From Collapse Theorems to Proof-Theoretic Arguments

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Abstract

On some views, we can be sure that parties to a dispute over the logic of 'exists' are not talking past each other if they can characterise 'exists' as the only monadic predicate up to logical equivalence obeying a certain set of rules of inference. Otherwise, we ought to be suspicious about the reality of their disagreement. This is what we call a *proof-theoretic argument*. *Pace* some critics, who have tried to use proof-theoretic arguments to cast doubts about the reality of disagreements about the logic of 'exists', we argue that proof-theoretic arguments can be deployed to establish the reality of several such disagreements. Along the way, we will also utilise this technique to establish similar results about some disagreements over the logic of identity.

Keywords: Existence, Identity, Quantification, Collapse Theorems, Verbal Disputes

1 Introduction

Whether or not an expression of the logical vocabulary obeys a certain law is often contentious. Perhaps the most well-known examples concern 'not' and 'if... then...' - does, e.g., 'not' obey Double Negation? But similar contentious issues have arisen, to our knowledge, for virtually every other standard logical expression. So too, there are disagreements about the logic of 'exists'. Such disagreements are the focus of this paper.

When theorising about the logic of 'exists', what needs to be decided, amongst other things, is whether anyone knowing that something has a property can safely infer that it thereby *exists*. What is at stake in this question is a putative principle of the logic of 'exists', dubbed by Williamson (1988) *Existence Principle* (EP), according to which something can have a property only if it exists. That 'exists' obeys EP is moot.

To delineate better the camps of friends and enemies of EP, we thought it useful to borrow from Berto (2013) the labels of *Parmenidean* and *Meinongians* theorists. The former, in line with the Quinean ruling that the ontological question - 'what is there?' (1948: 21) - is 'everything', have typically found EP an obviously correct principle of 'exists'. Meinongian theorists, by contrast, are in disagreement with Quine on his ruling about the ontological question. Many of them, unsurprisingly, have found that EP incurs counterexamples.

Of course, a number of distinctions need to be drawn within each camp. Thus for example, some Parmenideans have an account of 'exists' that is liberal enough to encompass things like Zeus, Vulcan and their ilk in its extension (Van Inwagen (1977), Braun (2005)). Other Parmenideans will disagree. Similarly, Meinongians of the positive free logic tradition have maintained that quantification always has an existential import (Leblanc and Thomason (1972), Bencivenga (2002: §7)). Meinongians of the noneist tradition, by contrast, have rejected a similar view (Routley (1980), Priest (2016), Berto (2013)).

Although it is important to be aware of these differences amongst Parmenideans and Meinongians, they will not particularly concern us here. All that matters for our purposes is that the validity of EP is not uncontroversial. And as such, at least *prima facie*, there would seem to be worthwhile debates to be had about the validity of EP. But though intuitive, this claim too is not beyond dispute. Indeed, Williamson (1988) has found it false: on Williamson's view, there *cannot* be genuine disagreements over the validity of EP. Williamson derives this conclusion from the alleged impossibility for proponents and opponents of EP to characterise 'exists' as the only monadic predicate up to logical equivalence obeying a certain set of rules of inference; a fact which, for Williamson, indicates that what one party means by 'exists' is not equivalent to what the other does. This is a concrete example of what we will call hereafter *proof-theoretic argument*.

This paper shows that Williamson's proof-theoretic argument concerning disagreements about EP fails: given Williamson's standards for real disagreement, and contrary to what Williamson predicted, *there are* genuine disagreements about EP. For, it is shown in §4 that if proponents and opponents of EP accept the equivalence of existence and self-identity, they can characterise 'exists' as Williamson demands. What is more, it is shown in §5 that, in these disagreements, the parties will also be able to prove an analogous result for identity. The next two sections illustrate the applicability of proof-theoretic arguments to the present (§3) and other (§2) disputes. Over the course of the paper, we will introduce three logical systems; the soundness of which is established in a **Technical Appendix**.

2 Background

On Williamson's telling, proof-theoretic arguments are a "technique for arguing that an apparent conflict is a real one" (1988: 110). They take off from results, known at least since Carnap (1943), which often go by the name of *collapse theorems*. Consider two logics, $\mathcal{L}_1$ and $\mathcal{L}_2$, differing only in that $\mathcal{L}_1$ includes a logical constant c_1 in its vocabulary and $\mathcal{L}_2$ a logical constant c_2, possibly obeying different rules of inference. Consider then a logic $\mathcal{L}_3$ whose vocabulary includes both c_1 and c_2. A collapse theorem for c_1 and c_2, in $\mathcal{L}_3$, is a proof that they are deductively equivalent. That is, let A be a formula containing some occurrence of c_1 and let B be the result of replacing every occurrence of c_1 in A with c_2; c_1 and c_2 are deductively equivalent just in case $A \leftrightarrow B$ is provable in $\mathcal{L}_3$. In that case, c_1 and c_2 are said to collapse.

With this in mind, an example of the effectiveness of proof-theoretic arguments, considered by Williamson (1988: 110-114), concerns the disagreement, between classical and intuitionist logicians, about whether negation obeys Double Negation (DN). On the face of it, the parties here would seem to express genuinely contradictory views. But could it not be that, as Quine (1970: 81) would have it, their disagreement turns in fact on an equivocation about the meaning of the word 'not', meaning one thing in the classicist's mouth, and another in the intuitionist's? A proof-theoretic argument, says Williamson, will rule this possibility out.

First, as Harris (1982) showed, in a system of natural deduction with two negation operators, one classical ($\neg$) and one intuitionist ($\rightharpoondown$), $\neg A \leftrightarrow \rightharpoondown A$ becomes provable, for any formula A. Thus, $\neg$ and $\rightharpoondown$ collapse. To get the proof-theoretic argument going, we now ought to ask ourselves whether: (i) there are rules of inference governing both $\neg$ and $\rightharpoondown$, and (ii) whether such rules could allow classical and intuitionist logicians to characterise negation as *the unique operator, up to logical equivalence, obeying those rules*. For a start, the answer to (i) is yes: both $\neg$ and $\rightharpoondown$ obey Ex Falso Quodlibet (EFQ) and the standard Introduction Rule for Negation, N-In. Let A, B be any formulae, and Γ, Δ sets of formulae. A monadic operator $\otimes$ obeys EFQ, N-In and N-El just in case the following two schemata are valid:

$$\frac{A \qquad \otimes A}{B}\ \text{EFQ} \qquad\qquad\qquad \frac{\begin{matrix}\overset{(n)}{A}\\ \vdots \\ \bot\end{matrix}}{\otimes A}\ (n)\ \text{N-In}$$

The vertical dots $\vdots$ indicate a derivation from an assumption or set of assumptions, whereas bracketed numerals (n) are used to mark discharged assumptions and indicate at which point in the derivation they are discharged.

The answer to (ii) is also yes: EFQ and N-In, it turns out, are strong enough to define up to logical equivalence any monadic operator obeying them. For, let $\otimes_1$ and $\otimes_2$ be any two monadic operators obeying those rules. The following derivation establishes the deductive equivalence of $\otimes_1$ and $\otimes_2$: $\vdash \otimes_1 P \leftrightarrow \otimes_2 P$. There is only one monadic operator, up to logical equivalence, obeying EFQ and N-In[1].

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{(1)}{P} \quad \overset{(2)}{\otimes_1 P}}{P}\,\text{EFQ} \qquad \dfrac{\overset{(1)}{P} \quad \overset{(2)}{\otimes_1 P}}{\otimes_2 P}\,\text{EFQ}}{\bot}\,\text{EFQ}}{\otimes_2 P}\,(1)\,\text{N-In}}{\otimes_1 P \rightarrow \otimes_2 P}\,(2) \rightarrow \text{I} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{(3)}{P} \quad \overset{(4)}{\otimes_2 P}}{P}\,\text{EFQ} \qquad \dfrac{\overset{(3)}{P} \quad \overset{(4)}{\otimes_2 P}}{\otimes_1 P}\,\text{EFQ}}{\bot}\,\text{EFQ}}{\otimes_1 P}\,(3)\,\text{N-In}}{\otimes_2 P \rightarrow \otimes_1 P}\,(4) \rightarrow \text{I}}{\otimes_1 P \leftrightarrow \otimes_2 P}\,\leftrightarrow \text{I}$$

As the answer to (i) and (ii) was positve, Williamson (1988: 111) proposes a proof-theoretic argument to the effect that the disagreement about DN, between classical and intuitionist logicians, is real and not merely verbal. If there is only one monadic operator, up to logical equivalence, obeying EFQ and N-In, surely this rules out that classical and intuitionist logicians are talking past each other when disagreeing about whether *it* obeys

[1]Over the course of the derivation, the classical Conditional Introduction rule is applied twice - namely, where assumptions (2) and (4) are discharged. Moreover, the classical Biconditional Introduction rule is applied in the last step of the derivation. Hence, strictly speaking, the result applies to any language *equipped with these rule of inference*.

DN. Either the intuitionist is right and the classicist wrong; or vice versa, the classicist is right and the intuitionist wrong. Either way, there cannot be a logic with two negation operators - one classical, one intuitionist - only one of which obeys DN[2].

This proof-theoretic argument says something important about Williamson's views on logical disagreement. Specifically, that a sufficient condition for deeming real a disagreement about whether a logical constant obeys a rule of inference is the possibility for the parties to characterise such logical constant up to logical equivalence. For our purposes, however, the most interesting claim is the converse. This amounts to the claim that the impossibility for the parties to characterise up to logical equivalence a logical constant is a sufficient condition for deeming any disagreement they may have about it *not real* (but merely verbal). Any two logicians who were to find themselves in a situation of this sort will not be able to come to a principled refutation that their disagreement is merely verbal, unless another method could help them achieve that much. Without any such method, Williamson says,

> any remaining belief in the non-equivocality of the dispute [...] would be little better than blind faith: for although there may well be an initial presumption that we mean the same by same-sounding words, a given instance of such a presumption hardly deserves to survive the failure to find evidence in its favour, if we have looked in earnest. (1988: 119)

Briefly, suppose the parties to a dispute, similar to the one described in this section, cannot establish the reality of their disagreement by a proof-theoretic argument. Absent alternative methods, it would be irrational to believe that they are not talking past each other[3].

What we have sketched just here very much corresponds to Williamson's verdict about disagreements over the validity of the Existence Principle (EP), the putative principle of 'exists' under focus in this paper. As anticipated, we have found that Williamson's conclusion is incorrect. But before presenting our reasons, we will have to present Williamson's reasons in the first place. The next section takes up this task.

3 Failure of Collapse

In this section we will first elaborate a bit on the positions at play over the validity of EP; then, we will get into the detail of Williamson's reasons for thinking that such positions are not genuinely in conflict.

To begin with, here is, formally, how Williamson (1988: 115) formulates EP. Where t is any term (open or closed), P any monadic predicate and $\mathsf{E!}$ a monadic predicate standing for 'exists', EP is the schema:

$$\frac{\mathsf{Pt}}{\mathsf{E!t}}\ \mathsf{EP}$$

[2]See Hossack (1990), Hand (1993), Raatikainen (2008), Murzi & Hjortland (2009) and especially Schechter (2011) for relevant discussion.

[3]Williamson's describing the belief in question as being "little better than blind faith" seems to point to its lack rational basis; whence the irrationality of holding this belief.

Informally, the principle allows one to infer that t exists, provided it enjoys a property expressed by a monadic predicate. More succinctly: having a property is sufficient to exist.

The validity of EP, we said, is a contentious matter. To see why, it is probably helpful to consider a vernacular instance of the principle, such as I1 below:

I1. Vulcan is an object of erroneous scientific theorising. *Therefore*, Vulcan exists.

The premise of I1 attributes a property to Vulcan; specifically, that of being an object of erroneous scientific theorising. Meinongians will typically take this premise at face value whilst rejecting the conclusion of I1: Vulcan is an object of erroneous scientific theorising *and* lacking existence. Consequently, I1 is invalid. Parmenideans by contrast balk at the idea of an object lacking existence. Their view can of course be made compatible with treating the conclusion of I1 as a falsehood, as 'Vulcan' may refer to nothing at all. Presumably then, the premise of the inference should also be taken as false; if 'Vulcan' does not refer to anything, for no property F could we truthfully say that Vulcan is F[4]. Thus, on this account I1 comes out valid, but unsound. Alternatively, Parmenideans may regard the conclusion of I1 as expressing a truth - in that case, a rather generous account of 'exists' would seem to be required. Such Parmenideans will regard the premise of I1 as true as well; the consequence being, that on their telling I1 would be both valid and sound.

The Meinongian reasoning leading to the invalidity of I1 may be seen as ruling out EP as a candidate principle governing 'exists': Vulcan has at least one monadic property, despite being a non-existent. The two Parmenidean lines of reasoning, by contrast, would seem to rule EP in: if no object as such lacks existence, then no object having a monadic property does. Meinongians and Parmenideans would appear to be in stark contrast over the validity of EP.

In a disagreement over the validity of EP, Williamson (1988: 118) asks, is what one party asserts really what the other side denies? A negative answer, for Williamson, seems to be a live possibility. As Williamson puts it:

> Both [parties] are likely to claim that they are using their words - in particular, the word 'exist' -in their everyday senses; but even if no unconscious philosophically motivated semantic change has occurred, the word 'exist' may have more than one everyday sense; these senses may vary in relative salience from one idiolect to another. It is certainly not clear that ordinary usage returns unequivocal answers to questions such as ['is it possible to have properties without existing?']. For part of what needs to be explained is how some intelligent people can find it a blindingly obvious piece of common sense that what doesn't exist just isn't there to have properties, while others can find it a blindingly obvious piece of common sense that the dead have the property of being dead and therefore don't exist, or that fictional characters have the property of being fictional characters and therefore don't exist. We can be sure that 'exist' will not turn out to have two wholly unrelated senses, in the way

[4] It needs to be observed that a similar line of thought is also available to Meinongians as well. For, Meinongianism is simply the view that an object may well be non-existent. As such, Meinongianism per se does not commit one to the view that any specific object (e.g. Vulcan) should be taken as a non-existent, as opposed to nothing at all. Similarly, some Meinongians and some Parmenideans may regard the premise of I1 as truth-valueless. I would like to thank Brian Ball for calling my attention on all these matters.

> that 'bank' has, but may it not turn out to have senses as different as those of 'true' in 'true statement' and 'true friend'? (1988: 118)

Proponents and opponents of the validity of EP, Williamson argues, will claim that their usage of 'exists' is the one corresponding to our ordinary usage. However, Williamson notices, there may well not be one true ordinary usage of 'exists' and, accordingly, no single truly commonsensical answer to the question whether it is possible to have properties without existing[5]. Perhaps then, Williamson goes on to suggest, friends of EP are using 'exists' in one sense; opponents of EP, in another.

If Williamson's suggestion were correct, it would be plausible to think that Parmenideans and Meinongians are talking past each other when disagreeing, for example, about whether 'exists' obeys EP. For surely, if there were two equally correct uses of 'exists', one Meinongian and one Parmenidean, then their disagreement would only have the appearance of a disagreement. The question, then, is whether Parmenideans and Meinongians really use 'exists' in the same sense. On the Williamsonian account illustrated in the previous section, they do just in case they could run a proof-theoretic argument. For this to happen, as we know, they will have to (i) agree on some rules of inference governing 'exists'; and (ii) show that such rules are strong enough to characterise 'exists' as the only monadic predicate up to logical equivalence obeying them.

For Williamson however, the possibility of their satisfying (i) and (ii) seems to be a remote one:

> [i]t does not seem likely that the proponent and the opponent of EP will share other assumptions about the logic of 'exist' of a kind that would allow them to agree on a unique characterization of it (up to logical equivalence) in terms of its logical properties (1988: 119-120).

It is unclear whether Williamson is claiming that the proponent and the opponent of EP do not share any other rule governing 'exists', or whether the rules they share are simply not strong enough to bring about the desired characterisation of 'exists' up to logical equivalence. But either way, the upshot is the same: it does not seem possible for the parties, so it is said, to establish the reality of their disagreement by means of a proof-theoretic argument. Williamson's conclusion is the one we would expect given the foregoing discussion: we have a case for thinking that "parties to a dispute over EP are characteristically using 'exists' in different and non-equivalent senses" (1988: 122).

The next section argues that Williamson's conclusion should be resisted. If proponents and opponents of EP were to share certain realistic assumptions about 'exists', they will be able to characterise it as the only monadic predicate obeying a certain set of rules of inference. Given the standards which Williamson has set, it follows that there can be genuine disagreements over the validity of EP.

4 Collapse and Unequivocation

Here is how we will reach the conclusion just mentioned. First, we will start in §4.1 by presenting a case of logical disagreement about 'exists' between two systems validating EP:

[5]See Williamson (1998: 259) and Williamson (2013: 22-25) for similar versions of this point.

classical logic "on its Parmenidean interpretation" (more on this shortly), and negative free logic. Importantly, existence in each of those systems is treated as equivalent to identity, in the sense that it is a theorem of both systems that anything exists if, and only if, it is self-identical. However, the equivalence between existence and self-identity is a neutral principle in the dispute between Parmenideans and Meinongians, which both parties are entitled to accept. For the latter, in particular, the result is a form of Meinongianism on which non-existents are taken as lacking self-identity.

We will consider three ways in which this form of Meinongianism may be further construed. The first (§4.2), sharing in part an intuition often associated with negative free logic, evaluates as false any atomic formula containing a term denoting a non-existent. The remaining two (§4.3) do not share a similar assumption. The systems of logic corresponding to these three forms of Meinongianism have striking consequences for our discussion. The former system validates EP: what is usually thought of as a Parmenidean principle of 'exists' actually turns out to be compatible with a Meinongian account thereof. The latter two systems do *not* validate EP. Nonetheless, *crucially*, they validate other principles which 'exists' also obeys in classical logic on its Parmenidean interpretation, as well as in negative free logic. And those principles are strong enough to provide the required proof-theoretic argument for 'exists'. Thus, one might be tempted to draw, *at this point*, the obvious conclusion that *pace* Williamson, proponents and opponents of EP can in some cases come to a principled refutation that they are talking past each other. We will, eventually, draw that conclusion; but not at the end of §4.3. For, we will first need to solve a pressing objection, which we will formulate in the next section. And since, in the next section, we will be able to solve that objection, we will, *at that point*, thereby draw the conclusion just mentioned.

4.1 Negative Free Logic

Classical logic on its Parmenidean interpretation is just the classical predicate calculus with the Quinean assumption that the quantifier $\exists$ is taken to express existence[6]. If one wanted, one could unproblematically make such existentially loaded account of $\exists$ explicit by adding to the language an 'exists' predicate $\mathsf{E!^Q}$ and stipulate that, for any term t, $\mathsf{E!^Q}t =_{\mathsf{Def}} \exists x(x = t)$ - the superscript 'Q' is an abbreviation for 'Quine'. Existence, on the Parmenidean account, reduces to a combination of quantification and identity. Of course, $\mathsf{E!^Q}$ obeys EP. But to be sure, several additional principles are part of the logic of $\mathsf{E!^Q}$, such as for instance the one we will call *Logical Necessity of Existence* (LNE) - for which the existence of an arbitrary thing is a logical theorem. Where t is any term, LNE is simply the following schema:

$$\frac{}{\mathsf{E!^Q}t}\ \mathsf{LNE}$$

Given that, classically, any term has a denotation in the domain of $\exists$, and given that such a domain is co-extensive with the range of $\mathsf{E!^Q}$, the Parmenidean interpretation of classical logic entails that there must exists at least one thing.

[6]It is therefore important, in this context, to specify that it is not classical logic per se that yields an account of $\exists$ as existentially loaded. There are indeed also non-Parmenidean, viz. Meinongian, interpretations of classical logic. See on this Routley (1980: 74-75) and Priest (2008). Hereafter, unless otherwise specified, by 'classical logic' we will always mean to refer to classical logic *on its Parmenidean interpretation*.

Of course, one may accept that 'exists' ought to obey EP but have serious reservations that it also ought to obey the more suspicious LNE[7]. If so, a possibility would be to maintain that the correct usage of 'exists' is captured by negative free logic. Whilst free logic in general is concerned with revising logic so as to detach it from its existential presuppositions[8], distinctive of the negative school of free logic are three philosophical ideas. First, that anything exists just in case it is self-identical. Second, that terms not referring to existents lack reference altogether. And third, that any atomic formula containing occurrences of such terms needs to be evaluated as false[9].

All this is achieved by requiring that, in the definition of a model, the valuation function from terms to the possibly empty domain of quantification is partial instead of total. The truth-condition for atomics is modified accordingly by imposing that $P(t_1, \ldots, t_n)$ is false when either the n-tuple $\langle t_1, \ldots, t_n \rangle$ is not in the extension of P *or* at least some of its members are not in the domain of the valuation function. The truth-condition for the 'exists' predicate, $E!^N$, simply stipulates that to exist is equivalent to being denoted by a term in the domain of the valuation function.

Formally, let $\mathcal{I}^N = \langle D, v \rangle$ be a model where D is a possibly empty set, v is a valuation function, and Dom is the domain of v such that: for each individual constant c, if $c \in \mathrm{Dom}(v)$, then $v(c) \in D$ and for each n-place predicate P^n, $v(P^n) \subseteq D^n$. Given a variable assignment g based on $\mathcal{I}^N$, the valuation function is extended to a function v^g whose domain is a subset of the union of the set of variables and the set of constants such that: for any constant c, $v^g(c) = v(c)$ if $c \in \mathrm{Dom}\ (v)$; and for any variable u $v^g(u) = g(u)$ if $u \in \mathrm{Dom}\ (v)$. Moreover, for each n-place predicate P^n, $v(P^n) = v^g(P^n) \subseteq D^n$. Satisfaction relative to a model $\mathcal{I}^N$ and assignment g based on that model is defined as follows for atomics, $E!^N$, $=$ and $\forall$ - the remaining clauses are as obvious:

$\mathcal{I}^N, v^g \models P(t_1, ..., t_n)$ iff $t_1, \ldots, t_n \in \mathrm{Dom}(v)$ and $\langle v^g(t_1), \ldots, v^g(t_n) \rangle \in v^g(P)$.

$\mathcal{I}^N, v^g \models E!^N t$ iff $t \in \mathrm{Dom}\ (v^g)$.

$\mathcal{I}^N, v^g \models t = u$ iff $t, u \in \mathrm{Dom}(v^g)$ and $v^g(t) = v^g(u)$.

$\mathcal{I}^N, v^g \models \forall x A$ iff $\mathcal{I}^N, v^g[d/x] \models A$ for each $d \in D$.

Three remarks are now in order.

Remark 1. On the semantics just presented, it can be shown that, for any t, the formula $E!^N t \leftrightarrow t = t$ is a logical truth. Thus, given that to say that t exists is equivalent to saying that t is identical to itself, existence and self-identity are, on this account, treated as equivalent. To establish the left-to-right part of the biconditional, we proceed as follows. Gratzl (2010) has provided a proof-theory which is sound and complete with respect to the semantics just presented. In this system, one is allowed to infer, from the premise that t exists, that t is identical to itself. That is, where t is any term, we have $E!^N t \vdash t = t$ as a rule. Whence, because the system also has the classical Conditional Introduction, we can get $\vdash E!^N t \rightarrow t = t$. Thus, by the soundness theorem, it follows that $\models E!^N t \rightarrow t = t$. For the right-to-left part, the

[7]See Oliver and Smiley (2013: 182-188) and Bencivenga (2002) for discussion.

[8]Such as that there must exist at least one thing, as per LNE.

[9]See Morscher & Simons (2001) and Lambert (2001) for philosophical discussion of negative free logic.

reasoning goes like this. Let $\mathcal{I}^{N}$ be a model and v^{g} a value assignment such that $\mathcal{I}^{N}, v^{g} \vDash t = t$. By the semantic clause for identity, we have $t \in \mathrm{Dom}(v^{g})$. Therefore, by the semantic clause for 'exists', it follows that $\mathcal{I}^{N}, v^{g} \vDash E!^{N}t$. Accordingly, since $\mathcal{I}^{N}$ and v^{g} were arbitrarily taken, it follows that $\vDash E!^{N}t \leftrightarrow t = t$.

Remark 2. LNE is clearly unsound with respect to the semantics given, and therefore $E!^{N}$ does not obey LNE: for some model $\mathcal{I}^{N}$, value assignment v^{g} and term t, $E!^{N}t$ may fail. Indeed, if $t \notin \mathrm{Dom}(v^{g})$, then $\mathcal{I}^{N}, v^{g} \nvDash E!^{N}t$.

Remark 3. EP is sound with respect to the semantics given, and therefore E^{N} obeys this rule. Take any model $\mathcal{I}^{N}$ and value assignment v^{g} such that $\mathcal{I}^{N}, v^{g} \vDash Pt$. By the semantic clause for atomics, one gets $t \in \mathrm{Dom}(v^{g})$. Thus, by the semantics clause for $E!^{N}$, it follows that $\mathcal{I}^{N}, v^{g} \vDash E!^{N}t$.

In negative free logic and classical logic, therefore, 'exists' is taken to obey EP. And, it turns out, EP characterises 'exists' up to logical equivalence. For, consider a language containing two predicates, say F and G, obeying EP. What we get is $\vdash \forall x(Fx \leftrightarrow Gx)$: any two predicates obeying EP are logically equivalent. The proof is immediate:

$$\cfrac{\cfrac{\cfrac{\cfrac{\overset{(1)}{Ft}}{Gt}\,\scriptstyle EP}{Ft \to Gt}\,\scriptstyle (1)\to I \qquad \cfrac{\cfrac{\overset{(2)}{Gt}}{Ft}\,\scriptstyle EP}{Gt \to Ft}\,\scriptstyle (2)\to I}{Gt \leftrightarrow Ft}\,\scriptstyle \leftrightarrow I}{\forall x(Fx \leftrightarrow Gx)}\,\scriptstyle \forall I$$

Thus, by a proof-theoretic argument *à la* Williamson, proponents of the two systems of logic discussed in this sub-section can therefore exclude that they are using 'exists' in different senses. Their disagreement about whether 'exists' obeys LNE, for example, is a genuine disagreement.

Having laid out the formal details of negative free logic, let us now observe how some of its main intuitions can generate the first form of Meinongianism discussed in this section.

4.2 Negative Free Logic, Meinongian Style

The gist of the negative free logic account revolves around three main tenets. First, the logical equivalence between existence and self-identity, as observed earlier in *Remark 1*. Second, the evaluation of any atomic formula containing a term referring to no existent as false. Third, the principle that any such term lacks reference altogether: terms can only refer to existents.

That terms can only refer to existents seems hard to accept from a Meinongian point of view. The whole point of Meinongianism is that objects non-trivially divide into two classes: those that exist, and those, such as Vulcan, Holmes, Zeus and the like, that do not. On this view, 'Vulcan' does get a referent: it is Vulcan[10]! On the contrary, nothing,

[10] Sainsbury (2009: 57-63) has argued that Meinongians face pressing problems when trying to fix reference to non-existents. Attempts to solve the puzzle raised by Sainsbury can be found in Priest (2016: §11.4) and Berto (2012; 2013: §9.4).

as far as we can see, should prevent a Meinongian from subscribing to the remaining two principles of negative free logic. The resulting form of Meinongianism is one for which non-existents are taken as objects lacking absolutely every property and standing in absolutely no relation, including self-identity[11]. In this subsection, we will make this idea precise by providing the semantics corresponding to such an account, and some rules of inference adequate with respect to the semantics. We will call the resulting Meinongian system $\mathcal{L}_{M^1}$ - 'M' for Meinongian.

The language of $\mathcal{L}_{M^1}$ comprises: individual constants a, b, c... and variables x, y, z... (countably many); n-place predicates P, Q, R... (countably many); the five connectives, the universal quantifier $\forall$; and the 'exists' predicate $E!^M$. The set of wffs is defined as usual. A model $\mathcal{I} = \langle D_O, D_I, v\rangle$ for $\mathcal{L}_{M^1}$ is a structure where: D_O (outer domain) is a non-empty set; D_I (inner domain) is a possibly empty set such that $D_I \subseteq D_O$; and v is a total function such that, for each individual constant c, $v(c) \in D_O$ and for each n-place predicate P^n, $v(P^n) \subseteq D_O^n$. Logical consequence is defined as in classical logic, so that a formula φ is a logical consequence of a possibly empty set of formulae Γ just in case every model of Γ is a model of φ.

Given a variable assignment g based on $\mathcal{I}$, the valuation function is extended to a function v^g whose domain is the union of the set of variables and the set of constants such that: for any constant c, $v^g(c) = v(c)$; and $v^g(u) = g(u)$ for any variable u. Moreover, for each n-place predicate P^n, $v(P^n) = v^g(P^n) \subseteq D_O^n$. Satisfaction relative to a model $\mathcal{I}$ and assignment g based on that model is defined as usual for the five connectives. The clauses for atomics, $E!^M$, $=$ and $\forall$ are defined as follows:

$\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^1}} (P(t_1, ..., t_n))$ iff $v^g(t_1), ..., v^g(t_n) \in D_I$ and $\langle v^g(t_1), ..., v^g(t_n)\rangle \in v^g(P)$.

$\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^1}} E!^M t$ iff $v^g(t) \in D_I$.

$\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^1}} t = u$ iff $v^g(t), v^g(u) \in D_I$ and $v^g(t) = v^g(u)$.

$\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^1}} \forall x A$ iff $\mathcal{I}, v^g[d/x] \vDash_{\mathcal{L}_{M^1}} A$ for each $d \in D_O$.

The clause for atomics deserves consideration. There are two ways for an atomic $P(t_1, \ldots, t_n)$ to come out false. One, of course, is when not all the referents of $\langle t_1, \ldots, t_n\rangle$ fall under the extension of P. Yet another is when not all of $t_1, \ldots, t_n$ are in the inner domain. In this case case, however, those member(s) of $t_1, \ldots, t_n$ not in the inner domain do not lack reference, as per negative free logic. For, v^g provides referent(s) to those terms in the outer domain.

The clause for identity deserves comment too. Since this implies that $t \neq t$ is true when the referent of t is not in the inner domain, identity here receives a non-standard treatment - just as it does in negative free logic. As a consequence, the proof theory of $\mathcal{L}_{M^1}$ can only admit a restricted form of Identity Introduction, as per rule RII below. (This fact is important, and will give rise to an objection against our main result, which we will discuss in §5). Importantly, also, self-identity and existence are here treated as equivalent in the sense that anything has the former just in case it has the latter - the inner domain comprises exactly those things which exist and are self-identical.

[11] An anonymous referee, whom I thank for pressing me on this point, has suggested that the view is best described as a form of epiphenomenalism about non-existents.

Proof-theoretically, the usual rules of inference for the five connectives are clearly adequate to semantics just provided, as are the classical Introduction and Elimination rules for $\forall$, i.e. $\forall$I and $\forall$E respectively - soundness for $\mathcal{L}_{M^1}$ is established in the **Technical Appendix**:

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ A[t/x]\end{matrix}}{\forall x A}\ \forall I \qquad\qquad \dfrac{\begin{matrix}\Gamma \\ \vdots \\ \forall x A\end{matrix}}{A[t/x]}\ \forall E$$

Provided t does not appear in Γ or A

In addition, given the equivalence between existence and self-identity, $\mathcal{L}_{M^1}$ will have two further rules. One of them, corresponding to a restricted Identity Introduction (RII), allows one to infer that, for arbitrary t, t is self-identical provided it exists. The other one restricts the introduction of an existential claim of the form Et to those cases where we are in possession of a premise that t is self-identical - we call this second rule REI, which abbreviates 'Restricted Existential Introduction. RII and REI are thus the following schemata:

$$\frac{E!^M t}{t = t}\ RII \qquad\qquad \frac{t = t}{E!^M t}\ REI$$

Importantly, another rule of inference sound with respect to the semantics of $\mathcal{L}_{M^1}$ is the classical Indiscernibility of Identicals $(Ind.Id.)$ - sometimes known as 'Identity Elimination'. That is:

$$\frac{Rtu \qquad A}{A'}\ Ind.\ Id.$$

A' is obtained from A by replacing one or more occurrences of t with u, or vice versa.

Moreover, EP is sound with respect to the Meinongian semantics for $\mathcal{L}_{M^1}$. Indeed, Pt is always false if t is a non-existent, and so truth would be trivially preserved if one were to infer $E!^M t$ from it. As far as we know, this represents a novelty: no other Meinongian system we are aware of can accommodate EP as a valid rule of inference. The fact that Meinongianism is compatible with a use of 'exist' obeying EP, we think, is a significant contribution of the present paper. One, however, could turn the point on its head: the fact that EP is sound with respect to the semantics for $\mathcal{L}_{M^1}$ actually shows that the system, contrary to what has just been said, is not Meinongian in spirit.

The best answer to this objection is to point to one fact about $\mathcal{L}_{M^1}$, namely, that $\forall x E^M!x$ is not a theorem of the system: not everything exists. Quantification, therefore, is taken as lacking existential import; a consequence which only Meinongian systems have[12]. That

[12]Which is not to say that all Meinongian systems take quantification as not existentially loaded. The school of positive free logic, for instance, despite maintaining that terms may refer to non-existents, still maintains that quantification is existentially loaded. Various systems of positive free logics can be found in Leblanc & Thomason (1972), Grandy (1972) and Cocchiarella (1966).

$\mathcal{L}_{M^1}$ has this consequence is a striking result. Indeed, given the validity of EP, one would have expected $\forall x E!^M x$ to be a theorem - as per classical logic and negative free logic. What makes $\forall x E!^M x$ fail is the combination of three features. First, that terms may refer to non-existents; second, that the domain of $\forall$ is the outer domain; third, that any atomic formula containing an occurrence of any such term gets evaluated as false. The result is that any model containing non-existents is a countermodel to $\forall x E!^M x$.

The attractiveness of this result is that it shows how a use of 'exists' obedient to EP can be demarcated from a use of 'exist' entailing that 'everything exists' is a theorem. In $\mathcal{L}_{M^1}$, one has the former but not the latter. Meinongians and Parmenideans, therefore, need not disagree over the validity of EP. $\mathcal{L}_{M^1}$-Meinongians will still disagree with Parmenideans, however, over whether 'everything exists' is a theorem, as well as over the validity of LNE. Yet, due to their accepting EP, they will also be in a position to define 'exists' as the only monadic predicate up to logical equivalence obeying this principle - as pointed out at the end of §4.1. By now familiar Williamsonian procedures, they can thereby come to a principled establishment of the reality of their disagreement.

We have seen how $\mathcal{L}_{M^1}$ is built around the equivalence of existence and self-identity. However, accepting this principle does not perforce commit one to the validity of EP. Meinongians could very well retain the former and reject the latter. And if they did, they would share with proponents of any other system discussed in this section enough rules of inference to characterise 'exists' up to logical existence. To these new accounts of non-existents, we now turn.

4.3 Collapse without the Existence Principle

The previous sub-section showed that Meinongians willing to subscribe to the equivalence of self-identity and existence can do so by accepting the validity of EP. What this section adds to the foregoing discussion is that the equivalence of existence and self-identity does not mandate acceptance of EP. EP might be thought of having intuitive counterexamples: Vulcan has the property of being an object of erroneous scientific theorising, but does not exist. Meinongians willing to hold on to this intuition may freely do so *and* treat existence and self-identity as equivalent.

Given the assumption that anything exists if, and only if, it is self-identical, there are two ways to make EP fail; each one resulting in a system differing from $\mathcal{L}_{M^1}$ only with respect to the semantic clause for atomics. One system, call it $\mathcal{L}_{M^2}$, will deliver an account of non-existents as having some properties (of course, other than existence) and standing in some relations (other than identity). The other system, call it $\mathcal{L}_{M^3}$, will deliver an account of non-existents as having every property (of course, other than existence) and standing in any relation (other than identity). So let us now inspect $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ in turn.

Let the language of $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ be that of $\mathcal{L}_{M^1}$. $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$-models are defined exactly in analogy with and $\mathcal{L}_{M^1}$ models, namely, as triples $\mathfrak{I} = \langle D_O, D_I, \nu \rangle$ whose elements are defined exactly as before. Logical consequence, as per $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$, is classical.

In $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$, satisfaction relative to a model and value assignment based on that model is defined as per $\mathcal{L}_{M^1}$ for the five connectives, the 'exists' predicate, $=$ and $\forall$.

In $\mathcal{L}_{M^2}$, the semantic clause for atomics is classical:

$$\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} P(t_1, ..., t_n) \text{ iff } \langle \nu^g(t_1), ..., \nu^g(t_n) \rangle \in \nu^g(P).$$

Three remarks about $\mathcal{L}_{M^2}$ are in order:

Remark 4. EP is unsound with respect to the semantics of $\mathcal{L}_{M^2}$. Let $\mathfrak{I}$ be a model and ν^g an assignment such that $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} Pt$. Nothing prevents that $\nu^g(t) \notin D_I$. If so, then $\mathfrak{I}, \nu^g \nvDash_{\mathcal{L}_{M^2}} E!^M t$.

Remark 5. Existence is equivalent to self-identity. First, let $\mathfrak{I}$ be a model and ν^g an assignment such that $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} t = t$. Then, $\nu^g(t) \in D_I$. And because any term t such that $\nu^g(t) \in D_I$ refers to an existent, we have $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} E!^M t$. Let now $\mathfrak{I}$ be a model and ν^g be an assignment such that $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} t \neq t$. Then, $\nu^g(t) \notin D_I$. And because any term t such that $\nu^g(t) \notin D_I$ refers to a non-existent, we have $\mathfrak{I}, \nu^g \nvDash_{\mathcal{L}_{M^2}} E!^M t$. Hence, for any model $\mathfrak{I}$, assignment ν^g, and term t, we have $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} t = t \leftrightarrow E!^M t$.

Remark 6. 'Everything exists' is not a logical truth, and so quantification is not existentially loaded. Let $\mathfrak{I}$ be a model and ν^g an assignment such that $\nu^g(t) \notin D_I$. Then, $\mathfrak{I}, \nu^g \nvDash_{\mathcal{L}_{M^2}} E!^M t$. By the semantic clause for $\forall$, it follows that $\mathfrak{I}, \nu^g \nvDash_{\mathcal{L}_{M^2}} \forall x E!^M x$.

Thus, as promised, in $\mathcal{L}_{M^2}$ EP is unsound (*Remark 4*), and existence and self-identity are treated as logically equivalent (*Remark 5*). Moreover, importantly, quantification is not existentially loaded (*Remark 6*).

Let us now turn to $\mathcal{L}_{M^3}$, where the clause for atomics is defined as follows:

$\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^3}} P(t_1, ..., t_n)$ iff $\nu^g(t_1), ..., \nu^g(t_n) \in D_I$ and $\langle \nu^g(t_1), ..., \nu^g(t_n) \rangle \in \nu^g(P)$ *or*, for some $t_i \in \langle t_1, ..., t_n \rangle$, $\nu^g(t_i) \notin D_I$.

Informally, an atomic formula $P(t_1, \ldots, t_n)$ is true just in case either all the referents of $\langle t_1, \ldots, t_n \rangle$ are in the inner domain *and* in the extension of P, or else some of them are simply not in the inner domain. In other words, $\mathcal{L}_{M^3}$ implements the assumption that any atomic formula containing a term referring to a non-existent is evaluated as true[13]. Notice that by 'atomic formula' here we mean any atomic formula *not* of the form $t = t$ or $E!^M t$. Indeed, by the semantic clauses for $=$ and $E!^M$, $t = t$ and $E!^M t$ are evaluated as false whenever t is not in the inner domain.

It is routine to check that *Remarks 4-6* carry over, *mutatis mutandis*, to $\mathcal{L}_{M^3}$: Existence is equivalent to self-identity, EP is unsound, and quantification not existentially loaded. It is also important to note that, although EP is unsound with respect to the semantics for $\mathcal{L}_{M^3}$, its negative version, call it NEP, is not. In other words, where P is any monadic predicate other than $E!^M$, and t any term, the following inference schema is valid on the semantics for $\mathcal{L}_{M^3}$ - a formal proof can be found in the last section of the **Technical Appendix**:

$$\frac{\neg Pt}{E!^M t} \text{ NEP}$$

That 'exist' is governed by NEP in $\mathcal{L}_{M^3}$ is an obvious consequence of the account of non-existents delivered by this systems. If non-existents are precisely those objects that

[13] $\mathcal{L}_{M^1}$ and $\mathcal{L}_{M^3}$, therefore, are built on contrary assumptions about whether all atomics containing some terms referring to non-existents are true.

enjoy any property, then knowing that something lacks a property is sufficient ground to infer that it exists.

Moreover, as argued in the **Technical Appendix**, all the previous rules for $\forall$ and $=$, sound with respect to the semantics for $\mathcal{L}_{M^1}$, are also sound with respect to the semantics of $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$. And crucially, given that, just like $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ too treat existence and self-identity as equivalent, rules RII and REI will be sound in $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ as well.

The equivalence between existence and self-identity thus holds in each one of the systems discussed in this section: classical logic, negative free logic, and the three Meinongian systems presented. Thus, 'exists' in each one of those systems will obey RII and REI. This common ground on the logic of 'exists' is strong enough to ensure the key result we promised at the beginning of this section. In any logic in which the introduction rules for $\rightarrow, \leftrightarrow$ and $\forall$ are classical, there is only one monadic predicate up to logical equivalence obeying RII and REI, as per **Theorem 1**.

Theorem 1. *Let $\mathcal{L}$ be any logic with classical Introduction rules for $\rightarrow$, $\leftrightarrow$ and $\forall$. Moreover, let* P *and* Q *be two monadic predicates of the language of $\mathcal{L}$ obeying* RII *and* REI. *Then,* $\vdash_{\mathcal{L}} \forall x(Px \leftrightarrow Qx)$.

The following simple derivation establishes **Theorem 1.**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{(1)}{Pt}}{t=t}\,\scriptstyle{RII}}{Qt}\,\scriptstyle{REI}}{Pt \rightarrow Qt}\,\scriptstyle{(1)\rightarrow I} \qquad \dfrac{\dfrac{\dfrac{\overset{(2)}{Qt}}{t=t}\,\scriptstyle{RII}}{Pt}\,\scriptstyle{REI}}{Qt \rightarrow Pt}\,\scriptstyle{(2)\rightarrow I}}{Pt \leftrightarrow Qt}\,\scriptstyle{\leftrightarrow I}}{\forall x(Px \leftrightarrow Qx)}\,\scriptstyle{\forall I}$$

In virtue of **Theorem 1**, it looks as though, *prima facie*, theorists willing to accept a use of 'exists' as obeying RII and REI will be able to characterise it as *the* monadic predicate obeying those rules (at least for logics of the sort described by **Theorem 1**). Some such theorists, as we know, will disagree about whether 'exists' also obeys EP. But in virtue of **Theorem 1**, it might be said, they will be able to provide a principled refutation that they are talking past each other. Hence, given the standards for genuine logical disagreement under assumption, it looks as though there are real disputes to be had about the validity of EP.

There is, however, a complication concerning the rules for identity, which are crucially required to successfully exploit **Theorem 1** in the way just illustrated. This complication gives rise to a serious objection which, if successful, might block the claim just made. We thus conclude by illustrating, and solving, this objection.

5 An Objection about Identity

Objection. **Theorem 1** might be used to run a proof-theoretic argument concluding that 'exists', in classical logic, $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ is not equivocal. Such a conclusion can be resisted by reasoning as follows. Consider, for example, a classical logician and an $\mathcal{L}_{M^1}$-Meinongian. A corollary of **Theorem 1**, it was argued, is that they can define 'exists' as the only monadic predicate up to logical equivalence obeying RII and REI. But these rules, besides existence, can also be thought of as governing identity. And identity, as we know, receives different treatments in classical logic and $\mathcal{L}_{M^1}$. For example, in $\mathcal{L}_{M^1}$, identity is not taken as a reflexive relation in that, as per *Remarks 5-6*, $\nvdash_{\mathcal{L}_{M^1}} \forall x(x = x)$. Therefore, the classical logician and the $\mathcal{L}_{M^1}$-Meinongian might be using identity in different senses. **Theorem 1**, by contrast, requires an unequivocal use of identity. Which, therefore, must be established[14]. Exactly analogous considerations apply to the remaining two systems presented earlier, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$. For, also in these systems identity receives a non-standard treatment. Hence, the problem under discussion here carries over to those systems as well. In sum, any two parties willing to exploit **Theorem 1** to characterise 'exists' up to logical equivalence, must also necessarily prove that they can characterise identity up to logical equivalence.

Reply. The answer to the objection is that such a proof can be produced. To be clear, the claim is that there are enough shared rules of inference governing identity, in classical logic and the three Meinongian systems considered here, to allow for a characterisation of identity as *the* only dyadic relation up to logical equivalence obeying those rules.

For a start, RII and REI will not be of any help here. For, such rules involve an 'exists' predicate. And, for all we know at this point, there may be many non-equivalent such predicates. Thus, RII and REI are not going to get us anywhere.

By contrast, Ind.Id. does not suffer from this problem. The vocabulary required to formulate such a rule, other than identity itself, is not at risk of equivocation in the way 'exists' is. Thus, relying on Ind.Id. to obtain the desired proof is certainly a good idea. The problem is, that Ind.Id. *alone* does not appear to be strong enough for our purposes. In other words, there may be many, not deductively equivalent dyadic relations obeying Ind.Id. Of course, it is pretty straightforward to prove that there can be only one dyadic *and reflexive* relation up to logical equivalence obeying Ind.Id. But this is irrelevant here since identity in $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ is not taken to be a reflexive relation. Thus, we will proceed on the assumption that Ind.Id. alone is not strong enough to deliver our desired result. If we are wrong, and it can be proven that there is in fact only one dyadic relation up to logical equivalence obeying Ind.Id., then this is welcome news. For, this would show that the result presented here can be even generalised.

An intuitive idea would be to help ourselves to the converse principle of Ind.Id., namely, the Identity of Indiscernibles (Id.Ind.), and see whether this principle (together with Ind.Id.) could be a step in the right direction. But here we run into a problem, having to do with the fact that Id.Ind. may not be expressible as a rule of inference in any of the languages considered here. For, the standard formulation of Id.Ind. requires a higher-order language, allowing for quantification into predicate position. In a higher-order language, Id.Ind. can be expressed as follows:

[14] If they wanted, classical logicians and $\mathcal{L}_{M^1}$-Meinongians could avoid the present objection by reasoning as follows. 'Exists', in classical logic and $\mathcal{L}_{M^1}$, obeys EP. And, as pointed out at the end of §4.1, there is only one monadic predicate up to logical equivalence obeying EP. Therefore, the classical logician and the $\mathcal{L}_{M^1}$-Meinongian may just as well agree to define 'exists' as *that* predicate, and thereby overcome the complications involving identity raised by the present objection. Still, given the importance of the present objection, it is important to show that it can also be directly refuted. That is, it is important to show that the classical logician and the $\mathcal{L}_{M^1}$-Meinongian have the resources to define identity up to logical equivalence.

Id. Ind. $\forall X \forall x \forall y((Xx \leftrightarrow Xy) \rightarrow x = y)$,

where X and Y are predicate variables.

Given that none of the languages under consideration here contains predicate variables or quantifiers bounding such variables, there is first of all a question as to how we could adequately represent Id.Ind. in a first-order language, if at all.

However, Read (2016: 416) has shown that there is a rule expressible in a first-order language, called by Read $= I'$, which adequately captures Id.Ind. To begin with, $= I'$ is the following schema, where F is a predicate variable ranging over monadic predicates, and δ_1, δ_2 derivations of Fu from Ft and of Ft from Fu respectively.

$$\dfrac{\begin{matrix}[Ft] \\ \vdots\ \delta_1 \\ Fu\end{matrix} \qquad \begin{matrix}[Fu] \\ \vdots\ \delta_2 \\ Ft\end{matrix}}{t = u} {\scriptstyle = I'}$$

provided over the course of δ_1 and δ_2 F does not occur in any side premises.

Informally, the rule says that, if we are in possession of a proof that t is F only if u is, and also of a proof that u is F only if t is, where F is an arbitrary monadic predicate (i.e. not occurring in any side premises), then we are entitled to infer that t is identical to u.

Even commentators such as Griffiths and Ahmed (2021: 1454) - who criticised Read's proposal for orthogonal reasons to the present issue[15] - are in agreement with Read that the predicate variables in the formulation of Read's $= I'$ do not take us beyond the resources available in a first-order language. For, only instances of F can appear within proofs, and so any concrete application of $= I'$ only relies on first-order resources.

Now, Read (2012), adapting a proof originally due to Kremer (2007), showed that a stronger version of $= I'$, call it $S{=}\ I'$ is sound with respect to the standard semantics for classical logic. Unlike $= I'$, the stronger $S{=}\ I'$ licenses the following reasoning: given just a derivation of Fu from Ft and possibly some side premises where F does not occur, infer $t = u$ and discharge Ft[16]. Given that $S{=}\ I'$ is sound in classical logic, so is *a fortiori* $= I'$.

The problem is that, although weaker than $S{=}\ I'$, $= I'$ is still too strong for the semantics of $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$. For, whilst none of their semantics validates the formula $\forall x(x = x)$, by applying $= I'$ first, and then $\forall I$, one could immediately prove $\vdash \forall x(x = x)$. Indeed, consider the deductively weakest of $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$, namely, $\mathcal{L}_{M^2}$; given that $Ft \vdash_{\mathcal{L}_{M^2}}$

[15]The bone of contention in the disagreement between Read and Griffiths & Ahmed is, roughly, whether the rules Ind.Id. and $= I'$ are harmonious in the sense of Dummett (1991). Informally, the question revolves around whether Ind.Id., which can be taken as an elimination rule for identity, allows us to infer no more and no less than can be deduced directly from the premises of $= I'$ (which can be taken as an introduction rule for identity). Similar issues are completely tangential to the one pursued here.

[16]Read's $S{=}\ I'$ is thus the following rule (where again, F cannot occur in any side premises):

$$\dfrac{\begin{matrix}[Ft] \\ \vdots \\ Fu\end{matrix}}{t = u} {\scriptstyle S= I'}$$

Ft and $Ft \vdash_{\mathcal{L}_{M^2}} Ft$, it follows by $= I'$ that one can discharge Ft and derive $t = t$, whence one can derive $\forall x(x = x)$ by $\forall I$. Thus, $\forall x(x = x)$ would be a theorem of $\mathcal{L}_{M^2}$. Given that anything deducible in $\mathcal{L}_{M^2}$ is also deducible in $\mathcal{L}_{M^1}$ and $\mathcal{L}_{M^3}$, the result just established carries over to those two systems as well. Read's $= I'$, therefore, needs to be weakened to ensure that one cannot prove more than the semantics of $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ validate.

To this end, there are many restrictions which one could introduce in $= I'$, capable of blocking the proof of $\vdash \forall x(x = x)$ in $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$. But not all of them will deliver the categorical result, sought after in this section, that any two dyadic relations, R_1 and R_2, obeying $Ind.Id.$ and the still to be determined restricted version of $= I'$, are deductively equivalent - i.e. a proof of $\vdash \forall x \forall y(R_1xy \leftrightarrow R_2xy)$. For example, to block the derivation of $\vdash \forall x(x = x)$, one could weaken $= I'$ by requiring two additional premises, $t = t$ and $u = u$, which remain undischarged when the rule is applied. The resulting restricted version of $= I'$, call it $R = I'$, has thus the following form[17]. Provided that, as with $= I'$, over the course of δ_1 and δ_2 F does not occur in any side premises, $R = I'$ is:

$$\dfrac{\begin{matrix}[Ft] \\ \vdots\ \delta_1 \\ Fu\end{matrix} \qquad \begin{matrix}[Fu] \\ \vdots\ \delta_2 \\ Ft\end{matrix} \qquad t = t \qquad u = u}{t = u}\ {\scriptstyle R = I'}$$

With $R = I'$ in place, the proof of $\vdash \forall x(x = x)$ is no longer possible (thanks to the two additional premises). Whilst this is certainly welcome news, the restriction introduced by means of the two additional premises appears to be too severe for our goals. Specifically, given that the two additional premises do not get discharged in an application of $R = I'$, it does not seem possible to establish the desired categorical result that $\vdash \forall x \forall y(R_1xy \leftrightarrow R_2xy)$, for any two dyadic relations R_1, R_2 obeying $Ind.Id.$ and $R = I'$. In case our prediction turned out to be wrong, and the categorical result could actually be established, this would be very welcome news; for, this would show that the restrictions introduced with $R = I'$ are fit for our goals, and that no further search is needed for an alternative restricted form of $= I'$. But in what follows, we will proceed on the assumption that such a result is not available with $R = I'$. Thus, Read's $= I'$ needs to be restricted in other ways.

The challenge is to single out which conditions, if any, could enable one to exploit the force of the derivations of Fu from Ft and Ft from Fu, in order to safely infer $t = u$ *without the need of additional premises* $t = t$ and $u = u$.

To this end, we can build upon the following result established by Griffiths (2014) in the context of a discussion of Read's S= I′. Where Γ is a possibly empty set of premises *not containing* F, and t and u are *distinct* terms, Griffiths showed that, *in classical logic*, the kinds of situations in which a derivation of Fu from the set $\{Ft\} \cup \{\Gamma\}$ is available reduce to *exactly* the following two[18].

(I) The first is when $=$ occurs in Γ in such a way as to allow a derivation of $t = u$. Consider, e.g., Ex. 1. below:

[17]The new additional premises of $R = I'$ are very much in the spirit of Milne (2007: 39), who restricted Read's S= I′ similarly to obtain a sound rule for the system of free logic presented by Tennant (1978: Ch. 7).

[18]Griffiths established the result both semantically (pp. 502-503) and syntactically (pp. 507-509).

$$\text{Ex. 1.}\quad \dfrac{\dfrac{\overset{(1)}{Ft} \quad \overset{(2)}{t = z}}{Fz}\text{ Ind.Id.} \qquad \overset{(3)}{z = u}}{\dfrac{Fu}{t = u}\text{ (1) S= I}'}\text{ Ind.Id.}$$

Notice, however, that the last step of the inference is an application of S= I′, and this rule is unsound in $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$. Therefore, the above derivation is disallowed in those systems, and so this example will not be particularly insightful for our goals. However, there are other derivations, not involving S= I′, which satisfy the condition set out in **(I)**. Ex. 2. below is one:

$$\text{Ex. 2.}\quad \dfrac{\dfrac{\overset{(1)}{t = z} \quad \overset{(2)}{z = u}}{t = u}\text{ Ind.Id.} \qquad \overset{(3)}{Ft}}{Fu}\text{ Ind.Id.}$$

This derivation, as we can see, contains only reasoning valid in $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$.

(II) The second kind of situation singled out by Griffiths is when the set $\{Ft\} \cup \{\Gamma\}$ contains a contradiction *not involving the predicate* F (given that Γ does not contain F); so that $\{Ft\} \cup \{\Gamma\} \vdash \bot$. When this situation obtains, indeed, Fu could be derived from $\{Ft\} \cup \{\Gamma\}$ by *Ex Falso Quodlibet*.

Griffiths' result thus establishes that, *in classical logic*, there is no derivation of Fu from $\{Ft\} \cup \{\Gamma\}$, for t and u distinct terms, if **(I)** identity does not occur in Γ and **(II)** $\{Ft\} \cup \{\Gamma\} \nvdash \bot$. Now, classical logic is of course an extension of all of $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$, given that all the rules of the latter remain valid in classical logic, but not vice versa. As such, Griffiths' result will carry over to $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ as well. This means that, in $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$, there is no derivation of Fu from $\{Ft\} \cup \{\Gamma\}$, for t and u distinct terms, if **(I)** identity does not occur in Γ and **(II)** $\{Ft\} \cup \{\Gamma\} \nvdash \bot$. Equivalently: if there is a derivation of Fu from $\{Ft\} \cup \{\Gamma\}$, then either $\{Ft\} \cup \{\Gamma\} \vdash \bot$ or identity does occur in Γ.

Now, consider again $\mathcal{L}_{M^2}$. Of course, if $\{Ft\} \cup \{\Gamma\} \vdash_{\mathcal{L}_{M^2}} \bot$, it follows, by *Ex Falso Quodlibet*, that $\{Ft\} \cup \{\Gamma\} \vdash_{\mathcal{L}_{M^2}} Fu$. So in this case the derivation of Fu from $\{Ft\} \cup \{\Gamma\}$ is rather trivial. On the other hand, if $\{Ft\} \cup \{\Gamma\} \nvdash_{\mathcal{L}_{M^2}} \bot$, then there is a derivation of Fu from $\{Ft\} \cup \{\Gamma\}$ only if the identity symbol occurs in Γ. And the identity symbol must occur in Γ in such a way as to allow for a derivation of $t = u$, as per our previous Ex. 2. For otherwise, the occurrence of the identity symbol in Γ is completely idle.

Putting all these considerations together, we obtain the following restriction of Read's = I′, whose soundness in $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ is established in the **Technical Appendix**. The label we have chosen for the rule is Id.Ind.*, as it is a restricted version of the Identity of Indiscernibles Principle, from which it descends - again, F here is a monadic predicate:

$$\dfrac{\begin{matrix}[Ft] \\ \vdots\ \delta_1 \\ Fu\end{matrix} \qquad \begin{matrix}[Fu] \\ \vdots\ \delta_2 \\ Ft\end{matrix}}{t = u}\text{ Id.Ind.}^{\star}$$

provided over the course of δ_1 and δ_2 F does not occur in any side premises, and t and u are distinct.

As can be seen, the key constraint here, which was absent in Read's $= I'$, is that t and u must be distinct. In this way, there is no risk that, in $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$, Id.Ind.* could ever allow us to prove $\vdash t = t$, for arbitrary t; nor therefore $\vdash \forall x(x = x)$. Moreover, it is thanks to the constraint that t and u must be distinct, that we are allowed not to require anymore the two additional premises $t = t$, $u = u$. For, by Griffiths' result, we can be sure that a derivation of Fu from $\{Ft\} \cup \{\Gamma\}$ will only be possible when $=$ occurs in Γ so as to allow for a derivation of $t = u$. And this, in turn, guarantees that if were to derive $t = u$ by an application of Id.Ind.*, and from there proceed to derive $t = t$ (or $u = u$), we would do so on the assurance that $t = t$ (or $u = u$) could already be derived, without the detour through Id.Ind.*, from some of the grounds for the application of Id.Ind.*. A consequence of this fact is the admissibility of Id.Ind.* (in $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ and classical logic): although Id.Ind.* cannot be derived using the other rules, its addition does not result in an expansion of what can be proven without it in each one of those systems.

Now for the crucial fact. It is provable that, in any logic in which the introduction rules for $\to$, $\leftrightarrow$ and $\forall$ are classical, any two dyadic relations obeying Ind.Id. andId.Ind.* are deductively equivalent (**Theorem 2** below). In other words, in such logics, there is one dyadic relation up to logical equivalence obeying Ind.Id. and Id.Ind.*. There are strong enough rules governing identity, in classical logic, $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ to characterise identity up to logical equivalence. So let us prove **Theorem 2**.

Theorem 2. *Let $\mathcal{L}$ be any logic with classical Introduction rules for $\to$, $\leftrightarrow$ and $\forall$. Moreover, let R_1, R_2 be two dyadic relations of the language of $\mathcal{L}$ obeying* Ind.Id. *and* Id.Ind.*. *Then* $\vdash_{\mathcal{L}} \forall x \forall y(R_1xy \leftrightarrow R_2xy)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{(1)}{R_1xy} \quad \overset{(2)}{Fx}}{Fy}\,{\scriptstyle \text{Ind.Id.}} \qquad \dfrac{\overset{(1)}{R_1xy} \quad \overset{(3)}{Fy}}{Fx}\,{\scriptstyle \text{Ind.Id.}}}{R_2xy}\,{\scriptstyle (2),(3)\,\text{Id.Ind.}^\star}}{R_1xy \to R_2xy}\,{\scriptstyle (1)\,\to I} \qquad \dfrac{\dfrac{\dfrac{\overset{(4)}{R_2xy} \quad \overset{(5)}{Fx}}{Fy}\,{\scriptstyle \text{Ind.Id.}} \qquad \dfrac{\overset{(4)}{R_2xy} \quad \overset{(6)}{Fy}}{Fx}\,{\scriptstyle \text{Ind.Id.}}}{R_1xy}\,{\scriptstyle (5),(6)\,\text{Id.Ind.}^\star}}{R_2xy \to R_1xy}\,{\scriptstyle (4)\,\to I}}{\dfrac{\dfrac{R_1xy \leftrightarrow R_2xy}{\forall y(R_1xy \leftrightarrow R_2xy)}\,{\scriptstyle \forall I}}{\forall x \forall y(R_1xy \leftrightarrow R_2xy)}\,{\scriptstyle \forall I}}\,{\scriptstyle \leftrightarrow I}$$

Let us take stock of what has just been shown. At the end of the previous section, we provided a result to the effect that 'exists' could be characterised, in sufficiently strong logics, as the only monadic predicate up to logical equivalence obeying RII and REI (**Theorem 1**). However, as we noted in this section, any two theorists willing to take **Theorem 1** as a refutation that they are equivocating on the meaning of 'exists' must be able to prove that they can also characterise identity up to logical equivalence. Our **Theorem 2** above shows that classical logicians and proponents of $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ can characterise identity as the only dyadic relation up to logical equivalence obeying Ind.Id. and Id.Ind.*. As such, they could legitimately take **Theorem 1** as a principled refutation that they are talking past each other in their disagreements on the logic of 'exists'. So in particular, in those disagreements on the validity of EP.

Technical Appendix

Three Lemmata

This appendix provides soundness proofs for the systems $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$. We will give the full proof only for the deductively weakest of the three, namely, $\mathcal{L}_{M^2}$. Indeed, whilst all the valid principles of $\mathcal{L}_{M^2}$ remain valid in $\mathcal{L}_{M^1}$ and $\mathcal{L}_{M^3}$, the converse is not true: there are principles valid in $\mathcal{L}_{M^1}$ which are invalid in $\mathcal{L}_{M^2}$ (for example, EP), and there are principles valid in $\mathcal{L}_{M^3}$ which are invalid in $\mathcal{L}_{M^2}$ (for example, NEP). Thus, our soundness proof for $\mathcal{L}_{M^1}$ will reduce to proving that EP is valid, and our soundness proof for $\mathcal{L}_{M^3}$ will reduce to proving that NEP is valid.

It is useful to start by establishing three lemmata which will be invoked over the course of the soundness proofs.

To begin with, notice that models for $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ are defined in exactly the same way. Any such model is a structure $\mathcal{I} = \langle D_O, D_I, v\rangle$, where: D_O (outer domain) is a non-empty set; D_I (inner domain) is a possibly empty set such that $D_I \subseteq D_O$; and v is a total function such that, for each individual constant c, $v(c) \in D_O$ and for each n-place predicate P^n, $v(P^n) \subseteq D_O^n$. In $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$, logical consequence is classical: a formula φ is a logical consequence of a possibly empty set of formulae Γ just in case every model of Γ is a model of φ.

Given a variable assignment g based on $\mathcal{I}$, the valuation function is extended to a function v^g whose domain is the union of the set of variables and the set of constants such that: for any constant c, $v^g(c) = v(c)$; and $v^g(u) = g(u)$ for any variable u. Moreover, for each n-place predicate P^n, $v(P^n) = v^g(P^n) \subseteq D_O^n$. Notice also that $\mathsf{Id.Ind.}^\star$, whose soundness in $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ will be established in this appendix, is formulated by resorting to a predicate variable, F, ranging over monadic predicates. Therefore, we will say that, if X^n is a predicate variable of arity n, then $v(X^n) = v^g(X^n) \subseteq D_O^n$.

Satisfaction relative to a model $\mathcal{I}$ of any one of the three systems and assignment g based on that model is defined as usual for the five connectives. The clauses for $E!^M$, $=$ and $\forall$ are defined as follows in all three systems:

$\mathcal{I}, v^g \vDash E!^M t$ iff $v^g(t) \in D_I$.

$\mathcal{I}, v^g \vDash t = u$ iff $v^g(t), v^g(u) \in D_I$ and $v^g(t) = v^g(u)$.

$\mathcal{I}, v^g \vDash \forall x A$ iff $\mathcal{I}, v^g[d/x] \vDash_{\mathcal{L}_{M^1}} A$ for each $d \in D_O$.

The $\mathcal{L}_{M^1}$ clause for atomics is the following:

$\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^1}} P(t_1, ..., t_n)$ iff $v^g(t_1), ..., v^g(t_n) \in D_I$ and $\langle v^g(t_1), ..., v^g(t_n)\rangle \in v^g(P)$.

The $\mathcal{L}_{M^2}$ clause for atomics is the following:

$\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^2}} P(t_1, ..., t_n)$ iff $\langle v^g(t_1), ..., v^g(t_n)\rangle \in v^g(P)$.

The $\mathcal{L}_{M^3}$ clause for atomics is the following:

$\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^3}} P(t_1, ..., t_n)$ iff $v^g(t_1), ..., v^g(t_n) \in D_I$ and $\langle v^g(t_1), ..., v^g(t_n)\rangle \in v^g(P)$ *or*, for some $t_i \in \langle t_1, ..., t_n\rangle$, $v^g(t_i) \notin D_I$.

Lemma 1. Let $\mathcal{I}$ be a model, t and t' terms, and ν^g an assignment based on $\mathcal{I}$. Let $\nu^{g'}$ be just like ν^g except that $\nu^{g'}(x) = \nu^g(t')$. Then: $\nu^g(t[t'/x]) = \nu^{g'}(t)$.

Proof. By induction on t.

If t is a constant, say c, then $t[t'/x] = c$ and $\nu^g(c) = c^{\mathcal{I}} = \nu^{g'}(c)$ (by definition of $\nu^{g'}$).

If t is a variable other than x, say y, then $t[t'/x] = y$ and $\nu^g(y) = \nu^{g'}(y)$ (by definition of $\nu^{g'}$).

If t is x, then $t[t'/x] = t'$ and $\nu^{g'}(x) = \nu^g(t')$ (by definition of $\nu^{g'}$).

□

Lemma 2. Let $\mathcal{I}$ be a model, A a formula, t a term, and ν^g an assignment based on $\mathcal{I}$. Let $\nu^{g'}$ be just like ν^g except that $\nu^{g'}(x) = \nu^g(t)$. Then: $\mathcal{I}, \nu^g \vDash A[t/x]$ iff, $\mathcal{I}, \nu^{g'} \vDash A$ - where $A[t/x]$ is the result of replacing each free occurrence of x in A by t.

Proof.

For the base case, where A is atomic, A can be either of the form: $P(t_1, ..., t_n)$, for P an n-place predicate and $t_1, ..., t_n$ terms; or of the form $t = u$, for t, u terms; or of the form $E!^M t$.

Suppose A is $P(t_1, ..., t_n)$, so that $A[t/x]$ is $P(t_1[t/x], ..., t_n[t/x])$. Then, there are three cases to check, in that the truth-conditions for atomics in $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ differ.

The case for $\mathcal{L}_{M^1}$ is as follows:

$$\begin{aligned} \mathcal{I}, \nu^{g'} \vDash_{\mathcal{L}_{M^1}} P(t_1, ..., t_n) &\Leftrightarrow \nu^{g'}(t_1), ..., \nu^{g'}(t_n) \in D_I \text{ and } \langle \nu^{g'}(t_1), ..., \nu^{g'}(t_n) \rangle \in \nu^{g'}(P) \\ &\Leftrightarrow \nu^g(t_1[t/x]), ..., \nu^g(t_n[t/x]) \in D_I \text{ and } \langle \nu^g(t_1[t/x]), ..., \nu^g(t_n[t/x]) \rangle \\ &\quad\ \in \nu^g(P) \text{ (By } \textbf{Lemma 1}) \\ &\Leftrightarrow \mathcal{I}, \nu^g \vDash_{\mathcal{L}_{M^1}} P(t_1[t/x], ..., t_n[t/x]) \end{aligned}$$

The case for $\mathcal{L}_{M^2}$ is as follows:

$$\begin{aligned} \mathcal{I}, \nu^{g'} \vDash_{\mathcal{L}_{M^2}} P(t_1, ..., t_n) &\Leftrightarrow \langle \nu^{g'}(t_1), ..., \nu^{g'}(t_n) \rangle \in \nu^{g'}(P) \\ &\Leftrightarrow \langle \nu^g(t_1[t/x]), ..., \nu^g(t_n[t/x]) \rangle \in \nu^g(P) \text{ (By } \textbf{Lemma 1}) \\ &\Leftrightarrow \mathcal{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} P(t_1[t/x], ..., t_n[t/x]) \end{aligned}$$

Finally, here is the case for $\mathcal{L}_{M^3}$:

$$\begin{aligned} \mathcal{I}, \nu^{g'} \vDash_{\mathcal{L}_{M^3}} P(t_1, ..., t_n) &\Leftrightarrow \nu^{g'}(t_1), ..., \nu^{g'}(t_n) \in D_I \text{ and } \langle \nu^{g'}(t_1), ..., \nu^{g'}(t_n) \rangle \in \nu^{g'}(P) \text{ or, for} \\ &\quad\ \text{some } t_i \in \langle t_1, ..., t_n \rangle, \nu^{g'}(t_i) \notin D_I \\ &\Leftrightarrow \nu^g(t_1[t/x]), ..., \nu^g(t_n[t/x]) \in D_I \text{ and } \langle \nu^g(t_1[t/x]), ..., \nu^g(t_n[t/x]) \rangle \\ &\quad\ \in \nu^g(P) \text{ or, for some } t_i \in \langle t_1[t/x], ..., t_n[t/x] \nu^g(t_i[t/x]) \rangle \notin D_I \\ &\quad\ \text{(By } \textbf{Lemma 1}) \\ &\Leftrightarrow \mathcal{I}, \nu^g \vDash_{\mathcal{L}_{M^3}} P(t_1[t/x], ..., t_n[t/x]) \end{aligned}$$

Suppose A is $t = u$, so that $A[t/x]$ is $t[t/x] = u[t/x]$. Then,

$$\begin{aligned}
\mathcal{I}, v^{g'} \models t = u &\Leftrightarrow v^{g'}(t), v^{g'}(u) \in D_I \text{ and } \langle v^{g'}(t)\rangle = \langle v^{g'}(u)\rangle \\
&\Leftrightarrow v^{g}(t[t/x]), v^{g}(u[t/x]) \in D_I \text{ and } \langle v^{g}(t[t/x])\rangle = \langle v^{g}(u[t/x])\rangle \\
&\ \text{(By } \textbf{Lemma 1}\text{)} \\
&\Leftrightarrow \mathcal{I}, v^{g} \models t[t/x] = u[t/x]
\end{aligned}$$

Suppose A is $E!^M t$, so that $A[t/x]$ is $E!^M t[t/x]$. Then, the case is in all similar to the previous one.

Now for the inductive step. Assume that $\mathcal{I}, v^g \models B[t/x]$ iff, $\mathcal{I}, v^{g'} \models B$ holds for all formulae B less complex than A. The induction step proceeds by cases determined by the main operator of A.

Suppose A is $\neg B$. Then,

$$\begin{aligned}
\mathcal{I}, v^{g'} \models A &\Leftrightarrow \mathcal{I}, v^{g'} \not\models B \\
&\Leftrightarrow \mathcal{I}, v^{g} \not\models B[t/x] \text{ (By Induction Hypothesis)} \\
&\Leftrightarrow \mathcal{I}, v^{g} \models A[t/x]
\end{aligned}$$

Suppose A is $B\&C$. Then,

$$\begin{aligned}
\mathcal{I}, v^{g'} \models A &\Leftrightarrow \mathcal{I}, v^{g'} \models B \text{ and } \mathcal{I}, v^{g'} \models C \\
&\Leftrightarrow \mathcal{I}, v^{g} \models B[t/x] \text{ and } \mathcal{I}, v^{g} \models C[t/x] \text{ (By Induction Hypothesis)} \\
&\Leftrightarrow \mathcal{I}, v^{g} \models A[t/x]
\end{aligned}$$

The cases for $\vee, \rightarrow, \leftrightarrow$ are as obvious. The case for $\forall$ is as follows:

Suppose A is $\forall x B$. Then,

$$\begin{aligned}
\mathcal{I}, v^{g'} \models \forall x B &\Leftrightarrow \mathcal{I}, v^{g'}[d/x] \models B \text{ for each } d \in D_O \\
&\Leftrightarrow \mathcal{I}, v^{g'}[d'/x] \models B \\
&\Leftrightarrow \mathcal{I}, v^{g}[d'/x] \models B[t/x] \text{ (By Induction Hypothesis)} \\
&\Leftrightarrow \mathcal{I}, v^{g}[d/x] \models B[t/x] \text{ for each } d \in D_O \text{ (Because } d' \text{ was arbitrarily chosen)} \\
&\Leftrightarrow \mathcal{I}, v^{g} \models \forall x B[t/x]
\end{aligned}$$

□

Lemma 3. Let A be a formula, $\mathcal{I}, \mathcal{I}'$ two models with $D_O^{\mathcal{I}} = D_O^{\mathcal{I}'}$, and v^g a variable assignment on $D_O^{\mathcal{I}} = D_O^{\mathcal{I}'}$. Suppose that $t^{\mathcal{I}} = t^{\mathcal{I}'}$ for every object in $D_O^{\mathcal{I}} = D_O^{\mathcal{I}'}$, and every term t occurring in A. Moreover, suppose that $R^{\mathcal{I}} = R^{\mathcal{I}'}$ for every n-tuple in $D_O^{\mathcal{I}n} = D_O^{\mathcal{I}'n}$ and n-place predicate R occurring in A. Then, $\mathcal{I}, v^g \models A$ iff $\mathcal{I}', v^g \models A$.

Proof. We write the assignment v^g based on $\mathcal{I}$ as $v^g_{\mathcal{I}}$, and first prove that, for every term t, it holds that $v^g_{\mathcal{I}}(t) = v^g_{\mathcal{I}'}(t)$. The proof is by induction on t.

Suppose t is a constant, say c. Then $v^g_{\mathcal{I}}(t) = c^{\mathcal{I}} = c^{\mathcal{I}'} = v^g_{\mathcal{I}'}(t)$.

Suppose t is a variable, say x, and let $x^{\mathcal{I}}_{v^g}$ be the object in $D^{\mathcal{I}}_O$ denoted by x under assignment v^g, based on $\mathcal{I}$. Then, $v^g_{\mathcal{I}}(x) = x^{\mathcal{I}}_{v^g} = x^{\mathcal{I}'}_{v^g} = v^g_{\mathcal{I}'}(x)$.

We now prove **Lemma 3** by induction on A.

For the base case, where A is atomic, A can be of the form: $P(t_1, ..., t_n)$, for P an n-place predicate and $t_1, ..., t_n$ terms; or of the form $t = u$, for t, u terms; or of the form $E!^M t$.

Suppose A is $P(t_1, ..., t_n)$. Then, there are three cases to check, in that the truth-conditions for atomics in $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ differ.

The case for $\mathcal{L}_{M^1}$ is as follows:

$$\begin{aligned}
\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^1}} P(t_1, ..., t_n) &\Leftrightarrow v^g_{\mathcal{I}}(t_1), ..., v^g_{\mathcal{I}}(t_n) \in D_I \text{ and } \langle v^g_{\mathcal{I}}(t_1), ..., v^g_{\mathcal{I}}(t_n)\rangle \in v^g_{\mathcal{I}}(P) \\
&\Leftrightarrow v^g_{\mathcal{I}}(t_1) = v^g_{\mathcal{I}'}(t_1), ..., v^g_{\mathcal{I}}(t_n) = v^g_{\mathcal{I}'}(t_n) \text{ (By the first part of } \textbf{Lemma 3}\text{)} \\
&\Leftrightarrow v^g_{\mathcal{I}'}(t_1), ..., v^g_{\mathcal{I}'}(t_n) \in D_I \text{ and } \langle v^g_{\mathcal{I}'}(t_1), ..., v^g_{\mathcal{I}'}(t_n)\rangle \in v^g_{\mathcal{I}'}(P) \\
&\Leftrightarrow \mathcal{I}', v^g \vDash_{\mathcal{L}_{M^1}} P(t_1, ..., t_n)
\end{aligned}$$

The case for $\mathcal{L}_{M^2}$ is as follows:

$$\begin{aligned}
\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^2}} P(t_1, ..., t_n) &\Leftrightarrow \langle v^g_{\mathcal{I}}(t_1), ..., v^g_{\mathcal{I}}(t_n)\rangle \in v^g_{\mathcal{I}}(P) \\
&\Leftrightarrow v^g_{\mathcal{I}}(t_1) = v^g_{\mathcal{I}'}(t_1), ..., v^g_{\mathcal{I}}(t_n) = v^g_{\mathcal{I}'}(t_n) \text{ (By the first part of } \textbf{Lemma 3}\text{)} \\
&\Leftrightarrow \langle v^g_{\mathcal{I}'}(t_1), ..., v^g_{\mathcal{I}'}(t_n)\rangle \in v^g_{\mathcal{I}'}(P) \\
&\Leftrightarrow \mathcal{I}', v^g \vDash_{\mathcal{L}_{M^2}} P(t_1, ..., t_n)
\end{aligned}$$

The case for $\mathcal{L}_{M^3}$ is as follows:

$$\begin{aligned}
\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^3}} P(t_1, ..., t_n) &\Leftrightarrow v^g_{\mathcal{I}}(t_1), ..., v^g_{\mathcal{I}}(t_n) \in D_I \text{ and } \langle v^g_{\mathcal{I}}(t_1), ..., v^g_{\mathcal{I}}(t_n)\rangle \in v^g_{\mathcal{I}}(P), \text{ or} \\
&\quad \text{for some } t_i \in \langle t_1, ..., t_n\rangle, v^g_{\mathcal{I}}(t_i) \notin D_I. \\
&\Leftrightarrow v^g_{\mathcal{I}}(t_1) = v^g_{\mathcal{I}'}(t_1), ..., v^g_{\mathcal{I}}(t_n) = v^g_{\mathcal{I}'}(t_n) \text{ (By the first part of } \textbf{Lemma 3}\text{)} \\
&\Leftrightarrow v^g_{\mathcal{I}'}(t_1), ..., v^g_{\mathcal{I}'}(t_n) \in D_I \text{ and } \langle v^g_{\mathcal{I}'}(t_1), ..., v^g_{\mathcal{I}'}(t_n)\rangle \in v^g_{\mathcal{I}'}(P) \text{ or} \\
&\quad \text{for some } t_i \in \langle t_1, ..., t_n\rangle, v^g_{\mathcal{I}'}(t_i) \notin D_I. \\
&\Leftrightarrow \mathcal{I}', v^g \vDash_{\mathcal{L}_{M^3}} P(t_1, ..., t_n)
\end{aligned}$$

Suppose A is $t = u$. Then,

$$\begin{aligned} \mathcal{I}, v^g \vDash t = u &\Leftrightarrow v^g_{\mathcal{I}}(t), v^g_{\mathcal{I}}(u) \in D_I \text{ and } \langle v^g_{\mathcal{I}}(t)\rangle = \langle v^g_{\mathcal{I}}(u)\rangle \\ &\Leftrightarrow v^g_{\mathcal{I}}(t) = v^g_{\mathcal{I}'}(t), v^g_{\mathcal{I}}(u) = v^g_{\mathcal{I}'}(u) \text{ (By the first part of } \textbf{Lemma 3}\text{)} \\ &\Leftrightarrow v^g_{\mathcal{I}'}(t), v^g_{\mathcal{I}'}(u) \in D_I \text{ and } \langle v^g_{\mathcal{I}'}(t)\rangle = \langle v^g_{\mathcal{I}'}(u)\rangle \\ &\Leftrightarrow \mathcal{I}', v^g \vDash t = u \end{aligned}$$

Suppose A is $E!^M t$. Then, the case is in all similar to the previous one.

Now for the inductive step. Assume that $\mathcal{I}, v^g \vDash A$ iff $\mathcal{I}', v^g \vDash A$ holds for all formulae B less complex than A. The induction step proceeds by cases determined by the main operator of A.

Suppose A is $\neg B$. Then,

$$\begin{aligned} \mathcal{I}, v^g \vDash A &\Leftrightarrow \mathcal{I}, v^g \nvDash B \\ &\Leftrightarrow \mathcal{I}', v^g \nvDash B \text{ (By Induction Hypothesis)} \\ &\Leftrightarrow \mathcal{I}', v^g \vDash A \end{aligned}$$

Suppose A is $B\&C$. Then,

$$\begin{aligned} \mathcal{I}, v^g \vDash A &\Leftrightarrow \mathcal{I}, v^g \vDash B \text{ and } \mathcal{I}, v^g \vDash C \\ &\Leftrightarrow \mathcal{I}', v^g \vDash B \text{ and } \mathcal{I}', v^g \vDash C \text{ (By Induction Hypothesis)} \\ &\Leftrightarrow \mathcal{I}', v^g \vDash A \end{aligned}$$

The cases for $\vee, \rightarrow, \leftrightarrow$ are as obvious. The case for $\forall$ is as follows:

$$\begin{aligned} \mathcal{I}, v^g \vDash \forall x B &\Leftrightarrow \mathcal{I}, v^g[d/x] \vDash B \text{ for each } d \in D_O \\ &\Leftrightarrow \mathcal{I}, v^g[d'/x] \vDash B \\ &\Leftrightarrow \mathcal{I}', v^g[d'/x] \vDash B \text{ (By Induction Hypothesis)} \\ &\Leftrightarrow \mathcal{I}', v^g[d/x] \vDash B \text{ for each } d \in D_O \text{ (Because } d' \text{ was arbitrarily chosen)} \\ &\Leftrightarrow \mathcal{I}', v^g \vDash \forall x B \end{aligned}$$

□

Having established these three lemmata for $\mathcal{L}_{M^1}$, $\mathcal{L}_{M^2}$ and $\mathcal{L}_{M^3}$ we now establish their soundness.

Soundness of $\mathcal{L}_{M^2}$

Theorem 5. (Soundness of $\mathcal{L}_{M^2}$). If $\Gamma \vdash_{\mathcal{L}_{M^2}} \varphi$, where Γ is a set of undischarged assumptions, then $\Gamma \vDash_{\mathcal{L}_{M^2}} \varphi$.

Proof. Let δ be a derivation of φ in $\mathcal{L}_{M^2}$. The proof is by induction on the number of inferences in δ. For the induction basis, suppose the number of inferences is 0. In this case, δ consists only of a single sentence φ, i.e., an assumption. That assumption is undischarged, since assumptions can only be discharged by inferences, and there are none. So, any structure $\mathfrak{I}$ and assignment ν^g that satisfy all of the undischarged assumptions of the proof also satisfy φ.

Now for the inductive step. By Induction Hypothesis, the premises of the lowermost inference follow from the undischarged assumptions of the sub-derivations ending in those premises. We have to show that the conclusion φ follows from the undischarged assumptions of the entire proof. We only consider the following cases: 1. where the lowermost inference is $\forall E$; 2. where it is $\forall I$; 3. where it is REI; 4. where it is RII; 5. where it is $Ind.Id.$; 6. where it is $= I'$. It is routine to check that the classical rules for the connectives are sound with respect to the semantics of $\mathcal{L}_{M^2}$.

Case 1. Suppose the lowermost inference is $\forall E$. Then, δ has the following form:

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\ \delta_1\\ \forall x A\end{array}}{A[t/x]}\ \forall E$$

Let $\mathfrak{I}$ be an $\mathcal{L}_{M^2}$ model and ν^g an assignment such that $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} \varphi$, for each $\varphi \in \Gamma$. Since, by Induction Hypothesis, $\Gamma \vDash_{\mathcal{L}_{M^2}} \forall x A$, we also have $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} \forall x A$. Let $\nu^g(t) = o$. If $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} \forall x A$, by the semantic clause for $\forall$, it follows that $\mathfrak{I}, \nu^g[o/x] \vDash_{\mathcal{L}_{M^2}} A$. By **Lemma 2**, $\mathfrak{I}, \nu^g[o/x] \vDash_{\mathcal{L}_{M^2}} A$ iff, $\mathfrak{I}, \nu^g[o/t] \vDash_{\mathcal{L}_{M^2}} A[t/x]$. So, $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} A[t/x]$. Hence, for any model $\mathfrak{I}$ and assignment ν^g, if $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} \Gamma \Rightarrow \mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} \forall x A$, it follows that $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} A[t/x]$.

Case 2. Suppose the lowermost inference is $\forall I$. In this case, δ has the following form:

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\ \delta_1\\ A[t/x]\end{array}}{\forall x A}\ \forall I$$

provided t does not appear in Γ or A.

Let $\mathfrak{I}$ and ν^g be an $\mathcal{L}_{M^2}$ model and an assignment respectively, such that $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} \varphi$, for each $\varphi \in \Gamma$. Let $\mathfrak{I}'$ be just like $\mathfrak{I}$ except that $\nu^g(t) = \nu^g(x) = o^{\mathfrak{I}'}$. Since t does not occur in Γ, it follows that $\mathfrak{I}', \nu^g \vDash_{\mathcal{L}_{M^2}} \varphi$, for each $\varphi \in \Gamma$. By Induction Hypothesis, $\Gamma \vDash_{\mathcal{L}_{M^2}} A[t/x]$. Thus, in particular, $\mathfrak{I}', \nu^g[o^{\mathfrak{I}'}/t] \vDash_{\mathcal{L}_{M^2}} A[t/x]$. By **Lemma 2**, $\mathfrak{I}', \nu^g[o^{\mathfrak{I}'}/t] \vDash_{\mathcal{L}_{M^2}} A[t/x]$ iff, $\mathfrak{I}', \nu^g \vDash_{\mathcal{L}_{M^2}} A$. Therefore, $\mathfrak{I}', \nu^g \vDash_{\mathcal{L}_{M^2}} A$. Since t does not occur in A, by **Lemma 3**,

$\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^2}} A$. And since $o^{\mathcal{I}'}$ was arbitrarily chosen, by the semantic clause for $\forall$, it follows that $\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^2}} \forall x A$.

Case 3. Suppose the lowermost inference is REI. In this case, δ has the following form:

$$\dfrac{\begin{matrix}\Gamma \\ \vdots\ \delta_1 \\ t = t\end{matrix}}{E!^M t}\ \mathrm{REI}$$

Let $\mathcal{I}$ and v^g be an $\mathcal{L}_{M^2}$ model and an assignment respectively, such that $\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^2}} \varphi$, for each $\varphi \in \Gamma$. Then, by Induction Hypothesis $\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^2}} t = t$. Suppose, for *reductio*, that $\mathcal{I}, v^g \nvDash_{\mathcal{L}_{M^2}} E!^M t$. Then, by the semantic clause for $E!^M$, it follows that $v^g(t) \notin D_I$; and therefore, by the clause for identity, it follows that $\mathcal{I}, v^g \nvDash_{\mathcal{L}_{M^2}} t = t$. Against our initial assumption. Hence, for any model $\mathcal{I}$ and assignment v^g, if $\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^2}} t = t$, then $\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^2}} E!^M t$.

Case 4. Suppose the lowermost inference is RII. In this case, δ has the following form:

$$\dfrac{\begin{matrix}\Gamma \\ \vdots\ \delta_1 \\ E!^M t\end{matrix}}{t = t}\ \mathrm{RII}$$

Let $\mathcal{I}$ and v^g be an $\mathcal{L}_{M^2}$ model and an assignment respectively, such that $\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^2}} \varphi$, for each $\varphi \in \Gamma$. Then, by Induction Hypothesis $\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^2}} E!^M t$. Suppose, for *reductio*, that $\mathcal{I}, v^g \nvDash_{\mathcal{L}_{M^2}} t = t$. Then, by the semantic clause for $=$, it follows that $v^g(t) \notin D_I$; and therefore, by the semantic clause for $E!^M$, it follows that $\mathcal{I}, v^g \nvDash_{\mathcal{L}_{M^2}} E!^M t$. Against our initial assumption. Hence, for any model $\mathcal{I}$ and assignment v^g, if $\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^2}} E!^M t$, then $\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^2}} t = t$.

Case 5. Suppose the lowermost inference is Ind.Id. In this case, δ has the following form:

$$\dfrac{t = u \qquad A}{A'}\ \mathrm{Ind.\ Id.}$$

A' is obtained from A by replacing one or more occurrences of t with u, or vice versa.

Let $\mathcal{I}$ and v^g be an $\mathcal{L}_{M^2}$ model and an assignment respectively, such that $\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^2}} t = u$, and $\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^2}} A$. Suppose, for *reductio*, that $\mathcal{I}, v^g \nvDash_{\mathcal{L}_{M^2}} A'$. If $\mathcal{I}, v^g \vDash_{\mathcal{L}_{M^2}} t = u$, then by the semantic clause for identity, it follows that $v^g(t), v^g(u) \in D_I$ and $v^g(t) = v^g(u)$. Thus in particular, $v^g(t) = v^g(u)$. Now, consider all the occurrences of u in A' which replaced occurrences of t in A, and let A'' be the result of replacing such occurrences of u in A' with t. Then, A'' is just A. Thus, since $v^g(t) = v^g(u)$, it follows that $\mathcal{I}, v^g \nvDash_{\mathcal{L}_{M^2}} A''$. But given that A'' is just A, it follows that $\mathcal{I}, v^g \nvDash_{\mathcal{L}_{M^2}} A$. Against our initial assumption. Thus, for any model $\mathcal{I}$ and assignment ν^g, if $\mathcal{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} t = u$ and $\mathcal{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} A$, then $\mathcal{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} A'$.

Case 6. Suppose the lowermost inference is Id.Ind.*. In this case, δ has the following form:

$$\dfrac{\begin{matrix}[Ft] \\ \vdots\ \delta_1 \\ Fu\end{matrix} \qquad \begin{matrix}[Fu] \\ \vdots\ \delta_2 \\ Ft\end{matrix}}{t = u}\ \text{Id.Ind.}^\star$$

provided over the course of δ_1 and δ_2 F does not occur in any side premises, and t and u are distinct.

What we need to show is that if $\Gamma, Ft \vDash_{\mathcal{L}_{M^2}} Fu$ and $\Delta, Fu \vDash_{\mathcal{L}_{M^2}} Ft$, then $\Gamma, \Delta \vDash_{\mathcal{L}_{M^2}} t = u$ - where Γ, Δ are possibly empty sets of formulae, F does not occur in them, and t and u are distinct. To simplify the reasoning, we will simply show that if $\Gamma, Ft \vDash_{\mathcal{L}_{M^2}} Fu$, then $\Gamma \vDash_{\mathcal{L}_{M^2}} t = u$. For then, *a fortiori*, it follows that if $\Gamma, Ft \vDash_{\mathcal{L}_{M^2}} Fu$, then $\Gamma, \Delta \vDash_{\mathcal{L}_{M^2}} t = u$. Whence, again *a fortiori*, it follows that if $\Gamma, Ft \vDash_{\mathcal{L}_{M^2}} Fu$ and $\Delta, Fu \vDash_{\mathcal{L}_{M^2}} Ft$, then $\Gamma, \Delta \vDash_{\mathcal{L}_{M^2}} t = u$. Suppose that $\Gamma, Ft \vDash_{\mathcal{L}_{M^2}} Fu$ and let $\mathcal{I}$ be a model and ν^g an assignment such that $\mathcal{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} \varphi$, for each $\varphi \in \Gamma$. In particular, given that $\mathcal{I}$ makes every formula of Γ true, Γ does not contain any contradiction. If so, in virtue of the result established by Griffiths (2014), $=$ must occur in Γ in such a way as to allow for a derivation of $t = u$. This means that for some term a, Γ must contain at least a formula of the form $t = a$. Because every member of Γ is true in $\mathcal{I}$ given ν^g, it follows that $\mathcal{I}, \nu^g \vDash_{\mathcal{L}_{M^2}} t = a$. Thus, by the truth-conditions for $=$ in $\mathcal{L}_{M^2}$, it follows that $\nu^g(t), \nu^g(a) \in D_I$, and $\nu^g(t) = \nu^g(a)$. Consider now another model $\mathcal{I}'$ and assignment $\nu^{g'}$ based on $\mathcal{I}'$ differing from $\mathcal{I}$ at most in that $\nu^{g'}(F) = \{\nu^g(t)\}$. Since F does not appear in Γ, and since $\mathcal{I}'$ differs from $\mathcal{I}$ at most with respect to the interpretation of F, it follows that $\mathcal{I}', \nu^{g'} \vDash_{\mathcal{L}_{M^2}} \varphi$, for each $\varphi \in \Gamma$. Now, since we have that $\nu^g(t) = \nu^{g'}(t) \in \nu^{g'}(F)$, it follows that $\mathcal{I}', \nu^{g'} \vDash_{\mathcal{L}_{M^2}} Ft$; and because we assumed that $\Gamma, Ft \vDash_{\mathcal{L}_{M^2}} Fu$, we also have $\mathcal{I}', \nu^{g'} \vDash_{\mathcal{L}_{M^2}} Fu$. But then, $\nu^{g'}(u) \in \nu^{g'}(F) = \{\nu^g(t)\}$; and hence, $\nu^{g'}(u) = \nu^g(u) = \nu^{g'}(t) = \nu^g(t)$. And because $\nu^g(t) \in D_I$, we also have that $\nu^g(u) \in D_I$. Thus, $\nu^g(t), \nu^g(u) \in D_I$ and $\nu^g(t) = \nu^g(u)$. Consequently, by the semantic clause for $=$, it follows that $\mathcal{I}, \nu^g \vDash t = u$. And since $\mathcal{I}$ and ν^g were arbitrarily chosen, we can infer that $\Gamma \vDash t = u$. Thus, if $\Gamma, Ft \vDash_{\mathcal{L}_{M^2}} Fu$, then $\Gamma \vDash_{\mathcal{L}_{M^2}} t = u$. *A fortiori*, it follows that if $\Gamma, Ft \vDash_{\mathcal{L}_{M^2}} Fu$ and $\Delta, Fu \vDash_{\mathcal{L}_{M^2}} Ft$, then $\Gamma, \Delta \vDash_{\mathcal{L}_{M^2}} t = u$. □

Soundness of $\mathcal{L}_{M^1}$

Theorem 6. (Soundness of $\mathcal{L}_{M^1}$). If $\Gamma \vdash_{\mathcal{L}_{M^1}} \varphi$, where Γ is a set of undischarged assumptions, then $\Gamma \vDash_{\mathcal{L}_{M^1}} \varphi$.

Proof. The induction base is as per the soundness proof for $\mathcal{L}_{M^2}$. As for the induction step, we only show one case: where the lowermost inference is EP. All the previous cases, which were shown over the course of the soundness proof for $\mathcal{L}_{M^2}$, carry over to $\mathcal{L}_{M^1}$ - given that $\mathcal{L}_{M^1}$ is an extension of $\mathcal{L}_{M^2}$. It is routine to check that the classical rules for the connectives are sound with respect to the semantics of $\mathcal{L}_{M^1}$.

Case 1. Suppose the lowermost inference is EP. In this case, δ has the following form:

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\ \delta_1\\ Pt\end{array}}{E!^{M}t}\ \mathsf{EP}$$

Let $\mathfrak{I}$ and ν^g be an $\mathcal{L}_{M^1}$ model and an assignment respectively, such that $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^1}} \varphi$, for each $\varphi \in \Gamma$. Then, by Induction Hypothesis, $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^1}} Pt$. By the semantic clause for atomics in $\mathcal{L}_{M^1}$, it follows in particular that $\nu^g(t) \in D_I$. Then, by the semantic clause for $E!^M$, it follows that $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^1}} E!^M t$. Hence, for any model $\mathfrak{I}$ and assignment ν^g if $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^1}} Pt$, then $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^1}} E!^M t$.

□

Soundness of $\mathcal{L}_{M^3}$

Theorem 7. (Soundness of $\mathcal{L}_{M^3}$). If $\Gamma \vdash_{\mathcal{L}_{M^3}} \varphi$, where Γ is a set of undischarged assumptions, then $\Gamma \vDash_{\mathcal{L}_{M^3}} \varphi$.

Proof. The induction base is as per the soundness proof for $\mathcal{L}_{M^2}$. As for the induction step, we only show one case: where the lowermost inference is NEP. All the previous cases, which were shown over the course of the soundness proof for $\mathcal{L}_{M^2}$, carry over to $\mathcal{L}_{M^3}$ - given that $\mathcal{L}_{M^3}$ is an extension of $\mathcal{L}_{M^2}$. Again, it is routine to check that the classical rules for the connectives are sound with respect to the semantics of $\mathcal{L}_{M^3}$.

Case 1. Suppose the lowermost inference is NEP. Then, δ has the following form:

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\ \delta_1\\ \neg Pt\end{array}}{E!^{M}t}\ \mathsf{NEP}$$

Let $\mathfrak{I}$ and ν^g be an $\mathcal{L}_{M^3}$ model and an assignment respectively, such that $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^3}} \varphi$, for each $\varphi \in \Gamma$. Then, by Induction Hypothesis, we have $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^3}} \neg Pt$. By the semantic clause for negation, it follows that $\mathfrak{I}, \nu^g \nvDash_{\mathcal{L}_{M^3}} Pt$. Thus, by the semantic clause for atomics in $\mathcal{L}_{M^3}$, it follows in particular that $\nu^g(t) \in D_I$. Consequently, by the semantic clause for $E!^M$, we have $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^3}} E!^M t$. Hence, for any model $\mathfrak{I}$ and assignment ν^g if $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^3}} \neg Pt$, then $\mathfrak{I}, \nu^g \vDash_{\mathcal{L}_{M^3}} E!^M t$.

□

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