77 (with no extra links between the elements shown).

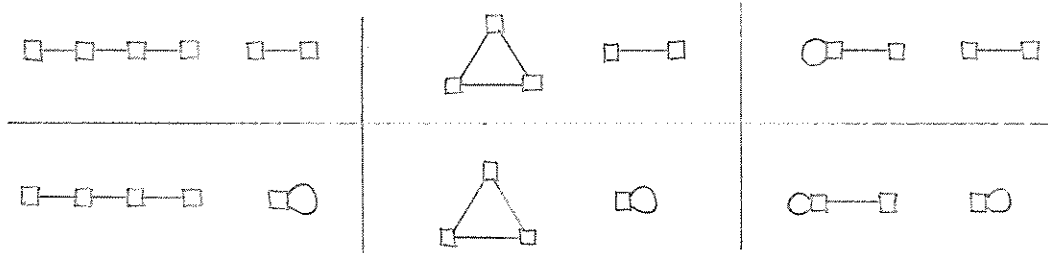


Figure 77.

So (4) excludes \emptyset and all cases of Fig. 71 and 72 (including the elliptic planes of orders 0 and 1), i.e. it requires an elliptic plane to have an order, which must be at least 2. But the same can be achieved by requiring a model to contain at least one of the two graphs in Fig. 78 (with no extra links between distinct elements). Hence (4) may be replaced by the shorter axiom:

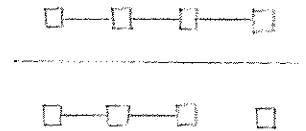


Figure 78.

$$(79) \quad (\exists a,b,c,d)(aIbIc \wedge \sim(a=c \vee b=d \vee aIc \vee bId)) .$$

No further such shortening is possible, as every 3-element irreflexive model for (3) can be embedded in a degenerate plane as in Fig. 72.

It may be wondered why non-trivial elliptic planes (satisfying (60), (3) and (79)) can be finite, in view of Karzel's theorem quoted by Bachmann [1, pp.123-4, Satz 20]. He shows that a finite elliptic plane cannot exist, by proving that the number of points in each orbit under the automorphism group would have to be $\frac{n^2+1}{2}$, which (for $n > 1$) is not a factor of n^2+n+1 . Since the proof uses mid-points, it might be thought to require Kordos's mid-points axiom [6, p.613, AX6]; but in fact any elliptic plane over a field of odd order has the particular mid-points required in Karzel's proof. The problem is that such a mid-point (x,y,z) may be reflexive, i.e. from (75) it may satisfy:

$$(80) \quad x^2 + y^2 + z^2 = 0 .$$

There is no proper reflection in such a point. This frustrates the argument that an orbit must have $\frac{n^2+1}{2}$ elements. Instead, a Pappus elliptic plane of odd order n has three orbits under its automorphism group: $n+1$ reflexive elements (points/tangents of the conic (80)), $\frac{n(n-1)}{2}$ irreflexive elements having no reflexive neighbours (interior points/non-secants of the conic), and $\frac{n(n+1)}{2}$ elements each having two reflexive neighbours (exterior points/secants of the conic). Over a field of characteristic 2, the conic (80) degenerates to the repeated line/point $(x+y+z)^2 = 0$; and for even order n the three orbits comprise: this single element $(1,1,1)$, its $n+1$ neighbours (the reflexive elements), and the remaining n^2-1 elements. (Cf. Fig. 76 for the cases $n = 2,3$.) Over the real field, (80) is the elliptic absolute, having no real points or tangents. If by analogy with this we want to banish the conic (80), we can add Kordos's AX2 [6, p.613], viz. the irreflexive law:

$$(81) \quad (\forall a) \sim (aIa) .$$

Although the axiom set $\{(3), (60), (79), (81)\}$ is weaker than Kordos's or Bachmann's, it still has no finite model, by the "Friendship Theorem" of graph theory [3, p.234, Th.6], cf. [2, p.337]. However this whole paragraph is a digression (with no proofs). Before (79) there was already enough material for the discussion of 7f in the next chapter.

V. GEOMETRICAL SIGNIFICANCE OF THE STRONGER SYSTEMS

The discussion of elliptic planes in Chapter IV arose from (59), the semantic condition for the modal system 8f. Similarly it may be asked what special properties a plane must have in order to be a model for any one of the eight extensions of 12g obtained in Chapter III.

Since much of the answer involves degenerate planes, axiom (4) will usually be ignored in this chapter. Axiom (2) is satisfied by any generated (i.e. connected) 12g-model. But axiom (3) needs to be borne in mind.

Fig. 49 shows that 0e, 2e, 8e and 7f all contain the basic elliptic system 8f, so all models for these e- and f-systems are elliptic planes. Projective planes are models for 12g but not for 8f, so the g-systems are relevant to these.

0e, 2e, 8e.

The e-systems can be dismissed fairly quickly. $L \rightarrow 1$ is an axiom for 8e (Table 50), so is also in the other two (cf. Fig. 49). But models for $L_p \supset p$ have I reflexive, so any two related elements would be as in Fig. 56 which violates (3) (cf. (62)). Hence the only non-empty model is the elliptic plane of order 0 (cf. Fig. 76), i.e. a single point/line incident with itself.

1g.

From Table 50, $1 \rightarrow L^2$ is an axiom for 1g. The corresponding semantic condition [8, p.59, A(i)(b)] is $(\forall a,b)(aI^2b \Rightarrow a = b)$, which makes each equivalence class for I^2 collapse to a singleton. Hence, by Def. (14), there are only one point and one line; i.e. the plane has order zero. Fig. 82 shows the two non-empty possibilities. The first is a

model for the stronger system 0e (= 1g + 8f : cf. Fig. 49). The second is a model for 1g but not for 0e.



Elliptic.



Projective.

Figure 82. Models for 1g.

4g.

From Table 50, $L \rightarrow L^3$ is an axiom for 4g. The corresponding semantic condition [8, p.59, A(i)] is $(\forall a,b)(aI^3b \Rightarrow aIb)$. From (15), this means that every point lies on every line. If there were more than one point and more than one line, this condition would violate (3); so at least one I^2 -class collapses to a singleton. Fig. 83 shows the

non-empty possibilities. The first is a model for the stronger system 2e (= 4g + 8f : cf. Fig. 49), indeed for 0e. The second is a model for 4g but not 2e, although the case $h = 1$ is a model for 1g (cf. Fig. 82).

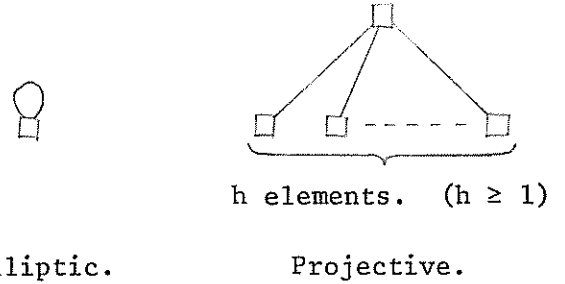


Figure 83. Geometrical models for 4g.

7f, 10g.

From Table 50, $L^2M \rightarrow M^2L$ is an axiom for 10g. Such axioms are notoriously difficult to handle semantically (cf. [8, pp.74-76]), but it is possible to cope with this one in the present context of geometrical models satisfying (3). It will be convenient to derive a variant of the axiom.

	$LLMp \supset MMLp$	
	$MM\sim Mp \vee MMLp$	PC, $M\sim$ for $\sim L$.
	$MM(\sim Mp \vee Lp)$	M-distribution [5, p.37, T7].
(84)	$MM(Mp \supset Lp)$	Substitution of equivalents.

As these steps are reversible, (84) may be added to 12g to give 10g; and hence also added to 8f to give 7f (= 10g + 8f : cf. Fig. 49).

A useful description of the modal semantics (cf. e.g. [8, p.22]) says that a formula α is true at x iff α is assigned a set X with $x \in X$. Then Lp is true at an element w iff p is true at every neighbour of w , and Mp is false at w iff p is false at every neighbour of w . Thus $Mp \supset Lp$

is true at w iff p has the same truth-value at every neighbour of w . In that situation (84) is true at every element u such that uI^2w , i.e. throughout the I^2 -class containing w . (In the elliptic case this class is the whole plane, by (60).) These principles will be used frequently in the following proofs.

(85) Every degenerate elliptic plane is a model for $7f$.

Proof. Consider first the degenerate plane of Fig. 71, with $k \geq 1$.

At each element, assign either truth-value to p .

Any irreflexive element w (on the lower level of the diagram) has only one neighbour.

$\therefore p$ has the same truth-value at every neighbour of w .

$\therefore Mp \supset Lp$ is true at w .

\therefore (84) is true at every element, whatever the assignment to p .

\therefore Fig. 71 gives a model for $7f$ whenever $k \geq 1$.

When $k = 0$ it is a model for $0e$, so certainly for $7f$ (cf.

Fig. 49). The same applies to the empty model.

Now treat Fig. 72 in a similar way.

If p has the same truth-value at every lower-level element, then $Mp \supset Lp$ is true at the upper element a .

Otherwise p has the same truth-value at a as at some lower-level element x , so $Mp \supset Lp$ is true at the element whose only neighbours are a and x .

In either case $Mp \supset Lp$ is true somewhere, so (84) is true everywhere.

\therefore Fig. 72 gives a model for $7f$.

(85) could also be obtained from the projective case (88), but the argument might be less clear.

Degenerate projective planes are shown in the next two figures, first by schematic geometrical diagrams and then by modal semantic graphs. The latter have their small squares black or white to show the two I^2 -classes.

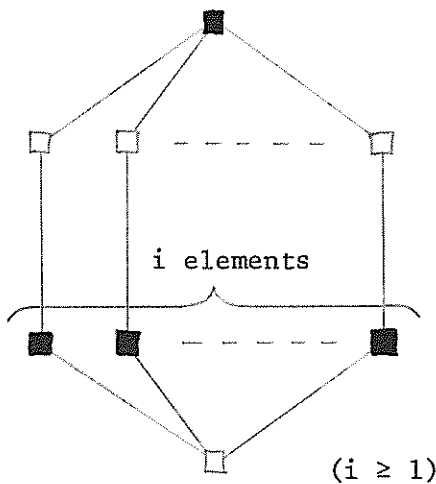
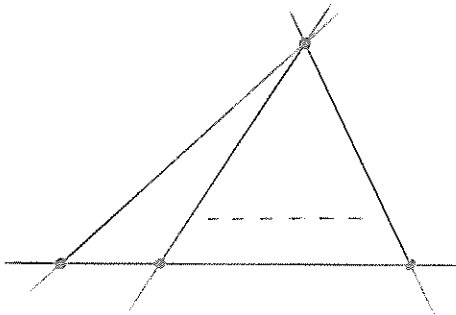


Figure 86.

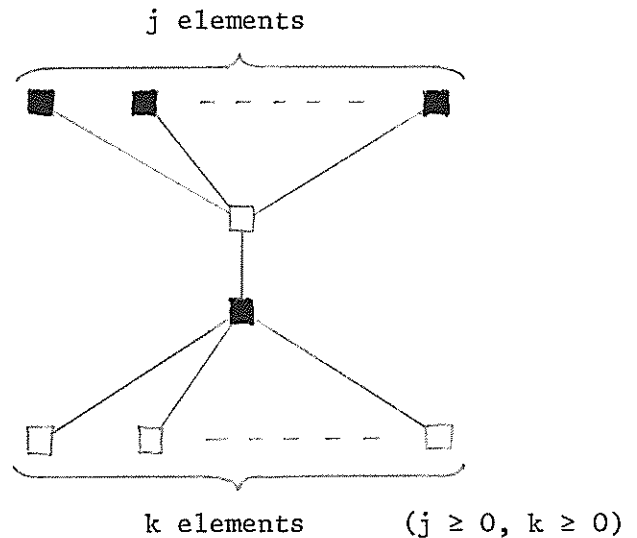


Figure 87.

(88) These degenerate projective planes are models for $10g$ iff $(j=0 \iff k=0)$.

Proof. First consider Fig. 86.

If p has the same truth-value at every upper white element, then $Mp \supset Lp$ is true at the top black element.

Otherwise p has the same truth-value at the bottom white element as at some upper white element, so $Mp \supset Lp$ is true at the intermediate black element.

In either case $Mp \supset Lp$ is true at some black element, so (84) is true at every black element.

Similarly (in fact, dually) (84) is true at every white element.

∴ Fig. 86 gives a model for 10g.

Next consider Fig. 87 with $j \geq 1$ and $k \geq 1$.

Any top-level black element has only one neighbour, so $M_p \supset L_p$ is true there.

∴ (84) is true at every black element, and similarly at every white element.

∴ Fig. 87 with $j \geq 1$ and $k \geq 1$ gives a model for 10g.

Finally consider Fig. 87 with $j = 0$ or $k = 0$.

This is the projective case of Fig. 83, with $h = k+1$ or $j+1$ respectively.

∴ it is a model for 4g.

Hence (cf. Fig. 49) it is a model for 10g iff it is a model for 1g (= 4g + 10g),

i.e. iff $h = 1$ (cf. Fig. 82),

i.e. iff $j = k = 0$.

(89) The projective plane of order 2 is a model for 10g.

The elliptic plane of order 2 is a model for 7f.

Proof. The next three figures are schematic geometrical diagrams of the projective plane of order 2. This becomes the elliptic plane of order 2 (Fig. 76) if each point is identified with a suitable (polar) line.

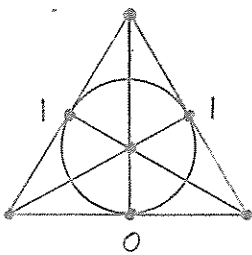


Figure 90.

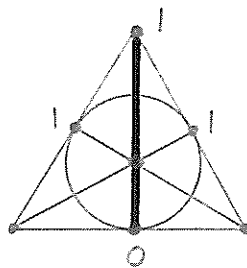


Figure 91.

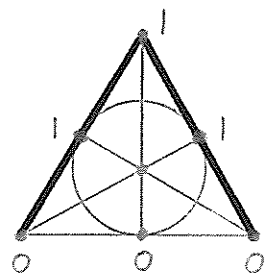


Figure 92.

The following argument shows that $Mp \supset Lp$ cannot be false at every line.

One of the lines is represented in Fig. 90 by a circle. Suppose $Mp \supset Lp$ is false at this line.

Then p does not have the same truth-value at all three points on it.

Without loss of generality, assume the truth-values shown in Fig. 90.

Suppose $Mp \supset Lp$ is false also at the vertical line (emphasized in Fig. 91).

Then since p is false at the bottom point on this line, it is true at some other point on it.

Without loss of generality, assume p is true at the top point (Fig. 91).

Suppose $Mp \supset Lp$ is false also at the other two lines through this point (emphasized in Fig. 92).

Then since p is true at two points on each of these lines, it is false at the third (Fig. 92).

But then p is false at every point on the bottom line, so $Mp \supset Lp$ is true at this line.

∴ however truth-values are assigned to p , $Mp \supset Lp$ is true at some line, so (84) is true at every line.

In the elliptic case this means that (84) is true at every element.

In the projective case a similar (dual) argument shows that (84) is true at every point as well as every line.

∴ the projective and elliptic planes of order 2 are models for 10g and 7f respectively.

(93) If a projective or elliptic plane has order at least 3, it is not a model for 10g or 7f.

Proof. Take any triangle in the plane. (The plane satisfies (4), and hence (18).) Let p be true at every point which lies

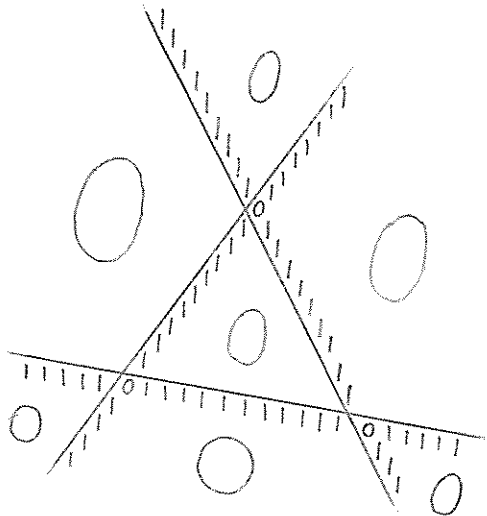


Figure 94.

on just one side (l-point), and false at every other point (o-point). Fig. 94 roughly indicates this.

Since the order of the plane is at least 3, each line passes through at least 4 points. (Cf. Def. 73.)

Thus any line passing through no vertex meets the three sides at l-points, and also has at least one o-point.

Also any line through just one vertex meets the opposite side at a l-point, and has at least three o-points.

Also any line through two vertices is a side of the triangle, having two o-points and at least two l-points.

∴ every line passes through at least one l-point (where p is true) and at least one o-point (where p is false).

∴ $Mp \supset Lp$ is false at every line.

∴ (84) is false at any line.

∴ the plane is not a model for 10g or 7f.

The only degenerate projective planes [4, p.232] not yet mentioned comprise an isolated point, line or both, i.e. one or two irreflexive elements; but these do not satisfy (2).

Summary of this chapter.

In the following list, each plane is allotted to the strongest system (highest in Fig. 49) for which it is a model.

Elliptic planes.

Geometrical models for 0e, 2e, 8e.

The empty model.

A single reflexive element (the elliptic plane of order 0).

Geometrical models for 7f but not for 0e.

All other degenerate elliptic planes : Fig. 71 with $k \geq 1$, and
Fig. 72. (85) (Fig. 72 includes the elliptic planes of
order 1 : Fig. 76.)

The elliptic plane of order 2 : Fig. 76. (89)

Geometrical models for 8f but not for 7f (or 8e).

Elliptic planes having order at least 3. (93)

Projective planes.

Geometrical model for 1g but not for 0e.

One point on one line (the projective plane of order 0) : Fig. 82.

Geometrical models for 4g but not for 1g (or 2e).

A line through at least 2 points, or a point on at least 2 lines :
Fig. 83 with $h \geq 2$. (One-dimensional models.)

Geometrical models for 10g but not for 1g or 7f.

All other degenerate projective planes : Fig. 86, and Fig. 87 with
 $j \geq 1$ and $k \geq 1$. (88) (Fig. 86 includes the projective plane
of order 1.)

The projective plane of order 2. (89)

Geometrical models for 12g but not for 4g or 8f or 10g.

Projective planes having order at least 3. (93)

VI. AXIOMS AND INDEPENDENCE

(17) and (18) make up a standard definition of projective plane [4, p.231], provided that the set of points is disjoint from the set of lines. (Cf. Def. 14.) Hence a projective plane may be defined by axioms (1), (2), (3), (4) and (95). The proofs of (96) and (97) require only (1), (2) and (95).

$$(95) \quad (\exists a) \sim (aI^3a) .$$

(96) There are two distinct equivalence classes for I^2 .

Proof. By (95), some element a satisfies $\sim(aI^3a)$.

By (13), the I^2 -classes are $\{x : aI^2x\}$ and $\{x : aI^3x\}$.

a is in the first class but not the second, so they are distinct.

Axiom (95) states that I^3 is not reflexive, but it may be strengthened to the universal statement that I^3 is irreflexive.

$$(97) \quad (\forall a) \sim (aI^3a) .$$

Proof. Let a be any element.

By (13), the I^2 -classes are $\{x : aI^2x\}$ and $\{x : aI^3x\}$.

a is in the first class, so by (96) it is not in the second.

$$\therefore \sim aI^3a .$$

Corollary : (81).

Definition (14) names the elements of the two I^2 -classes from (96). But for any model, it is quite arbitrary which I^2 -class is called the class of points and which the class of lines. This makes the principle of duality semantic. There are no dual theorems (unless each theorem is regarded as identical with its dual), as the theory itself treats the two classes in exactly the same way. The point/line distinction may be made either way in each particular model. For example, in Figs. 86 and 87, the black squares could represent either points or lines, and the white squares lines or points respectively.

For elliptic planes, the non-triviality axiom (4) could be replaced by the shorter (79). No such shortening is possible for projective planes, as every 5-element model for (3), (97) and $(\forall a) \sim (aI^5a)$ (a consequence of (1) and (97)) can be embedded in a degenerate plane as in Fig. 86. The two such 6-element graphs which cannot be embedded in Fig. 86 or 87, are shown in Fig. 98. (4) was obtained from the first of these in order to ease the proof of (18), but alternative 6-variable formulae could be obtained by taking account of the second.

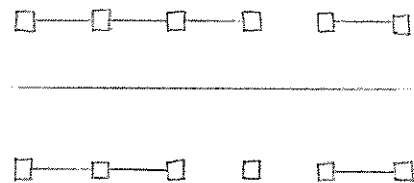


Figure 98.

Independence of the projective axioms.

The following models will prove the independence of (1), (2), (3), (4) and (95). (16) will be treated as a possible alternative axiom, as the cyclic order of variables in the antecedent of (3) may seem an artificial way of making I symmetric. Cf. [6, p.613, AX3 and AX5]. For a similar reason, (97) will be considered as well as the weaker (95).

(16) has already been proved from (1), (2) and (3); and (97) from (1), (2) and (95). Another such result should be noted.

(99) (2), (3), (97) \vdash (16) .

Proof. Let aIb .

$bI^2a \vee bI^3a$ by (2).

But $bI^2a \Rightarrow aIbI^2a \Rightarrow aI^3a$, contradicting (97).

$\therefore bI^3a$.

Proceed using (3) as in the original proof of (16).

Independence of (1).

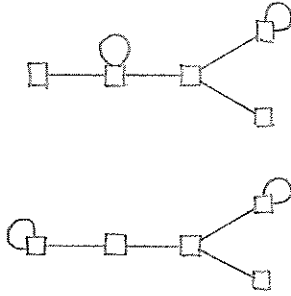


Figure 100.

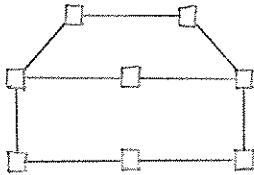


Figure 101.

Independence of (2).

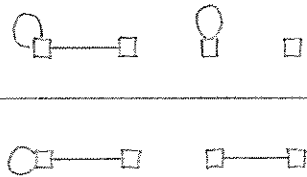


Figure 102.

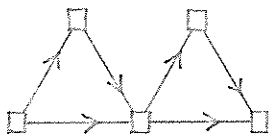


Figure 103.

Fig. 100 shows the two smallest models for (2), (3), (4) and (95). (62) may be helpful in checking (3). For (4), take $a=d$, $b=c$ and $e=f$. For (95), use the lower right-hand element. These models happen to satisfy (16) also. They do not satisfy (1), as the left-hand and right-hand elements are related by I^4 but not by I^2 . Hence (1) is independent of (2), (3), (4), (95), and (16).

Fig. 101 shows the smallest model for (2), (3), (4) and (97), and hence also (16), by (99). It does not satisfy (1), so (1) is independent of (2), (3), (4), (97) and (16).

The obvious model is two disjoint projective planes, but other models can have as few as 4 elements. Of such models, the two satisfying (16) are shown in Fig. 102. Hence (2) is independent of (1), (3), (4), (95) and (16).

Every model for (1), (3), (4) and (97) has at least 5 elements. One such model is shown in Fig. 103, where an arrow from x to y indicates that xIy but not yIx . For (4), take e and f to be the left-hand elements, and $f = a$. A symmetric model requires at least 6 elements, as in the upper part of Fig. 98. Hence (2) is independent of (1), (3), (4), (97) and (16).

Independence of (3).

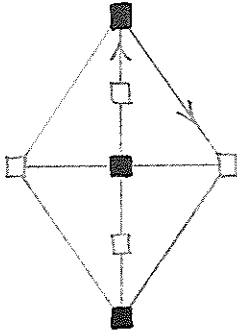


Figure 104.

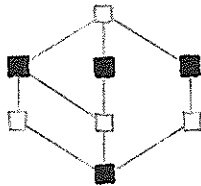


Figure 105.

Every model for (1), (2), (4), (95), and hence also (96) and (97), has at least 7 elements. One such model is shown in Fig. 104. For (4), take a and f both to be the top black element. Subgraphs as in Fig. 64 show that (3) is not satisfied. Hence (3) is independent of (1), (2), (4), (95) and (97).

A symmetric model requires at least 8 elements. One such model is shown in Fig. 105. Hence (3) is independent of (1), (2), (4), (97) and (16).

Independence of (4).

The smallest model for (1), (2), (3), (95), and hence also (16) and (97), is the projective plane of order 0 (Fig. 82). (4) is false in this, so (4) is independent of (1), (2), (3), (95), (97) and (16).

Independence of (95) or (97).

The negation of (95) or of (97) reduces the number of I^2 -classes to one, by (13). Thus the independence models are the non-degenerate elliptic planes, and the smallest such model is the elliptic plane of order 2 (Fig. 76). Hence (95) or (97) is independent of (1), (2), (3), (4) and (16).

Each of the earlier independence arguments included an irreflexive model, because (81) is a corollary of (97). In the present case there is no finite irreflexive model, as mentioned near the end of Chapter IV. A countable irreflexive model is the rational elliptic plane (defined using (75)), which shows that (95) or (97) is independent of (1), (2), (3), (4), (16) and (81). This might be of some interest if (81) were proposed as an axiom in the theory common to projective and elliptic planes. It could have been included in Chapter I, but would have served no useful purpose there.

Independence of the elliptic axioms.

(3), (60) and (79) are the axioms, but (16) and (81) will also be considered.

Independence of (3).

A model for (60) and (79) needs at least 3 elements, as for example in Fig. 106. (For (79), take $a=d$.) To satisfy either (16) or (81) as well, it needs at least 4 elements, as for example in Figs. 107, 108 respectively. Fig. 109 shows the two smallest models for (60), (79), (16) and (81). None of these models satisfies (3).

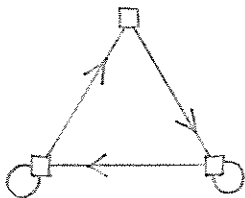


Figure 106.

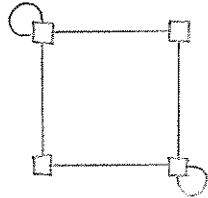


Figure 107.

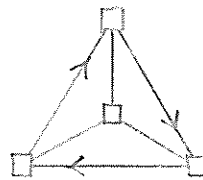


Figure 108.

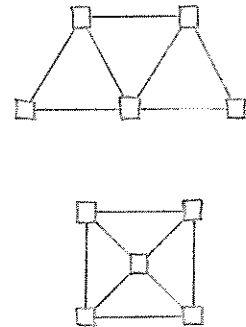


Figure 109.

Independence of (60).



Figure 110.

A model for (3) and (79) needs at least 3 elements. The example in Fig. 110 satisfies (81) also. A model for (3), (79) and (16) needs at least 4 elements. Fig. 78 shows the two such models which satisfy (81) as well. None of these models satisfies (60).

Independence of (79).

A model for (3) and (60) is a (possibly degenerate) elliptic plane, as discussed in Chapter IV. The smallest is the empty model, which also satisfies (16) and (81) but not (79). The smallest non-empty model with these same properties is the irreflexive elliptic plane of order 1 (self-polar triangle : cf. Fig. 76). The plane of order 0 has the same properties except for (81).

Further axioms.

Configurational axioms (e.g. Fano, Desargues, Pappus) may be added to the theory if suitably stated in the one-sorted language.

The Fano axiom states that the diagonal points of a quadrangle never

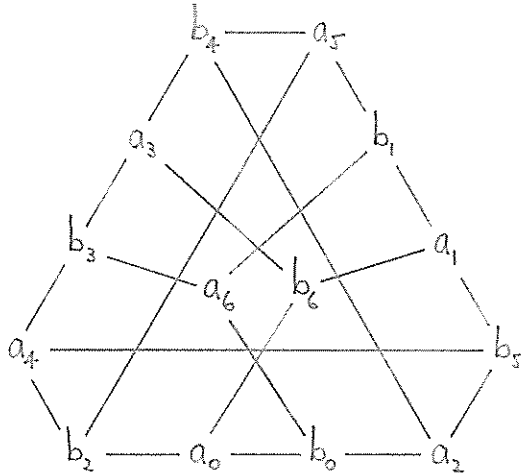


Figure 111. Projective plane of order 2.

colline, i.e. that there is no non-degenerate projective sub-plane of order 2. Thus if 14 elements are related as the points and lines of such a sub-plane (cf. Fig. 111), there must be some further incidence to make it degenerate. Any formula expressing this has to be rather clumsy, although with care it is possible to reduce the

consequent to three disjuncts, as follows:

$$(112) \quad (\forall a_0, a_1, a_2, a_3, a_4, a_5, a_6, b_0, b_1, b_2, b_3, b_4, b_5, b_6) ((a_6 \text{Ib}_0 \wedge a_6 \text{Ib}_1 \wedge a_6 \text{Ib}_3 \wedge a_5 \text{Ib}_1 \wedge a_5 \text{Ib}_2 \wedge a_5 \text{Ib}_4 \wedge a_4 \text{Ib}_2 \wedge a_4 \text{Ib}_3 \wedge a_4 \text{Ib}_5 \wedge a_3 \text{Ib}_3 \wedge a_3 \text{Ib}_4 \wedge a_3 \text{Ib}_6 \wedge a_2 \text{Ib}_4 \wedge a_2 \text{Ib}_5 \wedge a_2 \text{Ib}_0 \wedge a_1 \text{Ib}_5 \wedge a_1 \text{Ib}_6 \wedge a_1 \text{Ib}_1 \wedge a_0 \text{Ib}_6 \wedge a_0 \text{Ib}_0 \wedge a_0 \text{Ib}_2) \Rightarrow (a_4 \text{Ib}_4 \vee a_5 \text{Ib}_5 \vee a_6 \text{Ib}_6)).$$

In elliptic planes the same formula may be used, even though Fig. 111 does not represent an elliptic sub-plane unless certain pairs of elements coincide.

The Desargues and Pappus axioms may be expressed by formulae even clumsier than (112), which are better understood from diagrams.

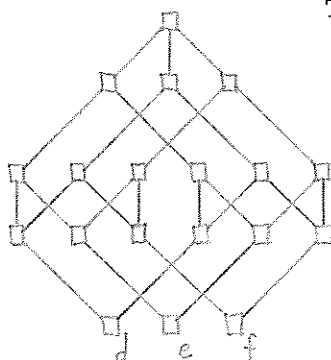
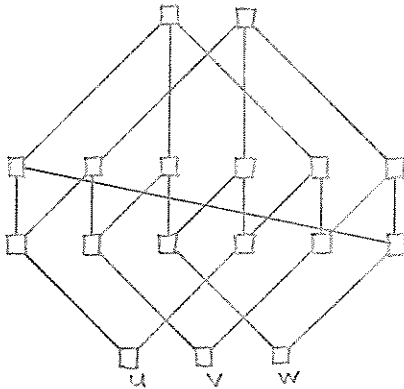


Figure 113. Desargues.

The Desargues axiom (or its dual) states that if 19 elements are related as in Fig. 113, then there exists an element related to all three of d, e, f. (The formula expressing this would also cover degenerate cases, in which some elements shown distinct in Fig. 113 would coincide.)

The Pappus axiom (or its dual) states that if 17 elements are related



as in Fig. 114, then there exists an element related to all three of u, v, w . (Degenerate cases would be included as in the Desargues axiom.)

Figure 114. Pappus.

VII. SHORTCOMINGS OF THE MODAL LANGUAGE

Chapter II set up the modal system 12g so that its models would satisfy (1), (7), (16) and hence certain other formulae from Chapter I. The lack of modal formulae corresponding to (2) and (4) is not important. To satisfy (2), restrict attention to generated (i.e. connected) 12g-models; and to satisfy (4), exclude the degenerate planes discussed in Chapters IV and V.

But several of the first-order formulae which cannot be so easily dismissed have no modal counterparts. The proof of this uses a metatheorem due to Segerberg [9]: the class of semantic models for any set of modal axioms is closed under homomorphisms. Such a model is a set W with a binary relation I , and a homomorphism from (W, I) to (W', I') is a mapping $\theta : W \rightarrow W'$ with the following two properties:

$$(115) \quad (\forall x, y \in W)(xIy \Rightarrow (x\theta) I'(y\theta))$$

$$(116) \quad (\forall x \in W)(\forall z' \in W')((x\theta)I'z' \Rightarrow (\exists y \in W)(xIy \wedge (y\theta = z'))).$$

Consider the special case where (W', I') comprises a single reflexive element (elliptic plane of order 0). From any non-empty (W, I) , the unique mapping $\theta : W \rightarrow W'$ trivially satisfies (115). It also satisfies (116) provided that $(\forall x \in W)(\exists y \in W)(xIy)$, i.e. provided that (W, I) satisfies (7). But any projective plane (W, I) , e.g. that in Fig. 111, satisfies (1), (2), (3), (4), (95), and hence also (97) and (81) as well as (7). Since (W', I') does not satisfy (81), (95) or (97), none of these properties is preserved under all homomorphisms, so none of them can be axiomatized by any modal formulae.

But (95) or (97), the irreflexiveness of I^3 , is the basic property of projective planes which distinguishes them from elliptic planes. And in Chapter V, modal axioms did make this distinction: the e- and f-systems (containing 8f) applied only to elliptic planes, whereas projective planes were models for 12g but not 8f. In this indirect way the modal language can cope with (95) by means of its negation:

$$(117) \quad (\forall a)(aI^3a) .$$

This semantic condition corresponds to the modal formula $L^3 \rightarrow 1$ [8, p.59, A(i)(a)], which Table 50 shows is indeed an axiom for 8f.

So far the modal language has coped with most of the geometrical axioms, even if indirectly. But there remains Axiom (3). In Chapter V, (3) was frequently used to select the geometrical planes from among the models of various modal systems. It would have been simpler to add to 12g one or more modal axioms corresponding to (3) itself. But no such axioms exist.

(118) The class of 8f-models satisfying (3) is not modally axiomatizable.

Proof. In any elliptic plane of order at least 3, label the points as in the proof of (93) (cf. Fig. 94).

A mapping θ from this plane onto the model of Fig. 56 maps every 1-point to w_1 , and every 0-point to w_2 .

Since the relation shown in Fig. 56 is universal, θ satisfies (115).

The proof of (93) showed that every line in the plane is incident with a 0-point and a 1-point.

Since every element is a line as well as a point, θ satisfies (116) also.

$\therefore \theta$ is a homomorphism.

But the elliptic plane satisfies (3), whereas its image (Fig. 56) does not. (62)

$\therefore 8f + (3)$ cannot be axiomatized by any modal formulae.

Corollary. The class of 12g-models satisfying (3) is not modally axiomatizable.

However, some indirect approach might enable the modal language to cope with (3). One such idea is to weaken (3) to (119), allowing each point and line to correspond to a set of elements rather than a single element of the graph.

$$(119) \quad (\forall a,b,c,d)((aIbIcIdIa) \Rightarrow ((\forall e)(aIe \Leftrightarrow cIe) \vee (\forall f)(bIf \Leftrightarrow dIf)))$$

Unfortunately:

$$(120) \quad \text{The class of } 8f\text{-models satisfying (119) is not modally axiomatizable.}$$

Proof. In any elliptic plane (of order at least 2),
 give any irreflexive element (point/line) the label 2,
 give every neighbour of this 2-element the label 1,
 and give every other element the label 0.

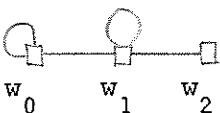


Figure 121.

A mapping θ from this plane onto the model of Fig. 121 maps every 0-element to w_0 , every 1-element to w_1 , and the 2-element to w_2 .

The method of labelling ensures that θ satisfies (115).

Every 0-point and every 1-point lies on a 1-line (which joins it to the 2-point) and a 0-line (any other); and every 1-point lies on the 2-line.

∴ θ satisfies (116) also, so it is a homomorphism.

The elliptic plane satisfies (3) and hence (119).

To see that its image (Fig. 121) does not satisfy (119), take

$$a = b = w_0, \quad c = d = w_1, \quad e = f = w_2.$$

∴ $8f + (119)$ cannot be axiomatized by any modal formulae.

Corollary. The class of 12g-models satisfying (119) is not modally axiomatizable.

It was possible to deal with (95) by means of a modal axiom for its negation (117). But in the case of (3) or (119), even that device fails.

(122) The class of 8f-models satisfying the negation of (3) is not modally axiomatizable.

The class of 8f-models satisfying the negation of (119) is not modally axiomatizable.

Proof. The models of Fig. 56 and 121 may each be mapped homomorphically to a single reflexive element (elliptic plane of order 0).

The proof of (118) shows that Fig. 56 satisfies the negation of (3).

The proof of (120) shows that Fig. 121 satisfies the negation of (119).

But their image trivially satisfies (3) and (119).

∴ neither $8f + \sim(3)$ nor $8f + \sim(119)$ can be axiomatized by any modal formulae.

Corollary. The class of 12g-models satisfying the negation of (3) is not modally axiomatizable.

The class of 12g-models satisfying the negation of (119) is not modally axiomatizable.

Axiom (3) is obviously of basic importance in plane geometry. This failure of the modal language to cope with (3) deals a severe blow to any hope that an extension of I_2 might shed much light on projective or elliptic planes.

VIII. EXTENSIONS, AND OPEN QUESTIONS

The following sections briefly sketch various possible modifications of the theory.

Affine planes.

Parallel lines (not related by I^2) can be admitted if (1) and (2) are replaced by:

$$(123) \quad (\forall a, b)(aI^5b \Rightarrow aI^3b)$$

$$(124) \quad (\forall a, b)(aI^3b \vee aI^4b) .$$

Then theorems like (8) to (12) show that I^4 is an equivalence relation giving two classes. But other axioms must distinguish the class of points (over which I^2 is universal) from the class of lines. There is no technical reason against this, but the lack of duality perhaps makes such a one-sorted theory seem rather artificial.

Spaces of higher dimension.

The axioms of Winternitz [11] suggest a one-sorted theory of projective 3-space, in which each variable may be interpreted as either a point or a plane. (2) and (95) can remain, but (1), (3) and (4) are replaced by (125), (126) and a more elaborate non-triviality condition.

$$(125) \quad (\forall a, b, c)((aI^2bI^2c) \Rightarrow (\exists d)(aId \wedge bId \wedge dIc))$$

$$(126) \quad (\forall a, b, c, d, e, f)((aIbIcIdIa \wedge aIeIc \wedge bIfId) \Rightarrow (a=c \vee b=d \vee eIf))$$

It appears possible to extend this idea to higher finite dimensions, although many of the formulae would become obscure if written in primitive notation. Elliptic spaces could be treated in a similar way.

Simplification of axioms.

In Chapter IV, the non-triviality axiom (4) was replaced by the shorter (79) for elliptic planes. This suggests the question whether some of the other axioms (notably Desargues, etc., mentioned in Chapter VI) could be shortened for elliptic planes. The Kordos elliptic theory might allow simplifications not possible in the general projective context where these axioms have chiefly been studied. For example, it might sometimes be possible to replace a general triangle (Fig. 127) by a self-polar triangle (Fig. 128), thus eliminating three variables.

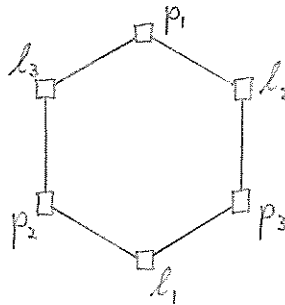


Figure 127.

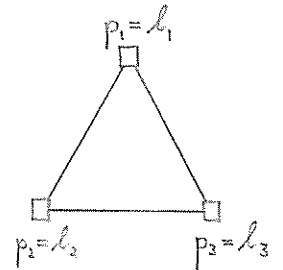


Figure 128.

Theory of incompleteness degree 2.

Every model of the theory of Chapter I satisfies either (95) or its negation, and is thereby either a projective or an elliptic plane. Hence (1), (2), (3) and (4) do in fact axiomatize the theory common to projective and elliptic planes. This common ground might be explored in more detail. In particular, Dr. L.W. Szczerba has suggested adding something like Hilbert's completeness axiom schema, in order to obtain the theory common to the real projective and elliptic planes.

The appropriate modal system.

Despite the damage done by Chapter VII, there is a well-defined system comprising the modal formulae valid in all projective and elliptic planes. Dr. K. Prażmowski has suggested the problem of axiomatizing this system. If it is stronger than I_2g , the additional axioms are not of the form $Ap \supset Bp$ investigated in Chapter III. Many known modal systems have proper axioms more complicated than these, but there is no general method of finding them. The search might require more effort than it is worth.

Finite elliptic planes.

The digression at the end of Chapter IV perhaps indicates the best way forward. The study of finite elliptic planes may be assisted by techniques of graph theory more than by those of modal logic, despite the graph theorists' apparent distaste for loops (reflexive elements). A set with a single binary relation seems a fairly primitive object, but quite different aspects of its theory have been developed by modal logicians and by graph theorists. It appears that some of these geometrical aspects are different again. Ideas from these three disparate sources may well interact fruitfully in the future.

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