Translating metainferences into formulae: satisfaction operators and sequent calculi

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Abstract

In this paper, we present a way to translate the metainferences of a mixed metainferential system into formulae of an extended-language system, called its associated σ-system. To do this, the σ-system will contain new operators (one for each satisfaction standard), called the σ operators, which represent the notions of "belonging to a (given) satisfaction standard". We first prove, in a model-theoretic way, that these translations preserve (in)validity. That is, that a metainference is valid in the base system if and only if its translation is a tautology of its corresponding σ-system. We then use these results to obtain other key advantages. Most interestingly, we provide a recipe for building unlabeled sequent calculi for σ-systems. We then exemplify this with a σ-system useful for logics of the **ST** family, and prove soundness and completeness for it, which indirectly gives us a calculus for the metainferences of all those mixed systems. Finally, we respond to some possible objections and show how our σ-framework can shed light on the "obeying" discussion within mixed metainferential contexts.

1 Introduction

In recent times validity has been extensively studied not only from an inferential perspective, but also from a metainferential one. That is, interest grew around inferences that have other inferences as premises and/or conclusions (which may themselves contain inferential components). The reasons for this interest are varied. Metainferences have recently come into focus as a useful way of distinguishing between various substructural solutions to semantic paradoxes (as shown in [21]), as a new way to characterize a logic (see for example [5]), as a way to analyze the debate between global and local validity [4], as a toolkit for understanding abstract features of consequence relations [27]), and as a key for a new version of the collapse argument against logical pluralism (as [2] shows).[1]

For example, one traditional characterization of a logic sees it as a dyadic relation between sets (or multisets/tuples) of formulae. Under that characterization, it was argued that the logic **ST** was identical to classical logic (**CL**), since they validate exactly the same inferences (for example, see [25], [26] or [11]). However, these two logics validate different sets of metainferences, metametainferences, and so on. Thus, if a logic is not just thought of as a relation between inferences, but also between metainferences at all levels (as [5] and [21] suggest), then **ST** and **CL** should not be considered the same logic.

On the other hand, one reason why **ST** is interesting is that it allows us to preserve all classically valid inferences, while at the same time being able to incorporate problematic vocabulary, such as a naive truth predicate (the failure of metainferential transitivity or Cut in **ST** is part of the reason why it can do this).

[1] And it is not implausible to think about metainferences also as a central feature of new solutions to vagueness-related phenomena.

From this point of view, taking a metainferential perspective is interesting because it allows us to get even closer to classical logic, while also being able to retain the problematic vocabulary in question. For instance, as [21] shows, the system **TS/ST** validates all classical inferences and metainferences, while also being able to incorporate a naive truth predicate. This system, however, will fail to validate some classically valid metametainferences. The system **STTS/TSST** can fix this, since it will validate every classical n-level inference up to $n = 3$, but will fail for some inferences of level $n > 3$ (where level 2 inferences are metainferences, level 3 inferences are meta-metainferences, etc). Generalizing this procedure and taking the union of the resulting systems, one can obtain what Pailos calls a "fully classical" logic characterized by substructural means.[2] This system will validate every n-level classical (meta)inference, and will also be able to contain a truth predicate.

In this paper we focus on a subclass of mixed metainferential logics, and provide some results for them. We will show a way to to collapse all n-level inferences to 0 level (i.e. into formulae), such that (in)validity is preserved in the translation. To accomplish this, we extend the language with new satisfaction operators, which will intuitively reflect the notion of "being satisfied according to a standard" inside the object language. In that way, we will be able to define systems where (some) formulae will be logically true if and only if the corresponding n-inference is valid.

This, in turn, will provide several advantages and further results. Most notable is the following. Metainferential and mixed metainferential logics of this kind have been studied mostly model-theoretically. The above results will allow us to give standard inferential calculi for checking the validity of metainferences, for both pure and mixed metainferential systems.

We shall proceed as follows. In section 2, we present some technical preliminaries, which will be necessary to understand our proposal in the sections that follow. Section 3 model-theoretically introduces our extended language σ-systems, as well as the apparatus that enables us to collapse (meta)inferences into formulae of those systems. In section 4, we present a proof theory for σ-systems in the form of n-sided sequent calculi, exemplified with a particular system and a proof of soundness and completeness for it. In section 5 we respond to some objections and consider some additional philosophical payoffs of our approach. Finally, we draw some conclusions.

2 Technical preliminaries

In this section, we lay out some basic vocabulary and results concerning the kinds of systems that we will be dealing with, that is, *many-valued mixed metainferential* systems. Let us explain what each of these terms refers to.

We begin by fixing some terminology. Let $\mathcal{L}$ be a propositional language, such that $\mathbf{FOR}(\mathcal{L})$ is the absolutely free algebra of formulae of $\mathcal{L}$, whose universe we denote by $FOR(\mathcal{L})$. We will let Γ, Δ, and other Greek capital letters represent *finite* sets of formulae, or sets of (meta)inferences. We will let context determine which one is relevant. γ, δ, and other lower-case Greek letters represent the members of Γ, Δ, that is, individual formulae, or (meta)inferences—once again, we hope context will make clear how they should be understood. We will also use some uppercase letters, such as A, B, C and D to represent arbitrary formulae.

The next step is to give a formal definition of both inferences and metainferences. Let $INF^0(\mathcal{L}) = FOR(\mathcal{L})$. Then:

Definition 2.1. A (meta)inference of (a finite) level n (for $1 \leq n < \omega$) is an ordered pair $\langle \Gamma, \Delta \rangle$, where $\Gamma, \Delta \subseteq INF^{n-1}(\mathcal{L})$ (written $\Gamma /_n \Delta$). $INF^n(\mathcal{L})$ is the set of all metainferences of level n

[2] Which is equivalent to what Scambler calls, in [28], "the twist logic $\mathbb{T}$".

on $\mathcal{L}$.

Thus, for example, $INF^1(\mathcal{L})$ will be the set of regular (level 1) inferences of $\mathcal{L}$, and we refer to its members using the notation $\Gamma/_1\Delta$. Each $\gamma \in \Gamma$ and $\delta \in \Delta$ will be a formula in that case. $INF^2(\mathcal{L})$ will be the set of (level 2) metainferences (written $\Gamma/_2\Delta$), where each $\gamma \in \Gamma$ and $\delta \in \Delta$ is a regular inference. More intuitively, a level 2 metainference can be understood as an inference between regular inferences (while metainferences of level 1 are just traditional inferences). Analogously, a level n metainference may be interpreted as an inference between $(n-1)$-level (meta)inferences. When the level of a metainference $\Gamma/_n\Delta$ is clear by context, we will more simply write Γ/Δ.

As said above, the logics we will be working with are *many-valued* and *mixed*. In [10], Chemlá, Egré and Spector introduce the notions of *mixed* and *impure* consequence relations. We shall say that a consequence relation (for a propositional language $\mathbf{L}$) is *mixed* if and only if, for every inference $\Gamma/_n\Delta$, Δ follows from Γ if and only if, for every valuation v, if $v(\gamma)$ meets some standard S_1 (for all $\gamma \in \Gamma$), then $v(\delta)$ meets some standard S_2 (for some $\delta \in \Delta$). If $S_1 \neq S_2$, then the mixed consequence relation is said to be *impure*. These notions can be made more precise with the following definitions.

Definition 2.2. A subset S_i of the entire set of truth values $\mathbf{V}$ is a 0-level standard (i.e. a standard for formulae, see below). A pair of $(n-1)$-level standards $\langle S_1, S_2\rangle$, which we write as $S_1/_nS_2$ (or, more simply, S_1/S_2 when the level is clear by context), is an n-level standard.

With this in mind, we can formally specify what it is for both a formula and a (meta)inference to meet some (simple or complex) standard, with the following recursive definition.

Definition 2.3. A 0-level inference (i.e. formula) A is *satisfied* by a valuation v according to a 0-level standard S_i if and only if $v(A) \in S_i$. (Equivalently, one can say that v satisfies A according to S_i). A (meta)inference $\Gamma/_n\Delta$ (of any finite level $n > 0$) is *satisfied* by a valuation v according to a standard $S_1/_nS_2$ if and only if, (if v satisfies every $\gamma \in \Gamma$ according to S_1 then v satisfies every $\delta \in \Delta$ according to S_2).

For example, take the set of truth values $\mathbf{V} = \{0, 1\}$ and the 0-level standards $F = \{0\}$ and $T = \{1\}$. The level-1 inference p/q is satisfied according to T/T by the valuation $v(p) = 0$ and $v(q) = 1$, but not by $v'(p) = 1$ and $v'(q) = 0$. The first is the case because the antecedent in the conditional (if v satisfies p according to T then v satisfies q according to T) is false; in the second case, the antecedent is true and the consequent false. If we change the standard to F/F then v' satisfies the inference while v does not. Similarly, a level-2 metainference $(p/q)/_2(r/s)$ will be satisfied by a valuation $v(p) = 1$, $v(q) = 1$, $v(r) = 0$, $v(s) = 1$ according to the standard $(T/T)/(T/T)$, but not according to the standard $(T/T)/(F/F)$. The first happens, because both the antecedent and the consequent in the conditional (if v satisfies p/q according to (T/T) then v satisfies r/s according to (T/T)) are true (to see this, one needs to apply the definition of satisfaction recursively in both the antecedent and the consequent). The second does not because, once again, the antecedent is true and the consequent is false (we leave it as an exercise to the reader to check that this is so).

Up until now, we have only been speaking of when an n-level standard satisfies an n-level inference. But the definition can be extended to cover cases where the level of the inference is either lower or greater than the level of the standard. As is common in the literature, we shall take a (meta)inference $\Gamma/_{(n-1)}\Delta$ of level $n-1$ to be satisfied by the standard $X/_nY$ just in case the n-inference $\emptyset/_n(\Gamma/_{(n-1)}\Delta)$ is satisfied by that standard; that is, when the conclusion standard Y satisfies the inference. Similarly, an $n-2$ inference $\Gamma/_{(n-2)}\Delta$ will be satisfied by the n-level standard $(X/Y)/_n(W/Z)$ just in case the n inference $\emptyset/_n(\emptyset/_{(n-1)}(\Gamma/_{(n-2)}\Delta))$ is satisfied by it; that is, when it is satisfied by Z, the conclusion standard of the conclusion standard. Ripley (in [27]) calls this "lowering".

For levels greater than n, we simply repeat the standard. A (meta)inference $\Gamma/_{(n+1)}\Delta$ of level $n+1$ is satisfied by the n-level standard $X/_nY$ if and only if it is satisfied by $(X/_nY)/_{n+1}(X/_nY)$. For an inference of level $n+2$ we duplicate this last standard. Ripley (in [27]) calls this "lifting". In that way, returning to the example above, we can state that both p/p and $((p/p)/_2(p/p))/_3((p/p)/_2(p/p))$ are satisfied by the valuation $v(p) = 1$ according to the level-2 standard T/T.

Once we know what it is for a (meta)inference to meet a standard according to a valuation (i.e. to be satisfied by it) we can define the mixed system $\mathbf{X/Y}$ based on the standard X/Y, in the following way.

Definition 2.4. If $X/_nY$ is an n-level standard, then $\mathbf{X/_nY}$ is the system in which the (meta)inference $\Gamma/_m\Delta$ (for any arbitrary level m) is valid if and only if every valuation v satisfies $\Gamma/_m\Delta$ according to $X/_nY$.

Returning once again to the previous example, the level-1 inference p/q is invalid in classical logic **CL** (i.e. $\mathbf{T/T}$). Although some valuations (for example $v(p) = 0$ and $v(q) = 1$) do satisfy that inference, there is at least one valuation ($v'(p) = 1$ and $v'(q) = 0$) that does not.

This way of characterizing the notion of metainferential validity is known as the "local conception of metainferential validity"([14]).[3] As it stands, the definition specifies what it takes for a particular metainference to be valid in a particular logic. Nevertheless, it can—and will—be used to specify when a metainferential *scheme* is valid in a logic **L**. In a nutshell, a scheme is valid in **L** if and only if every *instance* of it is valid.

It will also be useful for later on to define the notion of anti-validity introduced by Scambler in [28].

Definition 2.5. A (meta)inference $\Gamma/_m\Delta$ is anti-valid in the system $\mathbf{X/_nY}$ if and only no valuation v satisfies $\Gamma/_m\Delta$ according to $X/_nY$.

Before moving on, it will be useful to introduce some mixed 3-valued systems that have already been extensively studied in the literature, and that we will mention repeatedly below. Let $\mathbf{V}$ be the set of truth values $\{1, \frac{1}{2}, 0\}$, and let $S = \{1\}$ and $T = \{1, \frac{1}{2}\}$ be two standards on that set of truth values (usually referred to as the *strict* and *tolerant* standards, respectively). Using the Strong Kleene truth matrices for the connectives, we can define four systems by differently combining the premise and conclusion standards.

- The system $\mathbf{S/S}$, commonly known as $\mathbf{K_3}$.
- The system $\mathbf{S/T}$, commonly known as **ST**.
- The system $\mathbf{T/S}$, commonly known as **TS**.
- The system $\mathbf{T/T}$, commonly known as **LP**.

Of these, it is interesting to mention that **ST** and **TS** are not structural (**ST** is non-transitive and **TS** is non-reflexive). The following two results are also worth mentioning.

[3]Another relevant notion of metainferential validity is the *global* one, which is, essentially, preservation of validity. As [29] proves, both notions coincide for schemes if certain conditions are met—for example, if we have constants in the language for every truth value. For more about this notion, and the difference between a *local* and a *global* notion of metainferential validity, see [19].

Fact 2.6 ([18], [25])**.** **ST** *and classical propositional logic* **CL***—presented with the traditional bivalent deterministic valuations—have the same set of valid inferences.*

Fact 2.7 ([16])**.** **TS** *has no valid inferences.*

Note, however, that **ST** is not identical to **CL** for any metainferential level greater than 1. For example, it fails to validate the metainference $(\Gamma/_1\Delta), (\Delta/_1\Pi)/_2(\Gamma/_1\Pi)$, which is valid in classical logic. **TS**, on the other hand, even though it does not validate any inferences, will validate some metainferences (such as $(A/_1A)/_2(A/_1A)$, a form of meta-reflexivity). Finally, note that all these systems are subsystems of classical logic. This follows, at least partially, from the fact that, in the Strong Kleene matrices, classical inputs always return the corresponding classical output (Chemlá, Egré and Spector [10] call this property *bivalence-compliance*). All the systems we will consider from here on have this property.

Some people have also studied systems that are impure at some metainferential level (see, for example, [5], [21] or [3]). For example, the system **TS/ST** works as follows. A metainference $(\Gamma_1/_1\Delta_1), ..., (\Gamma_n/_1\Delta_n)/_2(\Sigma/_1\Pi)$ is valid in **TS/ST** if and only if, for every valuation v, if v satisfies every $\Gamma_i/_1\Delta_i$ *according to* **TS**, then v satisfies $\Sigma/_1\Pi$ *according to* **ST**; if and only if, either (i) there is a $\Gamma_i/_1\Delta_i$ such that $v(\Gamma_i) \in \{1, \frac{1}{2}\}$ and $v(\Delta_i) \in \{\frac{1}{2}, 0\}$, and/or (ii) $v(\Sigma) \in \{\frac{1}{2}, 0\}$ or $v(\Pi) = \{1, \frac{1}{2}\}$.

With these tools, [5] and [21] develop a hierarchy of metainferential logics based on **ST** (and **TS**). That hierarchy, which contains a system defined at each level n ($1 \leq n < \omega$), is such that for every i in the hierarchy, the logic defined at level i recovers every classically valid metainference of level i or less, but not every classical metainference of higher levels. [5] and [21] called each logic for level n in the hierarchy, $\mathbf{CM_n}$. We chose to call them $\mathbf{ST(n)}$, a neutral option that stresses its link with the inferential **ST**, which can be understood as the ground floor of the hierarchy.

Definition 2.8. The **ST** hierarchy can be recursively defined as follows:

$$\begin{aligned} &\mathbf{ST}(1) = \mathbf{S/_1T} \\ &\mathbf{TS}(1) = \mathbf{T/_1S} \\ &\mathbf{ST}(n) = \mathbf{TS}(n-1) /_n \mathbf{ST}(n-1) \\ &\mathbf{TS}(n) = \mathbf{ST}(n-1) /_n \mathbf{TS}(n-1) \end{aligned}$$

So, for example, $\mathbf{ST}(2) = (\mathbf{T/_1S}) /_\mathbf{2} (\mathbf{S/_1T})$ (or, more easily, **TS/ST**). A fully classical system $\mathbf{ST}(\omega)$ can be obtained by defining validity such that, for all n, $\Gamma /_n \Delta$ is valid if and only if it is valid in $\mathbf{ST}(n)$ (or equivalently, by taking the union of $\mathbf{ST}(n)$ for all $n \in \mathbb{N}$).

The following results summarize some of the key results presented in those papers.

Fact 2.9. *The General Collapse Result*

For every level n ($1 \leq n < \omega$), a metainference of level n $\Gamma/_n\Delta$ is valid in **CL** *if and only if it is valid in* $\mathbf{ST(n)}$

Fact 2.10. *For every n ($1 \leq n < \omega$),* $\mathbf{ST(n)}$ *invalidates infinitely many classically valid metainferences of level $n+1$. (And its metainferences of level $n+1$ are properly included in the $n+1$ classical validities.)*[4]

Fact 2.11. *The Absolute General Collapse Result*

For every level n ($1 \leq n < \omega$), a metainference of level n $\Gamma/_n\Delta$ is valid in **CL** *if and only if it is valid in* $\mathbf{ST}(\omega)$.

[4]Specifically, those logics invalidate higher-level forms of Cut, that the authors labelled as *Meta-Cut*$_n$.

One of the most attractive features of $\mathbf{ST}(\omega)$ is that it not only recovers every classical metainferential validity of every level (in fact, [21] refers to it as a *fully* classical logic), but it also supports a transparent truth predicate (as is also shown in [21], where it is also proved that each logic in the hierarchy can be safely extended with a transparent truth-predicate).

3 Collapsing (meta)-inferences into formulae

3.1 Translation Functions

As said above, our goal in this paper is to be able to deal with the notion of metainferential validity by translating metainferences into formulae, in such a way that the translation preserves (in)validity. To do that, it is useful to begin by examining some existing results in the literature. There are a number of papers that deal with translating inferences into formulae. For instance, Barrio et al. [7] define:

$$\tau(\Gamma/\psi) = \begin{cases} (\gamma_1 \wedge ... \wedge \gamma_n) \rightarrow \psi & \Gamma \neq \emptyset \\ \psi & \text{otherwise} \end{cases}$$

while Dicher and Paoli generalize this to a multiple conclusion setting ([14]; see also Pynko, [24])[5]:

$$\tau(\Gamma/\Delta) = \begin{cases} (\neg\gamma_1 \vee ... \vee \neg\gamma_n) \vee (\delta_1 \vee ... \vee \delta_n) & \Gamma \neq \emptyset \text{ and } \Delta \neq \emptyset \\ (\neg\gamma_1 \vee ... \vee \neg\gamma_n) & \Gamma \neq \emptyset \text{ and } \Delta = \emptyset \\ (\delta_1 \vee ... \vee \delta_n) & \Gamma = \emptyset \text{ and } \Delta \neq \emptyset \end{cases}$$

Using these translation functions, Barrio, Rosenblatt and Tajer (2015) argued that the logic **ST** was **LP**, since (using the above functions) the valid $(n+1)$-inferences of **ST** can be turned exactly into valid n-inferences of **LP**. In that way, a proof system for the inferences of **LP** indirectly gives us a proof system for the metainferences of **ST**, and a proof system for the metainferences of **LP** would indirectly give us a proof system for the metametainferences of **ST**. However, **LP** is not its own metainferential logic. For instance, the metainference $(/_1\phi), (\phi/_1)/_2(/_1\psi)$ is valid in **LP**; however, its translation, $\tau(/_1\phi), \tau(\phi/_1)/_1\tau(/_1\psi) = \phi, \neg\phi/_1\psi$ is invalid in that system. Therefore, a proof system for the inferences of **LP** can be used as a proof system for the metainferences of **ST**, but not for its metametainferences.

This divergence between the inferential and metainferential level logics (under standard translation functions) is part of what allows us to build theories (such as **ST**) which, for example, validate all classical inferences, but at the same time can non-trivially incorporate problematic vocabulary, such as truth predicates. However, this same phenomenon complicates the task of giving *inferential* calculi for meta n-level inferences, since it is typically not obvious which $n-1$ system will preserve (in)validity under a given translation function.

In the following subsection, we show a way to provide translation functions into an extended language system, which always preserves (in)validity.

3.2 Satisfaction Operators and σ-systems

What we shall do in this section is, given a language $\mathcal{L}$ and a system $\mathbf{X}/\mathbf{Y}$, construct a language $\mathcal{L}^\sigma$ and a system $\mathbf{X}/\mathbf{Y}^\sigma$, which add satisfaction operators to them. Intuitively, these operators

[5] The conditional Barrio et al used is a material one. Therefore, it is equivalent to Dicher and Paoli's when restricted to single conclusion inferences.

will represent (formula) valuation standards. For example, given the set of truth values $V = \{1, \frac{1}{2}, 0\}$ and the standards $S = \{1\}$ and $T = \{1, \frac{1}{2}\}$, we will define the operators σ_S and σ_T as follows:

A	$\sigma_S(A)$
1	1
$\frac{1}{2}$	0
0	0

A	$\sigma_T(A)$
1	1
$\frac{1}{2}$	1
0	0

Definition 3.1. More generally, given a set of truth values $V = \{v_1, ..., v_n\}$ and a standard $X \subseteq V$, the operator σ_X will be defined by:[6]

$$v(\sigma_X(A)) = \begin{cases} 1 & v(A) \in X \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{X}/_\mathbf{1}\mathbf{Y}$ be the logic defined from the level-1 standard $X/_1Y$ (call this logic **XY** for short), and σ_X, σ_Y be the satisfaction operators that represent those standards. We will now define a system $\mathbf{XY}^\sigma$, on the one hand, and a translation function from the (meta)-inferences of **XY** into formulas of $\mathbf{XY}^\sigma$, on the other.

Definition 3.2. The system $\mathbf{XY}^\sigma$ is easy to characterize. It contains the same language, truth values and truth functions as **XY**, extended with the operators σ_X and σ_Y. Both the premise and conclusion standards are equal to $\{1\}$; in other words, $\Gamma \vDash_{\mathbf{XY}^\sigma} \Delta$ (i.e. the inference from Γ to Δ is valid in the system $\mathbf{XY}^\sigma$) if an only if it preserves value 1 from premises to conclusion.[7]

Definition 3.3. The translation function $\tau_{\mathbf{XY}}$ is defined in the following way:[8]

$$\tau_{\mathbf{XY}}(\Gamma/_1\Delta) = \begin{cases} (\sigma_X(\gamma_1) \wedge ... \wedge \sigma_X(\gamma_i)) \rightarrow (\sigma_Y(\delta_1) \vee ... \vee \sigma_Y(\delta_j)) & \Gamma \neq \emptyset \text{ and } \Delta \neq \emptyset \\ \neg(\sigma_X(\gamma_1) \wedge ... \wedge \sigma_X(\gamma_i)) & \Gamma \neq \emptyset \text{ and } \Delta = \emptyset \\ (\sigma_Y(\delta_1) \vee ... \vee \sigma_Y(\delta_j)) & \Gamma = \emptyset \text{ and } \Delta \neq \emptyset \end{cases}$$

$$\tau_{\mathbf{XY}}(\Gamma/_{(n>1)}\Delta) = \begin{cases} (\tau_{\mathbf{XY}}(\gamma_1) \wedge ... \wedge \tau_{\mathbf{XY}}(\gamma_i)) \rightarrow (\tau_{\mathbf{XY}}(\delta_1) \vee ... \vee \tau_{\mathbf{XY}}(\delta_j)) & \Gamma \neq \emptyset \text{ and } \Delta \neq \emptyset \\ \neg(\tau_{\mathbf{XY}}(\gamma_1) \wedge ... \wedge \tau_{\mathbf{XY}}(\gamma_i)) & \Gamma \neq \emptyset \text{ and } \Delta = \emptyset \\ (\tau_{\mathbf{XY}}(\delta_1) \vee ... \vee \tau_{\mathbf{XY}}(\delta_j)) & \Gamma = \emptyset \text{ and } \Delta \neq \emptyset \end{cases}$$

[6] These operators, then, work as characteristic functions for each standard (in a given valuation), since standards are sets. We thank an anonymous reviewer for bringing this to our attention.

[7] As a reviewer points out, the definition as written allows for diagonalization, which would suffice to produce revenge paradoxes if something like a transparent truth-predicate were part of the language (see Section 5 for more on this). In that case, something would need to be done in order to avoid triviality. One option might be to somehow restrict the application of the operators, probably to grounded sentences. But there is another option available, which is to change the way in which we understand sentential equivalence to be expressed in the system. Instead of taking two sentences A and B to be equivalent if and only if the sentence $A \leftrightarrow B$ is true in every valuation, one could understand equivalence as $A \leftrightarrow B$ not being false in any valuation (or equivalently, it being non-false in every valuation). Note that, following this path, two sentences could be equivalent even if they have different truth values in some (or even all!) valuations. This is the path followed in [1], [6] and [20]. In the first, the authors defined systems with both a transparent truth-predicate and a consistency operator in a Strong-Kleene setting. In the second, they add a consistency operator to a truth-theory based on **ST**. In the third, the author presents two ways to add a validity operator to a transparent truth-theory based on **ST**.

[8] Note that we use a conditional, as Gentzen originally did **[17]**, but in the cases we are interested it is possible to read $A \rightarrow B$ as $\neg A \vee B$, and the definition would look similar to the one given by Dicher and Paoli above

Notice that this definition is recursive. In particular, in the second clause, the members of Γ and Δ can themselves be $(n-1)$-inferences. As an illustration, the following inferences and metainferences of **ST** would be translated into formulas of $\mathbf{ST}^\sigma$ as follows:

- $\tau_{\mathbf{ST}}(A/_1B)) = \sigma_S(A) \rightarrow \sigma_T(B)$
- $\tau_{\mathbf{ST}}(A, B/_1C, D)) = (\sigma_S(A) \wedge \sigma_S(B)) \rightarrow (\sigma_T(C) \vee \sigma_T(D))$
- $\tau_{\mathbf{ST}}((/_1A), (/_1\neg A)/_2(/_1B)) = (\tau_{\mathbf{ST}}(/_1A) \wedge \tau_{\mathbf{ST}}(/_1\neg A)) \rightarrow \tau_{\mathbf{ST}}(/_1B) = (\sigma_T(A) \wedge \sigma_T(\neg A)) \rightarrow \sigma_T(B)$
- $\tau_{\mathbf{ST}}((A/_1B)/_2(C/_1D)) = \tau_{\mathbf{ST}}(A/_1B) \rightarrow \tau_{\mathbf{ST}}(C/_1D) = (\sigma_S(A) \rightarrow \sigma_T(B)) \rightarrow (\sigma_S(C) \rightarrow \sigma_T(D))$

In the third case, notice that the metainference $(/_1A), (/_1\neg A)/_2(/_1B)$ is invalid in **ST** (the valuation that assigns $\frac{1}{2}$ to A and 0 to B is a counterexample). If we translated this metainference as Barrio et al. suggest, then we would get the formula $(A \wedge \neg A) \rightarrow B$ which is valid in **ST** (in particular, the valuation above assigns $\frac{1}{2}$ to the formula, which belongs to the conclusion standard). However, notice that our translation $(\sigma_T(A) \wedge \sigma_T(\neg A)) \rightarrow \sigma_T(B)$ has the same counterexample than the metainference above, receiving value 0 in that valuation.

With this apparatus in mind, we can now prove the following results.

Theorem 3.4. *For every mixed logic $\mathbf{X}/_\mathbf{1}\mathbf{Y}$ defined from a level-1 standard $X/_1Y$ (call this logic $\mathbf{XY}$ for short), every metainferential level n and every valuation v, if the connectives of $\mathbf{XY}$ behave classically for classical values, then v satisfies an n-inference $\Gamma/_n\Delta$ in $\mathbf{XY}$ if and only if v satisfies $\tau_{\mathbf{XY}}(\Gamma/_n\Delta)$ in $\mathbf{XY}^\sigma$*

Proof. We prove this result via an induction on the level of the inference n.

Base step: $n = 1$. $\Gamma/_1\Delta$ is a regular inference, and Γ and Δ are sets of formulae. Suppose that $\Gamma/_1\Delta$ is satisfied by a valuation v in $\mathbf{XY}$. Therefore, by Definition 2.3, if v satisfies γ_1 and ... and v satisfies γ_i then v satisfies δ_1 or ... or v satisfies δ_j. Thus, by Definitions 3.1 and 3.2, if $v(\sigma_X(\gamma_1)) = 1$ and ... and $v(\sigma_X(\gamma_i)) = 1$ then $v(\sigma_Y(\delta_1)) = 1$ or ... or $v(\sigma_Y(\delta_1)) = 1$. Given that the connectives of $\mathbf{XY}$ (and therefore of $\mathbf{XY}^\sigma$) behave classically for classical values, we have that $v((\sigma_X(\gamma_1) \wedge ... \wedge \sigma_X(\gamma_i)) \rightarrow (\sigma_Y(\delta_1) \vee ... \vee \sigma_Y(\delta_1))) = 1$. Thus, by Definition 3.3, $v(\tau_{\mathbf{XY}}(\Gamma/_1\Delta)) = 1$. And, once again by the definition of satisfaction, we have that $\tau_{\mathbf{XY}}(\Gamma/_1\Delta)$ is satisfied in $\mathbf{XY}^\sigma$. The right-to-left direction is identical, read backwards.

Inductive step: $n > 1$. $\Gamma/_n\Delta$ is a meta n-inference, and Γ and Δ are sets of $(n-1)$-inferences. Suppose that $\Gamma/_n\Delta$ is satisfied by a valuation v in $\mathbf{XY}$. Therefore (by Definition 2.3), if v satisfies γ_1 and ... and v satisfies γ_i then v satisfies δ_1 or ... or v satisfies δ_j (all in $\mathbf{XY}$). Since $\gamma_1, ..., \gamma_i$ and $\delta_1, ...\delta_j$ are inferences of level $n-1$, by the inductive hypothesis, we get that if v satisfies $\tau_{\mathbf{XY}}(\gamma_1)$ and ... and v satisfies $\tau_{\mathbf{XY}}(\gamma_i)$ then v satisfies $\tau_{\mathbf{XY}}(\delta_1)$ or ... or v satisfies $\tau_{\mathbf{XY}}(\delta_j)$ (all in $\mathbf{XY}^\sigma$). Given that $\mathbf{XY}^\sigma$ has $\{1\}$ as both the premise and conclusion standard, we get that if $v(\tau_{\mathbf{XY}}(\gamma_1)) = 1$ and ... and $v(\tau_{\mathbf{XY}}(\gamma_i)) = 1$ then $v(\tau_{\mathbf{XY}}(\delta_1)) = 1$ or ... or $v(\tau_{\mathbf{XY}}(\delta_1)) = 1$. The proof then continues in the same way as in the base step.

□

Corollary 3.1. *For every mixed logic $\mathbf{X}/_\mathbf{1}\mathbf{Y}$ defined from a level-1 standard $X/_1Y$ (call this logic $\mathbf{XY}$ for short) and every metainferential level n, if the connectives of $\mathbf{XY}$ behave classically for classical values and $\Gamma/_n\Delta$ is valid in $\mathbf{XY}$, then $\vDash_{\mathbf{XY}^\sigma} \tau_{\mathbf{XY}}(\Gamma/_n\Delta)$*

Corollary 3.2. *For every mixed logic $\mathbf{X}/_\mathbf{1}\mathbf{Y}$ defined from a level-1 standard $X/_1Y$ (call this logic $\mathbf{XY}$ for short) and every metainferential level n, if the connectives of $\mathbf{XY}$ behave classically for classical values and $\Gamma/_n\Delta$ is anti-valid in $\mathbf{XY}$, then $\tau_{\mathbf{XY}}(\Gamma/_n\Delta) \vDash_{\mathbf{XY}^\sigma}$*

Note that systems obtained from level-1 satisfaction standards will always be pure at every metainferential level (since "lifting" repeats the standard, see above). The previous definitions and results can be generalized to mixed logics obtained from standards of a level greater than 1, which can therefore be impure at metainferential levels (e.g. the system **TS/ST** introduced in section 2). In order to do that, the base case must be given at the inferential level of the standard from which the system is obtained.

Definition 3.5. The translation function for a logic $\mathbf{X/_{n>1}Y}$ based on an $(n > 1)$-level standard ($\mathbf{XY}$ for short) is as follows.

$$\tau_{\mathbf{XY}}(\Gamma/_n\Delta) = \begin{cases} (\tau_X(\gamma_1) \wedge ... \wedge \tau_X(\gamma_i)) \to (\tau_Y(\delta_1) \vee ... \vee \tau_Y(\delta_j)) & \Gamma \neq \emptyset \text{ and } \Delta \neq \emptyset \\ \neg(\tau_X(\gamma_1) \vee ... \vee \neg\tau_X(\gamma_i)) & \Gamma \neq \emptyset \text{ and } \Delta = \emptyset \\ (\tau_Y(\delta_1) \vee ... \vee \tau_Y(\delta_j)) & \Gamma = \emptyset \text{ and } \Delta \neq \emptyset \end{cases}$$

$$\tau_{\mathbf{XY}}(\Gamma/_{(m>n)}\Delta) = \begin{cases} (\tau_{\mathbf{XY}}(\gamma_1) \wedge ... \wedge \tau_{\mathbf{XY}}(\gamma_i)) \to (\tau_{\mathbf{XY}}(\delta_1) \vee ... \vee \tau_{\mathbf{XY}}(\delta_j)) & \Gamma \neq \emptyset \text{ and } \Delta \neq \emptyset \\ \neg(\tau_{\mathbf{XY}}(\gamma_1) \vee ... \vee \neg\tau_{\mathbf{XY}}(\gamma_i)) & \Gamma \neq \emptyset \text{ and } \Delta = \emptyset \\ (\tau_{\mathbf{XY}}(\delta_1) \vee ... \vee \tau_{\mathbf{XY}}(\delta_j)) & \Gamma = \emptyset \text{ and } \Delta \neq \emptyset \end{cases}$$

Notice that the first part of the definition is not recursive, since the translation function references *other* translation functions, which are different from the one being defined.

Once again, as an illustrative example, consider the logic **TS/ST**. The first clause of the base case in the above definition will be given by:

$$\tau_{\mathbf{TS/ST}}(\Gamma/_2\Delta) = (\tau_{\mathbf{TS}}(\gamma_1) \wedge ... \wedge \tau_{\mathbf{TS}}(\gamma_i)) \to (\tau_{\mathbf{ST}}(\delta_1) \vee ... \vee \tau_{\mathbf{ST}}(\delta_j))$$

Thus, a level 2 inference such as $(A/_1B)/_2(C/_1D)$ will be translated as:

$$\tau_{\mathbf{TS}}(A/_1B) \to \tau_{\mathbf{ST}}(C/_1D) = (\sigma_T(A) \to \sigma_S(B)) \to (\sigma_S(C) \to \sigma_T(D))$$

As one can readily see, every formula will be evaluated with the correct standard. With all this in mind, we can generalize the results given above to *metainferential* many-valued mixed logics.

Theorem 3.6. *For every metainferential mixed logic* $\mathbf{X/_mY}$ *defined from an arbitrary* m*-level standard (*$\mathbf{XY}$ *for short), every metainferential level* n *and every valuation* v*, if the connectives of* $\mathbf{XY}$ *behave classically for classical values, then* v *satisfies an* n*-inference* $\Gamma/_n\Delta$ *in* $\mathbf{XY}$ *if and only if* v *satisfies* $\tau_{\mathbf{XY}}(\Gamma/_n\Delta)$ *in* $\mathbf{XY}^\sigma$*.*

Proof. We prove this result via an induction on the level m of the standard from which the system is obtained. For the base step $m = 1$, the result is given by Theorem 3.4. We need only prove the inductive step, where $m > 1$. The inductive hypothesis thus claims that for every system $\mathbf{X/_kY}$ (with $k < m$) and every metainferential level n, v satisfies $\Gamma/_n\Delta$ in $\mathbf{X/_kY}$ if and only if v satisfies $\tau_{\mathbf{X/_kY}}(\Gamma/_n\Delta)$ in $\mathbf{X/_kY}^\sigma$ (call this Inductive Hypothesis 1). We now need to prove this for a logic defined at level m. We do this via another induction on the level n of the inference.

Base step: $n = m$. Note, first that if $n < m$ then $\Gamma/_n\Delta$ can be thought of as an m-inference with empty premises (and empty premises in the conclusion, and so on). Thus, by proving this result for $n = m$ we prove it for every $n < m$.

Suppose, then, that v satisfies the inference $\Gamma/_n\Delta$ in $\mathbf{X/_mY}$. Therefore (by the definition of satisfaction), if v satisfies γ_1 and ... and v satisfies γ_i (according to the premise standard X) then v satisfies δ_1 or ... or v satisfies δ_j (according to the conclusion standard Y). Now, since $m > 1$, both X and Y will be mixed standards, i.e. we can note them as $X_1/_{(m-1)}X_2$ and $Y_1/_{(m-1)}Y_2$. Thus, by Inductive Hypothesis 1 we get that if v satisfies $\tau_X(\gamma_1)$ and ... and v satisfies $\tau_X(\gamma_i)$ (all in $\mathbf{X}^\sigma$) then v satisfies $\tau_Y(\delta_1)$ or ... or v satisfies $\tau_Y(\delta_j)$ (all in $\mathbf{Y}^\sigma$). This implies that if v satisfies $\tau_X(\gamma_1)$ and ... and v satisfies $\tau_X(\gamma_i)$ then v satisfies $\tau_Y(\delta_1)$ or ... or v satisfies $\tau_Y(\delta_j)$ (all in $\mathbf{XY}^\sigma$). This is so because every $\tau_X(\phi)$ is a formula of $\mathbf{X}^\sigma$, and $\mathbf{XY}^\sigma$ contains the same truth values and connectives than $\mathbf{X}^\sigma$, and also contains the operator σ_X. Therefore, every valuation of a formula in $\mathbf{X}^\sigma$ will have the same value in $\mathbf{XY}^\sigma$. The same goes for $\mathbf{Y}^\sigma$.

Given that $\mathbf{XY}^\sigma$ has $\{1\}$ as both the premise and conclusion standards, and that the connectives of $\mathbf{XY}$ (and therefore of $\mathbf{XY}^\sigma$) behave classically for classical values, we have that $v((\tau_X(\gamma_1) \wedge ... \wedge \tau_X(\gamma_i)) \rightarrow (\tau_Y(\delta_1) \vee ... \vee \tau_Y(\delta_1))) = 1$.

Thus, by Definition 3.5, we get that $v(\tau_{\mathbf{XY}}(\Gamma/_n\Delta)) = 1$, and that $\Gamma/_n\Delta$ is satisfied in $\mathbf{XY}^\sigma$.

Inductive step: $n > m$. The proof is almost identical to the inductive step in Theorem 3.4, so we omit it here.

□

Corollary 3.3. *For every metainferential mixed logic $\mathbf{X}/_\mathbf{m}\mathbf{Y}$ defined from an arbitrary m-level standard ($\mathbf{XY}$ for short) and every metainferential level n, if the connectives of $\mathbf{XY}$ behave classically for classical values and $\Gamma/_n\Delta$ is valid in $\mathbf{XY}$, then $\vDash_{\mathbf{XY}^\sigma} \tau_{\mathbf{XY}}(\Gamma/_n\Delta)$*

Corollary 3.4. *For every metainferential mixed logic $\mathbf{X}/_\mathbf{m}\mathbf{Y}$ defined from an arbitrary m-level standard ($\mathbf{XY}$ for short) and every metainferential level n, if the connectives of $\mathbf{XY}$ behave classically for classical values and $\Gamma/_n\Delta$ is anti-valid in $\mathbf{XY}$, then $\tau_{\mathbf{XY}}(\Gamma/_n\Delta) \vDash_{\mathbf{XY}^\sigma}$*

In the next section, we show how to build proof-theoretic tools (specifically, sequent calculi) for σ-systems, which are sound and complete with respect to the semantics presented in this section. Given that, as we showed, the semantics of these σ-systems preserve the (in)validity of metainferences of their respective base systems (under translation), providing sound and complete sequent calculi for σ-systems will indirectly give us a proof theory for the metainferences of those base systems.

4 Proof Theories for σ-systems

In this section, we provide a recipe for generating n-sided sequent calculi for σ-systems, irrespective of the language, truth values and standards (i.e. σ operators) that they contain. We also provide an example of a particular system, which allows us determine metainferential validity for a wide range of known systems.

Before moving on to that, it is worth mentioning that [13] provides a traditional, two-sided and unlabelled sequent-calculi for every metainferential logic that can be defined through the notions of *strict* and *tolerant* satisfaction, namely, **ST**, **TS**, **LP** and $\mathbf{K_3}$. Nevertheless, these calculi are sound and complete for the *global* validities of these logics, and not for the local ones. On the other hand, [12] and [15] provide sound and complete sequent-calculi for the *local* validities of these logics. Nevertheless, they use *labelled* and *nested* sequent calculi. Our calculi will be sound and complete with respect to local validities, and will also be unlabelled and not nested.

Designing unlabelled n-sided sequent calculi for σ-systems (where n is the number of truth values of the many-valued logic, see below) is straightforward—provided we have an n-sided sequent calculi for some of the logics without the σ-operators. The key will be to add rules for each satisfaction operator added to the language, specifying when they receive values 1 and 0, respectively.

Let $V = \{v_1, \ldots, v_n\}$ be the set of truth-values used to specified the logic. Let $0, 1 \in V$, and let V be such that for every $i \in V$ such that $i \neq 0$ and $i \neq 1$, $0 < i < 1$. Moreover, let the order of truth-values in V be partial. Our n sided-sequents will have a disjunctive reading (as suggested in e.g. [25]).

Definition 4.1. Let Γ_i (for $1 \leq i \leq n$) be a (finite) set of formulas. A sequent of form $\Gamma_1 \mid \cdots \mid \Gamma_n$ is satisfied by a valuation v if and only if $v(\gamma_1) = v_1$ for some $\gamma_1 \in \Gamma_1$ or $\ldots$ or $v(\gamma_n) = v_n$ for some $\gamma_n \in \Gamma_n$. A sequent is valid if and only if it is satisfied by every valuation. A valuation is a counterexample to a sequent if the valuation does not satisfy the sequent.

The rules for a σ_X operator can be given as follows:

Definition 4.2. We need to consider two cases: (i) the cases in which $v_i \in X$, and (ii) the cases where $v_j \notin X$. The following will thus be the general form for the rules for a σ_X operator:

$$\frac{\Gamma_1 \mid \cdots \mid \Gamma_i, A \mid \cdots \mid \Gamma_n}{\Gamma_1 \mid \cdots \mid \Gamma_i \mid \cdots \mid \Gamma_n, \sigma_X(A)}\ (i) \qquad \frac{\Gamma_1 \mid \cdots \mid \Gamma_j, A \mid \cdots \mid \Gamma_n}{\Gamma_1, \sigma_X(A) \mid \cdots \mid \Gamma_j \mid \cdots \mid \Gamma_n}\ (ii)$$

Notice that if the standard X contains more than one truth value, then the formula A will appear in more than one side in the premise of rule (i), and if $V - X$ contains more than one truth value, A will appear in more than one side in the premise of rule (ii) (see below for an example).

Finally, since σ-systems have $\{1\}$ as both the premise and conclusion standards, we shall say that

Definition 4.3. $\Gamma \vdash \Delta$ if and only if $\Gamma \mid \cdots \mid \Gamma \mid \Delta$ is provable.

In what follows, we will exemplify these notions with a particular σ-system, namely, the $\mathbf{ST}^\sigma$ system. As shown in the previous sections, this system contains the truth values $\{1, \frac{1}{2}, 0\}$, and the σ_S and σ_T operators that represent the standards $S = \{1\}$ and $T = \{1, \frac{1}{2}\}$, respectively.

Notice that this system is the same as $\mathbf{TS}^\sigma$, $\mathbf{LP}^\sigma$, $\mathbf{TS}/\mathbf{ST}^\sigma$ and of any other either inferential or metainferential mixed logic that contains those truth values and standards. What will change in all these cases is the way in which the (meta)inferences are translated into formulae of the system. But since, as we have already proved in Corollary 3.3, their respective translations all preserve (in)validity, it will suffice to show that the sequent calculus for $\mathbf{ST}^\sigma$ (which we will call $\mathbf{LK}_\sigma$) is sound and complete with its semantics in order to (indirectly) obtain a proof-theory for the metainferences of all those systems.

According to Definition 4.2, the following would be the four rules of $\mathbf{LK}_\sigma$ for the σ_S and σ_T operators.

Definition 4.4. Rules for the σ_S and σ_T operators

$$\frac{\Gamma, A \mid \Sigma, A \mid \Delta}{\Gamma, \sigma_S(A) \mid \Sigma \mid \Delta}\ L\sigma_S \qquad \frac{\Gamma \mid \Sigma \mid \Delta, A}{\Gamma \mid \Sigma \mid \Delta, \sigma_S(A)}\ R\sigma_S$$

$$\frac{\Gamma, A \mid \Sigma \mid \Delta}{\Gamma, \sigma_T(A) \mid \Sigma \mid \Delta}\ L\sigma_T \qquad \frac{\Gamma \mid \Sigma, A \mid \Delta, A}{\Gamma \mid \Sigma \mid \Delta, \sigma_T(A)}\ R\sigma_T$$

As said above, our reading of the sequents will be disjunctive, thus the following will hold in the three-valued case.

Definition 4.5. A three-sided disjunctive sequent $\Gamma \mid \Sigma \mid \Delta$ is satisfied by a valuation v if and only if $v(A) = 0$ for some $A \in \Gamma$, or $v(B) = \frac{1}{2}$ for some $B \in \Sigma$, or $v(C) = 1$ for some $C \in \Delta$.

The proof system we are about to present includes some axioms and rules, as usual. A sequent is provable in $\mathbf{LK}_\sigma$ if and only if it follows from the axioms by some number (possibly zero) of applications of the rules. As we will be working with sets, the effects of the structural rules of Exchange and Contraction are built in, and Weakening is built into the axioms.

We will have three versions of a three-sided Cut rule, and also a Derived Cut rule (that can be inferred from the three basic rules of Cut)[9] and that will also play a key role in the completeness proof presented below, following [25]. Id is the only axiom-scheme. Cut 1, Cut 2, Cut 3 and Derived Cut are structural rules, while the rest are operational rules. Given that the connectives for which we do not give any rules ($\vee$, $\rightarrow$ and $\leftrightarrow$) can be defined in terms of the ones for which we do provide rules, this succinct presentation is just as good.

$$\frac{}{A, \Gamma \mid A, \Sigma \mid A, \Delta}\ \text{Id}$$

$$\frac{\Gamma, A \mid \Sigma \mid \Delta \quad \Gamma \mid \Sigma, A \mid \Delta}{\Gamma \mid \Sigma \ \mid \Delta}\ \text{Cut1} \qquad \frac{\Gamma \mid \Sigma \mid \Delta, A \quad \Gamma \mid \Sigma, A \mid \Delta}{\Gamma \mid \Sigma \ \mid \Delta}\ \text{Cut2} \qquad \frac{\Gamma, A \mid \Sigma \mid \Delta \quad \Gamma \mid \Sigma \mid \Delta, A}{\Gamma \mid \Sigma \ \mid \Delta}\ \text{Cut3}$$

$$\frac{\Gamma, A \mid \Sigma, A \mid \Delta \quad \Gamma \mid \Sigma, A \mid \Delta, A \quad \Gamma, A \mid \Sigma \mid \Delta, A}{\Gamma \mid \Sigma \ \mid \Delta}\ \text{Derived Cut}$$

$$\frac{\Gamma \mid \Sigma \mid \Delta, A}{\Gamma, \neg A \mid \Sigma \mid \Delta}\ L\neg \qquad \frac{\Gamma \mid \Sigma, A \mid \Delta}{\Gamma \mid \Sigma, \neg A \mid \Delta}\ M\neg \qquad \frac{\Gamma, A \mid \Sigma \mid \Delta}{\Gamma \mid \Sigma \mid \Delta, \neg A}\ R\neg$$

$$\frac{\Gamma, A, B \mid \Sigma \mid \Delta}{\Gamma, A \wedge B \mid \Sigma \mid \Delta}\ L\wedge \qquad \frac{\Gamma \mid \Sigma \mid \Delta, A \quad \Gamma \mid \Sigma \mid \Delta, B}{\Gamma \mid \Sigma \ \mid \Delta, A \wedge B}\ R\wedge$$

$$\frac{\Gamma \mid \Sigma, A \mid \Delta, A \quad \Gamma \mid \Sigma, B \mid \Delta, B \quad \Gamma \mid \Sigma, A, B \mid \Delta}{\Gamma \mid \Sigma, A \wedge B \mid \Delta}\ M\wedge$$

Finally, the following are the main results concerning $\mathbf{LK}_\sigma$.

Theorem 4.6 (Soundness)**.** *If a sequent* $\Gamma \mid \Sigma \mid \Delta$ *is provable in* $\mathbf{LK}_\sigma$*, then it is valid in* $\mathbf{LK}_\sigma$*.*

Proof. The axioms are valid, and validity is preserved by the rules, as can be checked without too much trouble. □

Theorem 4.7 (Completeness)**.** *If a sequent* $\Gamma \mid \Sigma \mid \Delta$ *is valid in* $\mathbf{LK}_\sigma$*, then it is provable in* $\mathbf{LK}_\sigma$*.*

Proof. We will use the method of reduction trees,[10] which will allow us to build, for any given sequent, either a proof of that sequent or a counterexample to it. The method also provides of a way of building the potential counterexample. We will introduce the notions of subsequent and sequent union, that will be used in the proof:

Definition 4.8. A sequent $S = \Gamma \mid \Sigma \mid \Delta$ is a *subsequent* of a sequent $S' = \Gamma' \mid \Sigma' \mid \Delta'$ (written $S \sqsubseteq S'$) if and only if $\Gamma \sqsubseteq \Gamma'$, $\Sigma \sqsubseteq \Sigma'$, and $\Delta \sqsubseteq \Delta'$.

Definition 4.9. A sequent $S = \Gamma \mid \Sigma \mid \Delta$ is the *sequent union* of a set of sequents $[\Gamma_i \mid \Sigma_i \mid \Delta_i]_{i \in I}$ (written $S = \sqcup[\Gamma_i \mid \Sigma_i \mid \Delta_i]_{i \in I}$) if and only if $\Gamma = \sqcup_{i \in I}\Gamma_i$, $\Sigma = \sqcup_{i \in I}\Sigma_i$ and $\Delta = \sqcup_{i \in I}\Delta_i$.

[9] Derived Cut would require two occurrences of at least two of the premises in order to derive from the given primitive rules. This will not affect anything in the later proofs, as we are working with sets of premises (and not multi-sets or sequences).

[10] For similar proofs, see [25] and [22].

The construction starts from a root sequent $S_0 = \Gamma_0 \mid \Sigma_0 \mid \Delta_0$, and then builds a tree in stages, applying at each stage all the operational rules that can be applied, plus Derived Cut "in reverse", i.e. from the conclusion sequent to the premise(s) sequent(s). For the proof, we use an enumeration of the formulae and an enumeration of names. We will reduce, at each stage, all the formulae in the sequent, starting from the one with the lowest number, then continuing with the formula with the second lowest number, and moving on in this way until the formula with the highest number in the sequent is reduced. In the case where a formula appears in more than one side of the sequent, we will start by reducing the formula that appears on the left side and then proceed to the middle and the right side, respectively. The final step, at each stage n of the reduction process, will be an application of the Derived Cut rule to the nth-formula in the enumeration. If we apply a multi-premise rule, we will generate more branches that will need to be reduced. If we apply a single-premise rule, we just extend the branch with one more leaf. We will only add formulae at each stage, without erasing any of them. As a result of the process just described, every branch will be ordered by the subsequent relation. Any branch that has an axiom as it topmost sequent will be closed. A branch that is not closed is considered open. This procedure is repeated until every branch is closed, or until there is an infinite open branch. If every branch is closed, then the resulting tree itself is a proof of the root sequent. If there is an infinite open branch Y, we can use it to build a counterexample to the root sequent. Thus, stage 0 will just be the root sequent S_0. If it is an axiom, the branch is closed. For any stage $n + 1$, one of two following things might happen:

1. For all branches in the tree after stage n, if the tip—e.g., the sequent that is being reduced— is an axiom, the branch is closed.

2. For open branches: For each formula A in a sequent position in each open branch, if A already occurred in that sequent position in that branch (i.e. A has not been generated during stage $n + 1$), and A has not already been reduced during stage $n + 1$, then reduce A as is shown below. There are three possible positions in which a formula can appear in a sequent: either (i) on the left side, or (ii) on the middle, or (iii) on the right side. We need to consider all these possible cases.

- If A is a negation $\neg B$, then: if A is in the left/middle/right position, extend the branch by copying its current tip and adding B to the right/middle/left position.

- If A is a conjunction $B \wedge C$, then: (i) if A is in the left position, extend the branch by copying its current tip and adding both B and C to the left position. (ii) If A is in the middle position, split the branch in three: extend the first by copying the current tip and adding B to both the middle and right positions; extend the second by copying the current tip and adding C to the middle and right positions; and extend the third by copying the current tip and adding both B and C to the middle position. (iii) If A is in the right position, split the branch in two: extend the first by copying the current tip and adding B to the right position; and extend the second by copying the current tip and adding C to the right position.

- If A is a formula of form $\sigma_S(B)$, then: (i) if A is in the left position, extend the branch by copying its current tip and adding B to the left and to the middle positions. (ii) If A is in the right position, extend the branch by copying its current tip and adding B to the right position. (iii) If A is in the middle position, then do nothing.

- If A is a formula $\sigma_T(B)$, then: (i) if A is in the left position, extend the branch by copying its current tip and adding B to the left position. (ii) If A is in the right position, extend the branch by copying its current tip and adding B to the middle and the right positions. (iii) If A is in the middle position, then do nothing.

We will also apply the Derived Cut rule at each step. Consider the nth formula in the enumeration of formulae and call it A. Now extend each open branch using the Derived Cut rule. For each open branch, if its tip is $\Gamma \mid \Sigma \mid \Delta$, split it in three and extend the new branches with the sequent $\Gamma, A \mid \Sigma, A \mid \Delta$, the sequent $\Gamma, A \mid \Sigma \mid \Delta, A$, and the sequent $\Gamma \mid \Sigma, A \mid \Delta, A$, respectively.

Now we need to repeat this procedure until every branch is closed, or, if that does not happen, until there is an infinite open branch. If the first scenario is the actual one, then the tree itself is a proof of the root sequent, because each step will be the result of an application of a structural or operational rule to the previous steps. If the second scenario is the actual one, we can use the infinite open branch to build a counterexample.[11]

If in fact there is an infinite open branch Y, then the Derived Cut rule will have been used infinitely many times. Thus, every formula will appear at some point in the branch for the first time, and will remain in every step afterwards. Now, we first collect all sequents of the infinite open branch Y into one single sequent $S_\omega = \Gamma_\omega \mid \Sigma_\omega \mid \Delta_\omega = \sqcup \{S \mid S \text{ is a sequent of } Y\}$.[12] Notice that, as Derived Cut has been applied infinitely many times in the construction of the branch, every formula will occur in exactly two places in S_ω.[13] Thus, there will be a valuation such that no formula in the sequent gets the value associated with the place where it occurs (i.e. 0 if the formula occurs in the left, $\frac{1}{2}$ if it occurs in the middle, 1 if it occurs in the right). Hence, for each formula A in the sequent, v will give to A a value different from the ones corresponding to the sides where A appears in the sequent. But that includes all the formulae in the initial and finite sequent S_0. That valuation, then, will also be a counterexample to S_0. Therefore that valuation will be a counterexample to the sequent being considered.

Thus, for atomic formulae A (propositional letters and truth assertions), $v(A) = 0$ or $\frac{1}{2}$ or 1, respectively, if and only if A does not appear in Γ_ω or Σ_ω or Δ_ω, respectively.

The rules for reducing formulae can be used to show by induction, that, if none of the components of complex formulae receive the value associated with any place in which they appear in S_ω, neither will the compound. We will not see, due to limitations of space, how this method works in detail. For conjunctions, negations and truth assertions, we proceed exactly as is shown in [25]. The new cases are that of formulae of the form $\sigma_S(A)$ and $\sigma_T(A)$. We will just check that assertions of the form $\sigma_S(A)$ can be reduced, because the reduction of $\sigma_T(A)$-type of assertions is very similar.

[11]As we have already mentioned, *the tip* is the sequent that is being reduced. If it is an axiom, then this branch will be closed. If it is not, then the reduction process will go on. We will present a toy example to understand exactly what *the tip* is. Take, for instance, the sequent $\oslash \mid \neg B \mid \neg A$. In this case, this is *the tip* we have mentioned. The first step of the reduction will give us the sequent $\oslash \mid B, \neg B \mid \neg A$. But this is not an axiom either. So the reduction process will keep on rolling. $\oslash \mid B, \neg B \mid \neg A$ is the new *tip*, and the next step of the reduction will give us the sequent $A \mid B, \neg B \mid \neg A$. Finally, as this sequent is not an axiom either, we will split this new *tip* in three when we apply the Derived Cut rule in reverse, and the reduction process will continue.

[12]As a reviewer noticed, S_ω cannot be part of this system because is has infinite formulas. But it is part of an extended system, $\mathbf{LK}^\infty_\sigma$ based on $\mathbf{LK}_\sigma$, that admits three-sided sequents with infinite formulas, but that otherwise works exactly as $\mathbf{LK}_\sigma$. In particular, as the definition of satisfaction of a sequent by a valuation does not (substantially) change, the only relevant difference is that there are many more sequents. The important thing, regarding the completeness proof, is that any counterexample to an infinite sequent is also a counterexample to a finite subsequent of it. And those finite subsequents are the sequent of the system we are working with. So, in particular, a counterexample to S_ω (in $\mathbf{LK}^\infty_\sigma$) is also a counterexample to S_0 (in $\mathbf{LK}_\sigma$), which is the root sequent we will be working with in this completeness proof.

[13]It cannot occur in the three places, because then there will be some finite stage n where the formula appears for the first time in the branch in the three sides. But then that sequent will be an axiom, and therefore the branch will be closed.

In the cases of formulae of the form $\sigma_S(A)$, no formula like this can appear both on the left and the right side, because then they would have appeared for the first time in the same leaf of the branch. When you reduced them at the next stage, you'll get a sequent with A at the three sides of the sequent. That sequent will be an axiom, and so will not be part of an infinite open branch. So (i) either $\sigma_S(A)$ is both in the left and the middle sides of the sequent, or (ii) it is both in the middle and the right side of the sequent.

Let us start with (i). At some point, $\sigma_S(A)$ will be reduced. It appears in the middle side of the sequent, so nothing is supposed to be done when this happens. But it also appears on the left side. If that happens, then A will appear on the left and on the middle sides of the sequent on the next stage of the construction. Therefore, A, by inductive hypothesis, will receive value 1, and so will $\sigma_S(A)$. Now consider (ii). Once $\sigma_S(A)$ is reduced, as it is on the right side of the sequent, that will get A on right side. At some point, an application of Derived Cut will put A either on the left or in the middle. In the later, A, by inductive hypothesis, will receive value 0, and so will $\sigma_S(A)$. In the former, by inductive hypothesis, A will receive value $\frac{1}{2}$, and then $\sigma_S(A)$ will get value 0. So, again, $\sigma_S(A)$ will not receive a value associated with the sides where it appears.

By completing the induction along these lines, we can show that we can construct a valuation such that no formula receives the value associated with any place where it appears in S_ω. But, as we know, that includes all the formulae in the initial and finite sequent S_0. That valuation, then, will also be a counterexample to S_0, which is what we were looking for. Thus, for any sequent S, either it has a proof or it has a counterexample. $\square$

5 Some Philosophical Consequences

In this section we examine the status of the σ-systems presented above, and draw some philosophical consequences from our approach. To do that, it will be useful to begin by examining two possible objections, namely: (*i*) That the new vocabulary is problematic in self-referential contexts (i.e. that there are "revenge" paradoxes); (*ii*) That they are not conservative over their base systems.

Regarding (*i*), it is easy to show that, in languages that possess the capacity for self-reference, some sentences containing the new vocabulary will turn out to be problematic. Let Tr be a transparent truth predicate—i.e., one such that, for every formula A and valuation v, $v(A) = v(Tr\ulcorner A\urcorner)$. For example, for a mixed system **XY**, consider the $\mathbf{XY}^\sigma$-sentence $\lambda \leftrightarrow \neg\sigma_Y Tr(\ulcorner\lambda\urcorner)$ (which states that its own truth is not satisfied by the conclusion standard), and a valuation v. If v satisfies λ, then λ must receive value 1 (since σ-systems have just $\{1\}$ as their conclusion standard). And since, in those systems, the biconditional behaves classically for classical values, $\neg\sigma_Y Tr(\ulcorner\lambda\urcorner)$ must also receive value 1. Again, since negation behaves classically for classical values, $\sigma_Y Tr(\ulcorner\lambda\urcorner)$ will have value 0 and $Tr(\ulcorner\lambda\urcorner)$ will not be satisfied. By transparency, this means that λ itself will not be satisfied. A similar reasoning applies to the case where one supposes that λ is not satisfied.

Notice that this objection is especially important for systems of the **ST** hierarchy, since, as we mentioned above, one of their proponents' main goals with them was to get as close to classical logic as possible *while still being able to accommodate problematic vocabulary* in self-referential contexts.

Our response to this objection is that we view the σ-systems as mere instruments for determining the (in)validity of metainferences. The workflow we assume is that you want to prove the (in)validity of some metainference in a system **XY**, you translate the metainference to a sentence of $\mathbf{XY}^\sigma$, you operate with the apparatus presented above, and then, via Corollary 3.3, and Theorems 4.7 and 4.6 you take back the result to **XY**. We do not claim there is anything philosophically substantial with $\mathbf{XY}^{\sigma}$ itself. Hence, if one wishes to avoid problematic sentences, it can be useful to restrict applications of the problematic vocabulary to sentences of the base language.

A similar point applies to objection (*ii*). Note that inferences such as $(A \wedge \neg A)/_1 B$ are invalid in systems like **LP**. However, since in $\mathbf{LP}^{\sigma}$ both standards are $\{1\}$ then the inference will be valid (even for inferences that do not contain the new operators), since no valuation will satisfy the premises. Hence, $\mathbf{LP}^{\sigma}$ is not conservative over its base language (the same applies to other systems). Once again, our reply would be that σ-systems are just tools for calculating the (in)validity of metainferences, and in that light, all we care about is that they are conservative over the inferences of original base system *under translation* -which is proved in Corollary 3.3.

Even though we conceive of σ-systems as mere instruments for metainferential validity calculation, they do provide us with other advantages. One key advantage is that they allow us to present metalogical results in an (extended) object language.

Consider an objection that metainferential mixed systems have received in the literature: that they are not closed under their own higher-order validities; or put in other words, that they do not obey the principles they contain. This objection was recently raised by Scambler [28], Ripley [27] and Porter [23]. A logic *contains* the principles it *accepts*—i.e., validates. But, what is *obeying*, according to them? The first to talk about this was Chris Scambler (he did not use the word *obedience* but rather talked about *closure*, but the underlying concept is essentially the same). According to Scambler, a logic "is closed under its own laws" ([28, p. 18])—or obeys a principle of level $\alpha + 1$, as Ripley, and we, prefer to call it—if and only if (i) that principle is valid in that logic, and (ii) the set of α-inferences is closed under the $\alpha + 1$ principle.

Consider, for example, the following instance of Cut without context, and with a $\frac{1}{2}$-constant (or a Liar-like sentence) as the Cut-formula:

$$\frac{\lambda \vDash \ , \ \vDash \lambda}{\vDash}$$

This metainference is (locally) valid in **TS/ST**. However, it is true that $\lambda \vDash_{\mathbf{TS/ST}}$ and that $\vDash_{\mathbf{TS/ST}} \lambda$, but not that the empty inference is valid in it. According to those critics, this would be a case of disobedience, since we have a valid (meta)inference pattern where the premises are valid but the conclusion is not. This also means that this instance of Cut is not *obeyed* in $\mathbf{ST}(\omega)$ and in every $\mathbf{ST}(n)$ with $n > 2$. What the critics are demanding for an obedient theory is that if a (meta)principle is valid in the theory, then whenever its premises are valid in the theory, so are its conclusions. Put more formally, if $\Gamma \vDash_{\mathbf{XY}} \Delta$ then (if $\vDash_{\mathbf{XY}} \Gamma$ then $\vDash_{\mathbf{XY}} \Delta$) (for any metainferential level n). In our example, this seems to not be the case.

We believe the spirit of the critics' demand is reasonable. However, the precise way in which they have formulated it is inadequate, for two different and independent reasons. The first relates to impure metainferential consequence relations, and the second to the global vs. local criterion of metainferential validity.

Let us begin with the first of these inadequacies. As [8] have argued, the formulation above does not interpret metainferences, in general, and the above instance of Cut, in particular, as *mixed metainferential* theories favour (particularly given that they can be impure at metainferential levels). In the example above, the right conditional would read as follows: if each premise of (the instance of) Cut is valid *according to* **ST**, then the conclusion is valid *according to* **ST**. Or again, more formally, that if $\Gamma \vDash_{\mathbf{TS/ST}} \Delta$ then (if $\vDash_{\mathbf{ST}} \Gamma$, then $\vDash_{\mathbf{ST}} \delta$) (since the inferences in the right conditional are of level 1 and thus have to be valid according to the conclusion standard of **TS/ST**).

But this is not how, for example, a supporter of $\mathbf{TS}/\mathbf{ST}$ (or of $\mathbf{ST}(\omega)$) would interpret this metainference. Supporters of mixed-metainferential theories claim that the standard for evaluating premises can be different from the standard for evaluating conclusions, which is just what this way of understanding obedience does not do. Scambler's Closure demands that if (i) if Γ is satisfied by a valuation according to **TS**, then Δ is satisfied by the valuation in **ST**, and (ii) Γ is valid according to **ST**, then (iii) Δ is valid in **ST**. But there is no reason why (iii) should follow from (i) and (ii).

To reiterate, in a mixed system $\mathbf{X}/_{\mathbf{n}}\mathbf{Y}$, the validity of an $(n-1)$-level inference will be evaluated with the conclusion standard Y. Thus, for spirit of the critics' demand to be adequately preserved in a mixed metainferential setting, one should not evaluate the premises of an n-inference $\Gamma/_n\Delta$ in $\mathbf{X}/_{\mathbf{n}}\mathbf{Y}$ itself, because that would amount to evaluating them in $\mathbf{Y}/_{\mathbf{n}}\mathbf{Y}$. In contrast, one should read the premises in the same way that one understands the premise part of $\Gamma \vDash_{\mathbf{XY}} \Delta$ (i.e. the premises are satisfied according to X)[14]

Thus, it would be more appropriate to characterize obedience differently for mixed systems. The following Mixed Obedience Principle is a more reasonable version of Scambler's Closure:

if $\Gamma \vDash_{\mathbf{XY}} \Delta$ then (if $\vDash_{\mathbf{X}} \Gamma$ then $\vDash_{\mathbf{Y}} \Delta$) (for the metainferential level n at which $\mathbf{XY}$ is defined, the criterion remains identical for inferences of level higher than n)

Here, unlike in Scambler's version, the $n-1$ level inferences in the right conditional will not both be evaluated with the conclusion standard Y. According to this mixed reading, even the truth-theory based on $\mathbf{ST}(\omega)$—which includes a sentence that gets the non-classical value in every valuation—obeys its metainferential validities. In the above example, this happens because it is not true that the premises of that instance of Cut are **TS**-valid.[15]

However, we believe that it is not enough to modify the obedience condition as shown above in order to fully respond to these critics. The second problem with their demand, related to the criterion of metainferential validity being used, can be introduced as follows. Notice that, even in the modified principle above, the antecedent ($\Gamma \vDash_{\mathbf{XY}} \Delta$) is a claim about the *local* validity of a (meta)inference, but its consequent (if $\vDash_{\mathbf{X}} \Gamma$ then $\vDash_{\mathbf{Y}} \Delta$) is the definition of *global* validity for that metainference. In other words, the modified principle essentially states that if a (meta)inference is locally valid then it is globally valid. Now, this does hold, because local validity is a stronger notion than global validity. But the demand that the principle is making is still conceptually strange, since it ties two different and independent notions of metainferential validity, which have no *a priori* reason to coincide. The notion of *obedience* is supposed to characterize the relation between $n+1$-level and n-level inferences, not between different criteria of metainferential validity.

Another way to formulate this issue is by discussing a problem related to the one the critics explicitly mention. So far, we have discussed whether these mixed metainferential logics obey the principles they contain—or validate. But, do they contain the principles they obey? (i.e.

[14]Note that, even though we have been using the local metainferential validity criterion in this article, this particular issue is independent of whether one adopts the global or local criteria. For example, the metainference $(p/p)/_2(p/q)$ is *globally* valid in $\mathbf{TS}/\mathbf{ST}$ (because the premise is invalid in **TS**). However, p/p is valid in $\mathbf{TS}/\mathbf{ST}$ (because, as a level-1 inference, it is evaluated with the conclusion standard, **ST**), and the criterion, as the critics formulate it, would thus allow us to infer p/q, which is invalid in $\mathbf{TS}/\mathbf{ST}$ (because it is invalid in **ST**). Also note that this case only affects the global reading, since that metainference is locally invalid.

[15]Some might say that nothing in this new criterion corresponds to the idea that the premises are valid in the theory, which is true. Thus, we can add that to the criterion, which will now look like this:

if $\Gamma \vDash_{\mathbf{XY}} \Delta$ then (if ($\vDash_{\mathbf{X}} \Gamma$ and $\vDash_{\mathbf{Y}} \Gamma$) then $\vDash_{\mathbf{Y}} \Delta$) (for the metainferential level n at which $\mathbf{XY}$ is defined, the criterion remains identical for inferences of level higher than n)

Notice that if $\vDash_{\mathbf{Y}} \Gamma$, then $\vDash_{\mathbf{XY}} \Gamma$.

are they closed under principles that they do not validate?).[16] This question corresponds to the conditional that results from switching sides between the antecedent and the consequent of our previous conditional, like this: in a mixed system $\mathbf{XY}$, $\Gamma \vDash_{\mathbf{XY}} \Delta$ *if* (if $\vDash_{\mathbf{X}} \Gamma$ then $\vDash_{\mathbf{Y}} \Delta$). The answer to this question is negative. For example, $(A/_1B)/_2(C/_1D)$ is not valid in $\mathbf{TS}/\mathbf{ST}$. Nevertheless, (if $\vDash_{\mathbf{TS}} A/_1B$, then $\vDash_{\mathbf{ST}} C/_1D$) is a true conditional statement, since its antecedent is plainly false, because nothing is valid in $\mathbf{TS}$.

We think a uniform criterion is needed. We should use either the local or the global metainferential criterion everywhere. Note that we are not arguing in favor of neither the global nor the local criterion. If the reader has independent reasons for choosing one or the other, then we do not dispute those reasons. All we are arguing for is that, whichever criterion one adopts, one should use it consistently in formulating one's meta-requisites. Since our concern in this article is particularly the local metainferential validity, we reformulate the adequacy condition, *now with a biconditional*, as follows (again, we only give it for the level n at which the system is defined):

For every valuation v, v satisfies $\Gamma/_n\Delta$ according to $\mathbf{XY}$ if and only if (if v satisfies $/_n\Gamma$ according to $\mathbf{X}$ then v satisfies $/_n\Delta$ according to $\mathbf{Y}$).

The advantage of our new σ-setting is that it makes it possible to evaluate this whole metalinguistic conditional in the extended object language of the σ theory. Provided that $\mathbf{XY}$ is defined from a satisfaction standard of n, this is the conditional we should be checking:

$$\vDash_{\mathbf{XY}^\sigma} \tau_{\mathbf{XY}}(\Gamma/_n\Delta) \leftrightarrow (\tau_{\mathbf{X}}(/_n\Gamma) \rightarrow \tau_{\mathbf{Y}}(/_n\Delta))$$

(Once again, if the level of the metainference is greater than n, then $\tau_{\mathbf{XY}}$ should be used everywhere). This is not only true in the example above, but it can also be easily proven to be valid in general from Definition 3.5.

To illustrate all this with the example above, the requisite would read: for every valuation v, v satisfies $(A/_1B)/_2(C/_1D)$ according to $\mathbf{TS}/\mathbf{ST}$ if and only if (if v satisfies $A/_1B$ according to $\mathbf{TS}$, then it satisfies $C/_1D$ according to $\mathbf{ST}$). And its σ-translation would be:

$$\vDash_{\mathbf{TS}/\mathbf{ST}^\sigma} \tau_{\mathbf{TS}/\mathbf{ST}}((A/_1B)/_2(C/_1D)) \leftrightarrow (\tau_{\mathbf{TS}}(A/_1B) \rightarrow \tau_{\mathbf{ST}}(C/_1D))$$

Thus, against what the critics have claimed, once we define obedience in a way that is appropriate to mixed logics of the kind we study, we can *prove* (using our σ-apparatus) that metainferential theories obey the principles they contain *and vice-versa*.

6 Conclusions

In this paper, we have introduced a framework that allows one to collapse an n-level metainference (for arbitrary n), belonging to any mixed many-valued metainferential system, into a formula of an extended-language σ-system. We model-theoretically proved that this translation preserves (in)validity.

All of this was exemplified with the logics of the $\mathbf{ST}$ family, namely, those that contain the truth values $\{1, 0, i\}$ and are built around combining the standards $S = \{1\}$ and $T = \{1, i\}$. This includes the well-studied systems $\mathbf{LP}$, $\mathbf{K_3}$, $\mathbf{ST}$ and $\mathbf{TS}$, but also other metainferential systems, such as $\mathbf{TS}/\mathbf{ST}$ and $\mathbf{ST}(\omega)$ which have gained some popularity in recent times.

[16] If we stick to Scambler's definition, the answer is indisputably yes, because the obedience condition includes the containment condition. What we are asking is: do metainferential logics validate every principle of level $n+1$ such that the set of n-validities is closed under it?

However, the results developed in this paper can be used with other systems, for instance, systems that contain more than three truth values (such as the logic **FDE**), systems where the connectives behave differently with non-classical inputs (e.g. Weak Kleene logic) and systems that incorporate other vocabulary into the language (such as some **LFI** systems, see [9]). Note that all the **ST**-family systems share a common σ-system (which we called $\mathbf{ST}^{\sigma}$ though they translate their metainferences into it differently), but in all these cases the resulting σ-system would be different from it.

We have also shown a general way to build sequent calculi for σ-systems, and exemplified this with a calculus for $\mathbf{ST}^{\sigma}$. Proving soundness and completeness for it indirectly gave us a calculus for the local metainferential validities of all the logics in the **ST**-family. This is important because (other than a few attempts we mentioned above) metainferential logics have been treated mostly model-theoretically in the literature. Our calculi are very simple to develop and use since we give a recipe for generating them, and since they are both unlabeled and non-nested.

In the final section, we responded to some possible objections, namely, that our σ-systems can introduce revenge paradoxes and that they are non-conservative over their base languages, by stating that they have only instrumental value. Finally, we showed how, despite this, they can be useful in the context of philosophical discussions, for instance, on the the "obeying" objection to mixed metainferential logics in the **ST**-family. Here σ-systems allow us to prove in the object language why the adequacy condition posed by critics (and more!), when formulated correctly, is fulfilled by those systems.

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