Models of Relevant Arithmetic

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Abstract

It is well known that the relevant arithmetic $\mathbf{R}^{\#}$ admits finite models whose domains are the integers *modulo* n rather than the expected natural numbers. Less well appreciated is the fact that the logic of these models is much more subtle than that of the three-valued structure in which they are usually presented. In this paper we investigate the propositional structure of relevant arithmetic over finite domains by considering the De Morgan monoids in which $\mathbf{R}^{\#}$ can be modelled, deriving a fairly complete account of those modelling the stronger arithmetic $\mathbf{RM}^{\#}$ *modulo* n and a partial account for the case of $\mathbf{R}^{\#}$ *modulo* a prime number. The more general case in which the modulus is arbitrary is shown to lead to infinite propositional structures even with the additional constraint that '$0 = 1$' implies everything.

1 Introduction

The relevant arithmetic $\mathbf{R}^{\#}$ [6, 9] is obtained by adding some very natural versions of the Peano-Dedekind axioms for natural number theory to a basis not of classical logic but of the relevant logic $\mathbf{R}$. The result is a first order theory of the natural numbers, but, as has often been observed [6, 9, 10, 13], relevant arithmetic is not classical arithmetic. In particular, the paraconsistency of $\mathbf{R}$ allows it to have inconsistent models, the most startling of which have finite domains. Classically, a false equation such as $0 = 3$ implies everything, because one of the axioms says $0 \neq 3$ so the equation is contrary to the theory and there is no more to be said. Relevantly, however, we may ask what is *really* involved in supposing 0 to equal 3 and what is not. According to $\mathbf{R}^{\#}$, if 0 were equated with 3, numerical identity would be indistinguishable from with congruence *modulo* 3. That would be inconsistent, but not trivial. A model in the integers *modulo* 3 satisfies the equation $12 + 7 = 1$, for instance, but it is easily seen not to satisfy $12 + 7 = 26$. In the same way, models can be constructed to show in a purely finitary way—Gödel notwithstanding—that relevant number theory is *reliable* in the sense that it never delivers a wrong result of any calculation.

As axioms for $\mathbf{R}^{\#}$ we may add to first order relevant logic[1] the following:

$$\begin{array}{l}
\forall x(x = x) \\
\forall x\forall y\forall z(x = y \to (x = z \to y = z)) \\
\forall x\forall y(x = y \to s(x) = s(y)) \\
\forall x\forall y(s(x) = s(y) \to x = y) \\
\neg\exists x(s(x) = 0) \\
\forall x(x + 0 = x) \\
\forall x(x + s(y) = s(x + y)) \\
\forall x(x \times 0 = 0) \\
\forall x(x \times s(y) = (x \times y) + x) \\
(A_{x\leftarrow 0} \wedge \forall x(A \to A_{x\leftarrow s(x)})) \to \forall x A
\end{array}$$

That is, we take numerical equality to be an axiomatised arithmetical relation, not a logical constant: the first two axioms make it an equivalence relation, while the third ensures that it is a congruence with respect to the function symbols and the fourth that the successor function is injective. The last "axiom" is actually an axiom scheme delivering induction for all properties expressible in the notation of the theory. Note that negation occurs only in the axiom saying that zero is not a successor; the result of dropping that axiom and removing negation from the underlying logic is the positive arithmetic $\mathbf{R}^{\#}_{+}$.

As noted, $\mathbf{R}^{\#}$ may be nontrivially extended by adding an axiom $0 = n$, conflating equality with congruence *modulo* n. We shall refer to such an extension as $\mathbf{R}^{\#}$ *modulo* n.

A model of relevant arithmetic requires not only a domain and an interpretation of function symbols defined on it, but also a reading of the predicate symbol '=' and of the logical connectives and quantifiers. As an example—almost the only example in most of the literature on $\mathbf{R}^{\#}$, in fact[2]—here is a picture of $\mathbf{R}^{\#}$ *modulo* 2 in the three-valued structure characteristic for the

[1] The model theory of first order relevant logic is problematic in general [4, 5, 7] because the "obvious" axioms are incomplete for the "obvious" semantics. That kind of incompleteness, however, is not germane to the present paper, which is concerned with relevant Peano arithmetic rather than the pure logic, and with algebraic models rather than Routley-Meyer frames. Moreover, the focus here is on *finite* models, in which quantifiers play no essential part.

[2] The 1984 paper of Meyer and Mortensen [10] is an honourable exception, including discussion of some larger cases and even one non-idempotent model.

logic **RM3**. The three values are T (just true), F (just false) and c (true and false, or confused). The matrices for the connectives are:

A	$\neg A$
T	F
c	c
F	T

$\wedge$	T	c	F
T	T	c	F
c	c	c	F
F	F	F	F

$\rightarrow$	T	c	F
T	T	F	F
c	T	c	F
F	T	T	T

The domain over which the quantifiers range is $\{0, 1\}$. Equations $t = u$ are assigned the value c if t and u designate the same element, and F if they are different. All theorems of $\mathbf{R}^{\#}$ take values in $\{T, c\}$, but 0=1 (for instance) does not. Evidently, this construction may be carried out with any modulus, showing that no false ground equations are provable in $\mathbf{R}^{\#}$.

2 The algebra of relevant arithmetic

The algebraic structures modelling **R** are De Morgan monoids (DMMs) [3, 11][3] which may be presented as sextuples $\langle \mathcal{A}, \circ, \wedge, \vee, ^{-}, t\rangle$ where $\langle \mathcal{A}, \circ, t\rangle$ is a commutative monoid with identity t, $\langle \mathcal{A}, \wedge, \vee\rangle$ is a distributive lattice defining a lattice order $\leq$ on the elements, $^{-}$ is an involutive dual automorphism on the lattice, and for all elements a, b, etc.:

$$a \leq a \circ a$$
$$a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$$
$$a \circ b \leq \bar{c} \text{ iff } a \circ c \leq \bar{b}$$

It is usual, and helpful, to make use of some definitions:

$$f \;=_{\mathrm{df}}\; \bar{t}$$
$$a \rightarrow b \;=_{\mathrm{df}}\; \overline{a \circ \bar{b}}$$
$$a^2 \;=_{\mathrm{df}}\; a \circ a$$

A model $\mathcal{M}$ of **R** on a DMM $\mathcal{D}$ is an epimorphism[4] in the obvious sense from the formula algebra of **R** onto $\mathcal{D}$. That is, if the formula t is present, it is mapped to the monoid identity t, and for all formulae A and B:

[3]Many of the facts about DMMs noted below may be found in the recent paper of Moraschini, Raftery and Wannenburg [11] which is to be credited with sharpening my interest in these questions. They are reproduced here to make the present paper more self-contained.

[4]More usually, any homomorphism is allowed to count as an interpretation, but for the purposes of this paper it is convenient to take the mapping to be surjective. For example, it is simpler to refer to the greatest element of $\mathcal{D}$, rather than that of the subalgebra of $\mathcal{D}$ constituted by the range of $\mathcal{M}$.

$$\begin{aligned}
\mathcal{M}(\neg A) &= \overline{\mathcal{M}(A)} \\
\mathcal{M}(A \wedge B) &= \mathcal{M}(A) \wedge \mathcal{M}(B) \\
\mathcal{M}(A \vee B) &= \mathcal{M}(A) \vee \mathcal{M}(B) \\
\mathcal{M}(A \rightarrow B) &= \mathcal{M}(A) \rightarrow \mathcal{M}(B)
\end{aligned}$$

It is easy to see that in any DMM

$$a \leq b \text{ iff } t \leq a \rightarrow b$$

and in fact that $t \rightarrow a = a$, and $a \rightarrow f = \overline{a}$. A formula A is true on interpretation $\mathcal{M}$ iff $t \leq \mathcal{M}(A)$. The formulae true on all interpretations on all DMMs are exactly the theorems of **R** [2].

It is useful to note the notion of a *Dunn monoid*: the $\{\wedge, \vee, \circ, \rightarrow, t\}$ reduct of a DMM. For this purpose, the third postulate defining DMMs must be replaced by the residuation equivalence

$$a \circ b \leq c \text{ iff } a \leq b \rightarrow c.$$

In a good sense, Dunn monoids capture the positive fragment of the theory of De Morgan monoids, and it is well known [2] that they correspond to the negation-free fragment of **R**.

A DMM is said to be *prime* iff the element t is join-irreducible (i.e. iff there are no elements a and b both strictly below t such that $a \vee b = t$). A DMM is finitely subdirectly irreducible iff it is prime, and since in this paper we are concerned mainly with finite DMMs, the prime ones here are exactly the subdirectly irreducible ones. Every De Morgan monoid considered here is thus a subdirect product of prime ones.

Although $a \leq a^2$ generally, the converse is not the case. We say that an element a is *idempotent* if $a^2 = a$, and that a DMM is idempotent iff every element is.

Fact 1. *A DMM $\mathcal{D}$ with identity t is idempotent iff the element f is idempotent.*

Proof. By residuation, $f^2 = f$ iff $f \leq f \rightarrow f$, but $f \rightarrow f = t$ so f is idempotent iff $f \leq t$. If $f \leq t$ then $f \circ \overline{a} \leq t \circ \overline{a}$. But $a \circ \overline{a} \leq f$ and $t \circ \overline{a} = \overline{a}$, so if $f \leq t$ then $a \circ \overline{a}^2 \leq \overline{a}$. By square-increasing, $\overline{a} \leq \overline{a}^2$, so $a \circ \overline{a} \leq a \circ \overline{a}^2$. Hence if f is idempotent then $a \circ \overline{a} \leq \overline{a}$, whence by the third postulate listed in the definition of a DMM, $a^2 \leq \overline{\overline{a}} = a$. □

Moraschini *et al* [11] define a *Sugihara monoid* to be an idempotent DMM. Here we define a *Sugihara chain* to be a structure isomorphic to the integers or the integers omitting zero, or to a bounded subset $[-n, +n]$, again with or without zero. To construe a Sugihara chain as a DMM, identify the lattice operations with numerical minimum and maximum, the De Morgan complement with additive inverse, and t with 0 if that is present or with $+1$ if it is not. $a \circ b$ is then $a \wedge b$ if it is false (i.e. if $a \leq -b$) and $a \vee b$ otherwise.

The logic **RM** is **R** with the additional axiom $A \to (A \to A)$ corresponding to the idempotence of fusion.

Fact 2. **RM** *is modelled by Sugihara chains.*

Proof. See Dunn's seminal work on RM [2] or Meyer's account in Anderson and Belnap's first volume on relevant logic [1]. □

A *bounded* DMM is one with top and bottom elements $\top$ and $\bot$ respectively. Obviously, all finite DMMs are bounded. As a consequence of residuation, in any bounded DMM, $\bot \circ a = \bot$ for every element a. If in addition, for any $a \neq \bot$, $\top \circ a = \top$ then the DMM is said to be *rigorously compact.*

Fact 3 ([11], Thm 5.3)**.** *Every prime, bounded DMM is rigorously compact.*

Proof. Let a be an element of such a DMM. By primeness, either $t \leq \top \circ a$ or else $\top \circ a \leq f$. In the former case, $\top \circ t \leq \top \circ \top \circ a$, but $\top \circ t = \top$ because t is the identity, and $\top \circ \top \circ a = \top \circ a$ because $\top$ is idempotent. In the latter case, by residuation $a \leq \top \to f = \overline{\top} = \bot$. □

Fact 4. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be bounded DMMs generated by subsets G_1 and G_2 of their elements respectively. Let R be a doubly serial relation between G_1 and G_2—i.e. $\forall x \in G_1 \exists y \in G_2\, Rxy$ and $\forall y \in G_2 \exists x \in G_1\, Rxy$. Then the direct product $\mathcal{D}_1 \times \mathcal{D}_2$ is generated by R (i.e. by $\{\langle x, y\rangle : Rxy\}$) iff the extreme pairs $\langle \top, \bot\rangle$ and $\langle \bot, \top\rangle$ are so generated.*

Proof. This is a simple fact about any algebraic structures with lattice operations. Consider any elements a and b of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. The pair $\langle a, \top\rangle$ is generated, as it is $\langle a, x\rangle \vee \langle \bot, \top\rangle$ for some x, and similarly $\langle \top, b\rangle$ is $\langle x, b\rangle \vee \langle \top, \bot\rangle$ for some x. Then $\langle a, b\rangle$ is the meet of these two $\langle a, \top\rangle \wedge \langle \top, b\rangle$. □

In what follows, we do not discuss DMMs in general, but focus on those modelling $\mathbf{R}^{\#}$. Since $\mathbf{R}^{\#}$ is a special theory rather than a logic, we should begin by noting that the basic constructions required for such modelling and the simple completeness results extend from **R** to $\mathbf{R}^{\#}$. We take an $\mathbf{R}^{\#}$ theory to be a set of formulae in the language of $\mathbf{R}^{\#}$ closed under a simple consequence relation: we say $\Gamma \vdash_{\mathrm{R}\#} A$ iff there exist formulae $B_1, \ldots, B_n$ such that $(B_1 \wedge \ldots \wedge B_n) \to A$ is a theorem of $\mathbf{R}^{\#}$. An $\mathbf{R}^{\#}$ theory is *regular* iff it contains all theorems of $\mathbf{R}^{\#}$, and *prime* iff it does not contain any disjunction without also containing one of its disjuncts. Two formulae A and B are *equivalent* under a theory T iff both $A \to B$ and $B \to A$ are in T. It is easy to see that if T is a regular $\mathbf{R}^{\#}$ theory then this is indeed an equivalence relation, and moreover a congruence on the formula algebra since all connectives respect it. Fairly clearly, the quotient under such a congruence is a DMM, and the function mapping each formula to its equivalence class is a model of $\mathbf{R}^{\#}$ on which the formulae mapped into the positive cone are exactly the members of T.

For a slightly more informative completeness theorem, note that where A is any non-theorem of $\mathbf{R}^{\#}$, there is a regular $\mathbf{R}^{\#}$ theory θ which is *maximal* under the constraints of being an $\mathbf{R}^{\#}$ theory and excluding A. This version of Lindenbaum's lemma may be established by the standard Henkin construction, or in two lines by Zorn's lemma. Such a maximal A-consistent theory is prime, since if neither B nor C is in θ, then by maximality there is a formula D in θ such that both $B, D \vdash_{\mathrm{R}\#} A$ and $C, D \vdash_{\mathrm{R}\#} A$; but then $B \vee C, D \vdash_{\mathrm{R}\#} A$ so $B \vee C$ is also not in θ. It follows that any non-theorem of $\mathbf{R}^{\#}$ is refuted by a model in some prime DMM. $\mathbf{R}$ itself satisfies the stronger condition that every non-theorem is refuted in a DMM which is prime and *consistent*. This, however, is *not* true of $\mathbf{R}^{\#}$, as was shown by Meyer and Friedman [6], though their result is not germane to the present paper which concerns inconsistent extensions of $\mathbf{R}^{\#}$.

Without loss of generality, we can take any DMM modelling $\mathbf{R}^{\#}$ to be generated by the elements taken as values by equations $a = b$ where a and b are numerals. We call these the *equational elements* of the DMM on the model in question. Again without loss of generality, we may confine attention to equations of the form $0 = c$, since any $a = b$ is equivalent to one such just by subtracting $|a - b|$ from both sides. It will be convenient to have a notation for the element $\mathcal{M}(0 = n)$ where $\mathcal{M}$ is a given interpretation. We shall refer to this element as $\perp_n^{\mathcal{M}}$, or, where $\mathcal{M}$ is understood, simply as $\perp_n$. Of particular importance in what follows is the element $\perp_1$. While we are fixing notation, let $\top_n$ be defined as $\perp_n \to \perp_n$. Again, $\top_1$ is particularly significant.

In some, but not all, DMM models of $\mathbf{R}^{\#}$, $\perp_1$ is the bottom element $\perp$, in which case $\top_1$ is the top element $\top$. We say that such a model is *equationally bounded* (EB). In any case:

Fact 5. *Let $\mathcal{M}$ be a model of $\mathbf{R}^{\#}$ on a DMM $\mathcal{D}$. Then $\mathcal{D}$ is bounded with bottom element $\overline{\perp_1} \to \perp_1$ and top element $\overline{\perp_1}^2$.*

Proof. The interval $[\overline{\perp_1} \to \perp_1, \overline{\perp_1}^2]$ obviously contains all elements $\perp_n$ and is closed under the lattice operations and $^-$. It is also closed under $\circ$: where a

and b are in the interval, $(\overline{\perp_1} \to \perp_1)^2 \leq a \circ b \leq \overline{\perp_1}^4$, but both $\overline{\perp_1} \to \perp_1$ and $\overline{\perp_1}^2$ are idempotent. □

To model $\mathbf{R}^\#$ generally, the algebra is expected to be complete as a lattice: at least it needs the infinite meets and joins corresponding to quantified formulae if both $\mathcal{A}$ and the numerical domain are infinite. This restricts the class of DMMs somewhat, but not in a way that changes the set of propositional formulae true in all models. The models in the integers *modulo* n, of course, have finite domains (even if the algebras are infinite), and so for those the universal and existential quantifiers amount just to certain conjunctions and disjunctions, making the theory essentially propositional rather than first order.

The equations $0 = n$ exhibit some regularities.

Fact 6. *The relation of implication in* $\mathbf{R}^\#$ *orders equations by divisibility:* $0 = i$ *provably implies* $0 = j$ *if and only if* i *divides* j *[9, 13].*

Proof. The implication $a = b \to na = nb$ is easily proved[5] in $\mathbf{R}^\#$ by induction on n. For any $a < b$ and $n > 1$, the converse implication is refuted by the **RM3** model in the integers *modulo* $n(b - a)$. □

In particular $0 = 1$ implies every equation, and every equation implies $0 = 0$.

Fact 6 can be strengthened, following a couple of observations:

Observation 1. *Let* S *be a set of natural numbers with at least one member greater than 0, and let* S *be closed under absolute difference. Then the smallest non-zero member of* S *divides every member of* S*.*

Proof. Let a be the smallest positive number in S and suppose b is the smallest member of S which is not a multiple of a. Then b is $ka + c$ for some $k > 0$ and $0 < c < a$. But $b - a$ is in S, which is to say $(k - 1)a + c$ is in S; but this is smaller than b and not a multiple of a, contrary to the supposition. □

Observation 2. *For any filter* $\mathcal{F}$ *of a DMM modelling* $\mathbf{R}^\#$*, the set of* k *such that* $\perp_k \in \mathcal{F}$ *is closed under absolute difference.*

Proof. $((0 = a) \wedge (0 = b)) \to (a = b)$ is a theorem of $\mathbf{R}^\#$, but $a = b$ is equivalent to $0 = |a - b|$. □

[5] This proof relies critically on contraction (the square increasing postulate). In the analogous arithmetic based on relevant logic without contraction, it does not hold [13].

Theorem 1. *Let $\mathcal{D}$ be a DMM modelling $\mathbf{R}^{\#}$. Let $\mathcal{F}$ be any filter of $\mathcal{D}$. Then the set of numbers k such that $\perp_k \in \mathcal{F}$ is either empty or consists of all and only the multiples of some number. Equivalently, for any numbers i and j, and for any model on a DMM, $\perp_i \wedge \perp_j = \perp_{gcd(i,j)}$.*

Proof. Immediate from the two observations above. □

A corollary of Theorem 1 is that $\perp_i \leq \perp_j$ iff k divides j where k is the smallest number such that $\perp_i = \perp_k$. A corollary of *that* is that whenever $\perp_i \leq \perp_j$, we must have $\perp_i = \perp_{\mathrm{gcd}(i,j)}$.

Another consequence, sufficiently useful to be a Fact in its own right, is:

Fact 7. *In any model $\mathcal{M}$ of $\mathbf{R}^{\#}$ on a DMM, for any number $n > 0$, the number of equational elements x such that $x \leq \perp_n^{\mathcal{M}}$ is finite.*

Proof. Every such equational element is $\perp_k^{\mathcal{M}}$ for some k which divides n, and n has only finitely many divisors. □

A few more well known facts are worth noting at this point:

Fact 8. *In the DMM generated by the values of equations, $\perp_0$ is the identity t. It follows that $\top_0$ is also t.*

Proof. The identity of any DMM is the generalised lattice meet of all elements of the form $a \to a$. The meet of the $g \to g$ for generators g is below $a \to a$ for any other a, as can be verified by easy induction on the the generation of a. $0 = 0$ implies $(0 = a) \to (0 = a)$ for every a, by the transitivity and symmetry of identity, so the result is immediate. □

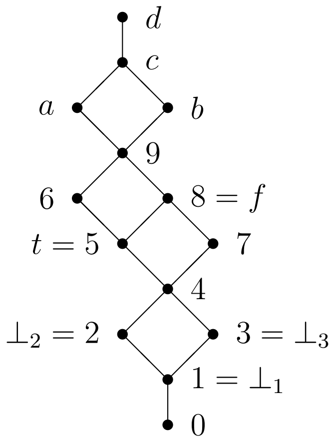
Fact 9. *All elements $\perp_i$ (on any model) are idempotent.*

Proof. All such elements satisfy $x \leq t$, and any element below t is idempotent because $x \circ x \leq t \circ x = x$. □

Fact 10. *Although $A \to (B \to A)$ is not, of course, a theorem scheme of $\mathbf{R}$ the special case $A \to (a = b \to A)$ is provable in $\mathbf{R}^{\#}$.*

Proof. In $\mathbf{R}$, A implies $t \to A$, and in $\mathbf{R}^{\#}$ as just noted, $a = b$ implies t, so $t \to A$ implies $a = b \to A$. □

Fact 11. *For any interpretation $\mathcal{M}$ of $\mathbf{R}^{\#}$ on a DMM $\mathcal{D}$ and any numeral n, the relation $\lambda x, y\, (\perp_n^{\mathcal{M}} \leq (x \to y) \wedge (y \to x))$ is a congruence on $\mathcal{D}$.*



Domain = {**0**, **1**, **2**, **3**, **4**, **5**}

=	**0**	**1**	**2**	**3**	**4**	**5**
0	5	1	2	3	2	1
1	1	5	1	2	3	2
2	2	1	5	1	2	3
3	3	2	1	5	1	2
4	2	3	2	1	5	1
5	1	2	3	2	1	5

A	$\neg A$
0	d
1	c
2	b
3	a
4	9
5	8
6	7
7	6
8	5
9	4
a	3
b	2
c	1
d	0

$\to$	0	1	2	3	4	5	6	7	8	9	a	b	c	d
0	d	d	d	d	d	d	d	d	d	d	d	d	d	d
1	0	6	6	6	6	6	6	c	c	c	c	c	c	d
2	0	3	6	3	6	6	6	b	b	b	c	b	c	d
3	0	2	2	6	6	6	6	a	a	a	a	c	c	d
4	0	1	2	3	6	6	6	9	9	9	a	b	c	d
5	0	1	2	3	4	5	6	7	8	9	a	b	c	d
6	0	1	2	3	4	4	6	7	7	9	a	b	c	d
7	0	0	0	0	0	0	0	6	6	6	6	6	6	d
8	0	0	0	0	0	0	0	4	5	6	6	6	6	d
9	0	0	0	0	0	0	0	4	4	6	6	6	6	d
a	0	0	0	0	0	0	0	3	3	3	6	3	6	d
b	0	0	0	0	0	0	0	2	2	2	2	6	6	d
c	0	0	0	0	0	0	0	1	1	1	2	3	6	d
d	0	0	0	0	0	0	0	0	0	0	0	0	0	d

Figure 1: Model of $\mathbf{R}^{\#}$ *mod* 6

Proof. The congruences are determined by the filters containing t, which includes the principal filter determined by any $\bot_n$. Clearly the relation is an equivalence ($\bot_n \leq t$ is necessary here for reflexivity), and that it respects all of the operations is shown by cases and poses no difficulty. Again, $\bot_n \leq t$ is needed for the cases of the lattice operations. $\square$

The 3-valued propositional structure used above to model $\mathbf{R}^\#$ modulo 2 is sufficient to make the point about the existence of finite models, but is hardly typical. Even without a specification of the domain (i.e. with any modulus or none) it validates such non-theorems as:

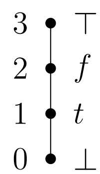
$$0 \neq 0 \rightarrow 0 = 0$$
$$\neg\exists x \exists y(x = y)$$

and indeed all instances of:

$$0 = 1 \rightarrow A$$
$$A \rightarrow (A \rightarrow A)$$
$$(A \rightarrow B) \vee (B \rightarrow A)$$
$$(A \wedge \neg A) \rightarrow (B \vee \neg B)$$
$$A \vee (A \rightarrow B)$$

In the interests of a less distorted view of relevant logic and relevant number theory, Figure 1 shows another propositional structure modelling $\mathbf{R}^\#$ *modulo* 6. Bold face has been used for the numbers (i.e. the integers *modulo* 6) in order to distinguish them from the elements of the De Morgan monoid. The dotted lines in the table are just for readability. The tables for conjunction and disjunction (lattice meet and join) can be read off the Hasse diagram in the usual way. This structure is in fact the smallest prime DMM modelling $\mathbf{R}^\#$ *modulo* 6 in such a way that $0 = 1$ does not imply everything, $\bot_2$ and $\bot_3$ are independent of each other, and $0 = 1 \rightarrow 0 = 1$ is not false.

Notice that the set $\{1, 2, 3, 4, 5, 6\}$ contains the identity element and is closed under the lattice operations, $\circ$ and $\rightarrow$. It is therefore a Dunn monoid, a subalgebra of the positive reduct of the DMM. The positive formulae are all interpreted in this Dunn monoid, which is generated by the elements assigned to equations. This is a general feature of models of $\mathbf{R}^\#$: the equations are all implied by $0 = 1$ and all imply $0 = 0$, so their values lie in the interval between $\bot_1$ and t, and it is easy to show by induction on structure that the positive formulae are bounded below by $\bot_1$ and above by $\top_1$. Clearly, the relation $\lambda x, y(\bot_1 \leq x \leftrightarrow y)$ is a congruence on any DMM modelling $\mathbf{R}^\#$ and it sets all of the interval from $\bot_1$ to $\top_1$ congruent to t. If the model is generated by the equations then the quotient algebra under this congruence is generated by the identity alone—i.e. it has no proper subalgebra. It is also inconsistent, because $0 = 1$ is always false. There are, up to isomorphism, only two inconsistent, 0-generated DMMs: the trivial algebra with only one element (which is uninteresting) and C4:



$\to$	0	1	2	3
0	3	3	3	3
1	0	1	2	3
2	0	0	1	3
3	0	0	0	3

Theorem 2 ([12], p.121)**.** *In any EB model (i.e. where $\bot_1$ is $\bot$) the quotient under the congruence determined by $\bot_1$ is the trivial algebra, while in all other cases the quotient is C4.*

Fact 12. *Any interpretation of* $\mathbf{R}^{\#}$ *on a DMM is EB iff* $\overline{\bot_1}$ *is idempotent.*

Proof. Half of this equivalence is trivial: if $\bot_1$ is at the bottom, then $\overline{\bot_1}$ is at the top, so by square-increasing it is idempotent. For the converse, note that if $\overline{\bot_1}$ is idempotent, then $\overline{\bot_1} \leq \overline{\bot_1} \to \overline{\bot_1}$, which is to say $\overline{\bot_1} \leq \bot_1 \to \bot_1$ and hence $\bot_1 \leq \overline{\bot_1} \to \bot_1$, but $\overline{\bot_1} \to \bot_1 = \bot$ on any interpretation. □

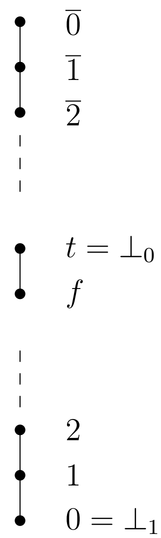
DMMs which have C4 as a homomorphic image exhibit some strong regularities. There is only ever one epimorphism from a DMM $\mathcal{D}$ onto C4, and it often happens, as in Figure 1 above, that the pre-images of 0 and 3 (i.e. $\bot$ and $\top$) are singletons. In that case, because its structure is so rigid and clear, we say that $\mathcal{D}$ is "crystalline".

Theorem 3 ([12], p.122)**.** *Every subdirectly irreducible DMM which has C4 as a homomorphic image is crystalline.*

Proof. Let h be a homomorphism from a DMM $\mathcal{D}$ onto C4, and suppose a and b are distinct elements of $\mathcal{D}$ such that $h(a) = \bot = h(b)$. Without loss of generality, suppose $a \not\leq b$, so $a \to b \leq f$. Then $h(a) \to h(b) \leq h(f)$, which is to say $\bot \to \bot \leq h(f)$ or $\top \leq h(f)$, whence $h(t) = \bot$ which is impossible. □

This has logical consequences: for instance, $\mathbf{R}^{\#}$ has as theorems all formulae of the form $(0 = 1 \to A) \vee (A \to B)$, because in any crystalline model $\bot_1$ is the unique atom while in any EB model it is $\bot$, so the disjunction is true in all prime models, and hence in all models.

A subdirect product of DMMs one of which is crystalline obviously also has C4 as a homomorphic image; we shall refer to such DMMs as "sub-crystalline". The Lindenbaum algebra of $\mathbf{R}^{\#}$, for example, is sub-crystalline (though not, of course, crystalline).



$$a \circ b = \begin{cases} a \wedge b & \text{if } a \leq \overline{b} \\ a \vee b & \text{otherwise} \end{cases}$$

$$\bot_{2^k(2n+1)} = k$$
$$\mathcal{M}(a = b) = \bot_{|a-b|}$$

Figure 2: $\mathbf{RM}^{\#}$ model invalidating $\exists x(x > 0 \wedge \forall y(y = 0 \vee (0 = y \rightarrow 0 = x)))$. The lower part of the chain consists of the natural numbers $\mathcal{N}$ and the upper part is its mirror image. Every positive natural number p is $2^k(2n+1)$ for some (unique) n and k, and the model has $\bot_p = k$ in that case.

3 Models with $\bot_1 = \bot$

So prime models in which $\overline{\bot_1} < \overline{\bot_1}^2$ are crystalline. About EB models, in which $\bot_1$ is the bottom element, less is known. In the remainder of this paper, we open the attack on them by describing the small cases in detail. The simplest such models are the idempotent ones characterising **RM**. We may refer to the arithmetic based upon those as $\mathbf{RM}^{\#}$.

As noted, nothing is lost if the prime DMMs modelling the pure propositional logic **RM** are taken to be Sugihara chains. The same is *not* true, however, of models of $\mathbf{RM}^{\#}$. For a counter-example, consider the formula

$$\exists x(x > 0 \wedge \forall y(y = 0 \vee (0 = y \rightarrow 0 = x)))$$

This is true in every model on a Sugihara chain, since any Sugihara chain consists of its generators and their De Morgan complements, meaning that every element below t is equational. This includes the element immediately below t, which by Fact 7 is of finite height, whence every Sugihara chain model is finite. It is not a theorem of $\mathbf{RM}^{\#}$ however (unless *per impossibile* the latter is ω-inconsistent) as it fails in the structure shown in Figure 2. $\mathbf{RM}^{\#}$ therefore lacks the finite model property. This contrasts sharply with

RM itself, which has the much stronger Scroggs property: every proper extension of it has a finite characteristic algebra [2].

Next, in virtue of Theorem 1:

Fact 13. *In a totally ordered model of* $\mathbf{R}^{\#}$*, the generating equations form a discrete sequence* $\langle \bot_{k_1}, \ldots, \bot_{k_i}, \ldots \bot_0 \rangle$*, starting at the bottom with* $k_1 = 1$ *and going up, with* $0 = 0$ *as the identity* t*. For any* $i > 1$*,* k_i *is a proper multiple of* k_{i-1}*, and if the domain of the model is finite then each positive* k_i *divides the modulus.*

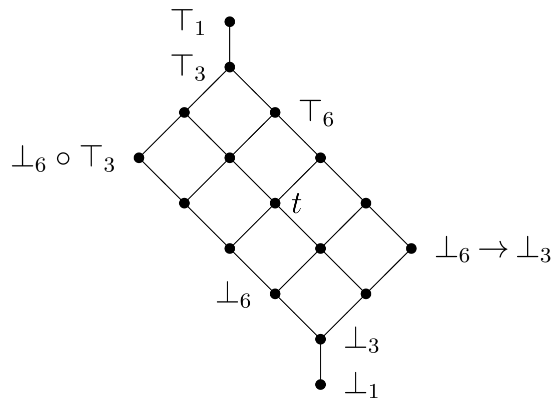
Fact 13 severely restricts the range of possible models, particularly in the finite case to which we now turn.

$\mathbf{RM}^{\#}$ modulo a prime p is essentially **RM3**: the 3-element Sugihara algebra is its only non-trivial model. This is immediate from Fact 13 as clearly there exists no k_2 which properly divides p, so all equations $0 = n$ are equivalent to $0 = 1$ unless n is a multiple of p.

As an example where the modulus is not prime, consider the subdirectly irreducible models of $\mathbf{RM}^{\#}$ modulo 12. As noted, these are Sugihara chains consisting of the elements modelling equations and their negations. Every equation takes as value $\bot_k$ for some $k < 12$. Following the prescription of Fact 13, we know $\bot_k = \bot_1 = \bot$ where k is not a multiple of a divisor of 12 (i.e. for $k \in \{5, 7, 11\}$), that $\bot_{12-k} = \bot_k$, and of course $\bot_0 = t$. For the other possible generators, there are 6 possibilities:

1. $\bot_6 = \bot_4 = \bot_3 = \bot_2 = \bot$; Sugihara chain of order 3.
2. $\bot_6 = \bot_4 = \bot_2 > \bot_3 = \bot$; Sugihara chain of order 5.
3. $\bot_4 > \bot_6 = \bot_2 > \bot_3 = \bot$; Sugihara chain of order 7.
4. $\bot_6 > \bot_4 = \bot_2 > \bot_3 = \bot$; Sugihara chain of order 7.
5. $\bot_6 = \bot_3 > \bot_4 = \bot_2 = \bot$; Sugihara chain of order 5.
6. $\bot_6 > \bot_3 > \bot_4 = \bot_2 = \bot$; Sugihara chain of order 7.

The direct product of these six is *not* generated by the equations, as for instance any formula of $\mathbf{RM}^{\#}$ whose value in model 5 is $\bot$ has the value $\bot$ in model 6 as well, because they are both rigorously compact and all generators with value $\bot$ in one have value $\bot$ in the other. The equation-generated subdirect product of models 5 and 6, in fact, consists of the direct products of their inner subalgebras (i.e. excluding $\top$ and $\bot$), with the top and bottom elements added:



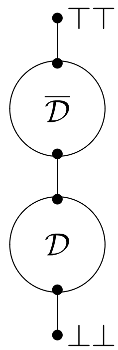
We shall call this algebra $\mathcal{A}_{5-6}$. The equation-generated subdirect product $\mathcal{A}_{2-4}$ of models 2, 3 and 4 is likewise the direct product of their inner subalgebras with $\top$ and $\bot$ above and below respectively—since this structure has 77 elements, it is not drawn explicitly here. The direct product $\mathcal{A}_{2-4} \times \mathcal{A}_{5-6}$ is generated by the equations, as its extreme pairs are $\bot_2 \to \bot_3$ and $\bot_3 \to \bot_2$ respectively. Furthermore, the direct product of all of this with model 1 is equation-generated, with extreme pairs $\bot_6 \to \bot_1$ and $\bot_6 \circ \top_1$. Hence $\mathbf{RM}^\#$ *modulo* 12 has, up to provable equivalence, exactly $77 \times 17 \times 3 = 3927$ propositions.

The number 12 is just an example, of course, but it is reasonably clear that a similar construction can be applied to any given modulus, so we declare the finite models of $\mathbf{RM}^\#$ sufficiently analysed for present purposes.

$\mathbf{R}^\#$ is a different matter. As an indication of how little control we have, consider:

Theorem 4. *Let $\mathcal{D}$ be a Dunn monoid generated by the two elements t and $\bot$. Then $\mathcal{D}$ can be embedded in a De Morgan monoid on which there is a model of* $\mathbf{R}^\#$ modulo *any prime p.*

Proof. The required model of $\mathbf{R}^\#_+$ is defined by setting $\mathcal{M}(a = b) = \bot_{|a-b|}$, where $\bot_n = t$ if n is a multiple of p and $\bot_n = \bot$ otherwise. It is easy to verify that all axioms of $\mathbf{R}^\#_+$ are true for $\mathcal{M}$. To extend to the whole of $\mathbf{R}^\#$, we embed $\mathcal{D}$ is a DMM by the construction used by Meyer [8] to show conservative extension for negation in $\mathbf{R}$. That is, let $\overline{\mathcal{D}}$ consist of elements $\overline{d}$ corresponding to the $d \in \mathcal{D}$ but with the lattice order reversed, let $\top\top$ and $\bot\bot$ be two new elements, and order everything thus:



$\circ$	$\bot\bot$	b	$\overline{b}$	$\top\top$
$\bot\bot$	$\bot\bot$	$\bot\bot$	$\bot\bot$	$\bot\bot$
a	$\bot\bot$	$a \circ b$	$\overline{a \to b}$	$\top\top$
$\overline{a}$	$\bot\bot$	$\overline{b \to a}$	$\top\top$	$\top\top$
$\top\top$	$\bot\bot$	$\top\top$	$\top\top$	$\top\top$

(where a and b are elements of $\mathcal{D}$)

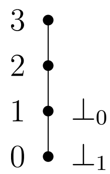
The monoid operation is defined on the extended structure as shown in the table. It is routine to check that the DMM postulates hold, and that all of $\mathbf{R}^\#$ is modelled by $\mathcal{M}$. $\square$

Note that $\mathcal{D}$ could be *any* Dunn monoid generated by the two specified elements. There are many such structures, some of which are infinite [12]. There is therefore no end to the possibilities for modelling $\mathbf{R}^\#$ *modulo* a prime, or indeed to $\mathbf{R}^\#$ with the additional axiom

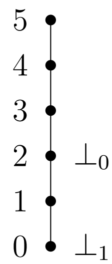
$$\forall x((0 = x + 1) \to (0 = 1)).$$

Those large models, however, are all sub-crystalline. At least in the simple case of $\mathbf{R}^\#$ *modulo* a prime, the EB models remain finite.

Since every equation is equivalent in a model of $\mathbf{R}^\#$ *modulo* a prime either to t or to $\bot$, it is clear that such a model is a DMM generated by those two elements. As observed in previous work [12] there are only two ways in which this can happen in a prime DMM: either $\bot$ is identical with the lowest element of the subalgebra generated from t alone, or else it is properly below the t-generated subalgebra (and its negation above) and it generates nothing else because of rigorous compactness. As noted, there are only two 0-generated DMMs which are inconsistent and subdirectly irreducible: the trivial algebra and C4. There are thus only four subdirectly irreducible EB models of $\mathbf{R}^\#$ modulo a prime: the trivial algebra (which can be disregarded), the 3-element Sugihara model (which can also be disregarded as it is a homomorphic image of one of the others) and these two:



$\to$	0	1	2	3
0	3	3	3	3
1	0	1	2	3
2	0	0	1	3
3	0	0	0	3



$\to$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	0	4	4	4	4	5
2	0	1	2	3	4	5
3	0	1	1	2	4	5
4	0	1	1	1	4	5
5	0	0	0	0	0	5

The algebra on the left is C4, with $\bot_1$ set to 0 instead of 1. On the right is the same extended by the addition of new top and bottom elements. Let us call the left model C4^0 and the right one "extended C4", or eC4.

Theorem 5. *The maximal EB model of* $\mathbf{R}^\#$ modulo *a prime (i.e. the Lindenbaum algebra of* $\mathbf{R}^\#$ *with additional axioms* $0 = 1 \to (0 \neq 1 \to 0 = 1)$ *and* $0 = p$ *for a prime* p*) is the direct product C4*^{0} $\times$ *eC4 and so has exactly 24 elements.*

Proof. The direct product C4^0 $\times$ eC4 is generated by the two elements $\langle t, t\rangle$ and $\langle \bot, \bot\rangle$, since the extreme pairs $\langle \bot, \top\rangle$ and $\langle \top, \bot\rangle$ are so generated: $(\bar{t} \to t) \to \bot$ is $\langle \top, \bot\rangle$, and $\langle \bot, \top\rangle$ is its negation. The result then follows by Fact 4. □

There are infinitely many prime crystalline DMMs generated by t and $\bot_1$, so we cannot display them all in detail, but we know the following:

Theorem 6. *There is a crystalline model* $\mathcal{C}$ *such that the direct product* $\mathcal{C} \times$ *C4*^{0} $\times$ *eC4 is characteristic for* $\mathbf{R}^\#$ *with the additional axiom* $0 = p$ *where* p *is any prime.*

Proof. Let Σ be a set consisting of one member of each isomorphism class of subdirectly irreducible non-EB models of $\mathbf{R}^\#$*modulo* a prime. Let Σ be well-ordered somehow,[6] and let Π be the direct product of Σ in that order. The required $\mathcal{C}$ is the subalgebra of Π generated by the equations (i.e. by $\langle t, \ldots, t, \ldots\rangle$ and $\langle \bot_1, \ldots, \bot_1, \ldots\rangle$). To see this, note first that $\mathcal{C}$ is crystalline because all of the subdirectly irreducible models of which it is a subdirect product are crystalline; in every case, the generators are in the pre-image of element 1 on every homomorphism into C4, so every polynomial whose value maps to 0 on any such homomorphism maps to 0 on all of them. It follows that the pre-image of 0 on a homomorphism from their product into C4 is a product of singletons, and so is itself a singleton. Note also that in every

[6] Since all models are countable, the appeal to choice and well ordering here is not extreme.

subdirectly irreducible case, the pre-image of 1 is the interval $[\bot_1, \top_1]$, so this too is true of $\mathcal{C}$.

The direct product of $\mathcal{C}$ with $\mathrm{C4}^0 \times \mathrm{eC4}$ is generated by $\{t, \bot_1\}$ in accordance with Fact 4, because the extreme pairs are so generated: $\langle \top, \bot \rangle$ is $\overline{\bot_1}^2 \circ \bot_1$ while $\langle \bot, \top \rangle$ is $\overline{\bot_1} \to \top_1$.

Now let A and B be any two formulae which are not equivalent in the theory $\mathbf{R}^\#$ plus the axiom $0 = p$. By the completeness theorem for $\mathbf{R}^\#$ there is a subdirectly irreducible DMM $\mathcal{D}$ in which they have different values on assigning all equations $m = n$ the element t if m and n are congruent *modulo* p and $\bot_1$ otherwise. This is either EB, in which case A and B are separated by the order 24 DMM of Theorem 5, or else it is one of the crystalline structures whose product is Π, in which case A and B are separated by $\mathcal{C}$. In either case, they are distinct in $\mathcal{C} \times \mathrm{C4}^0 \times \mathrm{eC4}$ as required. $\square$

$\mathbf{R}^\#$ *modulo* a composite number, however, is under less control. In fact, every such arithmetic has many EB models, some of them infinite, and little is known about their structure as DMMs. Consider, for example, the simple case of $\mathbf{R}^\#$ extended with the axiom $0 = 4$. We may construct an EB model of this from any (non-EB) crystalline model of $\mathbf{R}^\#$ satisfying $0 = p$ for a prime p. Let $\mathcal{P}$ be such a model on a DMM $\mathcal{D}$. Now let the domain of another model $\mathcal{M}$ be the integers *modulo* 4 and interpret the identity relation thus:

$=$	0	1	2	3
0	t	$\bot$	$\bot_1^{\mathcal{P}}$	$\bot$
1	$\bot$	t	$\bot$	$\bot_1^{\mathcal{P}}$
2	$\bot_1^{\mathcal{P}}$	$\bot$	t	$\bot$
3	$\bot$	$\bot_1^{\mathcal{P}}$	$\bot$	t

The successor, addition and multiplication functions, of course, are as standard in arithmetic *modulo* 4. We know that $\mathcal{D}$ is generated by $\{\bot_1^{\mathcal{P}}, t\}$, so to show that $\mathcal{M}$ is a model of $\mathbf{R}^\#$ it suffices that all the postulates are verified. That all the theorems of $\mathbf{R}$ hold is given, and most of the arithmetical axioms of $\mathbf{R}^\#$ are trivial. That includes the induction axiom, which holds because the domain is finite. The only axiom really requiring an argument is that stating the transitivity and symmetry of equality:

$$a = b \to (a = c \to b = c)$$

Note that there are only three possible values for an equation, so we may write out the table for implications between these:

$\rightarrow$	$\bot$	$\bot_1^{\mathcal{P}}$	t
$\bot$	$\top$	$\top$	$\top$
$\bot_1^{\mathcal{P}}$	$\bot$	$\top_1^{\mathcal{P}}$	$\top_1^{\mathcal{P}}$
t	$\bot$	$\bot_1^{\mathcal{P}}$	t

Clearly, any possible failure of the above axiom must have a congruent to b *mod* 2, and similarly a congruent to c *mod* 2, since otherwise the whole formula has value $\top$. Suppose, then, that a, b and c are all congruent *mod* 2. The value of $a = c \rightarrow b = c$ is $\bot_1^{\mathcal{P}}$ or higher, so the axiom cannot fail unless a and b are congruent *mod* 4. Similarly, it cannot fail unless a and c are also congruent *mod* 4, which makes a and b congruent *mod* 4 as well, in which case the whole has the value t. Therefore the transitivity axiom holds on $\mathcal{M}$, which is therefore a model of $\mathbf{R}^\#$.

4 Summary

The models of $\mathbf{R}^\#$ with finite domains are easily described from the purely arithmetical point of view: they amount simply to the integers *modulo* n for any positive n. The propositional structures determining the relevant logic of such models, however, are harder to describe. As demonstrated above, the stronger arithmetic $\mathbf{RM}^\#$ is under good control. The insights gained there extend partially to finite models of $\mathbf{R}^\#$ where the modulus n is prime. In those cases, the EB models are finite and fully described here, while the subdirectly irreducible non-EB models are all crystalline and generated by idempotent elements below the identity t. The good news, however, stops here. Not only do we know little of such crystalline DMMs, except that they may be arbitrarily complex and even infinite, but we also find that as soon as the modulus is composite, even the EB models run similarly out of control. The present paper has at least exposed the richness of the field, and may act as a corrective to the idea that finite models of $\mathbf{R}^\#$ can be understood through acquaintance with a single three-valued structure. [7]

[7]The author wishes to thank the anonymous reviewer for numerous suggestions which have improved this paper at many points.

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