

INFORMATION FLOW IN LOGICS IN THE VICINITY OF **BB**

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ABSTRACT. Situation theory in general, and channel theory in particular, have been used to provide motivational accounts of the ternary relation semantics of relevant, substructural, and various non-classical logics. Among the constraints imposed by channel-theory, one must posit a certain existence criterion for situations which result from the composites of multiple channels (this is used in modeling *information flow*). In logics obeying a certain associativity condition, it is relatively straightforward to show that a certain such condition is met, but the problem is trickier in non-associative logics. Following Tedder (2017), where it was shown that the conjunction-conditional fragment of the logic **B** admits the existence of composite channels, I present a generalised version of the previous argument, appropriate to logics with disjunction, using the neighbourhood ternary relation semantic framework. I close by suggesting that the logic $\mathbf{BB}_{(\wedge)}^+$, which falls in between Lavers' system \mathbf{BB}^+ and the standard 'minimal' relevant logic \mathbf{B}^+ , satisfies the conditions for the general argument to go through.

1. INTRODUCTION

The tradition of situation semantics, coming out of work of Barwise and Perry in the early 80s, most famously [7], has long provided the means to interpret and motivate the ternary relation semantics for relevant and substructural logics. This connection, noted even in 1985 by Barwise and Perry [7, p.xvii], was discussed by relevant logicians in [1, p.333]. Since then, something like the situation-theoretic reading has been widely adopted for **FDE**, where the major departure from classical logic is in permitting both incomplete theories and non-trivially inconsistent theories.¹ Extensions of the situation-theoretic picture with conditional information have been used to motivate/interpret the distinctive truth conditions for conditional formulas used in the ternary relation semantics. Most notable is the tradition

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¹Indeed, this interpretation of **FDE** was presaged before the advent of situation semantics in Belnap's [10].

of channel theory, introduced by Barwise [4], which seeks to explain conditional information in terms of *channels*. These are objects (perhaps more situations, perhaps not) which form connections between pairs of situations, allowing information to flow from one to the other. Barwise along with a number of co-authors, and many in the relevant logic tradition, have sought to adapt channel theory or some other kind of situation-theoretic account of conditional information (such as Israel and Perry’s theory) to the relational semantics of relevant and substructural logics – for examples, see [6, 22, 23, 29].

Among the features of such accounts of conditional information are various *constraints* on channels, or situations supporting conditional information. For my purposes here, the most salient property concerns the *serial composition* of conditional-supporting channels/situations. Barwise [4] introduces the constraint that for any two channels, there exists a channel composing them, which must satisfy certain properties. Tedder [39] proved that the weak relevant logic \mathbf{B}^\wedge is complete with respect to those of its ternary relation models in which such composites exist, using some machinery developed by [2], but the method used there does not extend to \mathbf{B}^+ , because of the interaction between disjunction and the conditional.

There are two aims of this paper. The first is to provide a more general version of the argument of [39] using the Neighbourhood Ternary relation (NTR) semantics developed by Sylvan (née Routley) and Meyer [32, 33]. This will provide a collection of *sufficient conditions* for a logic to be complete with respect to the class of its NTR models satisfying a strong *composites-existence* property. For this purpose, I’ll first present those elements of the NTR semantics which are needed for this purpose (alongside the usual TR semantics for comparison) and briefly discuss some of the existing work on how to understand composition (and a related principle) in the TR semantic framework. In so doing, I’ll argue that we should not, *pace* Restall [29], require frames to satisfy the defining condition for the axiom form of suffixing.² Then, I’ll discuss a logic $\mathbf{BB}_{(\wedge)}^+$, extending \mathbf{B}^\wedge by slightly weaker disjunction principles than \mathbf{B}^+ , and present some evidence that it satisfies the sufficient conditions for completeness with respect to its NTR models in which channel composites exist. The upshot will be that there are reasons to think that logics in the vicinity of \mathbf{BB} can be endowed with NTR frames appropriate to model the interaction between channels and situations, at least as concerns serial composition.

2. TERNARY RELATION SEMANTICS: TWO DIFFERENT ONES

2.1. Languages. I’ll be working throughout in a small handful of propositional languages, since I’ll be discussing properties of logics which are sensitive to language extensions. To that end, the following definition is general, but basically standard.

- \mathbb{P} is a set of atomic sentence letters – these are represented by lower-case letters of the Latin alphabet.

²Similar reasoning applies to the constraint defining the axiom form of prefixing, though I won’t dwell on that.

- The basic set of connectives C has the elements \wedge, \rightarrow , each of arity 2. $C_{[\vee]}$ extends this by the binary connective \vee , and similar naming conventions apply for \circ, \leftarrow, \neg (arity 2, 2, 2, and 1, respectively) where they come up.

\mathcal{L}_0 is the set constructed from \mathbb{P} and C in the usual manner, and $\mathcal{L}_{[\vee]}$ substitutes $C_{[\vee]}$ for C , and similarly for other sets of connectives. When context will distinguish, or where the differences between these various languages don't matter, I'll use \mathcal{L} as a variable over these. Throughout, I'll use $\&$ and \Rightarrow as metalinguistic connectives – these will behave as classical conjunction and material implication – and furthermore all uses of quantifiers are metalinguistic. Universally quantified metalanguage expressions will usually just be written with bald variables. Finally, \wedge, \vee bind more strongly than \rightarrow .

2.2. Ternary Relation (TR) Semantics. Introduced by Sylvan (née Routley) and Meyer in [31], following on earlier work by Sylvan and Plumwood (née Routley) [36] and further developed in [35, 12] among other places, the ternary relation (or ‘‘Routley-Meyer’’) semantics operates on frames of the following type (I omit the * used in interpreting negation):

$$\mathfrak{F} = \langle W, N, R \rangle$$

where these elements are:

- $\emptyset \neq N \subseteq W$
- $R \subseteq W^3$

In addition to these, an order $\leq \subseteq W^2$ is defined:

$$\alpha \leq \beta := \exists \gamma \in N (R\gamma\alpha\beta)$$

and frame elements are subject to the following constraints:

- $\langle W, \leq \rangle$ is a partially ordered set
- $N \in \wp(W)^\uparrow = \{X \subseteq W \mid (\alpha \in X \ \& \ \alpha \leq \beta) \Rightarrow \beta \in X\}$
- $(R\alpha\beta\gamma \ \& \ \alpha' \leq \alpha \ \& \ \beta' \leq \beta \ \& \ \gamma \leq \gamma') \Rightarrow R\alpha'\beta'\gamma'$

Intuitively a frame includes a set of *situations*, a some of which, N , support all the logical truths, and a ternary R which interprets the implication – I'll get into this shortly. The defined order \leq expresses *containment* between situations – i.e. whenever $\alpha \leq \beta$, β will support any proposition supported by α , similarly to the heredity constraint in the frame semantics for intuitionist logic. The elements of $\wp(W)^\uparrow$, upward closed sets of situations, model propositions.

Given a frame \mathfrak{F} , the full complex algebra on \mathfrak{F} is defined:

$$\langle \wp(W)^\uparrow, N, \cap, \cup, \rightarrow, \leftarrow, \circ \rangle$$

where for all $X, Y \subseteq W$:

- $X \rightarrow Y = \{\alpha \in W \mid \forall \beta, \gamma \in W ((R\alpha\beta\gamma \ \& \ \beta \in X) \Rightarrow \gamma \in Y)\}$
- $X \leftarrow Y = \{\alpha \in W \mid \forall \beta, \gamma \in W ((R\beta\alpha\gamma \ \& \ \beta \in Y) \Rightarrow \gamma \in X)\}$
- $X \circ Y = \{\alpha \in W \mid \exists \beta, \gamma \in W (R\beta\gamma\alpha \ \& \ \beta \in X \ \& \ \gamma \in Y)\}$

The following results are standard:

Proposition 2.1. When $X, Y \in \wp(W)^\uparrow$, so is $X \otimes Y$ for $\otimes \in \{\rightarrow, \leftarrow, \circ\}$.

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Proposition 2.2. For $X, Y \in \wp(W)^\uparrow$, $N \subseteq X \rightarrow Y \iff X \subseteq Y$

A model M on a frame \mathfrak{F} is a valuation function $V : \mathbb{P} \rightarrow \wp(W)^\uparrow$ – when M is on frame \mathfrak{F} , elements of \mathfrak{F} are picked out by W_M, N_M, R_M as needed (where context makes this obvious, the convention of explicitly noting the model may be dropped). This is extended to a valuation on the full language $\llbracket \cdot \rrbracket_M : \mathcal{L} \rightarrow \wp(W)^\uparrow$ as follows:

- $\llbracket p \rrbracket_M = V(p)$ for $p \in \mathbb{P}$
- $\llbracket A \wedge B \rrbracket_M = \llbracket A \rrbracket_M \cap \llbracket B \rrbracket_M$
- $\llbracket A \vee B \rrbracket_M = \llbracket A \rrbracket_M \cup \llbracket B \rrbracket_M$
- $\llbracket A \otimes B \rrbracket_M = \llbracket A \rrbracket_M \otimes \llbracket B \rrbracket_M$ for $\otimes \in \{\rightarrow, \leftarrow, \circ\}$

$\alpha \in \llbracket A \rrbracket_M$ is often written as $M, \alpha \Vdash A$, or some variation thereon. Finally, A is *true-in- M* if and only if $N_M \subseteq \llbracket A \rrbracket_M$. A is furthermore *valid-in- \mathfrak{F}* iff true-in- M for every M on \mathfrak{F} , and where \mathcal{F} is a set of frames, A is *valid-in- \mathcal{F}* iff it is valid in all the frames in \mathcal{F} . These are written $\Vdash_\dagger A$ where \dagger is the name of a model, frame, or set of frames.³

2.3. Neighbourhood Ternary Relation (NTR) Semantics. The Neighbourhood TR semantics can be understood as capitalising on the central feature that $N \subseteq X \rightarrow Y \iff X \subseteq Y$, while generalising the TR framework. Some of the complexity of the TR semantics arises from treating of this feature *via* situations, as it were. By generalising just a bit, it can be captured more directly and simply. Sylvan and Meyer introduced this framework in [32, 33] (a particularisation of yet more general semantics [34]) in order to provide a general account of implication connectives. I'll consider the framework as also incorporating the connectives \wedge, \vee (in their 'normal' varieties as per [32]).⁴ Here an NTR frame is a structure:

$$\mathfrak{F} = \langle W, N, R, Prop \rangle$$

where W, N are as before, and furthermore:

- $Prop \subseteq \wp(W)$
- $R \subseteq W \times \wp(W) \times \wp(W)$

satisfying the constraints:

(C0) $Prop$ is closed w.r.t. \cap, \cup , and \rightarrow , where

$$X \rightarrow Y = \{\alpha \in W \mid R\alpha XY\}$$

(C1) $N \subseteq X \rightarrow Y \iff X \subseteq Y$, for all $X, Y \in Prop$

(C2) $X = X', Y = Y'$ only if $X \rightarrow Y = X' \Rightarrow Y'$

³There are a number of options available for defining relations of consequence (Tarski, Scott, or otherwise) of relevant logics, but some are better than others, depending on the purpose – see [1, p.169] for some discussion. For the sake of simplicity, I'll deal here just with theoremhood, and leave considerations of the correct notion of *relevant* logical consequence to the side.

⁴Goble [17] further developed (a variant on) the NTR framework, proving adequacy and decidability results for a number of systems similar to those discussed in this paper.

The inclusion of *Prop* is not strictly necessary, and involves a bit of additional complexity in the definitions – having said that, it makes the canonical model construction part of the completeness argument simpler, so I’ll keep it.

Here, (C1) is the only substantial constraint – reflecting this basic commitment concerning conditionals encoded in Prop 2.2. In fact, this is closely connected to the justification of the basic properties of conditionals given by Sylvan and Meyer [32] (following a comment from Suszko).

A model M on an NTR frame is a $V : \mathbb{P} \longrightarrow \wp(W)$, with $\llbracket \cdot \rrbracket_M$ being defined as before, excepting that now:

$$\llbracket A \rightarrow B \rrbracket_M = \llbracket A \rrbracket_M \rightarrow \llbracket B \rrbracket_M = \{\alpha \mid R\alpha \llbracket A \rrbracket_M \llbracket B \rrbracket_M\}$$

The definition of the various \vDash ’s are as before. The basic NTR logic **F** is axiomatisable as follows – I shall use an “r” to distinguish the rules from the axioms, as similar names will recur again in the next section.⁵ The \Rightarrow used in the statement of the rules should be understood to express the fact that these are rules of proof – i.e. $A_1, \dots, A_n \Rightarrow B$ reads “from derivations of A_1, \dots, A_n , one may derive B .”

- (Id) $A \rightarrow A$
- (\wedge E) $A \wedge B \rightarrow A, A \wedge B \rightarrow B$
- (\vee I) $A \rightarrow A \vee B, B \rightarrow A \vee B$
- (Dist) $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$
- (rMP) $A \rightarrow B, A \Rightarrow B$
- (rAdj) $A, B \Rightarrow A \wedge B$
- (r \wedge I) $A \rightarrow B, A \rightarrow C \Rightarrow A \rightarrow B \wedge C$
- (r \vee E) $A \rightarrow C, B \rightarrow C \Rightarrow A \vee B \rightarrow C$
- (rWB) $A \rightarrow B, B \rightarrow C \Rightarrow A \rightarrow C$
- (rCong) $A' \leftrightarrow A, B' \leftrightarrow B \Rightarrow (A \rightarrow B) \rightarrow (A' \rightarrow B')$

Given a logic **L** defined by some axiomatic presentation, let $\vdash_{\mathbf{L}}$ express theoremhood of that logic as usual – i.e. every axiom of **L** is a theorem of **L**, and any formula obtained from theorems of **L** by the use of rules of **L** is a theorem of **L**.

2.4. Some NTR Frame Constraints. A key part of this semantics, as in all frame semantics, is the association between valid axioms/rules, and frame constraints, closely tying logics to certain NTR frames (for our purposes, the one which most matters is that of the canonical model).

In the NTR semantics, this is pretty trivial, at least as concerns the usual axiomatic extensions of the basic relevant logics. One need only rewrite the formula you want in the language of the frame, and **F** will be extended to obtain the logic

⁵The name **F** is used in [32]. This system is called **Min** by Goble [17], and a closely related system is discussed under the name “**B**” in [9, p. 93].

of those frames by the addition of the axiom matching the condition. Logics resulting from **F** by extensions by the following axioms and rules are important for our purposes.⁶

- (rB) $A \rightarrow B \Rightarrow (C \rightarrow A) \rightarrow (C \rightarrow B)$
- (rB') $A \rightarrow B \Rightarrow (B \rightarrow C) \rightarrow (A \rightarrow C)$
- (\wedge I) $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
- (\vee E) $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$
- (WB) $(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$
- (B) $(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$
- (B') $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$

In keeping with the triviality claim, here are two examples of NTR frame constraints defining these rules/axioms (the others follow the same pattern):

- (rB) $X \subseteq Y \Rightarrow Z \rightarrow X \subseteq Z \rightarrow Y$
- (\wedge I) $(X \rightarrow Y) \cap (X \rightarrow Z) \subseteq X \rightarrow (Y \cap Z)$

The basic idea here is that, as generally holds in relevant logics and their NTR models, a conditional formula $A \rightarrow B$ is valid just in case every element of an NTR model will satisfy A (or “is a member of A ’s proposition”) only if it satisfies B . That is, $A \rightarrow B$ will be valid iff for every M , $\llbracket A \rrbracket_M \subseteq \llbracket B \rrbracket_M$, as set out in (C1). So in order to guarantee that (\wedge I) is valid, we can just ensure that whenever $\alpha \in \llbracket (A \rightarrow B) \wedge (A \rightarrow C) \rrbracket_M$ then $\alpha \in \llbracket A \rightarrow B \wedge C \rrbracket_M$, which is what the homophonic NTR frame constraint delivers. In the case of rules, the only difference is that we have to consider (meta-level) implications between inclusions to model rules allowing us to infer implication formulas from other implication formulas. So the NTR frame constraint for (rB) just recapitulates that rule at the level of implications between inclusion statements between propositions.

The logic resulting from adding the first four principles listed above, i.e. (rB)–(\vee E), to **F** is **B**, the logic of all TR frames and the ‘basic affixing system’.⁷ The addition of only the first two is **BB**, a logic motivated by Peter Lavers [20, 21] as a particularly nice extension of **FDE** better capturing *tautological entailments*, as originally presented by Anderson and Belnap. I’ll write the name of a logic with a subscripted name of one of these axioms in order to name a system of interest – for my purposes, the most important one here is **BB**_(\wedge I).

Proposition 2.3. If **L** extends **BB** in the language $\{\rightarrow, \wedge\}$ and has (\wedge I), then for any NTR frame for **L**, and $\{X_i\}_{i \in I}, \{Y_i\}_{i \in I} \subseteq Prop$ (for finite I):

$$\bigcap_{i \in I} (X_i \rightarrow Y_i) \subseteq \bigcap_{i \in I} X_i \rightarrow \bigcap_{i \in I} Y_i$$

⁶It should be noted that I include the combinator names because they are commonly used in the literature on relevant and substructural logics, but that I don’t mean by their inclusion to suggest anything deeper about the logics (for instance, I don’t mean to claim that they include \circ in the language, or can otherwise represent the central application operation of combinatory logic).

⁷Whenever I write generally about logics, I’ll always mean extensions of **F**.

Proof. Note that $\bigcap_{i \in I} (X_i \rightarrow Y_i) \subseteq X_j \rightarrow Y_j$ for each $j \in I$, and furthermore $X_j \rightarrow Y_j \subseteq \bigcap_{i \in I} X_i \rightarrow Y_j$ follows from $(\wedge E)$ and (rB') when $j \in I$, so $\bigcap_{i \in I} (X_i \rightarrow Y_i) \subseteq \bigcap_{i \in I} X_i \rightarrow Y_j$ for each Y_j . It follows that $\bigcap_{i \in I} (X_i \rightarrow Y_i) \subseteq \bigcap_{j \in I} (\bigcap_{i \in I} X_i \rightarrow Y_j)$, and so $\bigcap_{i \in I} (X_i \rightarrow Y_i) \subseteq \bigcap_{i \in I} X_i \rightarrow \bigcap_{i \in I} Y_i$ follows by $(\wedge I)$. \square

As a final note, while the first four axioms/rules are valid in all TR frames, the others are not. To state their TR frame correspondents, it is easiest to employ the following standard notational conventions:

$$\begin{aligned} R\alpha\beta\gamma\delta &:= \exists\epsilon(R\alpha\beta\epsilon \ \& \ R\epsilon\gamma\delta) \\ R\alpha(\beta\gamma)\delta &:= \exists\epsilon(R\beta\gamma\epsilon \ \& \ R\alpha\epsilon\delta) \end{aligned}$$

Then the corresponding TR frame constraints are:

- (B) $R\alpha\beta\gamma\delta \Rightarrow R\alpha(\beta\gamma)\delta$
- (B') $R\alpha\beta\gamma\delta \Rightarrow R\beta(\alpha\gamma)\delta$

For further details on frame correspondence in TR semantics, see [35, 30], and for NTR see [32, 33, 17].

2.5. Canonical NTR Models. Below I'll be working with the prime theories of various logics, squeezed into an NTR structure – i.e. I'll be working with the *canonical models* of these logics. This section collects some important facts about these structures.

Definition 2.1. α is an **L**-theory iff

- $\vdash_{\mathbf{L}} A \rightarrow B$ and $A \in \alpha$ imply $B \in \alpha$
- $A, B \in \alpha$ implies $A \wedge B \in \alpha$

α is *prime* iff $A \vee B \in \alpha$ only if one of A, B is in α .

α is *normal* iff $\vdash_{\mathbf{L}} A$ implies $A \in \alpha$.

Given a set of formulas Γ and a logic **L**, let Γ 's *theory-extension* be:

$$[\Gamma] = \{B \mid \exists A_1, \dots, A_n \in \Gamma (\vdash_{\mathbf{L}} \bigwedge_{1 \leq i \leq n} A_i \rightarrow B)\}$$

Note that $[\Gamma]$ is always at least an **L**-theory, but there is no guarantee of primeness or normality. The NTR canonical model of a logic **L** is a tuple

$$M_{\mathbf{L}} = \langle W_{\mathbf{L}}, N_{\mathbf{L}}, R_{\mathbf{L}}, Prop_{\mathbf{L}}, V_{\mathbf{L}} \rangle$$

where:

- $W_{\mathbf{L}}$ is the set of theories of **L** – when the language of **L** includes disjunction, it is the set of all *prime* theories of **L**.⁸
- $N_{\mathbf{L}} \subseteq W_{\mathbf{L}}$ is the set of normal elements of $W_{\mathbf{L}}$.

Definition 2.2. Given an $M_{\mathbf{L}}$, $\llbracket A \rrbracket_{\mathbf{L}} := \{\alpha \in W_{\mathbf{L}} \mid A \in \alpha\}$.

⁸Note that in the terminology of [32, 33] it is only logics with *normal* disjunction that are considered here.

- $R_L \alpha XY \iff \exists B, C (X = \llbracket B \rrbracket_L \ \& \ Y = \llbracket C \rrbracket_L \ \& \ B \rightarrow C \in \alpha)$
- $Prop_L = \{X \subseteq W_L \mid \exists A (X = \llbracket A \rrbracket_L)\}$
- $V_L(p) = \llbracket p \rrbracket_L$

Proposition 2.4. The following facts are true of canonical NTR models – the proofs are either available in [32] or are immediate:

- $\forall A \exists X \in Prop_L (X = \llbracket A \rrbracket_L)$
- $\forall X \in Prop_L \exists A (X = \llbracket A \rrbracket_L)$
- $\vdash_L A \rightarrow B \iff \llbracket A \rrbracket_L \subseteq \llbracket B \rrbracket_L$

The next fact is important for later:

Proposition 2.5. The following are equivalent, for any formulas A, B and any (prime) theories α, β :

- $\exists C \in \mathcal{L} (A \rightarrow C \in \alpha \ \& \ C \rightarrow B \in \beta)$
- $\exists X \in Prop_L (R_L \alpha \llbracket A \rrbracket_L X \ \& \ R_L \beta X \llbracket B \rrbracket_L)$

As a shorthand, I'll write the latter as $R_L \beta (\alpha \llbracket A \rrbracket_L) \llbracket B \rrbracket_L$, following the pattern of $R\alpha(\beta\gamma)\delta$ in the TR semantics.

Finally, the key result is the following.

Proposition 2.6 (Adequacy). If L is any logic extending F by any of the axioms/rules listed above, then:

$$\models_{M_L} A \text{ iff } \vdash_L A.$$

A more general result is proved in [32] (they consider more frame correspondents, as well as logics with negation).

3. CHANNEL COMPOSITION

Barwise and co-authors, in a couple of papers in the early 90s [4, 6], developed channel theory in terms of a ternary relation semantics which is, more or less, an extension of TR semantics. So the central ingredient is the relation $R\alpha\beta\gamma$ representing the state of affairs that α is a channel from β to γ , though he represents this ternary relation using (some variation on) the notation $\beta \xrightarrow{\alpha} \gamma$. In particular, he stipulates the existence of two kinds of composite channels, though I'll only be discussing *serial composition*. The serial composite of channels α, β is a channel $\alpha; \beta$ such that, for any situations γ, δ :

$$R(\alpha; \beta)\gamma\delta \iff R\beta(\alpha\gamma)\delta$$

This is meant to represent/simulate the effect of linking two channels up in series, and allowing information to flow along the first, and then the second, from the input situation to the output. Given the shorthand above, this does seem to be capturing something like the right idea – there is some situation ϵ such that α is a channel from γ to ϵ , and furthermore β is a channel from ϵ to δ .

In light of this development, both Barwise (and his co-authors Gabbay and Hartonas) [6] and Restall [29] (in the same volume) note that the work of serial composition is easily modeled by the twisted associativity principle (B') when in this

framework, we identify the serial composition of α with β with the result of applying binary ‘operation’ encoded by the ternary relation to them.⁹ This cannot behave simply functionally, if situations are to be modeled by prime theories, for well known reasons having to do with combining relevant conditionals with distributive lattice operations. So instead we have to understand $R\alpha\beta\gamma$ as the image of the operation which, at the level of propositions, is represented by $A \circ B$. Following Restall [29], I’ll write this ‘operation’ on α, β as:

$$\alpha \circ \beta = \{\gamma \mid R\alpha\beta\gamma\}$$

and say that $\alpha \circ \beta$ satisfies a formula A when $\alpha \circ \beta \subseteq \llbracket A \rrbracket$ (and the appropriate analogue for any $X \subseteq \wp(W)^\uparrow$).

Noteworthy about $\alpha \circ \beta$ is that, given the truth condition for \rightarrow , we more or less immediately have:

$$\alpha \in X \rightarrow Y, \beta \in X \Rightarrow \alpha \circ \beta \subseteq Y$$

Suggesting that \circ is a natural candidate for modeling the result of *applying* the conditional information supported by the channel α to that information supported by β .¹⁰ A central insight of Barwise, Gabbay, Hartonas, and Restall, is that under the various associativity assumptions (which are desirable for some other reasons), it is a sensible strategy to *identify* $\alpha; \beta$ with $\alpha \circ \beta$. It can be shown:

Proposition 3.1. If L is complete w.r.t. some class of TR frames satisfying (B’), then for any model M and any $X, Y \in \wp(W^M)^\uparrow$:

$$\alpha \in X \rightarrow Z, \beta \in Z \rightarrow Y \Rightarrow \alpha \circ \beta \subseteq X \rightarrow Y$$

Proof. Suppose that $R\alpha\beta\gamma$, and furthermore that $R\gamma\delta\epsilon$ and $\delta \in X$. Thus $R\alpha\beta\delta\epsilon$, so by (B’), $R\beta(\alpha\delta)\epsilon$, and thus there is a ζ such that $R\alpha\delta\zeta$ and $R\beta\zeta\epsilon$. The former implies that $\zeta \in Z$ and the latter thus implies that $\epsilon \in Y$. So $\gamma \in X \rightarrow Y$. □

Of course, this point can be seen just by working with Barwise’s original definition, assuming that $;$ is functional. With that definition B’ and its converse *dual* combinator b' provide the desired equivalence:

$$R\alpha\beta\gamma\delta \iff R\beta(\alpha\gamma)\delta$$

The imposition of these structural rules also provides Barwise et al the means to build their target *information network* structures out of algebraic models of the associative Lambek calculus.

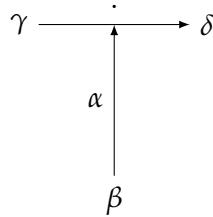
⁹The reason why it is B’ and not B here is that I employ the convention that in the truth condition for \rightarrow using the ternary R , the situation at which the conditional formula is evaluated appears in the first argument position. If one instead uses the middle argument position (reserving the first position for that at which the antecedent of the conditional is evaluated), then the transitivity principle at work here is B. There is no substantial difference between these options, and I use the convention I do as it is more standard in the relevant logic literature.

¹⁰A natural upshot of the failure of the semilattice semantics of [40] is that in the usual TR setting, these sorts of *actions on prime theories* do not, generally, produce more *prime theories*.

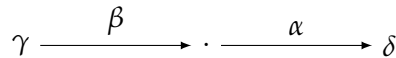
3.1. **Some Reasons to Dislike (B) and (B')**. As we've seen, in Restall and Barwise's work the axioms (B) and (B') play a substantial role. The corresponding frame constraints allow channel composition to be modeled along the same lines as application of a channel to a site. I want to survey some reasons to think that, on the channel-theoretic story, these axioms, and their corresponding frame constraints, are theoretically undesirable.

These axioms, and their corresponding frame constraints, have been criticised in some recent work by Sebastian Sequoiah-Grayson providing epistemic interpretations of the TR semantic framework and closely related structures [37, 38]. His reasons may also, to an extent, be extended to provide reasons to reject them in this setting. Indeed, the idea, pursued below, of considering actions on channels/situations is influenced by Sequoiah-Grayson's notion of an *epistemic action*. For the most part, he considers abstract counterexamples, but I also think we might be able to get some traction with more concrete counterexamples.

To aid in this, I'll use some diagrams. I'll represent $R\alpha\beta\gamma$, following Barwise [4, 6], as an indexed arrow $\beta \xrightarrow{\alpha} \gamma$, representing the fact that α is a channel between situations β, γ . Using this, we can represent the defined combination $R\alpha\beta\gamma\delta$ with the following diagram – using '.' to pick out the existentially quantified position (in this case, the arrow (channel) itself):



and we can represent $R\alpha(\beta\gamma)\delta$ as:



This makes it clear how this latter kind composition of ternary relations models composition of arrows. So the frame constraint for (B) allows one to infer the latter from the former, and that for (B') allows us to infer from the former the variation on the latter where the positions of α, β are permuted. For the second, note that we must treat β as behaving like a site in the first diagram, but as a channel in the second. It has come under some discussion [6, 29] whether we should understand channels as separate kinds of thing from sites. Treating them as separate would seem to me to provide good reason to rule out (B') immediately.

However, even if we take channels and sites just to all be situations, it is still substantial to posit these *systematic* interactions between situations. On a plausible way of understanding versions of the account where channels and sites are of the same kind, we should talk of situations *acting* as a channel in some circumstance (as, for instance, when we discuss applying conditional information holding in that

situation to some other situation) or talk of them acting as sites – the context will determine what situation plays what role where. Positing (B') amounts to demanding that whenever α , acting as a channel, applied to β produces some situation which acts as a channel from γ to δ , it must also be that β and α , *acting as channels*, can be serially composed to get from γ to δ . Perhaps there's some argument to think that situations *must* behave this way, but it strikes me as very far from obvious.

I take it that the proponent of (B') has the onus to justify the inclusion of their desired principle, but it would still be nice to have a compelling counterexample. To that end, consider the following:

Counterexample Consider a cable α taking some instructions β to a motor which, upon receiving them, will open a window, thus forming a channel allowing for information from the situation immediately outside the window γ to flow into the room δ . This is a connection of of the form $R\alpha\beta\gamma\delta$.

Question: if we take the instructions β and compose them with the cable α , do we get a channel taking γ to δ ? In other words, if we apply (the conditional information) in β to any appropriate information in γ , and take the resulting situation and apply the conditional information in α to it, do we get just δ ? That is, does it follow that $R\alpha(\beta\gamma)\delta$?

One consideration in favour of 'No' is that it's not obvious what it would mean to compose some instructions and a cable which carries those instructions. One can consider the result of applying α to β by *sending* β to the resulting channel, but this doesn't seem to involve conditional information supported by β at all, as *composing* seems to call for. The connection $R\alpha\beta\gamma\delta$ makes perfect sense, but, on the face of it, the connection $R\beta(\alpha\gamma)\delta$ doesn't make sense, without filling in more information about the information supported by β . The important point is just that the latter doesn't *follow* from the former.

3.2. Another Way to Proceed. The method in [39] suggests another way to characterise channel composition – in terms of the conditional information supported by channels, rather than by relations among channels themselves.

Definition 3.1. L admits channel composition just in case for any pair α, β of (prime) L -theories, there is another, γ , such that for any formulas A, B :

$$A \rightarrow B \in \gamma \iff \exists C(A \rightarrow C \in \alpha \ \& \ C \rightarrow B \in \beta)$$

Note that if L admits channel composition, then while not all of its models must include composite points, it is complete with respect to the set of its models which do.

Note that this definition is stronger than that initially proposed in [4], where only the right-to-left condition is required. However, it seems to me natural to make this
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more stringent demand, as if such γ 's can be shown to exist, then we'll have pinned down something closer to *the* channel composites of the various α, β .¹¹

4. SOME SUFFICIENT CONDITIONS FOR CHANNEL COMPOSITION

In this section, I'll display two sufficient conditions for a logic \mathbf{L} to admit channel composition in the sense of § 3.2.¹²

4.1. Bubbling. The first is that the logic satisfy the *Bubbling Lemma*, initially proved [2] for $\mathbf{B}^{\wedge\top}$, the extension of \mathbf{B}^{\wedge} by the addition of the propositional constant \top , as discussed in [30]. This lemma is property-ified as follows:

Definition 4.1. A logic \mathbf{L} (whose language extends $\{\rightarrow, \wedge\}$) *bubbles* iff whenever

$$\vdash_{\mathbf{L}} \bigwedge_{i \in I} (A_i \rightarrow B_i) \rightarrow (C \rightarrow D)$$

(and $\not\vdash_{\mathbf{L}} \top \rightarrow D$, if the language of \mathbf{L} contains \top) then there is a non-empty $J \subseteq I$ such that:

$$\vdash_{\mathbf{L}} C \rightarrow \bigwedge_{j \in J} A_j \quad \text{and} \quad \vdash_{\mathbf{L}} \bigwedge_{j \in J} B_j \rightarrow D.$$

Briefly, when \mathbf{L} bubbles, one can *decompose* certain validities containing nested conditionals into other validities containing at least one fewer such nesting. This is a very nice property which provides the grist for natural theory constructions. The canonical application of the bubbling lemma for $\mathbf{B}^{\wedge\top}$ is in providing nice *filter* models of the λ -calculus (with intersection types) and combinatory logic. In the relevant logic literature, Meyer and a number of co-authors over the years (see [16, 14, 28]) used this fact to build nice ternary relation models of combinatory logic.¹³

4.2. ECP.

Definition 4.2. A logic \mathbf{L} in a language extending $\{\rightarrow, \wedge, \vee\}$ has the *entailment-consequent-primeness* property (or ECP) just in case whenever:

$$\vdash_{\mathbf{L}} \bigwedge_{i \in I} (A_i \rightarrow B_i) \rightarrow \bigvee_{j \in J} (C_j \rightarrow D_j)$$

then there is a $j \in J$ s.t.

$$\vdash_{\mathbf{L}} \bigwedge_{i \in I} (A_i \rightarrow B_i) \rightarrow (C_j \rightarrow D_j)$$

This property is a special instance of a more natural property delivered by the combination of the *disjunction property*:

¹¹Restall [29] obtains a similar 'as unique as possible' construction for channel composites (that is, unique with respect to the *theory*, though not necessarily with whatever *prime* theories extend that theory) in the context with associativity.

¹²For my purposes the logics of interest from here on all extend \mathbf{B}^{\wedge} , so I take \mathbf{L} to range over just those axiomatic extensions.

¹³Note that in [3, p.624], bubbling logics are called *β -sound*.

Whenever $\vdash_{\mathbf{L}} A \vee B$, either $\vdash_{\mathbf{L}} A$ or $\vdash_{\mathbf{L}} B$

and the admissibility of a rule familiar from Harrop [18] (where A is required to contain no *relevant* instance of disjunction):

$$A \rightarrow B \vee C \Rightarrow (A \rightarrow B) \vee (A \rightarrow C)$$

4.3. Putting the Sufficient Conditions to Work.

Theorem 4.1. Let \mathbf{L} be a logic extending \mathbf{B}^\wedge with (possibly) a larger set of connectives. Then if \mathbf{L} bubbles, then for any pair of \mathbf{L} -theories α, β (prime if the language of \mathbf{L} extends $\mathcal{L}_{[\vee]}$), there is a theory $\alpha; \beta$ such that:

$$A \rightarrow B \in \alpha; \beta \iff R_{\mathbf{L}}\beta(\alpha[[A]]_{\mathbf{L}})[[B]]_{\mathbf{L}}$$

Proof. Define $(\alpha; \beta)_0 := \{A \rightarrow B \mid R_{\mathbf{L}}\beta(\alpha[[A]]_{\mathbf{L}})[[B]]_{\mathbf{L}}\}$, and let $\alpha; \beta = [(\alpha; \beta)_0]$. It is immediate that $\alpha; \beta$ is a theory, and furthermore if $R_{\mathbf{L}}\beta(\alpha[[A]]_{\mathbf{L}})[[B]]_{\mathbf{L}}$, then $A \rightarrow B \in \alpha; \beta$, so I prove the converse.

Suppose that $A \rightarrow B \in \alpha; \beta$. Then there are formulas $C'_1, \dots, C'_n \in (\alpha; \beta)_0$ such that $\vdash_{\mathbf{L}} \bigwedge_{i \in I} C'_i \rightarrow (A \rightarrow B)$ where $I = \{1, \dots, n\}$. By definition, each C'_i is of the form $C_i \rightarrow D_i$ where $R_{\mathbf{L}}\beta(\alpha[[C_i]]_{\mathbf{L}})[[D_i]]_{\mathbf{L}}$. Thus, since \mathbf{L} bubbles, there is a non-empty $J \subseteq I$ such that:

$$\vdash_{\mathbf{L}} A \rightarrow \bigwedge_{j \in J} C_j \quad \text{and} \quad \vdash_{\mathbf{L}} \bigwedge_{j \in J} D_j \rightarrow B$$

It follows that $[[A]]_{\mathbf{L}} \subseteq [[\bigwedge_{j \in J} C_j]]_{\mathbf{L}} = \bigcap_{j \in J} [[C_j]]_{\mathbf{L}}$ and also that $\bigcap_{j \in J} [[D_j]]_{\mathbf{L}} = [[\bigwedge_{j \in J} D_j]]_{\mathbf{L}} \subseteq [[B]]_{\mathbf{L}}$.

Furthermore, for every $C_j \rightarrow D_j$ there is an E_j such that $C_j \rightarrow E_j \in \alpha$ and $E_j \rightarrow D_j \in \beta$. By Prop 2.3, and the adequacy theorem, $\alpha \in \bigcap_{j \in J} [[C_j]]_{\mathbf{L}} \rightarrow \bigcap_{j \in J} [[E_j]]_{\mathbf{L}}$ and $\beta \in \bigcap_{j \in J} [[E_j]]_{\mathbf{L}} \rightarrow \bigcap_{j \in J} [[D_j]]_{\mathbf{L}}$. Let $X = \bigcap_{j \in J} [[E_j]]_{\mathbf{L}}$. Since $[[A]]_{\mathbf{L}} \subseteq \bigcap_{j \in J} [[C_j]]_{\mathbf{L}}$, by (rB'):

$$\bigcap_{j \in J} [[C_j]]_{\mathbf{L}} \rightarrow X \subseteq [[A]]_{\mathbf{L}} \rightarrow X$$

Similarly, since $\bigcap_{j \in J} [[D_j]]_{\mathbf{L}} \subseteq [[B]]_{\mathbf{L}}$, by (rB):

$$X \rightarrow \bigcap_{j \in J} [[D_j]]_{\mathbf{L}} \subseteq X \rightarrow [[B]]_{\mathbf{L}}$$

Thus, $\alpha \in [[A]]_{\mathbf{L}} \rightarrow X$ and $\beta \in X \rightarrow [[B]]_{\mathbf{L}}$, and thus $R_{\mathbf{L}}\beta(\alpha[[A]]_{\mathbf{L}})[[B]]_{\mathbf{L}}$. □

Theorem 4.2. Let \mathbf{L} be a logic in a language expanding $\mathcal{L}_{[\vee]}$, and furthermore suppose that \mathbf{L} bubbles and has ECP. Then there is a prime theory $(\alpha; \beta)'$ such that:

$$A \rightarrow B \in (\alpha; \beta)' \iff R_{\mathbf{L}}\beta(\alpha[[A]]_{\mathbf{L}})[[B]]_{\mathbf{L}}$$

Proof. We can use the construction of the previous theorem to build a theory $\alpha; \beta$ satisfying the target property. Now to obtain a prime $(\alpha; \beta)' \supseteq \alpha; \beta$, we can employ Belnap's Pair Extension lemma (details are available in [30]). For note that the following is an *independent pair* of sets of formulas:

$$\langle \alpha; \beta, \{C \rightarrow D \mid \overline{R_L} \beta(\alpha \llbracket C \rrbracket_L) \llbracket D \rrbracket_L\} \rangle$$

For suppose that there were $A'_1, \dots, A'_{m'} \in \alpha; \beta$ and $C_1 \rightarrow D_1, \dots, C_n \rightarrow D_n$ such that $\vdash_L \bigwedge_{1 \leq i \leq m'} A'_i \rightarrow \bigvee_{1 \leq j \leq n} (C_j \rightarrow D_j)$. By construction, each A'_i is entailed by a finite conjunction of conditionals, so using (rB), (rB'), it follows that:

$$\vdash_L \bigwedge_{1 \leq i \leq m} (A_i \rightarrow B_i) \rightarrow \bigvee_{1 \leq j \leq n} (C_j \rightarrow D_j)$$

By ECP, there is a $j \in J$ such that $\vdash_L \bigwedge_{1 \leq i \leq m} (A_i \rightarrow B_i) \rightarrow (C_j \rightarrow D_j)$, and thus $C_j \rightarrow D_j \in \alpha; \beta$ and so, by Theorem 4.1, $R_L \beta(\alpha \llbracket C_j \rrbracket_L) \llbracket D_j \rrbracket_L$, contrary to the assumption. Therefore, the above pair is a *pair*, and so by the pair extension lemma, can be extended to a full pair with the left element $(\alpha; \beta)'$ a prime theory extending $\alpha; \beta$ but containing no $C \rightarrow D$ such that $\overline{R_L} \beta(\alpha \llbracket C \rrbracket_L) \llbracket D \rrbracket_L$. So $(\alpha; \beta)'$ is the desired prime theory. □

Theorem 4.3. The following hold of the construction of $\alpha; \beta$ from Thm 4.1:

- (1) If $\alpha, \beta \in N$ then $\alpha; \beta \in N$
- (2) If $A \rightarrow B \in \alpha$ and $\beta \in N$ then $A \rightarrow B \in \alpha; \beta$
- (2a) If $\alpha \in N$ and $A \rightarrow B \in \beta$ then $A \rightarrow B \in \alpha; \beta$
- (3) $A \rightarrow B \in \alpha; (\beta; \gamma)$ iff $A \rightarrow B \in (\alpha; \beta); \gamma$

Proof. (1) For this, note that if $\alpha, \beta \in N$, then if $\vdash A \rightarrow B$ then $A \rightarrow B \in \alpha$ and $B \rightarrow B \in \beta$, and thus $A \rightarrow B \in \alpha; \beta$.

(2), (2a) are left to the interested reader to verify.

(3) For this, we show that $(\alpha; (\beta; \gamma))_0 = ((\alpha; \beta)_0; \gamma)_0$. This follows from the following sequence of equivalences – recall that we have shown $A \rightarrow B \in \alpha; \beta \iff A \rightarrow B \in (\alpha; \beta)_0$.

$$\begin{aligned} A \rightarrow B \in \alpha; (\beta; \gamma) &\iff A \rightarrow B \in (\alpha; (\beta; \gamma))_0 \\ &\iff \exists C (A \rightarrow C \in \alpha \ \& \ C \rightarrow B \in \beta; \gamma) \\ &\iff \exists C (A \rightarrow C \in \alpha \ \& \ C \rightarrow B \in (\beta; \gamma)_0) \\ &\iff \exists C, D (A \rightarrow C \in \alpha \ \& \ C \rightarrow D \in \beta \ \& \ D \rightarrow B \in \gamma) \\ &\iff \exists D (A \rightarrow D \in (\alpha; \beta)_0 \ \& \ D \rightarrow B \in \gamma) \\ &\iff \exists D (A \rightarrow D \in \alpha; \beta \ \& \ D \rightarrow B \in \gamma) \\ &\iff A \rightarrow B \in ((\alpha; \beta); \gamma)_0 \\ &\iff A \rightarrow B \in (\alpha; \beta); \gamma \end{aligned}$$

In short, while it may not be that $\alpha; (\beta; \gamma)$ and $(\alpha; \beta); \gamma$ agree on every formula, they do agree everywhere we care about – in the implication formulas. □

5. CANDIDATE CHANNEL-ADMITTING LOGICS

This leaves the question of which logics satisfy these sufficient conditions. Sadly there are almost none that do, but I'll present some evidence (though not a proof) that there really are *almost* none. For indeed, there is one promising candidate!

5.1. $\mathbf{BB}_{(\wedge I)}^+$. Recall that $\mathbf{BB}_{(\wedge I)}^+$ extends \mathbf{BB}^+ by the axiom $(\wedge I)$, and so extends the basic logic \mathbf{F}^+ by this axiom, as well as by (rB) and (rB') .

Theorem 5.1. $\mathbf{BB}_{(\wedge I)}^+$ has ECP.

Proof. By an application of the method of metavaluations [24, 13], it can be shown that $\mathbf{BB}_{(\wedge I)}^+$ is *metacomplete*, and thus has the disjunction property.¹⁴ It suffices, then, to show that the following rule is admissible in $\mathbf{BB}_{(\wedge I)}^+$:

$$\bigwedge_{i \in I} (A_i \rightarrow B_i) \rightarrow \bigvee_{j \in J} (C_j \rightarrow D_j) \Rightarrow \bigvee_{j \in J} (\bigwedge_{i \in I} (A_i \rightarrow B_i) \rightarrow (C_j \rightarrow D_j))$$

To that end, let $E = \bigwedge_{i \in I} (A_i \rightarrow B_i)$, and suppose that there is a $\mathbf{BB}_{(\wedge I)}^+$ model M such that $\not\models_M \bigvee_{j \in J} (E \rightarrow (C_j \rightarrow D_j))$. So then where $M = \langle W, N, R, Prop, V \rangle$, we have some $\alpha_1, \dots, \alpha_n \in \llbracket E \rrbracket_M$ ($J = \{1, \dots, n\}$) where $\alpha_j \notin \llbracket C_j \rightarrow D_j \rrbracket_M$ holds for each $j \in J$. The aim is to build a model M' with a point α such that $\alpha \in \llbracket E \rrbracket_{M'}$ and $\alpha \notin \bigcup_{j \in J} \llbracket C_j \rightarrow D_j \rrbracket_{M'}$, hence showing that $E \rightarrow \bigvee_{j \in J} (C_j \rightarrow D_j)$ is not a $\mathbf{BB}_{(\wedge I)}^+$ -theorem.

So, let's build M' out of M as follows:

- $W' = W \cup \{\alpha\}$
- $N' = N$

The other elements of M' will have to be defined inductively.

Let V' be the least function of type $\mathbb{P} \rightarrow \wp(W')$ such that:

- (a) For $\beta \in W$, $\beta \in V'(p)$ if $\beta \in V(p)$
- (b) $\alpha \in V'(p)$ if $\forall j \in J (\alpha_j \in V(p))$

Now the trick is to build R' and $Prop'$ to ensure that these satisfy the model constraints, and have a couple extra properties. These we construct by induction.

Fix $Prop'_0 = \{X \subseteq W' \mid X = V'(p)\}$.

Fix $\langle \beta, X, Y \rangle \in R'_0$ if $X, Y \in Prop'_0$ (so fix $X = \llbracket p \rrbracket_{M'}$ and $Y = \llbracket q \rrbracket_{M'}$) and either:

- (c) $\beta \in W$ and $R\beta \llbracket p \rrbracket_M \llbracket q \rrbracket_M$
- (d) $\beta = \alpha$ and $\forall j \in J (R\alpha_j \llbracket p \rrbracket_M \llbracket q \rrbracket_M)$

Then, fix:

$Prop'_{n+1} = Prop'_n \cup \{X \subseteq W' \mid \exists Y, Z \in Prop'_n (X = Y \cup Z, Y \cap Z, \text{ or } Y \rightarrow Z)\}$

and further, let $\langle \beta, X, Y \rangle \in R'_{n+1}$ if $X, Y \in Prop'_n$ (so there are $A, B \in \mathcal{L}$ such that $X = \llbracket A \rrbracket_{M'}$, $Y = \llbracket B \rrbracket_{M'}$) and either:

- (c') $\beta \in W$ and $R\beta \llbracket A \rrbracket_M \llbracket B \rrbracket_M$
- (d') $\beta = \alpha$ and $\forall j \in J (R\alpha_j \llbracket A \rrbracket_M \llbracket B \rrbracket_M)$

¹⁴While $\mathbf{BB}_{(\wedge I)}^+$ is not among the systems directly covered in [24], it is straightforward to adapt the arguments to show that it is metacomplete – the only rule not accounted for there is (rVE) , but the argument needed for this rule is similar to those given in Meyer's paper.

and set $Prop' = \bigcup_{n \in \mathbb{N}} Prop'_n$, $R' = \bigcup_{n \in \mathbb{N}} R'_n$.

Note that it follows immediately that $V' : \mathbb{P} \rightarrow Prop'$ and that $Prop'$ is closed w.r.t. \cup, \cap, \rightarrow . So (C0) is satisfied. Furthermore, we have the following lemmata more or less immediately.

Lemma 5.2. $\forall X \in Prop' \exists A \in \mathcal{L} (X = \llbracket A \rrbracket_{M'})$

Lemma 5.3. If $\beta \in W$ then $\beta \in \llbracket A \rrbracket_{M'} \iff \beta \in \llbracket A \rrbracket_M$

Proof. From the definition of V' it follows that $\beta \in V'(p) \iff \beta \in V(p)$ for all $p \in \mathbb{P}$. Furthermore, note that $\beta \in \llbracket A \rightarrow B \rrbracket_{M'} \iff R'\beta \llbracket A \rrbracket_{M'} \llbracket B \rrbracket_{M'}$, which holds only in case $R'_n \beta \llbracket A \rrbracket_{M'} \llbracket B \rrbracket_{M'}$, and thus $R\beta \llbracket A \rrbracket_M \llbracket B \rrbracket_M$. These are the two cases of an induction which pose any problem – the inductive hypothesis takes care of the rest. \square

Lemma 5.4. $\alpha \in \llbracket A \rrbracket_{M'} \iff \forall j \in J (\alpha_j \in \llbracket A \rrbracket_M)$

Proof. As before, we can prove this by induction. The base case is taken care of by the definition of V' , and the case where A is a conditional formula is taken care of by the definition of R' . The other cases are similar. \square

With these, we turn to the rest of the verification of the model constraints. For (C1) – $N' \subseteq X \rightarrow Y \iff X \subseteq Y$ for all $X, Y \in Prop'$ – note that if $X, Y \in Prop'$ there are $A, B \in \mathcal{L}$ such that $X = \llbracket A \rrbracket_{M'}$, $Y = \llbracket B \rrbracket_{M'}$. First, suppose that $\llbracket A \rrbracket_{M'} \subseteq \llbracket B \rrbracket_{M'}$. In this case, by Lem. 5.2, it follows immediately that $\llbracket A \rrbracket_M \subseteq \llbracket B \rrbracket_M$, and so $N \subseteq \llbracket A \rightarrow B \rrbracket_M$. But if $\beta \in N'$ then $\beta \in N$, so then $\beta \in \llbracket A \rightarrow B \rrbracket_M \iff \beta \in \llbracket A \rightarrow B \rrbracket_{M'} = \llbracket A \rrbracket_{M'} \rightarrow \llbracket B \rrbracket_{M'}$, as desired.

Next, suppose that $N' \subseteq \llbracket A \rightarrow B \rrbracket_{M'}$, and let $\beta \in \llbracket A \rrbracket_{M'}$. Again, we have for all $\gamma \in N$, that $\gamma \in N'$ and that $\gamma \in \llbracket A \rightarrow B \rrbracket_M \iff \gamma \in \llbracket A \rightarrow B \rrbracket_{M'}$. Thus, $N \subseteq \llbracket A \rightarrow B \rrbracket_{M'}$ and so $\llbracket A \rrbracket_{M'} \subseteq \llbracket B \rrbracket_{M'}$. Suppose that $\beta \in W$, then the result is an immediate consequence of Lem. 5.2. Suppose $\beta = \alpha$, then by Lem 5.4, it follows that:

$$\beta \in \llbracket A \rrbracket_{M'} \iff \forall j \in J (\alpha_j \in \llbracket A \rrbracket_M) \Rightarrow \forall j \in J (\alpha_j \in \llbracket B \rrbracket_M) \iff \beta \in \llbracket A \rrbracket_{M'}$$

and the result follows.

What remains are the constraints specific to $\mathbf{BB}_{(\wedge I)}^+$. These are (rB), (rB'), and ($\wedge I$). The first two are similar, so I'll only present one of those, and sketch the argument for the third.

For (rB), suppose that $X, Y, Z \in Prop'$, so that there are $A, B, C \in \mathcal{L}$ such that $X = \llbracket A \rrbracket_{M'}$, $Y = \llbracket B \rrbracket_{M'}$, and $Z = \llbracket C \rrbracket_{M'}$. Suppose $\llbracket A \rrbracket_{M'} \subseteq \llbracket B \rrbracket_{M'}$, and let $\beta \in \llbracket C \rightarrow A \rrbracket_{M'}$. If $\beta \in W$, this holds iff $\beta \in \llbracket C \rightarrow A \rrbracket_M$, and thus $\beta \in \llbracket C \rightarrow B \rrbracket_M$ since M is a $\mathbf{BB}_{(\wedge I)}^+$ model, which in turn holds iff $\beta \in \llbracket C \rightarrow B \rrbracket_{M'}$. If $\beta = \alpha$, then $\beta \in \llbracket C \rightarrow A \rrbracket_{M'}$ iff $\forall j \in J (\alpha_j \in \llbracket C \rightarrow A \rrbracket_M)$, in which case $\forall j \in J (\alpha_j \in \llbracket C \rightarrow B \rrbracket_M)$, and thus $\beta \in \llbracket C \rightarrow B \rrbracket_{M'}$, as desired.

Finally, for $(\wedge I)$, suppose that X, Y, Z are as before, and furthermore that $\beta \in (X \rightarrow Y) \cap (X \rightarrow Z)$. As before, if $\beta \in W$, then $\beta \in X \rightarrow (Y \cap Z)$ can be obtained by Lem 5.2, and if $\beta = \alpha$ the result follows by Lem 5.4.

So M' is a $\mathbf{BB}_{(\wedge I)}^+$ model, and furthermore the construction guarantees that $\alpha \in \llbracket E \rrbracket_{M'}$ and $\alpha \notin \bigcup_{j \in J} \llbracket C_j \rightarrow D_j \rrbracket_{M'}$, so $\not\models_{M'} \bigwedge_{i \in I} (A_i \rightarrow B_i) \rightarrow \bigvee_{j \in J} (C_j \rightarrow D_j)$. So the desired rule is indeed admissible in $\mathbf{BB}_{(\wedge I)}^+$.

It follows from this and the fact that $\mathbf{BB}_{(\wedge I)}^+$ has the disjunction property that $\mathbf{BB}_{(\wedge I)}^+$ has ECP. □

This leaves the last part, which is still open:

Conjecture 5.5. $\mathbf{BB}_{(\wedge I)}^+$ bubbles.

I don't have a proof to provide here, but I will provide some reasons to think that this conjecture is true.

First, an alternative axiomatisation of $\mathbf{BB}_{(\wedge I)}^+$ is helpful. The system is a *binary assertional* system in the terminology of [15] or a FMLA-FMLA system in that of [19]. A *sequent* here is a pair $\langle A, B \rangle \in \mathcal{L}^2$, which I'll write $A \leq B$. I'll occasionally refer to formula in the position of A as the 'precedent' and that in the position of B as the 'succedent.' The following system is adapted from that for $\mathbf{ITD}/\mathbf{B}^{\wedge \top}$ given in [2, 14] and elsewhere. It consists of axioms (A1)–(A5) and rules (R1)–(R6):

- (A1) $A \leq A$
- (A2) $A \wedge B \leq A, A \wedge B \leq B$
- (A3) $(A \rightarrow B) \wedge (A \rightarrow C) \leq A \rightarrow B \wedge C$
- (A4) $A \leq A \vee B, B \leq A \vee B$
- (A5) $A \wedge (B \vee C) \leq (A \wedge B) \vee (A \wedge C)$

- (R1) $A \leq C, C \leq B \Rightarrow A \leq B$
- (R2) $A \leq A', B \leq B' \Rightarrow A \wedge B \leq A' \wedge B'$
- (R3) $A \leq A', B \leq B' \Rightarrow A \vee B \leq A' \vee B'$
- (R4) $A' \leq A, B \leq B' \Rightarrow A \rightarrow B \leq A' \rightarrow B'$
- (R5) $A \leq B, A \leq C \Rightarrow A \leq B \wedge C$
- (R6) $A \leq C, B \leq C \Rightarrow A \vee B \leq C$

I'll write $\vdash_{BA} A \leq B$ to indicate that there is a proof of the sequent from the above proof system ("BA" for "binary assertional"). The following result is straightforward.

Proposition 5.6. $\vdash_{BA} A \leq B$ iff $\vdash_{\mathbf{BB}_{(\wedge I)}^+} A \rightarrow B$.

This fact, in addition to the following:

- $\vdash_{\mathbf{BB}_{(\wedge I)}^+} A \wedge B \iff \vdash_{\mathbf{BB}_{(\wedge I)}^+} A$ and $\vdash_{\mathbf{BB}_{(\wedge I)}^+} B$
- $\vdash_{\mathbf{BB}_{(\wedge I)}^+} A \vee B \iff \vdash_{\mathbf{BB}_{(\wedge I)}^+} A$ or $\vdash_{\mathbf{BB}_{(\wedge I)}^+} B$

shows that the binary assertional system captures the same theorems as the Hilbert system. This is because any theorem is either a conditional formula (which is covered by the proposition) or is a conjunction or disjunction thereof, which we can simulate, via the above facts, by collections of provable implication formulas. Finally, we can characterise bubbling in this system as the property that whenever:

$$\bigwedge_{i \in I} (A_i \rightarrow B_i) \leq C \rightarrow D$$

then there is a nonempty $J \subseteq I$ such that:

$$C \leq \bigwedge_{j \in J} A_j \quad \text{and} \quad \bigwedge_{j \in J} B_j \leq D$$

While I won't prove this result (for lack of such a proof), I will provide some evidence which, I suspect, when assembled the right way does provide a proof. On its own, I take the following facts as evidence that $\mathbf{BB}_{(\wedge)}^+$ bubbles.

Proposition 5.7. No instance of any axiom of the binary assertional system is a counterexample to bubbling.

Proof. By inspection. □

Proposition 5.8. No application of (R2)–(R6) produces a counterexample to bubbling.

Proof. This takes a bit more spelling out, but is mostly by inspection. First note that (R2), (R3), (R5), and (R6) have conclusion sequents which have no instances of the form $\bigwedge_{i \in I} (A_i \rightarrow B_i) \leq C \rightarrow D$, so none of these can produce counterexamples.

As for (R4), we can obtain something of the right form, but where I is a singleton: $A \rightarrow B \leq C \rightarrow D$, and we can only get there when we've proved $C \leq A$ and $B \leq D$, which works. □

So, as usual, (R1) is the troublemaker. It seems that some kind of cut-elimination style argument is called for here, but the details seem to be devilish. What I'll present here are some 'low-in-the-induction' cases, showing that very shorts proof in the binary assertional system don't produce counterexamples to bubbling. I'll consider, first, the application of (R1) to a pair of instances of axioms, showing that no counterexample can arise there, and then go on to consider a case where one of the premises to (R1) is the result of applying (R1) to axioms, and so that also there no counterexamples can arise.

In order to state these results more clearly, I'll need some notation. We are after applications of (R1) which can produce conclusions of the form $(\text{Con}) \bigwedge_{i \in I} (A_i \rightarrow B_i) \leq C \rightarrow D$, so we are interested in a left premise (Lp), a right premise (Rp), cut together on a cut formula, as below:

$$(\text{Lp}) \bigwedge_{i \in I} (A_i \rightarrow B_i) \leq E \quad \text{and} \quad (\text{Rp}) E \leq C \rightarrow D$$

Proposition 5.9. Wherever (Lp) and (Rp) are both instances of axioms, no counterexample to bubbling is provable.

Proof. Cases are numbered by pairs $\langle i, j \rangle$ where $1 \leq i, j \leq 5$, and these numbers stand in for the similarly numbered axiom. So, for instance $\langle 4, 5 \rangle$ provide premises for (R1) where Lp is an instance of (A4) and Rp an instance of (A5).

First note that merely in virtue of the forms that we need for Lp and Rp, a number of cases are ruled out. For instance, (A4) has a succedent where the main connective is a disjunction, but the only right premises we are concerned with have a conditional as the main connective. Hence $\langle 1, 4 \rangle$ is of the wrong form as (A4) has succedents of the form $A_1 \vee B_1$ whereas our Rp must have a succedent in which the main connective is a conditional. Another way for a pair of axioms to be ruled out is that they could not form an instance of cut, because E could not have both of the forms required. Hence, $\langle 3, 2 \rangle$ is ruled out, as then E would have to be both of the form $A_1 \rightarrow (B_1 \wedge C_1)$ for (A3) and of the form $A_2 \wedge B_2$ for (A2), and these are incompatible. These two considerations – having the wrong form to produce an instance of (Con), which cases I'll call (WF), or having no cut formula, which I'll call (NCF) – rule out of the following cases:

- WF: $\langle 1, 4 \rangle, \langle 1, 5 \rangle, \langle 2, 4 \rangle, \langle 2, 5 \rangle, \langle 4, 4 \rangle$ and $\langle 5, n \rangle$ for any $1 \leq n \leq 5$.
- NCF: $\langle 3, 2 \rangle, \langle 3, 3 \rangle, \langle 3, 5 \rangle, \langle 4, 3 \rangle, \langle 4, 5 \rangle$

I'll briefly consider some of the remaining cases, to show that no counterexample can arise (the others are similar to those considered here).

- $\langle 1, 1 \rangle$ In this case, both the left and right premise must be $C \rightarrow D \leq C \rightarrow D$, and we have that $C \leq C$ and $D \leq D$.
- $\langle 1, 2 \rangle$ In this case, the right premise has the form $A' \wedge (C \rightarrow D) \leq C \rightarrow D$ and in order for the left premise to have the desired form, it must be that A' is a conjunction of conditionals. So the left premise must be of the form $\bigwedge_{i \in I'} (A_i \rightarrow B_i) \wedge (C \rightarrow D) \leq \bigwedge_{i \in I'} (A_i \rightarrow B_i) \wedge (C \rightarrow D)$ and the right premise the form $\bigwedge_{i \in I'} (A_i \rightarrow B_i) \wedge (C \rightarrow D) \leq C \rightarrow D$. Note in this case, again, that $C \leq C$ and $D \leq D$.
- $\langle 1, 3 \rangle$ In order for $C \rightarrow D$ to be the succedent of the right premise, it must be that $D = E \wedge F$, so the right premise of the form $(C \rightarrow E) \wedge (C \rightarrow F) \leq C \rightarrow E \wedge F$. But then, the left premise is still an instance of (A1), in which case note that $C \leq C \wedge C$ and $E \wedge F \leq E \wedge F$.
- $\langle 2, 1 \rangle$ Lp must be of the form $\bigwedge_{i \in I'} (A_i \rightarrow B_i) \wedge (C \rightarrow D) \leq C \rightarrow D$, and the right premise the correct instance of (A1) for this to fit the desired pattern. In this case, $C \leq C$ and $D \leq D$ suffice again.
- $\langle 2, 2 \rangle$ To fit the required form, we would need for the left premise to be of the form $\bigwedge_{i \in I} (A_i \rightarrow B_i) \wedge (C \rightarrow D) \leq \bigwedge_{i \in I'} (A_i \rightarrow B_i) \wedge (C \rightarrow D)$ and for the right premise to be $\bigwedge_{i \in I'} (A_i \rightarrow B_i) \wedge (C \rightarrow D) \leq C \rightarrow D$, where $I' \subset I$, in which case, we have that $C \leq C$ and $D \leq D$.

- $\langle 2, 3 \rangle$ In order for the right premise to fit the form, we would need that $D = E \wedge F$, to obtain $(C \rightarrow E) \wedge (C \rightarrow F) \leq C \rightarrow E \wedge F$. In this case, the left premise would have to be of the form $\bigwedge_{i \in I} (A_i \rightarrow B_i) \wedge (C \rightarrow E) \wedge (C \rightarrow F) \leq (C \rightarrow E) \wedge (C \rightarrow F)$. But we know that $C \leq C \wedge C$ and $E \wedge F \leq E \wedge F$.
- $\langle 3, 1 \rangle$ Similar to $\langle 1, 4 \rangle$.

□

Finally, I present one more suggestive preliminary result.

Proposition 5.10. There can be no counterexample to bubbling resulting from the application of (R1) to a pair R_p, L_p where R_p is an instance of an axiom and L_p is the result of applying (R1) to axioms.

Proof. Suppose that $E \leq C \rightarrow D$ is the result of applying some rule to axioms, and that $\bigwedge_{i \in I} (A_i \rightarrow B_i) \leq E$ is an axiom. I'll employ the same naming convention for cases as before, but now the right index of $\langle i, j \rangle$ is the number of an *axiom*, while the left is that of a *rule*, and the cases will index the application of (R1) to obtain (Con) by means of the pair of first the axiom of which R_p is an instance, and the rule the application of which L_p is a result.

As before, we can immediately cut down the search space by structural considerations. In particular, we can remove any pair $\langle 5, n \rangle$ for $1 \leq n \leq 6$, as (A5) has a precedent of which $\bigwedge_{i \in I} (A_i \rightarrow B_i)$ cannot be an instance. Furthermore, we can ignore any pair whose left element is between 2 and 6, as all of these rules result in sequents with succedents which cannot have $C \rightarrow D$ as instances. So that leaves eight cases to consider: those $\langle m, n \rangle$ where $1 \leq m \leq 4$ and $n = 1$ or $n = 4$.

Consider those cases of the form $\langle m, 4 \rangle$. For one of these to obtain, it must be that E is of the form $C_1 \rightarrow D_1$, where $C \leq C_1$ and $D_1 \leq D$. But note then that for (R1) to apply to L_p and R_p , it must be that $\bigwedge_{i \in I} (A_i \rightarrow B_i) \leq C_1 \rightarrow D_1$ is an axiom, and so we know that $C_1 \leq \bigwedge_{j \in J} A_j$ and $\bigwedge_{j \in J} B_j \leq D_1$ for some nonempty $J \subseteq I$. From this it follows that $C \leq \bigwedge_{j \in J} A_j$ and $\bigwedge_{j \in J} B_j \leq D$, as desired.

Finally, we must consider those cases of the form $\langle m, 1 \rangle$, and so we must concern ourselves with axiom-pairs which can be cut together to obtain a sequent of the form $E \leq C \rightarrow D$. Let us call $E \leq F L^R p$, and $F \leq C \rightarrow D L^L p$. The only axioms which fit the desired form for $L^R p$ are (A1), (A2), and (A3). I'll consider these cases in turn.

Case: $L^R p$ is an instance of (A1) – i.e. is $C \rightarrow D \leq C \rightarrow D$. Now we must move down a step again and consider subcases according to the axioms of which $E \leq C \rightarrow D$ could be an instance, again showing in all that the cut on L_p and R_p cannot be a counterexample to bubbling.

- (1) $L^L p$ is an instance of (A1). Immediate.

- (2) $L^L p$ is an instance of (A2). In this case E is of the form $G \wedge (C \rightarrow D)$, in which case Rp is $\bigwedge_{i \in I} (A_i \rightarrow B_i) \leq G \wedge (C \rightarrow D)$. This can only happen when Rp is an instance of (A1) or (A2). Let us consider the latter. If Rp is an instance of (A2), then $F = A_k \rightarrow B_k$, $C = A_l$, and $D = B_l$ for some $k, l \in I$, but then there's no counterexample.
- (3) $L^L p$ is an instance of (A3). In this case $D = D_1 \wedge D_2$ and $E = (C \rightarrow D_1) \wedge (C \rightarrow D_2)$, but this can only happen if Rp is (A1) or (A2), and the former case is trivial. As for the latter, then $C = A_k = A_l$, $D_1 = B_k$, and $D_2 = B_l$ for some $k, l \in I$, and once again we're done.

Case: $L^R p$ is an instance of (A2) with the form $G \wedge (C \rightarrow D) \leq C \rightarrow D$. This argument again proceeds by considering what $L^L p$ could be, and hence what Rp could be, and is similar to previous arguments.

Case: $L^R p$ is an instance of (A3) with the form $(C \rightarrow D_1) \wedge (C \rightarrow D_2) \leq C \rightarrow D_1 \wedge D_2$. If this is the case, then $L^L p$ could be (A1), in which case we're done, or it could be (A2). In this case, Rp can either be (A1) or (A2), and in either case we're done.

□

It is tedious, but not difficult, to prove the natural partner to Prop 5.10, that no counterexample to bubbling can result from an application of (R1) to Lp (an instance of an axiom) and Rp (the result of applying some rule to axioms). These results display some nice patterns which could, it seems, be made to coagulate into an inductive argument, but I don't, at the time of writing, see how to do it. So I'll leave off with the conjecture that $\mathbf{BB}_{(\wedge I)}^+$ bubbles, and this some evidence for it.

It also seems that the extension of $\mathbf{BB}_{(\wedge I)}^+$ by weak DeMorgan negation, that is a connective just satisfying double negation equivalence and the rule form of contraposition, shouldn't pose substantial further difficulties. So if the bubbling argument can be made to go through, the result of extending the system with such a negation would seem to also succumb to such an argument. Call this system $\mathbf{BB}_{(\wedge I)}$ – this is another seemingly good candidate for composition admission.

5.2. Almost No Logics Bubble. It is known that \mathbf{B}^\wedge bubbles, there is reason to think that $\mathbf{BB}_{(\wedge I)}^+$ bubbles, and to suspect that $\mathbf{BB}_{(\wedge I)}$ does as well. This leads to a natural further question: which other logics bubble? Sadly the title of this section gives away the answer. The following list of axioms (and one rule) have instances (produce conclusions) which counterexample bubbling:

- ($\vee E$) $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$
- (Cont) $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$
- (WB) $(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$
- (WI) $(A \rightarrow B \wedge A) \rightarrow B$
- (CI) $A \rightarrow ((A \rightarrow B) \rightarrow B)$
- (B) $(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$
- (K) $A \rightarrow (B \rightarrow A)$

$$\begin{aligned} (\circ\text{I}) \quad & A \rightarrow (B \rightarrow (A \circ B)) \\ (\text{r}\leftarrow) \quad & A \rightarrow (B \rightarrow C) \Leftrightarrow B \rightarrow (C \leftarrow A) \end{aligned}$$

I'll fill in the details for counterexamples fitting a couple of the above axioms, leaving the others for the interested reader. For $(\vee\text{E})$, note that $\not\vdash p \vee q \rightarrow p$, so $\not\vdash p \vee q \rightarrow p \wedge q$, though with this axiom we have $\vdash (p \rightarrow r) \wedge (q \rightarrow r) \rightarrow (p \vee q \rightarrow r)$. For $(\circ\text{I})$, note that while $\vdash (p \rightarrow q) \rightarrow (r \rightarrow ((p \rightarrow q) \circ r))$, in no reasonable logic do we have $\vdash r \rightarrow p$. As for $(\text{r}\leftarrow)$, the extension of \mathbf{BB}^+ by this rule has $(\vee\text{E})$ as a theorem.

So while it seems highly likely that $\mathbf{BB}_{(\wedge\text{I})}^+$ bubbles, it seems that almost no other logics bubble. So this rare, fragile property, while useful for performing surgery on prime theories, is really not to be looked for outside of very minimal extensions to distributive lattice logic.

6. CONCLUSION

In this paper I have provided a collection of reasonably simple and natural sufficient conditions for a logic extending \mathbf{B}^\wedge to admit channel composition. In addition, I have suggested that the logic $\mathbf{BB}_{(\wedge\text{I})}^+$ is an obvious candidate for a system admitting channel composites, by means of this argument. Should it turn out that $\mathbf{BB}_{(\wedge\text{I})}^+$ bubbles, this provides a new, interesting motivation for a potentially interesting logic; one not, to my knowledge, picked out for attention in the past.

The results here suggest a potential unexplored avenue for constructing models of combinatory logics in logics with disjunction. It has long been known that the set of all $\mathbf{B}^{+\top}$ theories does not form a model of combinatory logic. In the past, such models have been constructed using Harrop theories in place of prime theories [14]. Should $\mathbf{BB}_{(\wedge\text{I})}^+$ prove to bubble, it seems highly likely that one will be able to find, among (a nice subset of) its prime theories, a model of combinatory logic, enriched with union types. This is just a guess, but it strikes me as a potentially interesting avenue for new research on this topic.

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