A note on formalizing discussive logic

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Abstract

Discussive logic was introduced by Jaśkowski as a logic of discussion. In this note we show that some natural translation-based formalizations of discussive logic in modal logic do not yield a paraconsistent logic but rather classical logic. Some alternative modal formalizations of discussive logic that avoid the collapse into classical logic are put forward.

1 Introduction

Discussive logic was introduced by Stanisław Jaśkowski, the founding father of paraconsistent logics (cf. [6, 7]).[1] According to Jaśkowski (cf. [7, p.43]), one of the motivations is to deal with discussions in which several participants may have conflicting opinions. One may understand discussive logic as a formalization of the minutes of the discussion, if we follow a suggestion made by Max Urchs (cf. [20, p.236]). One natural way to try to formalize this idea is using the language of modal logic. Indeed, this was the strategy taken by Jaśkowski in which the modal logic **S5** was deployed. This note aims at looking at some formalizations different from the one given by Jaśkowski.

First, here are some standard preliminaries. The languages in this note consist of a finite set S of propositional connectives and a countable set Prop of propositional variables. We refer to the languages as $\mathcal{L}$, $\mathcal{L}_1$ and $\mathcal{L}_2$ when the sets of connectives are $\{\sim, \vee, \wedge, \to\}$, $\{\sim, \vee, \wedge, \to, \Diamond\}$ and $\{\sim, \vee, \wedge, \to, \Diamond_1, \Diamond_2\}$ respectively. Furthermore, we denote by Form, Form_1 and Form_2 the set of formulas defined as usual in $\mathcal{L}$, $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Moreover, we denote a formula by A, B, C, etc. and a set of formulas by Γ, Δ, Σ, etc.

In view of the motivation for discussive logic of Jaśkowski, in [7, p.43], we may understand that the idea behind discussive logic is to replace 'p is true' with 'p is held as an opinion in a discussion'. It is then natural to construe the latter as 'some participant of the discussion holds that p'. Combining this

[1]Some of the important earlier work on discussive logic includes [10, 9, 11, 12, 3, 1]. Somewhat more recent work includes [22, 13]. For an up to date survey, see [4].

consideration with Jaśkowski's intuitive reading of the possibility operator as "in accordance with the opinion of one of the participants in the discussion" ([7, p.43]), it is rather natural to make these precise by means of the following translation into the language of modal logic.

Definition 1.1. We define a translation $\tau : \mathsf{Form} \longrightarrow \mathsf{Form}_1$ as follows:

- $\tau(p) = \Diamond p$ for $p \in \mathsf{Prop}$,
- $\tau({\sim}A) = {\sim}\tau(A)$,
- $\tau(A * B) = \tau(A) * \tau(B)$, where $* \in \{\vee, \wedge, \rightarrow\}$.

In fact, this extremely natural and intuitively plausible translation was put forward by Graham Priest and Koji Tanaka in a previous version [17] of their *Stanford Encyclopedia of Philosophy* entry on paraconsistent logic. The entry has since been revised (see [18]) and the translation removed in response to some of the points raised in the present note. We would like to stress that the motivation of the present note is *not* to point out shortcomings of an obsolete version of an SEP entry — our aim is to take a more general look at representations of ideas inherent in discussive logic by means of translations into modal logic, and the translation originally suggested by Priest and Tanaka is a natural place to start given its simplicity and intuitive plausibility.

Definition 1.2. For $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, we define $\models_{\mathbf{SD}}$ (**SD** for **S**imple **D**iscussive logic) as follows:

$$\Gamma \models_{\mathbf{SD}} A \text{ iff } \Diamond\tau(\Gamma) \models_{\mathbf{S5}} \Diamond\tau(A)$$

where $\Diamond\tau(\Gamma) = \{\Diamond\tau(B) : B \in \Gamma\}$.

Then, it is initially plausible that $\models_{\mathbf{SD}}$ is a good gloss on Jaśkowski's idea, and so one would expect the logic thus generated to be paraconsistent. Surprisingly, perhaps, this is not the case. We will see the details of this collapse in the next section, namely §2, and reflect upon this negative result in §3. The last section, namely §4, summarizes the observations and open problems discussed in this note.

Before turning to the negative result, here are two remarks on $\models_{\mathbf{SD}}$.

Remark 1.3. First, there are two differences between the above translation introduced by Priest and Tanaka, and the original translation of Jaśkowski (cf. [6, 7]).[2] First, Jaśkowski did *not* translate propositional atoms as complex formulas.[3] Second, Jaśkowski translated $A \rightarrow B$ into $\Diamond\tau(A) \rightarrow \tau(B)$. Note also that in [8], Jaśkowski introduced another translation in which $A \wedge B$ is translated into $\tau(A) \wedge \Diamond\tau(B)$. For some recent discussions on the discussive logics related to different languages, see [16].

[2]To be precise, Jaśkowski did *not* introduce the translation explicitly, but making use of the translation seems the most natural to make his ideas explicit. Indeed, this is the way formulated and discussed in the literature.

[3]Note that Horacio Arló-Costa, in [2, p.600], considers a translation that maps atoms to complex formulas, even before Priest and Tanaka.

Second, note that we generalized the original definition given by Priest and Tanaka by allowing the set of premises to be infinite. If we set Γ to be a finite set $\{B_1, \dots, B_n\}$ and make use of the local deduction theorem valid for extensions of the modal logic **K**, the above definition will give us the original definition of $\models_{\mathbf{SD}}$ which was given as follows:

$$B_1, \dots, B_n \models_{\mathbf{SD}} A \text{ iff } \models_{\mathbf{S5}} \Diamond\tau(B_1) \to (\cdots \to (\Diamond\tau(B_n) \to \Diamond\tau(A))\cdots).$$

This generalization will not influence the main point of this note.

2 A negative result

First, we prepare a lemma related to the translation introduced by Priest and Tanaka. To this end, we make use of following theses and rules of **S5**, where $\Box A$ is defined as ${\sim}\Diamond{\sim}A$ given the choice of the language $\mathcal{L}_1$.

$$\begin{array}{llll}
{\sim}\Box A \leftrightarrow \Diamond{\sim}A & \text{(T1)} & \dfrac{A\leftrightarrow B}{{\sim}A\leftrightarrow{\sim}B} & \text{(R2)} \\
\Box\Diamond A \leftrightarrow \Diamond A & \text{(T2)} & \dfrac{A\leftrightarrow B \quad B\leftrightarrow C}{A\leftrightarrow C} & \text{(R3)} \\
\Diamond(\Diamond A\wedge\Diamond B)\leftrightarrow(\Diamond A\wedge\Diamond B) & \text{(T3)} & \dfrac{A\leftrightarrow B \quad C\leftrightarrow D}{(A\wedge C)\leftrightarrow(B\wedge D)} & \text{(R4)} \\
\Diamond(\Diamond A\to\Diamond B)\leftrightarrow(\Diamond A\to\Diamond B) & \text{(T4)} & & \\
\dfrac{A\leftrightarrow B}{\Diamond A\leftrightarrow\Diamond B} & \text{(R1)} & \dfrac{A\leftrightarrow B \quad C\leftrightarrow D}{(A\to C)\leftrightarrow(B\to D)} & \text{(R5)}
\end{array}$$

Lemma 2.1. *For all $A \in \mathsf{Form}$, $\models_{\mathbf{S5}} \Diamond\tau(A)\leftrightarrow\tau(A)$.*

Proof. Even if the present lemma is crucial, it is perhaps not necessary to give a full proof. However, we include it anyway for the sake of a non-expert reader. We proceed by induction on the complexity of the formula A.

- If $A = p \in \mathsf{Prop}$, then $\tau(A) = \Diamond p$. Since we have $\models_{\mathbf{S5}} \Diamond\Diamond p\leftrightarrow\Diamond p$, the base case is proved.
- If $A = {\sim}B$, then we need to establish $\models_{\mathbf{S5}} \Diamond{\sim}\tau(B)\leftrightarrow{\sim}\tau(B)$.

1	$\Diamond\tau(B)\leftrightarrow\tau(B)$	[IH]
2	$\Diamond{\sim}\tau(B)\leftrightarrow\Diamond{\sim}\Diamond\tau(B)$	[1, (R2), (R1)]
3	$\Diamond{\sim}\Diamond\tau(B)\leftrightarrow{\sim}\Box\Diamond\tau(B)$	[(T1)]
4	${\sim}\Box\Diamond\tau(B)\leftrightarrow{\sim}\Diamond\tau(B)$	[(T2), (R1)]
5	${\sim}\Diamond\tau(B)\leftrightarrow{\sim}\tau(B)$	[1, (R2)]
6	$\Diamond{\sim}\tau(B)\leftrightarrow{\sim}\tau(B)$	[2–5, (R3)]

- If $A = B\wedge C$, then we need to establish $\models_{\mathbf{S5}} \Diamond(\tau(B)\wedge\tau(C))\leftrightarrow(\tau(B)\wedge\tau(C))$.

1	$\Diamond\tau(B)\leftrightarrow\tau(B)$	[IH]
2	$\Diamond\tau(C)\leftrightarrow\tau(C)$	[IH]
3	$(\tau(B)\wedge\tau(C))\leftrightarrow(\Diamond\tau(B)\wedge\Diamond\tau(C))$	[1, 2, (R4)]
4	$\Diamond(\tau(B)\wedge\tau(C))\leftrightarrow\Diamond(\Diamond\tau(B)\wedge\Diamond\tau(C))$	[3, (R1)]
5	$\Diamond(\tau(B)\wedge\tau(C))\leftrightarrow(\tau(B)\wedge\tau(C))$	[3, 4, (T3), (R3)]

- If $A{=}B{\to}C$, then we need to establish $\models_{\mathbf{S5}} \Diamond(\tau(B){\to}\tau(C)){\leftrightarrow}(\tau(B){\to}\tau(C))$.

1	$\Diamond\tau(B){\leftrightarrow}\tau(B)$	[IH]
2	$\Diamond\tau(C){\leftrightarrow}\tau(C)$	[IH]
3	$(\tau(B){\to}\tau(C)){\leftrightarrow}(\Diamond\tau(B){\to}\Diamond\tau(C))$	[1, 2, (R5)]
4	$\Diamond(\tau(B){\to}\tau(C)){\leftrightarrow}\Diamond(\Diamond\tau(B){\to}\Diamond\tau(C))$	[3, (R1)]
5	$\Diamond(\tau(B){\to}\tau(C)){\leftrightarrow}(\tau(B){\to}\tau(C))$	[3, 4, (T4), (R3)]

- If $A{=}B{\vee}C$, then it is again straightforward and leave the details for the readers.

This completes the proof. □

In view of the above lemma, we obtain the following which shows that the "diamond effect", a key feature of Jaśkowski's discussive logic, is lost.

Proposition 2.2. *For* $\Gamma \cup \{A\}{\subseteq}\mathsf{Form}$*,* $\Gamma \models_{\mathbf{SD}} A$ *iff* $\tau(\Gamma) \models_{\mathbf{S5}} \tau(A)$*, where* $\tau(\Gamma){=}\{\tau(B) : B{\in}\Gamma\}$.

With the help of this proposition, we are ready to prove the following result (we refer to the semantic consequence relation of classical logic as $\models_{\mathbf{CL}}$).

Proposition 2.3. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \models_{\mathbf{CL}} A$ *then* $\Gamma \models_{\mathbf{SD}} A$*.*

Proof. The proposition follows from the fact that $\models_{\mathbf{CL}}$ is contained in $\models_{\mathbf{S5}}$ which is closed under substitutions. □

Since the other way around is obvious, we obtain the following result.[4]

Theorem 2.4. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_{\mathbf{SD}} A$ *iff* $\Gamma \models_{\mathbf{CL}} A$*.*

3 Reflections

3.1 The assumptions behind the negative result

Keeping the simple motivation for discussive logic in mind, then, what lessons can we learn from this negative result? The crucial assumptions we made behind the observation are:

(A1) the use of the diamond-based translation suggested by Priest and Tanaka, and

(A2) the use of the modal logic **S5**.

[4] Based on Theorem 2.4, communicated by one of the present authors to Graham Priest, Koji Tanaka and Zach Weber, the problematic translation is removed in the latest version of the SEP entry on paraconsistent logic [18].

Both assumptions can be seen as problematic from the point of view of formalizing opinions put forward in a discussion. This point will be discussed in the present subsection. It will be shown in the following two subsections that tweaking the assumptions to avoid the problems gives rise to non-collapsing consequence relations.

Concerning (A1), let us first recall the intuitive reading of the possibility operator suggested by Jaśkowski himself who read it as “in accordance with the opinion of one of the participants in the discussion” ([7, p.43]).[5] The possibility operator does not carry information about the identity of the discussant whose opinion is being assumed: Jaśkowski speaks simply of one (meaning “at least one”) of the participants concerned. We may thus say that the sources of opinions put forward in discussions are anonymized by the diamond-based translation of Priest and Tanaka. However, replacing $\Diamond p$ by, for instance, $\Diamond_1 p \vee \Diamond_2 p$ does not by itself avoid collapse, as the reader can easily verify.

More importantly, the use of the diamond operator itself seems to come with rather implausible assumptions concerning the nature of opinions put forward in a discussion. Speaking semantically, the diamond-based translation assumes that opinions of individual discussants are represented by individual possible worlds in a Kripke model and $\Diamond p$, read as “p is the case according to the opinion of one of the participants in the discussion”, holds anywhere in the model if some of those opinions satisfy p. However, this assumes the opinions of the discussants to be complete and consistent theories. This is clearly too strong an assumption as the following consideration shows. Clearly, $p \to (q \vee {\sim}q)$ is a theorem of any extension of **K** which implies that so is $\Diamond p \to (\Diamond q \vee \Diamond{\sim}q)$. This means that if it is the opinion of any discussant that p, then there is an opinion (for or against) held in the discussion concerning any proposition q whatsoever. If the discussants have an opinion on *something*, then they have an opinion on *everything*.

Concerning (A2), the use of modal logic **S5** implies that “p is the case according to the opinion of one of the participants in the discussion” has a number of rather strong properties. One example is the “positive introspection” axiom of **S5**, namely $\Diamond\Diamond p \to \Diamond p$, which on the diamond-based reading translates into “If it is an opinion of a discussant that someone in the discussion holds the opinion that p, then someone in the discussion holds the opinion that p”.

In the rest of this section, we will discuss these problematic features in more detail and we will propose alternative formalizations that avoid them. It will be shown that the alternative formalizations avoid collapse into classical logic.

[5]Note that Jaśkowski is paying attention to theses of the system, *not* all sentences of the system.

3.2 First strategy: replace S5

In what follows, we will consider a generalization of $\models_{\mathbf{SD}}$ defined as follows.

Definition 3.1. Let $\mathbf{M}$ be a modal logic formulated in $\mathcal{L}_1$. Then, for $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, we define $\models_{\mathbf{SD(M)}}$ as follows:

$$\Gamma \models_{\mathbf{SD(M)}} A \text{ iff } \Diamond\tau(\Gamma) \models_{\mathbf{M}} \Diamond\tau(A)$$

where $\Diamond\tau(\Gamma) = \{\Diamond\tau(B) : B \in \Gamma\}$.

If we try to keep the diamond-based translation, then we will need to replace **S5** by something else. Note that if we give up **S5**, then collapse is avoided. For example, it is easy to observe the following.

- $\not\models_{\mathbf{S4}} \Diamond\Diamond p \to (\Diamond{\sim}\Diamond p \to \Diamond\Diamond q)$, i.e. $p, {\sim}p \not\models_{\mathbf{SD(S4)}} q$;
- $\not\models_{\mathbf{T}} \Diamond\Diamond p \to (\Diamond\Diamond q \to \Diamond(\Diamond p \wedge \Diamond q))$, i.e. $p, q \not\models_{\mathbf{SD(T)}} p \wedge q$;
- $\not\models_{\mathbf{K}} \Diamond(\Diamond p \vee {\sim}\Diamond p)$, i.e. $\not\models_{\mathbf{SD(K)}} p \vee {\sim}p$.

Based on these, it might be of some interest to explore systems obtained by considering other weaker modal logics.[6] We will leave this as an open problem to interested readers. We only note here one more observation related to these variants.

Proposition 3.2. *Let* $\mathbf{M}$ *be a modal logic contained in* **Triv** *such that*

(i) $\mathbf{M}$ *contains classical logic,*
(ii) $\mathbf{M}$ *is closed under substitution and*
(iii) $\mathbf{M}$ *is closed under the following rule: if* $\models_{\mathbf{M}} A$ *then* $\models_{\mathbf{M}} \Diamond A$.

Then, we obtain that for $A \in \mathsf{Form}$, $\models_{\mathbf{SD(M)}} A$ *iff* $\models_{\mathbf{CL}} A$.

Proof. The left-to-right direction is obvious by considering the case in which possibility is trivialized in the sense of **Triv**. In particular, $\not\models_{\mathbf{CL}} A$ iff $\not\models_{\mathbf{Triv}} \Diamond\tau(A)$ only if $\not\models_{\mathbf{M}} \Diamond\tau(A)$ only if $\not\models_{\mathbf{SD(M)}} A$. For the other direction, assume $\models_{\mathbf{CL}} A$. Then, by the assumptions (i) and (ii) for $\mathbf{M}$, we obtain that $\models_{\mathbf{M}} \tau(A)$ since $\tau(A)$ is just a substitution instance of A. Finally, by applying the assumption (iii), we obtain $\models_{\mathbf{M}} \Diamond\tau(A)$ which, by the definition of $\models_{\mathbf{SD(M)}}$, amounts to: $\models_{\mathbf{SD(M)}} A$, and this is the desired result. □

Remark 3.3. This result reminds us of the famous result on Logic of Paradox (**LP**) that the set of valid formulas is identical with the set of tautologies in classical logic. The difference lies in the rules of inference. In our case, if $\mathbf{M} = \mathbf{S5}$, then $\models_{\mathbf{SD(M)}}$ and $\models_{\mathbf{CL}}$ are identical as consequence relations, as we observed in Theorem 2.4. For other cases, such as **S4** and **T**, the set of valid formulas is identical with the set of tautologies in classical logic, but not the rules of inference, just like the case of **LP**. And if we go even weaker, such as **K**, then the set of valid formulas is a proper subset of tautologies in classical logic. However, the reason is not so exciting – **K** has no validities of the form $\Diamond A$.

[6] It should be noted that a similar problem is well-explored for the original discussive logic. See e.g. [14, 15].

3.3 Second strategy: replace the translation

We may also try to keep the modal logic **S5**, and in this case, we know that some translations will avoid the collapse, such as the more standard ones in the literature on discussive logic, considering the discussive conditional as in [6, 7], as well as the discussive conjunction as in [8].

There can be even more, and finding suitable translations based on different motivations still seems to be an interesting problem to tackle. We have already noted in §3.1 that the diamond operator is a problematic means to formalize "in accordance with the opinion of one of the participants in the discussion" as its use implies that if the discussants have an opinion on something, then they have an opinion on everything. An alternative rendering of "in accordance with the opinion of one of the participants in the discussion" that avoids this problem builds on the approach of modal epistemic logic [21], where information available to agents – or their opinions, if you will – is represented by a box instead of a diamond. The semantic intuition behind this is that opinions of discussants are usually not complete theories and so they should be represented by a set of (complete) possible worlds that are consistent with these opinions and this set is usually not a singleton. On this view, p holds according to a discussant's opinion if it is true in all possible worlds that are consistent with the discussant's opinions.

When adopting a box-based translation, it is essential to avoid the anonymity of the diamond operator and "keep track" of the discussants by using indices on modal operators. To keep things simple, we will confine ourselves to two discussants.

To put together a translation reflecting these considerations, we will use the language $\mathcal{L}_2$ with modalities indexed by 1 and 2, indices intuitively corresponding to two participants of a discussion. Based on our considerations discussed above, "p holds in accordance with the opinion of one of the participants in the discussion" will be formalized as $\Box_1 p \vee \Box_2 p$.[7]

Definition 3.4. We define a translation $\sigma : \mathsf{Form} \longrightarrow \mathsf{Form}_2$ as follows:

- $\sigma(p) = \Box_1 p \vee \Box_2 p$ for $p \in \mathsf{Prop}$,
- $\sigma({\sim}A) = {\sim}\sigma(A)$,
- $\sigma(A * B) = \sigma(A) * \sigma(B)$, where $* \in \{\vee, \wedge, \rightarrow\}$.

We will also write $\mathsf{D}A$ for $\Box_1 A \vee \Box_2 A$.

Then, we define the semantic consequence relation as follows, following Priest and Tanaka.

Definition 3.5. For $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, we define $\models_{\Box\mathbf{SD(S5)}}$ as follows:

$$\Gamma \models_{\Box\mathbf{SD(S5)}} A \text{ iff } \mathsf{D}\sigma(\Gamma) \models_{\mathbf{S5}\times\mathbf{S5}} \mathsf{D}\sigma(A),$$

[7]Note that a very similar idea is considered by Arló-Costa in [2, §10]. Moreover, the anonymous referee directed our attention to yet another idea that can be applied in the present context. The idea, in brief, builds on [19, p.157] due to Schotch and Jennings.

where $\mathsf{D}\sigma(\Gamma) = \{\mathsf{D}\sigma(B) \mid B \in \Gamma\}$.

Then, we can establish that $\models_{\Box\mathbf{SD(S5)}}$ does *not* collapse. In particular, we obtain the following.

Proposition 3.6. $p, q \not\models_{\Box\mathbf{SD(S5)}} p \wedge q$.

Proof. Take the **S5** Kripke model depicted below (reflexive arrows are omitted):

$$\begin{array}{ccccc} v & \overset{1}{\text{———}} & w & \overset{2}{\text{———}} & u \\ p & & p, q & & q \end{array}$$

In this model, $w \not\models \Box_1(\Box_1 q \vee \Box_2 q)$ and similarly $w \not\models \Box_2(\Box_1 p \vee \Box_2 p)$. Hence, $w \not\models \Box_1\sigma(p \wedge q)$ and $w \not\models \Box_2\sigma(p \wedge q)$. This implies that $w \not\models \mathsf{D}\sigma(p \wedge q)$. However, $w \models \Box_1(\Box_1 p \vee \Box_2 p)$ since $w \models \Box_1 p$ and similarly $w \models \Box_2(\Box_1 q \vee \Box_2 q)$ since $w \models \Box_2 q$. Hence, $w \models \mathsf{D}\sigma(p) \wedge \mathsf{D}\sigma(q)$. □

On the other hand, we can prove a generalization of Proposition 3.2, for which we need the following definition.

Definition 3.7. Let $\mathbf{M}$ be any extension of $\mathbf{K}$. For $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, we define $\models_{\Box\mathbf{SD(M)}}$ as follows:

$$\Gamma \models_{\Box\mathbf{SD(M)}} A \text{ iff } \mathsf{D}\sigma(\Gamma) \models_{\mathbf{M}\times\mathbf{M}} \mathsf{D}\sigma(A)\,.$$

Proposition 3.8. *Let* $\mathbf{M}$ *be any extension of* $\mathbf{K}$*. If* $\models_{\mathbf{CL}} A$ *then* $\models_{\Box\mathbf{SD(M)}} A$*.*

Proof. If $\models_{\mathbf{CL}} A$, then $\models_{\mathbf{M}\times\mathbf{M}} \sigma(A)$ since $\mathbf{M} \times \mathbf{M}$ contains $\mathbf{CL}$ and is closed under substitutions. But then $\models_{\mathbf{M}\times\mathbf{M}} \mathsf{D}\sigma(A)$ using the Necessitation rule. □

In the remainder of the section, we consider a combination of the two approaches mentioned above. This combination involves an indexed-box translation using a modal logic weaker than **S5**. We have explained our reasons for both of these avenues separately, but combining them would actually make good sense. If discussants' opinions are modelled by box operators of any logic extending **T** then, taking the the reflexivity axiom $\Box p \to p$ into account, what is being modelled is quite a strong notion of opinion. In fact, this notion is closer to *knowledge* than to opinion as usually understood. The latter, we submit, is closer to the notion of (rational) *belief.* Without going too much into epistemological details, let us consider the consequence relation $\models_{\Box\mathbf{SD(D)}}$, using the basic doxastic logic $\mathbf{D}$. It can be shown that $\models_{\Box\mathbf{SD(D)}}$ does not collapse into classical logic and that it is paraconsistent:

Proposition 3.9. $p, {\sim}p \not\models_{\Box\mathbf{SD(D)}} q$.

Proof. Take the **D** Kripke model depicted below:

$$\begin{tikzcd} v \arrow[loop left, "1,2"] & w \arrow[l, "1"'] \arrow[r, "2"] & u \arrow[loop right, "1,2"] \\ p & & \end{tikzcd}$$

In this model, $w \models \Box_1\Box_1 p$ and so $w \models \Box_1\sigma(p)$, which entails $w \models \mathsf{D}\sigma(p)$. Similarly, $w \models \Box_2(\sim\Box_1 p \wedge \sim\Box_2 p)$ and so $w \models \mathsf{D}\sim\sigma(p)$, which means $w \models \mathsf{D}\sigma(\sim p)$. However, we have both $w \models \Diamond_1(\sim\Box_1 q \wedge \sim\Box_2 q)$ and $w \models \Diamond_2(\sim\Box_1 q \wedge \sim\Box_2 q)$, which means that $w \not\models \mathsf{D}\sigma(q)$. $\Box$

4 Concluding remarks

What we hope to have established is that there are a lot of problems to be discussed for discussive logics, inspired by the translation introduced by Priest and Tanaka in [17]. In particular, we tried to analyze the result that the consequence relation introduced by Priest and Tanaka collapses into classical logic. We observed that there are at least two ways to avoid the collapse: either by replacing **S5** by a weaker modal logic, or by revisiting the intuition of Jaśkowski and make use of multi-modal languages and translations using box operators. This will give rise to a number of open problems which include:

- a detailed study of $\models_{\mathbf{SD(M)}}$ and $\models_{\Box\mathbf{SD(M)}}$ for a wide range of modal logics **M**;
- a detailed study of the general n-participant case of $\models_{\Box\mathbf{SD(M)}}$;
- axiomatizations of the consequence relations discussed in this note.

Note that for the last item, some results established by Lloyd Humberstone in [5] are related. Moreover, there is also some room to explore the intuition behind $\models_{\Box\mathbf{SD(M)}}$ by making use of other modal languages that are more expressive, such as hybrid logics.

It remains to be seen how rich the old idea of Jaśkowski is, and we hope some readers will be motivated to join the authors to continue with the development of discussive logics.

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