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Immune Logics

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Abstract

This article is concerned with an exploration of a family of systems—called immune logics—whose main properties are, in some sense, related to those of the well-known family of infectious logics. The distinctive feature of the semantic of infectious logics is the presence of a certain "infectious" semantic value, i.e. a value which is a zero element for all the operations in the underlying algebraic structure. On the other hand, what is characteristic of the semantic of immune logics is to have a certain "immune" value, i.e. an identity element for the binary operations in the underlying algebraic structure. In this article, we will define these structures, focusing on the 3-element case, discuss the relations between immune and infectious elements, and provide technical results regarding them, and the various logical systems defined using such semantics.

1 Introduction

In the past few years, there has been an increasing amount of attention paid to a family of many-valued systems called *infectious logics*. These systems usually have a truth-value behaving in an infectious way, by which we mean that this value is a *zero element* (alternatively, annihilating, or absorbing element) for all the operations in the underlying algebraic structure.

The aim of this article is to explore some structures which are somewhat related to infectious logics, leading to systems that we will call *immune logics* for obvious reasons. Thus, with the purpose of conducting this investigation, the article is structured as follows. In section 2, we present algebras with infectious elements and discuss some of their characteristic properties. In section 3, we devote ourselves to the examination of the 3-element algebras with an universal idempotent immune element, leading us to discuss operations already introduced in other contexts by Sobociński, Ebbinghaus, Cooper, and others. In section 4, we consider different logical systems that can be defined over the infectious and the immune semantics, also exploring some applications that can be entertained as regards subject-matter preservation. Finally, in section 5 we wrap up our investigation with some concluding remarks and directions for future work.

2 Infectious Logics

In this section, we discuss the formal details underlying the so-called infectious logics. As we briefly said before, systems of this sort are induced by logical matrices whose algebraic reduct is an algebra having an infectious or absorbing element. This element can be seen as a universal zero element, as well as an element working in a value-in-value-out fashion. In fact, these two different approximations can be seen to be equivalent as noted in the definitions below, taken from [31] and [10].

Definition 2.1. An algebra $\mathbf{A}$ has *an infectious element* m if and only if for every n-ary operation ¶ of $\mathbf{A}$, and every $\{a_1, \ldots, a_n\} \subseteq A$:

$$\text{if } m \in \{a_1, \ldots, a_n\}, \text{ then } \P^{\mathbf{A}}(a_1, \ldots, a_n) = m$$

In this article we will be focusing on algebras having the 2-element Boolean algebra as a proper subalgebra. For this purpose, it is interesting to have in mind the definition of an extension of an arbitrary algebra with an infectious element.

Definition 2.2. Given an algebra $\mathbf{A}$, the algebra $\mathbf{A}[m]$ is its *extension with an infectious element* $m \notin A$, such that for all n-ary operations ¶ of $\mathbf{A}[m]$ and all $\{a_1, \ldots, a_n\} \subseteq A \cup \{m\}$:

$$\P^{\mathbf{A}[m]}(a_1, \ldots, a_n) = \begin{cases} m & \text{if } m \in \{a_1, \ldots, a_n\} \\ \P^{\mathbf{A}}(a_1, \ldots, a_n) & \text{otherwise} \end{cases}$$

In this vein, it is key to focus on the smallest structure containing an infectious element which also extends the 2-element Boolean algebra. This structure is usually referred to as the week Kleene algebra (**WK** algebra, for short), whose operations can be depicted in the truth-tables appearing in Figure 1—also, see [22].

	$\neg_{\mathbf{WK}}$
t	**f**
u	**u**
f	**t**

$\wedge_{\mathbf{WK}}$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**u**	**u**
f	**f**	**u**	**f**

$\vee_{\mathbf{WK}}$	**t**	**u**	**f**
t	**t**	**u**	**t**
u	**u**	**u**	**u**
f	**t**	**u**	**f**

Figure 1: The weak Kleene truth-tables

As regards the algebraic reduct of a logical matrix of any sort, it is usually customary to study what kind of structure it constitutes. For example, the algebraic structure underlying the two-valued semantics for Classical Logic is a Boolean algebra, as we said before—that is to say, a complemented bounded distributive lattice. In this respect, it is interesting to highlight that the weak Kleene algebra is a *(generalized) involutive bisemilattice*—see [24] and references therein.

By this we mean, a structure with a unary operation (which we may call $\neg$) and two binary operations (which we may call $\wedge$ and $\vee$) behaving in the following way. First, both $\wedge$ and $\vee$ induce partial orders (which we may refer to as $\leq^{\wedge}_{\mathbf{WK}}$ and $\leq^{\vee}_{\mathbf{WK}}$), and $\neg$ is an involution in both of these orders. Secondly, the absorption laws do not hold for these operations or, equivalently, these orders are not each other's inverse. Thus, conjunction can be seen as the minimum of an order, because $x \leq^{\wedge}_{\mathbf{WK}} y$ iff $x \wedge_{\mathbf{WK}} y = x$. Also, disjunction can be seen as the maximum of an order, because $x \leq^{\vee}_{\mathbf{WK}} y$ iff $x \vee_{\mathbf{WK}} y = y$. But these orders are not the same. Thus, the order $\leq^{\wedge}_{\mathbf{WK}}$ associated with the **WK** conjunction has a maximum, it just happens that this maximum is not the **WK** disjunction. Similarly, the order $\leq^{\vee}_{\mathbf{WK}}$ associated with the **WK** disjunction has a minimum, it just happens that this minimum is not the **WK** conjunction. A graphical depiction of the 3-element **WK** algebra as a structure of this sort can be observed in the Figure 2.

One of the important things for us in this work is to consider logics built on top of the **WK** algebra. For this purpose, we will analyze two well-known infectious logics that are subsystems of Classical Logic—the logics $\mathsf{K}^{\mathsf{w}}_3$ and PWK—as well as two less known systems—the logics wST and wTS—which

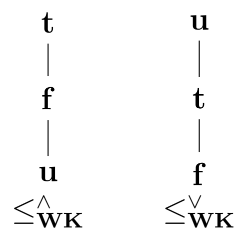

Figure 2: The **WK** algebra as a generalized involutive bisemilattice

are respectively non-transitive and non-reflexive. We will do this however in a later section, where we will also compare them to systems built on the algebras with immune elements that we will discuss in the next section.

Infectious logics have received many interpretations in the literature. Although in the recent years novel epistemic, computational, topic-related, and even metaphysical readings have been discussed, the most standard one is semantic—see [18], [17], [14], [4], [15]. It dates from the early discussions of the last century, revolving around set-theoretic and semantics paradoxes.

Thus, according to some scholars—among which we saliently find Bochvar [5] and Halldén [20], but also Goddard and Routley [19]—the statements comprising the paradoxes constitute nonsensical or rather meaningless sentences. That is to say, expressions which despite being grammatical did not convey a meaning or express a proposition at all. It was the opinion of these authors that conjugating meaningless sentences with a meaningful sentence only resulted in a meaningless sentence because it was impossible for something to be meaningful and have a component lacking meaning. In fewer words, meaninglessness has an infectious behavior. It is easy to observe that the operations resulting from a system allowing for true, false (in either case, meaningful) and meaningless sentences, are faithfully represented by the operations of the weak Kleene algebra.

With the technical and conceptual remarks on infectious logics over, we now move on to the main topic of our article: immune logics.

3 Immune logics

There seems to be an equally interesting class of logics, related to that of the infectious logics, which (apart from a few exemptions, which we will mention) have been somehow neglected in the contemporary literature—we will call them *immune logics*.

While infectious logics are characterized by having an all-purpose zero element, immune logics are by having an *all-purpose identity element*, or at least an element that can be thus regarded for all binary operations. In the case of binary operations, whatever elements are conjugated with an identity element, the result of that operation is the remaining element. Notice that this implies that when the inputs of some binary operation are only the identity element, the output is the identity element. If, additionally, we assume that idempotency holds for all unary operations—and we only have these kind of operations in the algebra in question—this justifies calling the involved elements *universally idempotent immune* elements.[1]

Definition 3.1. An algebra **A** has *a universally idempotent immune element* m if and only if for every n-ary operation ¶ of **A**, and every $\{a_1, \ldots, a_n\} \subseteq A$:

$$\text{if } \{m\} \subsetneq \{a_1, \ldots, a_n\}, \text{ then } *(a_1, \ldots, a_n) = a_j \neq m$$

$$\text{if } \{m\} = \{a_1, \ldots, a_n\}, \text{ then } *(a_1, \ldots, a_n) = m$$

In what follows, we will be dealing with 3-element algebras. Thus, one would like to conceive a method to extend arbitrary algebras with universally idempotent immune elements, as we did in the Definition 2.2 for the infectious elements. However, as we will see this is not as easy as it initially seems. Assume we have any algebra **A**, and we would like to obtain the algebra which is its *extension with a universally-idempotent immune element*, call it $\mathbf{A}[m]$ (with $m \notin A$). The most natural way to do it would be to define the n-ary operations ¶ of $\mathbf{A}[m]$ and all $\{a_1, \ldots, a_n\} \subseteq A \cup \{m\}$ in the following way:

$$\P^{\mathbf{A}[m]}(a_1, \ldots, a_n) = \begin{cases} a_j \neq m & \text{if } \{m\} \subsetneq \{a_1, \ldots, a_n\} \\ m & \text{if } \{m\} = \{a_1, \ldots, a_n\} \\ \P^{\mathbf{A}}(a_1, \ldots, a_n) & \text{otherwise} \end{cases}$$

[1] As we will see, we will not assume that the idea of an all-purpose identity element can be generalized so as to apply to algebras having arbitrary n-ary operations. Everything we claim in this article can be understood as restricted to algebras containing, at most, binary operations.

However, notice that $\mathbf{A}[m]$ is not well-defined, since the first condition states that $¶^{\mathbf{A}[m]}(a_1, \ldots, a_n) = a_j \neq m$ if $\{m\} \subsetneq \{a_1, \ldots, a_n\}$. However, if we have a ternary operation $\circ(a_1, m, a_2)$ with $a_1 \neq m \neq a_2$, the instruction does not determine one single algebra, since it only requires that $\circ(a_1, m, a_2) \neq m$, but it does not decide between a_1 and a_2.[2] This point, of course, can be generalized to arbitrary n-ary operations, when $n \geq 3$. For these reasons, in this article we will not define extensions of arbitrary algebras with universally-idempotent immune elements, but we will only restrict our investigations to the 3-element case with unary and binary operations.[3]

Below we depict the unique 3-element algebra with an universally idempotent immune element having the 2-element Boolean algebra as a subalgebra. This structure was presented by Sobociński in [28], and, for this reason, we call it the 3-element Sobociński algebra, appearing in Figure 3. Incidentally, these operations are also discussed in works by Ebbinghaus, and Finn and Griogolia—see [28], [13] and [16].

	$\neg_{\mathbf{S}}$
t	**f**
u	**u**
f	**t**

$\wedge_{\mathbf{S}}$	**t**	**u**	**f**
t	**t**	**t**	**f**
u	**t**	**u**	**f**
f	**f**	**f**	**f**

$\vee_{\mathbf{S}}$	**t**	**u**	**f**
t	**t**	**t**	**t**
u	**t**	**u**	**f**
f	**t**	**f**	**f**

Figure 3: The Sobociński truth-tables

It is interesting to observe a couple of things with regard to the ordered structure induced by these operations. Firstly, if we take conjunction to be the minimum of the order $\leq^{\wedge}_{\mathbf{S}}$ and disjunction to be the maximum of the order $\leq^{\vee}_{\mathbf{S}}$ it constitutes another generalized involutive bisemilattice, as portrayed in Figure 4. This is similar to what happened in the case of the weak Kleene algebra. Secondly, this generalized involutive bisemilattice is such that the $\leq^{\wedge}_{\mathbf{S}}$ order here corresponds to the $\leq^{\vee}_{\mathbf{WK}}$ order in the **WK** algebra, whereas the $\leq^{\vee}_{\mathbf{S}}$ order here corresponds to the $\leq^{\wedge}_{\mathbf{WK}}$ order in the **WK** algebra. So, in a way these structures are closely intertwined and can be properly seen as symmetric.

Before closing this section, let us say a few words on possible interpretations of these operations. The first interpretation—which is novel to this

[2] We would like to thank an anonymous reviewer for pointing this out to us.

[3] A possible solution to this problem would be to use non deterministic algebras. Although interesting, we leave this exploration for future research.

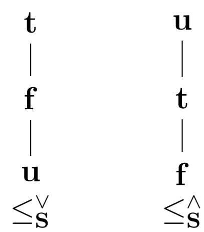

Figure 4: The **S** algebra as a generalized involutive bisemilattice

article, as far as we know—is inspired in the meaninglessness interpretation provided for the weak Kleene algebra. The idea is the following. While the interpretation based on **WK** considers that nonsensical sentences are infectious, we could argue that nonsensical sentences are innocuous regarding the semantics. From a semantical point of view, they are like noise or interference. So, nonsensical sentences are evaluated as semantically ineffective, in the sense that they do not add semantic content to the meaningful sentences which they are combined with. In the context of the **WK** algebra we can interpret meaninglessness as a phenomenon that propagates. One nonsensical sentence is enough to contaminate, to infect the whole context. On the other hand, the proposal we're suggesting consists in considering meaninglessness as a phenomenon that should be encapsulated. We should omit nonsensical parts when evaluating the semantic value of a compound sentence. Of course, here we're simply outlining a possible new application for these operations and although interesting, a fully developed story of this application should wait for another occasion.

An established interpretation, perhaps of the sort that we outlined before, was proposed by Humberstone. In [21, p.1051] the author mentions that the conjunction and disjunction operations of the Sobociński algebra can be motivated in the context of the theory of emotivism, i.e., the theory that claims that moral judgments lacks of cognitive meaning—and are therefore neither true nor false—but only have emotive meaning. In particular, using a passage from Ayer [1], he motivates this meaninglessness reading, claiming their lack of truth conditions does not deprive the complex sentence thereof. So, for example, the truth conditions of a conjunctive sentence like "Today is Sunday and stealing is wrong" are identical to those of the first conjunct—and similarly, if it was a disjunction. Naturally, if all components of a complex sentence one of the sort, the sentence will not express a proposition and therefore it will not be true or false. It is easy to observe that the operations resulting from a system allowing for true, false and neither true nor false sentences of this sort, are faithfully represented by the operations of the Sobociński algebra.

Going back to our original plan, in the next section we will consider some consequence relations that can be defined over the algebras we have been considering, and we will mention some of their salient features.

4 Consequence relations

The aim of this section is to analyze the different logics induced by considering different consequence relations over 3-element algebras with infectious or, respectively, with immune elements. For this purpose, we will consider the framework laid out by Cobreros, Égré, Ripley, van Rooij, Chemla, Spector, and others, where so-called mixed consequence relations are defined—see, e.g., [6], [7], and [11].

Consequence relations of this form generalize the usual regular matrix consequence relations, where logical consequence is defined in terms of the preservation of a set of designated values. In the mixed case, there are pairs of sets of designated values and logical consequence is not understood in terms of the preservation of the members of any of them. Instead, the idea is that if all of the premises receive values in the set of designated values for premises, then some of the conclusions must receive a truth-value in the set of the designated values for conclusions. We take the following definitions from [6].

Definition 4.1. A *mixed consequence relation* is a relation noted $\vDash_{\mathcal{D}_p,\mathcal{D}_c}$, such that $\Gamma \vDash_{\mathcal{D}_p,\mathcal{D}_c} \Delta$ if and only if for every valuation v, $v(\gamma) \notin \mathcal{D}_p$ for some $\gamma \in \Gamma$ or $v(\delta) \in \mathcal{D}_c$ for some $\delta \in \Delta$.

So, depending on how we choose the sets $\mathcal{D}_p, \mathcal{D}_c$ we can make further distinctions.

Definition 4.2. We will call a mixed consequence relation *pure* if and only if $\mathcal{D}_p = \mathcal{D}_c$. Otherwise, we'll call it *impure*.

Also, we will focus not simply on mixed consequence relations, but on the intersection between them.

Definition 4.3. An *intersective mixed consequence relation* $\vDash_{\mathcal{D}_p^1,\mathcal{D}_c^1 \,\cap\, \dots \,\cap\, \mathcal{D}_p^n,\mathcal{D}_c^n}$ is an intersection of mixed consequence relations (i.e., for each i, $\vDash_{\mathcal{D}_p^i,\mathcal{D}_c^i}$ is a mixed consequence relation).

So, these definitions are quite general. In particular, usually a set of designated values is not any subset of the set of semantic values, but an upset. In the 3-element case, using the order $\mathbf{f} \leq \mathbf{u} \leq \mathbf{t}$, we could think as a set of designated values a set including $\mathbf{t}$ and not including $\mathbf{f}$.[4] However, here we are not restricting ourselves to this condition, since the consequence relations that do not respect it show very interesting properties. In view of this, the plan of the section is as follows. In the Section 4.1 we will present the logics that respect this condition. Next, in Section 4.2 we will consider some consequence relations that do not obey this restriction. In both sections, we will compare the resulting consequence relations over each of the algebras previously discussed.

4.1 Restricting the sets of designated values

When working with a 3-element algebra that has the 2-element Boolean algebra as a subalgebra, and assuming that the sets of designated values contain $\mathbf{t}$ but not $\mathbf{f}$, one can consider two possible sets of designated values $s = \{\mathbf{t}\}$ and $t = \{\mathbf{t}, \mathbf{u}\}$. With this in mind, it is immediate to define four mixed consequence relations. In terms of [7], two pure and two impure: $\vDash_{s,s}$, $\vDash_{t,t}$, $\vDash_{s,t}$ and $\vDash_{t,s}$, or more simply ss, tt, st, and ts consequence relations—referred to in this way for obvious reasons. The first two of these are usually understood as the preservation of truth, and preservation of non-falsity respectively.

Let us now proceed to inspect the four different logics resulting from considering these consequence relations over the weak Kleene algebra **WK**. To begin with, we can first focus on the logic induced by the ss and tt consequence relations over this algebra. these logics already quite well-known and have received a lot of attention at least in the past years. Respectively, they are called the paracomplete weak Kleene logic $\mathsf{K}_3^{\mathsf{w}}$ and the paraconsistent weak Kleene logic PWK—see, e.g., [22], [20], [8], [10].

These systems have quite distinctive features. The paracomplete one is such that the Law of Excluded Middle fails in it, as well as the rather intuitive rule of Addition, or Disjunction Introduction. In a sort of dual manner, the

[4]Actually, in [6] the authors consider the set of designated values in this way.

paraconsistent one is such that Explosion is invalid in it, while also the highly intuitive rule of Simplification, or Conjunction Elimination is.

$$\psi \nvDash^{\mathbf{WK}}_{ss} \varphi \vee \neg\varphi \quad \varphi \nvDash^{\mathbf{WK}}_{ss} \varphi \vee \psi \quad \varphi \wedge \neg\varphi \nvDash^{\mathbf{WK}}_{tt} \psi \quad \varphi \wedge \psi \nvDash^{\mathbf{WK}}_{tt} \varphi$$

In fact, it is known that these features can be rather seen as emerging symptoms of a deeper comprehension of these logics. A number of different scholars have worked out exhaustive characterizations of the set of valid inferences of these logics, which make it easier to understand what holds in them—for references, see [33], [8], [9], [10], [23].

$$\varphi \vDash^{\mathbf{WK}}_{ss} \psi \textit{ iff } \begin{cases} \varphi \vDash_{\mathsf{CL}} \psi \text{ and } Var(\psi) \subseteq Var(\varphi), \text{ or} \\ \varphi \vDash_{\mathsf{CL}} \chi \text{ for all } \chi \end{cases}$$

$$\varphi \vDash^{\mathbf{WK}}_{tt} \psi \textit{ iff } \begin{cases} \varphi \vDash_{\mathsf{CL}} \psi \text{ and } Var(\varphi) \subseteq Var(\psi), \text{ or} \\ \chi \vDash_{\mathsf{CL}} \varphi \text{ for all } \chi \end{cases}$$

Turning now to the properly mixed consequence relations defined over the weak Kleene algebra, it is important to notice that the logics induced by these considerations have not received a similar amount of attention in the specialized literature. To be more precise, the logic induced by considering the *st* consequence relation over this algebra only is a subject of debate in less than a handful of articles, while the logic induced by considering the *ts* consequence relation has not been subject to discussion at all. This being said, let us proceed to examine some of the interesting properties of these systems.

Starting with the *st* consequence relation leads us to consider the logic dubbed weak ST or wST in the recent literature—see [32]. Before going into its characteristic features, it is important to observe that the set of valid inferences on this logic is exactly the set of valid inferences of Classical Logic. A proof of this fact can be easily grasped by noticing the following two facts. Firstly, if an inference $\Gamma \vDash \Delta$ has a counterexample in wST then it is classically invalid. The reason for this is that a **WK**-valuation v is a counterexample of $\Gamma \vDash \Delta$ if and only if $v(\gamma) = \mathbf{t}$ for every $\gamma \in \Gamma$ and $v(\delta) = \mathbf{f}$ for every $\delta \in \Delta$. Since the value $\mathbf{u}$ is infectious and the **WK**-valuations restricted to the classical inputs behave as the classical valuations, it is easy to see that this valuation can be transformed into a classical counterexample to the same inference. On the other hand, if an inference $\Gamma \vDash \Delta$ has a classical counterexample, there must be a classical valuation v such that $v(\gamma) = \mathbf{t}$ for every $\gamma \in \Gamma$ and $v(\delta) = \mathbf{f}$ for every $\delta \in \Delta$. By inspecting the algebra **WK** it is easy to check that this valuation is a counterexample to the same inference in wST.

Be that as it may, it thus seems this logic has no particular quirks in what pertains to its set of valid inferences, whence we may seek for some of its peculiarities in higher inferential levels—particularly, at the metainferential level. Metainferences are inferences between inferences themselves, and as such can be regarded as valid or invalid in a system—see, e.g., [27], [2]. In what follows to evaluate whether a metainference is valid or not we will use the so-called *local metainferential validity* standard discussed for example in [12] and [2], defined as follows:

Definition 4.4. A metainference is locally valid in a given logic L if and only if every valuation v which is a counterexample of the conclusion is also a counterexample of some of the premises.

So, as the notion of counterexample depends on the logic L, different logics can locally validate different metainferences. In this regard, as in the case of the system induced by taking the *st* consequence relation over the strong Kleene algebra, the rule of Cut is also invalid in the infectious logic wST:

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

This can be observed by taking a formula φ receiving the intermediate value $\mathbf{u}$. Furthermore, it is immediate to observe that the two elimination metainferences akin to the ones invalid in PWK are invalid in wST. These are the left disjunction elimination metainferences, and the right conjunction elimination metainference.[5]

$$\frac{\Gamma, \varphi \vee \psi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta} \quad \frac{\Gamma, \varphi \vee \psi \Rightarrow \Delta}{\Gamma, \psi \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}{\Gamma \Rightarrow \varphi, \Delta} \quad \frac{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}{\Gamma \Rightarrow \psi, \Delta}$$

[5] Any **WK**-valuation such that $v(\varphi) = \mathbf{t}$ and $v(\psi) = \mathbf{u}$ is a counterexample to the first metainference, and any valuation such that $v(\varphi) = \mathbf{u}$ and $v(\psi) = \mathbf{t}$ is a counterexample to the second one. Similarly, it's easy to find counterexamples to the third and the fourth ones.

We will comment on the fact that this may happen as the result of a deeper characterization of the notion of metainferential validity for wST, at the end of this section.

Moving on now to the logic induced by considering the *ts* consequence relation over the weak Kleene algebra, resulting in a system we call wTS, we must stress the number of caveats. First of all, in the case of the logic induced by considering these consequence relation over the strong Kleene algebra, called TS in [11], the resulting system has no inferential validities. These can be easily noticed by pointing out the fact that the valuation giving the intermediate value to every formula is a counterexample to every inference formulated in the language. This further implies the fact that the rule of Reflexivity is invalid in this system. Again, the standard for assessing the validity of metainferences is the local one.

$$\varphi \Rightarrow \varphi$$

On top of these considerations, it is interesting to note that further metainferences are also invalid in this system. To wit, the right disjunction introduction metainference, as well as the left conjunction introduction metainference are invalid in this logic.[6]

$$\frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \lor \psi, \Delta} \quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \land \psi \Rightarrow \Delta}$$

Once more, we will comment on the fact that this may happen as the result of a deeper characterization of the notion of metainferential validity for wTS, connecting it to $\mathsf{K}_3^{\mathsf{w}}$, at the end of this section.

Let us turn our attention to the different logics that can be defined over the Sobociński algebra **S**, by considering the four consequence relations previously mentioned. As far as we know, some of these logics were not considered so far in the literature. So, first, let's begin with the pure consequence relations, *ss* and *tt*. To remark their salient features, we can call them respectively the paracomplete Sobociński logic So_3 and the paraconsistent Sobociński PSo_3.

As in the weak Kleene systems, of course, in the paracomplete case the Law of Excluded Middle fails, and in the paraconsistent one, the rule of

[6]To see this, any **WK**-valuation such that $v(\varphi) = \mathbf{t}$ and $v(\psi) = \mathbf{u}$ is a counterexample to the first metainference. Also, any **WK**-valuation such that $v(\varphi) = \mathbf{f}$ and $v(\psi) = \mathbf{u}$ is a counterexample to the second one.

Explosion fails. However, contrary to what happens with the weak Kleene case, it is in the paraconsistent logic in which the rule of Addition fails, whereas in the paracomplete case the rule of Simplification does not hold[7]:

$$\psi \nvDash^{\mathbf{S}}_{ss} \varphi \vee \neg\varphi \quad \varphi \wedge \psi \nvDash^{\mathbf{S}}_{ss} \varphi \quad \varphi \wedge \neg\varphi \nvDash^{\mathbf{S}}_{tt} \psi \quad \varphi \nvDash^{\mathbf{S}}_{tt} \varphi \vee \psi$$

So, one could be inclined to think that as in the weak Kleene logics, these systems characterize some syntactically interesting fragment of Classical Logic. For instance, from the inspection of the above examples, one could think that $\mathsf{So_3}$ and $\mathsf{PSo_3}$ are defined by the following:

$$\varphi \vDash^{\mathbf{S}}_{ss} \psi \textit{ iff } \begin{cases} \varphi \vDash_{\mathsf{CL}} \psi \text{ and } Var(\varphi) \subseteq Var(\psi), \text{ or} \\ \varphi \vDash_{\mathsf{CL}} \chi \text{ for all } \chi \end{cases}$$

$$\varphi \vDash^{\mathbf{S}}_{tt} \psi \textit{ iff } \begin{cases} \varphi \vDash_{\mathsf{CL}} \psi \text{ and } Var(\psi) \subseteq Var(\varphi), \text{ or} \\ \chi \vDash_{\mathsf{CL}} \varphi \text{ for all } \chi \end{cases}$$

However, the following inference: $\psi \wedge \varphi \vDash \varphi \vee \neg\psi$ is a counterexample to this conjecture, since it is invalid both in $\mathsf{So_3}$ and $\mathsf{PSo_3}$, but nonetheless is classically valid and the propositional variables in the premise and in the conclusion are the same (take any $\mathbf{S}$-valuation v, such that $v(\varphi) = \mathbf{u}$ and $v(\psi) = \mathbf{t}$). Of course, this does not imply that a syntactic characterization of these systems is not possible, but only that we are not aware of it.

Let's move on to the mixed consequence relations defined over $\mathbf{S}$. The first one to consider is the *st* consequence relation, which we can call sST. It's worth noticing that contrary to the weak Kleene case, the set of valid inferences of this logic is a proper subset of the set of classically valid inferences. So, although it's easy to check that the following inferences are valid in sST:

$$\psi \vDash^{\mathbf{S}}_{st} \varphi \vee \neg\varphi \quad \varphi \wedge \psi \vDash^{\mathbf{S}}_{st} \varphi \quad \varphi \wedge \neg\varphi \vDash^{\mathbf{S}}_{st} \psi \quad \varphi \vDash^{\mathbf{S}}_{st} \varphi \vee \psi$$

the following for instance are not[8]:

[7]Any $\mathbf{S}$-valuation such that $v(\varphi) = \mathbf{u}$ and $v(\psi) = \mathbf{t}$ is a counterexample to the first and the second metainferences, and any $\mathbf{S}$-valuation such that $v(\varphi) = \mathbf{u}$ and $v(\psi) = \mathbf{f}$ is a counterexample to the third and the fourth ones.

[8]To see this, take a $\mathbf{S}$-valuation v, such that $v(\varphi) = \mathbf{u}$, $v(\chi) = \mathbf{t}$ and $v(\psi) = \mathbf{f}$.

$$\chi \wedge (\varphi \wedge \neg\varphi) \not\models^{\mathbf{S}}_{st} \psi \quad \chi \models^{\mathbf{S}}_{st} \psi \vee (\varphi \vee \neg\varphi)$$

And thus, sST does not coincide with Classical Logic. However, at the level of the metainferences there is some symmetry to the weak Kleene case. First, it's trivial to note that since the logic is non-transitive (as wST) the following metainference is invalid:

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

However, it is worth noticing that the elimination metainferences which were invalid in wST (and valid in wTS) are valid in sST. And surprisingly, the introduction metainferences (which are invalid in wTS, and valid in wST) are also invalid in this logic[9]:

$$\frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta} \quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}$$

As we said before, below we will comment on the fact that this may happen as the result of a deeper characterization of the notion of metainferential validity for sST.

Let's consider now the logic induced by taking the *ts* consequence relation on the Sobociński algebra. This logic, which we will call sTS was never considered in the literature. It's easy to check that as in the wTS, the logic sTS has no valid inferences. In particular, reflexivity, or identity, fails:

$$\varphi \Rightarrow \varphi$$

But, what about the metainferences of this logic? Here we have a similar situation as the described for the case of sST. The following metainferences which are wST invalid (but wTS valid) are sTS invalid[10]:

$$\frac{\Gamma, \varphi \vee \psi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta} \quad \frac{\Gamma, \varphi \vee \psi \Rightarrow \Delta}{\Gamma, \psi \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}{\Gamma \Rightarrow \varphi, \Delta} \quad \frac{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}{\Gamma \Rightarrow \psi, \Delta}$$

[9] The valuation v such that $v(\varphi) = v(\psi) = \mathbf{u}$, $v(\varphi \wedge \psi) = \mathbf{t}$ and $v(\varphi \vee \psi) = \mathbf{f}$ is a counterexample to both of them.

[10] The valuation v, $v(\varphi \vee \psi) = \mathbf{f}$, and $v(\varphi) = \mathbf{u}$ is a counterexample to the first one. Dually, the valuation v, $v(\varphi \vee \psi) = \mathbf{f}$, and $v(\psi) = \mathbf{u}$ is a counterexample to the second one. The valuation v, $v(\varphi \wedge \psi) = \mathbf{t}$, and $v(\varphi) = \mathbf{u}$ is a counterexample to the third one. And lastly, the valuation v, $v(\varphi \wedge \psi) = \mathbf{t}$, and $v(\psi) = \mathbf{u}$ is a counterexample to the fourth one.

So, the metainferences that represent elimination rules for the connectives are invalid. Is this a symptom of a greater ongoing phenomenon? Let's discuss this not only as pertains to sTS, but all the other *st* and *ts*-based logics.

Some very interesting observations have been made in the recent couple of years by researchers investigating the *st* consequence relation over three-valued logics, and some highly plausible conjectures have been concomitantly floated regarding the *ts* consequence relation over similar systems. These observations concern the characterization of those metainferences valid in the *st* and *ts* systems. By this we mean, description of the sufficient and necessary conditions for a metainference to be valid in these systems, in terms of conditions which do not make reference to their obvious definitions. Some works even refer to these characterizations as representation theorems of sorts.

The stepping point for these reflections is the discussion of the set of valid metainferences of the logic induced by taking the *st* notion of logical consequence over the strong Kleene algebra—thus rendering the system usually known as ST. Researchers—among them, saliently [3], [2], and [12]—have pointed out that the set of valid metainferences of this logic can be characterized, via some appropriate translations, in terms of the set of valid inferences of the logic induced by taking the *tt* notion of logical consequence over the strong Kleene algebra. In the wake of these results, some have conjectured that there is a similar relation between the set of valid metainferences of the logic induced by taking the *ts* notion of logical consequence over the strong Kleene algebra (thus rendering the system usually known as TS) and the set of valid inferences of the logic induced by taking the *ss* notion of logical consequence over the set algebra—although no exhaustive proof is found in the literature.

Given these results, one may conjecture that the logics induced by applying these standards for logical consequence over there weak Kleene algebra and, respectively, the Sobociński algebra will be similarly related. To this extent, we may confirm that such a connection between wST and PWK has been recently established in [25]. Whether it is possible to produce similar proofs for the case of the remaining systems is as of today, yet to be determined.

4.2 Relaxing the sets of designated values

In the last section, we have defined some mixed consequence relations over the weak Kleene and the Sobociński algebra, and we have remarked some of their salient properties. In particular, we have shown some similarities and differences between the logics based on these two algebras. One of the main differences consists in the fact that logics based on the weak Kleene algebra have shown to be especially suitable to formulate or to capture some fragments of Classical Logic—for instance, as we have seen, this is true for $\vDash^{\mathbf{WK}}_{ss}$ and $\vDash^{\mathbf{WK}}_{tt}$. However, we have seen that $\vDash^{\mathbf{S}}_{ss}$ and $\vDash^{\mathbf{S}}_{tt}$ are not adequate in the same way.[11]

In this section, we will show that using some mixed consequence relations other than those reviewed in the last section we can obtain a positive result regarding the notion of (logical) *subject-matter preservation* and, furthermore, of *content inclusion*. In particular, we will show that by relaxing the restriction for the sets of designated values of containing $\mathbf{t}$ and not containing $\mathbf{f}$, we can define logics over the weak Kleene and the Sobociński algebras which allow us to capture exactly the valid inferences of any logic that respect content inclusion from premises to conclusions, and a similar phenomenon from conclusions to premises.

For the purpose of this analysis, we will have in mind an account of content inclusion due to Yablo in [34, p. 1-4]. He argues that content inclusion as a relation between propositions is characterized in terms of the joint satisfaction of truth preservation and subject-matter preservation between them. Granting the assumption that the (logical) subject-matter of a proposition is to be formally represented by the set of propositional variables appearing in it, these considerations can have a formal life of their own. In fact, subject-matter preservation from φ to ψ could be represented by the claim that $Var(\psi) \subseteq Var(\varphi)$—otherwise known as Parry's Proscriptive Principle, for which see [26].[12] Furthermore, since paradigmatically the notion of logical consequence incarnated in Classical Logic can be understood in terms of truth preservation, the joint satisfaction of classical entailment and the Pro-

[11] In the sense that, as we mentioned, for instance the following inference $\psi \wedge \varphi \vDash \varphi \vee \neg\psi$ is invalid both in So_3 and PSo_3, although it is classically valid and premises and conclusion share the propositional variables.

[12] By no means do we intend to claim that variable-sharing or variable-inclusion is a necessary or sufficient condition for relevance. We only wanted to report that some researchers thought that variable-inclusion could be a guide, and perhaps even a necessary condition, for the formalization of the notion of content inclusion from premises to conclusion.

scriptive Principle characterizes a Yablovian approach to *content inclusion* in Classical Logic. However, this can be further generalized. If the notion of logical consequence of a given logic L can be allegedly understood in terms of truth preservation, content inclusion in L could be characterized as follows: $\varphi \vDash^{\mathsf{L}} \psi$ and $Var(\psi) \subseteq Var(\varphi)$. Although these philosophical remarks do not straightforwardly carry over to the case of subject-matter preservation from conclusion to premises, the previous reflections are interesting enough for similar and related systems to be intriguing in their own right.

Let us now go ahead and see how all these relates to infectious and immune logics. So, let's first define the following two sets of truth-values: $d = \{\mathbf{t}, \mathbf{f}\}$, and $\overline{d} = \{\mathbf{u}\}$. With these two sets we can define new mixed consequence relations: $\vDash_{d,d}$, $\vDash_{\overline{d},\overline{d}}$, $\vDash_{d,\overline{d}}$,$\vDash_{\overline{d},d}$, or more simply dd, $\overline{dd}$, $d\overline{d}$ and $\overline{d}d$ consequence relations. In what follows we will not consider the consequence relations $d\overline{d}$ and $\overline{d}d$ defined over both algebras, because they result in empty logics.[13] Regarding the other consequence relations we can obtain the following facts:

Fact 4.5. *Variable inclusion can be defined over weak Kleene as follows:*

$$\varphi \vDash^{\mathbf{WK}}_{dd} \psi \quad \textit{iff} \quad Var(\psi) \subseteq Var(\varphi)$$

$$\varphi \vDash^{\mathbf{WK}}_{\overline{dd}} \psi \quad \textit{iff} \quad Var(\varphi) \subseteq Var(\psi)$$

Proof. We first prove that: $\varphi \vDash^{\mathbf{WK}}_{dd} \psi$ iff $Var(\psi) \subseteq Var(\varphi)$.

If: By reductio, assume $Var(\psi) \subseteq Var(\varphi)$ but $\varphi \nvDash^{\mathbf{WK}}_{dd} \psi$. Then, there is a **WK**-valuation v, such that $v(\varphi) \in \{\mathbf{t}, \mathbf{f}\}$ but $v(\psi) = \mathbf{u}$. It's straightforward to prove by induction that if $v(\psi) = \mathbf{u}$ there must be some propositional variable $p \in Var(\psi)$ such that $v(p) = \mathbf{u}$. But since $Var(\psi) \subseteq Var(\varphi)$, hence $p \in Var(\varphi)$, and therefore because of the infectiousness of $\mathbf{u}$, $v(\varphi) = \mathbf{u}$, which contradicts our initial assumption.

[13] In the case of $\vDash^{\mathbf{WK}}_{d\overline{d}}$ and $\vDash^{\mathbf{S}}_{d\overline{d}}$ these consequence relations are empty since any valuation which assigns to every propositional variable a classical value will be a counterexample of any inference, since no formula will receive the intermediate value (and therefore satisfy the standard for conclusions). On the other hand, $\vDash^{\mathbf{WK}}_{\overline{d}d}$ and $\vDash^{\mathbf{S}}_{\overline{d}d}$ are empty consequence relations, since the valuation which assigns to every propositional variable the intermediate value is a counterexample for any inference (since no formula receives a classical value in this valuation).

Only if: By reductio, assume $\varphi \vDash_{dd}^{\mathbf{WK}} \psi$ but $Var(\psi) \nsubseteq Var(\varphi)$. Hence, there must be some propositional letter p, such that $p \in Var(\psi)$ but $p \notin Var(\varphi)$. Now, take any **WK**-valuation v, such that $v(q) \in \{\mathbf{t}, \mathbf{f}\}$, for every $q \in Var(\varphi)$ and $v(p) = \mathbf{u}$. Since all of the operations of **WK** are such that the output of classical inputs is a classical input and $\mathbf{u}$ is an infectious element, it's easy to check that the valuation v will be a counterexample to $\varphi \vDash_{dd}^{\mathbf{WK}} \psi$, which contradicts what we have assumed.

Following this, we prove that: $\varphi \vDash_{\underline{dd}}^{\mathbf{WK}} \psi$ iff $Var(\varphi) \subseteq Var(\psi)$.

If: By reductio, assume $Var(\varphi) \subseteq Var(\psi)$ but $\varphi \nvDash_{\underline{dd}}^{\mathbf{WK}} \psi$. Hence, there is a **WK**-valuation v, such that $v(\varphi) = \mathbf{u}$ but $v(\psi) \in \{\mathbf{t}, \mathbf{f}\}$. Then, there must be some propositional letter $p \in Var(\varphi)$, such that $v(p) = \mathbf{u}$ but because of the infectiousness of $\mathbf{u}$, $v(q) \in \{\mathbf{t}, \mathbf{f}\}$, for every $q \in Var(\psi)$. So $p \notin Var(\psi)$, which contradicts our initial assumption.

Only if: By reductio, assume $\varphi \vDash_{\underline{dd}}^{\mathbf{WK}} \psi$ but $Var(\varphi) \nsubseteq Var(\psi)$. Hence, there must be some propositional letter p, such that $p \in Var(\varphi)$ but $p \notin Var(\psi)$. Now, take any **WK**-valuation v, such that $v(q) \in \{\mathbf{t}, \mathbf{f}\}$, for every $q \in Var(\psi)$ and $v(p) = \mathbf{u}$. Since all of the operations of **WK** are such that the output of classical inputs is a classical input and $\mathbf{u}$ is an infectious element, it's easy to check that the valuation v will be a counterexample to $\varphi \vDash_{\underline{dd}}^{\mathbf{WK}} \psi$, which contradicts what we have assumed.

□

Fact 4.6. *Variable inclusion can be defined over the Sobociński's algebra as follows:*

$$\varphi \vDash_{dd}^{\mathbf{S}} \psi \quad \textit{iff} \quad Var(\varphi) \subseteq Var(\psi)$$

$$\varphi \vDash_{\underline{dd}}^{\mathbf{S}} \psi \quad \textit{iff} \quad Var(\psi) \subseteq Var(\varphi)$$

Proof. We first prove that: $\varphi \vDash_{dd}^{\mathbf{S}} \psi$ iff $Var(\varphi) \subseteq Var(\psi)$.

If: By reductio, assume $Var(\varphi) \subseteq Var(\psi)$ but $\varphi \nvDash_{dd}^{\mathbf{S}} \psi$. Then, there must be a **S**-valuation v, such that $v(\varphi) \in \{\mathbf{t}, \mathbf{f}\}$ but $v(\psi) = \mathbf{u}$. It's easy to check that if $v(\psi) = \mathbf{u}$ then $v(q) = \mathbf{u}$, for every propositional variable $q \in Var(\psi)$. But since $Var(\varphi) \subseteq Var(\psi)$, $v(p) = \mathbf{u}$ for every $p \in Var(\varphi)$, and therefore $v(\varphi) = \mathbf{u}$, which contradicts our initial assumption.

Only if: By reductio, assume $\varphi \vDash^{\mathbf{S}}_{dd} \psi$ but $Var(\varphi) \nsubseteq Var(\psi)$. Hence, there must be some propositional letter p, such that $p \in Var(\varphi)$ but $p \notin Var(\psi)$. Now, take any **S**-valuation v, such that $v(q) = \mathbf{u}$, for every $q \in Var(\psi)$ and $v(p) \in \{\mathbf{t}, \mathbf{f}\}$. Because of the immune behavior of $\mathbf{u}$, it's straightforward to check that the valuation v will be a counterexample to $\varphi \vDash^{\mathbf{S}}_{dd} \psi$, which contradicts what we have assumed.

Following this, we prove that: $\varphi \vDash^{\mathbf{S}}_{\overline{dd}} \psi$ iff $Var(\psi) \subseteq Var(\varphi)$.

If: By reductio, assume $Var(\psi) \subseteq Var(\varphi)$ but $\varphi \nvDash^{\mathbf{S}}_{\overline{dd}} \psi$. Hence, there is a **S**-valuation v, such that $v(\varphi) = \mathbf{u}$ but $v(\psi) \in \{\mathbf{t}, \mathbf{f}\}$. Then, because of the properties of the immune element in **S**, $v(p) = \mathbf{u}$ for every propositional letter $p \in Var(\varphi)$. But since $Var(\psi) \subseteq Var(\varphi)$, $v(q) = \mathbf{u}$ for every propositional letter $q \in Var(\psi)$. Hence, $v(\psi) \notin \{\mathbf{t}, \mathbf{f}\}$, which contradicts our initial assumption.

Only if: By reductio, assume $\varphi \vDash^{\mathbf{S}}_{\overline{dd}} \psi$ but $Var(\psi) \nsubseteq Var(\varphi)$. Hence, there must be some propositional letter p, such that $p \in Var(\psi)$ but $p \notin Var(\varphi)$. Now, take any **S**-valuation v, such that $v(q) = \mathbf{u}$, for every $q \in Var(\varphi)$ and $v(p) \in \{\mathbf{t}, \mathbf{f}\}$. Now, it's straightforward to check that this valuation v will be a counterexample to $\varphi \vDash^{\mathbf{S}}_{\overline{dd}} \psi$, which contradicts what we have assumed.

□

So, the valid inferences in each of these logics are exactly those that respect the corresponding variable inclusion. Thus, we have a semantic characterization of this syntactic property. With this in place it's evident that the intersective mixed consequence relation between one of these logics and a logic L will characterize exactly the fragment of L which respect the corresponding variable inclusion. This is what the following corollary claim:

Corollary 4.7. *Given a logic* L*, its set of valid inferences such that* $Var(\psi) \subseteq Var(\varphi)$ *coincides with the intersective mixed consequence relation:* $\vDash^{\mathbf{S}}_{\overline{dd}} \cap \vDash^{\mathsf{L}}$*, and with* $\vDash^{\mathbf{WK}}_{\overline{dd}} \cap \vDash^{\mathsf{L}}$*.*

Corollary 4.8. *Given a logic* L*, its set of valid inferences such that* $Var(\varphi) \subseteq Var(\psi)$ *coincides with the intersective mixed consequence relation:* $\vDash^{\mathbf{S}}_{dd} \cap \vDash^{\mathsf{L}}$*, and with* $\vDash^{\mathbf{WK}}_{\overline{dd}} \cap \vDash^{\mathsf{L}}$*.*

In these cases the proofs are immediate from the Facts 4.5 and 4.6. In particular, since in the last section we have seen that the only thing coming between $\vDash_{ss}^{\mathbf{WK}}$ and the fragment of Classical Logic that respects subject-matter preservation from premise to conclusion are the antitheorems, and the relations $\vDash_{dd}^{\mathbf{WK}}$ and $\vDash_{dd}^{\mathbf{S}}$ semantically characterize subject-matter preservation from premise to conclusion, it is easy to see that there intersective consequence relations $\vDash_{ss}^{\mathbf{WK}} \cap \vDash_{dd}^{\mathbf{WK}}$ and $\vDash_{ss}^{\mathbf{WK}} \cap \vDash_{dd}^{\mathbf{S}}$ characterize the desired fragment of Classical Logic.

Similarly, since in the last section we have seen that the only thing coming between $\vDash_{tt}^{\mathbf{WK}}$ and the fragment of Classical Logic that respects subject-matter preservation from conclusion to premise are the theorems, and the relations $\vDash_{dd}^{\mathbf{WK}}$ and $\vDash_{dd}^{\mathbf{S}}$ semantically characterize subject-matter preservation from conclusion to premise, it is easy to see that there intersective consequence relations $\vDash_{tt}^{\mathbf{WK}} \cap \vDash_{dd}^{\mathbf{WK}}$ and $\vDash_{tt}^{\mathbf{WK}} \cap \vDash_{dd}^{\mathbf{S}}$ characterize the desired fragment of Classical Logic.

Finally, it can be further observed that the intersection of any of these intersective consequence relations leads to the fragment of Classical Logic where all the propositional values of the premises are equal to those of the proposition variables in the conclusions, which can be seen as the fragment of Classical Logic that respects subject-matter equality between premises and conclusion. This will give a semantic characterization of this system in terms of the intersection of consequence relations, which is independent of other characterizations that have been given in the literature, e.g., in [9].

5 Conclusion

In this article we have introduced and discussed a family of logics that we referred to as immune logics—whose properties can be seen as related, in a way, to the infectious logics. We noticed that immune logics are characterized by containing a truth-value that behaves as an all-purpose identity element for all the binary operations of the underlying algebra, and which is also idempotent for all unary operations. In the 3-element case, we have called the algebra uniquely determined by these ideas that extends the 2-element Boolean algebra, the Sobociński algebra. Also, we have shown that these immune systems are adequate to characterize some variable-inclusion fragments of any logic, in particular, by taking the intersection of some mixed consequence relations defined over the Sobociński algebra.

In these explorations, we have sadly left out some very interesting topics. Firstly, it would be interesting to consider algebras with more than three truth-values, containing not only an immune truth-value but also a truth-value behaving in an infectious way. In this vein, it seems that the works by Paoli [23] and Szmuc [29, 30] can be seen as witnesses of the fruitfulness of considering systems of these sorts. Secondly, we have defined immune elements for binary operations, and we have shown the problems of extending it to n-ary operations, when $n \geq 3$. We leave open whether these problems can be solved and we hope to investigate them in future work.

Finally, it could also be interesting to consider some other systems related to infectious logics, which could equally deserve the title of immune logics—although for different reasons. Instead of considering algebras that have a truth-value behaving in a value-in-value-out fashion, we could consider for instance structures containing a truth-value that behaves in a *value-in-different-value-out* manner. These may require considering different algebras extending the 2-element Boolean algebra, and it may or may not be possible to collect them altogether in a non-deterministic set of truth-tables, or a non-deterministic algebra. We hope to investigate these and other related topics in future work.

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