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The Logic of Internal Rational Agent

Abstract: In this paper, we introduce a new four-valued logic which may be viewed as a variation on the theme of Kubyshkina and Zaitsev's Logic of Rational Agent **LRA** [16]. We call our logic **LIRA** (Logic of Internal Rational Agency). In contrast to **LRA**, it has three designated values instead of one and a different interpretation of truth values, the same as in Zaitsev and Shramko's bi-facial truth logic [42]. This logic may be useful in a situation when according to an agent's point of view (i.e. internal point of view) her/his reasoning is rational, while from the external one it might be not the case. One may use **LIRA**, if one wants to reconstruct an agent's way of thinking, compare it with respect to the real state of affairs, and understand why an agent thought in this or that way. Moreover, we discuss Kubyshkina and Zaitsev's necessity and possibility operators for **LRA** definable by means of four-valued Kripke-style semantics and show that, due to two negations (as well as their combination) of **LRA**, two more possibility operators for **LRA** can be defined. Then we slightly modify all these modalities to be appropriate for **LIRA**. Finally, we formalize all the truth-functional n-ary extensions of the negation fragment of **LIRA** (including **LIRA** itself) as well as their basic modal extension via linear-type natural deduction systems.

Keywords: Logic of rational agent, logic of internal rational agency, four-valued logic, logic of generalized truth values, modal logic, natural deduction, correspondence analysis.

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1 Introduction

Kubyshkina and Zaitsev [16] have recently presented the **L**ogic of **R**ational **A**gent **LRA**. It is a four-valued logic with one designated value and it is a member of the family of logics of generalized truth values (see, for example, Shramko and Wansing's book [36] about them). One may treat it as a truth-functional epistemic logic. **LRA** has two different negations (called ontological and epistemic ones) which due to the interpretation of the truth values of **LRA** model the behavior of an epistemic operator "the agent $\mathcal{A}$ knows that $\mathcal{X}$". The truth values are as follows: `T1` (the proposition is true and its value is known to the agent), `T0` (the proposition is true and its value is not known to the agent), `F1` (the proposition is false and its value is known to the agent), `F0` (the proposition is false and its value is not known to the agent). The only designated value is `T1`, i.e. the propositions that are true and the agent knows their values are tautologies of this logic. As Kubyshkina and Zaitsev have shown, **LRA** avoids the well-known knowability paradox (Church-Fitch's paradox). They have presented an adequate axiomatization of their logic and have supplied it with necessity and possibility operators defined by **S5**-style Kripke relational semantics.

In this paper, we introduce the logic **LIRA** (**L**ogic of **I**nternal **R**ational **A**gency)[1] which is a modification of **LRA**. It has the same set of truth values, but with a different interpretation, a different set of designated values, the same negations, but different conjunction and disjunction. This logic was inspired by a situation, when from an agent's (internal) point of view her/his reasoning is rational, while from an external point of view it might not the case. One may use this logic to reconstruct an agent's way of thinking, compare it with respect to the real state of affairs, and understand why an agent thought in this or that way. Following Zaitsev and Shramko [42], we interpret the truth components `1` and `0`, respectively, as "the agent accepts the proposition" and "the agent rejects the proposition". All the values, except `F0` (the proposition is false and the agent rejects it), are designated. Hence, the valid propositions of **LIRA** are either true or accepted by

[1] I would like to thank Petr Cintula who suggested this name during the conference ManyVal 2019.

the agent. The logic **LRA** allows us to analyse an agent's knowledge about the real state of affairs, but it is not precise enough with respect to the situations when an agent is mistaken. Our logic allows us to investigate how an agent accepts or rejects the real state of affairs taking into account the fact that an agent can be aberrant (e.g., because of an incompleteness of her/his knowledge or using unreliable sources of information).

We present Jaśkowski-Fitch-style natural deduction for the negation fragment of **LIRA** and all its truth-functional n-ary extensions, including **LIRA** itself, by Kooi and Tamminga's machinery of correspondence analysis [18]. Moreover, we consider a **K**-style modal extension of **LIRA** and a natural deduction system for it. Let us point out an interesting feature of **LRA** and **LIRA**: since each of these logics has two negations, so we have three distinct possibility operators for one necessity operator, by the standard definition of possibility operator via necessity and negation.

The structure of this paper is as follows. In Section 2, we describe the semantics of **LRA**. In Section 3, we introduce the logic **LIRA**. In Section 4, by correspondence analysis, we formalize via natural deduction systems the negation fragment of **LIRA** and all its truth-functional n-ary extensions (and, as a particular case, **LIRA** itself). In Section 5, we discuss the correspondence analysis for **LRA** from [27] and generalize it for a larger class of logics. In Section 6, we discuss some four-valued connectives already mentioned in the literature which may be added to **LRA** or **LIRA** and formalized via correspondence analysis. In Section 7.1, we discuss Kubyshkina and Zaitsev's modal operators for **LRA** and introduce two new possibility operators. In Section 7.2, we provide alternative modal operators for **LIRA**. In Section 7.3, we develop a natural deduction system for **K**-style modal logic based on **LIRA**. Section 8 contains a description of future research.

2 The semantics of LRA

LRA [16] is build over a propositional language $\mathscr{L}$ the alphabet of which contains propositional variables $p, q, r, s, p_1, \ldots$, two negations ($\neg$ and $\sim$), conjunction ($\wedge$), disjunction ($\vee$) as well as left and right parentheses. Let $\mathcal{P}$ be the set of all propositional variables. The set $\mathscr{F}$ of all formulas of the language $\mathscr{L}$ is defined in the standard inductive way. A valuation is a function V such that $V(P) \in \mathscr{V}_4$, for each $P \in \mathcal{P}$, and the values of $V(\neg A)$, $V(\sim A)$, $V(A \wedge B)$, and $V(A \vee B)$, for each $A, B \in \mathscr{F}$, are defined as follows:

A	$\neg$	$\sim$
T1	F1	T0
T0	F0	T1
F1	T1	F0
F0	T0	F1

$\wedge$	T1	T0	F1	F0
T1	T1	T0	F1	F0
T0	T0	T0	F1	F0
F1	F1	F1	F1	F1
F0	F0	F0	F1	F0

$\vee$	T1	T0	F1	F0
T1	T1	T1	T1	T1
T0	T1	T0	T0	T0
F1	T1	T0	F1	F0
F0	T1	T0	F0	F0

The elements of the set $\mathscr{V}_4 = \{\mathtt{T1}, \mathtt{T0}, \mathtt{F1}, \mathtt{F0}\}$ of truth values are interpreted as follows, where V is a valuation and $A \in \mathscr{F}$:

- $V(A) = \mathtt{T1}$ — A is true and is known by the agent,
- $V(A) = \mathtt{T0}$ — A is true and is not known by the agent,
- $V(A) = \mathtt{F1}$ — A is false and is known by the agent,
- $V(A) = \mathtt{F0}$ — A is false and is not known by the agent.

The components T and F of truth values are ontological truth and falsehood, respectively. The components 1 and 0 are epistemic states of an agent, known and unknown, respectively. The negation

$\neg$ is said to be *ontological* one, since it changes the components T and F of truth values. The negation $\sim$ is said to be *epistemic* one, since it changes epistemic states of the agent (1 and 0).

Kubyshkina and Zaitsev [16] do not mention which order relation can be defined on the set $\mathscr{V}_4$. However, in an earlier paper [41], Zaitsev considers the $\{\neg, \wedge, \vee\}$-fragment of **LRA** and gives the following ordering: $\texttt{F1} \leqslant \texttt{F0} \leqslant \texttt{T0} \leqslant \texttt{T1}$. As we can see, conjunction and disjunction are minimum and maximum operations, respectively, according to this order relation.

The entailment relation in **LRA** is defined as follows, for any $\Gamma \subseteq \mathscr{F}$ and $A \in \mathscr{F}$:

$$\Gamma \models_{\mathbf{LRA}} A \text{ iff } V(G) = \texttt{T1}, \text{ for any } G \in \Gamma, \text{ implies } V(A) = \texttt{T1}, \text{ for any valuation } V.$$

If $\Gamma = \emptyset$, then A is called a tautology or a valid formula. Clearly, in **LRA**, the set of designated values is $\{\texttt{T1}\}$. The tautologies of **LRA** are those formulas which are true and the agent knows it.

3 The semantics of LIRA

In this section, we semantically present our own logic, **LIRA**. First of all, **LIRA** has the same language and truth values as **LRA**. However, the interpretation of these truth values is different (V is a valuation and $A \in \mathscr{F}$):

- $V(A) = \texttt{T1}$ — A is both ontologically and epistemically true, i.e. it is objectively true and accepted by the agent;
- $V(A) = \texttt{T0}$ — A is ontologically, but epistemically false, i.e. it is objectively true, but rejected by the agent;
- $V(A) = \texttt{F1}$ — A is ontologically false and epistemically true, i.e. it is objectively false, but accepted by the agent,
- $V(A) = \texttt{F0}$ — A is both ontologically and epistemically false, i.e. it is objectively false and rejected by the agent.

This interpretation of truth values was used by Zaitsev and Shramko [42] for their bi-facial truth logic.

The ordering of truth values also differs from the case of **LRA**: $\texttt{F0} \leqslant' \texttt{F1} \leqslant' \texttt{T0} \leqslant' \texttt{T1}$. Both negations are defined in the same way as in **LRA**, while conjunction and disjunction are different (they are minimum and maximum, respectively, according to $\leqslant'$).

$\wedge$	T1	T0	F1	F0
T1	T1	T0	F1	F0
T0	T0	T0	F1	F0
F1	F1	F1	F1	F0
F0	F0	F0	F0	F0

$\vee$	T1	T0	F1	F0
T1	T1	T1	T1	T1
T0	T1	T0	T0	T0
F1	T1	T0	F1	F1
F0	T1	T0	F1	F0

The definition of the entailment relation is yet another distinction between **LRA** and **LIRA**:

$$\Gamma \models_{\mathbf{LIRA}} A \text{ iff } V(G) \neq \texttt{F0}, \text{ for any } G \in \Gamma, \text{ implies } V(A) \neq \texttt{F0}, \text{ for any valuation } V.$$

Let us say a few more words about the truth values and the entailment relation of **LIRA**. Ontological components of truth values (T and F) are independent from the epistemic ones (1 and 0). Thus, for example, the following situation is possible: a statement is ontologically false, but an agent accepts it as a true (e.g., because of an incompleteness of her/his knowledge). So, a statement can be ontologically false, but epistemically true. For example, Ptolemy accepted an ontologically false proposition "the sun revolves around the earth". At the same time an agent can accept some other ontologically true statement and there can be a situation such that an agent infers an ontologically false proposition from an ontologically true one, because both of them are accepted by her/him. The agent just does not know their ontological values and believes that these statements are true and that her/his reasoning is right. As observers we have more information and know that agent's reasoning is not correct.

We believe that **LIRA** might be useful, if we want to reconstruct an agent's reasoning (even if it is mistaken from an ontological point of view) and to try to understand how the agent has thought, why the agent has thought in this or that way. The epistemic components of truth values can represent an agent's truth value assignment of propositions. The ontological ones can represent the real state of affairs. It allows us to check the relevancy of agent's knowledge with respect to the actuality.

4 Natural deduction for LIRA

In this section, using correspondence analysis we formalize **LIRA** itself as well as a large class of related logics. Correspondence analysis was developed by Kooi and Tamminga [18] in order to formalize some three-valued logics. One may find in [19, 20] a survey of the current stage of the development of correspondence analysis. We just mention that for **LRA** it was applied in [27]. However, **LRA** itself has been already axiomatized in [16].

We must say that correspondence analysis is not the only method of construction of natural deduction systems for many-valued logics. For example, Baaz, Fermüller, and Zach [6] provide the technique of automated generation of *labeled* natural deduction systems for many-valued logics which is based on their earlier work on sequent and tableau calculi [5]. However, we are interested in the development of the *non-labeled*, purely syntactic calculi, so we will not follow Baaz, Fermüller, and Zach's manner. Yet another way to obtain a natural deduction system for a many-valued logic is to present a sequent calculus for this logic, following Avron's [3] approach (see also a later paper by Avron, Ben-Naim, and Konikowska [4]) and transform this sequent calculus to the natural deduction system. A comparison of correspondence analysis and Avron, Ben-Naim, and Konikowska's method one may find in [26].

The idea of the technique of correspondence analysis is as follows. We consider a functionally incomplete set of connectives and present a natural deduction system for it (a basic system). Then using these connectives we show how to rewrite semantical conditions for the other truth-functional connectives into inference rules. Here are some examples. Kooi and Tamminga [18, 39] dealt with the $\{\neg, \wedge, \vee\}$-language for the basic system which was supplied with unary operations $\sim_1, \ldots, \sim_m$ and binary ones $\circ_1, \ldots, \circ_n$. In [28], it was shown that for the same logics (and some others) it is enough to consider the $\{\neg\}$-language for the basic system. In [27], basic systems were built in the $\{\neg, \sim, \wedge, \vee\}$-language. In our case, the $\{\neg, \sim\}$-language (which is obviously functionally incomplete) is powerful enough to be the language of the basic system.[2] In Section 5, we show that it is powerful enough for the case of **LRA** as well (thus, the results from [27] can be generalized).

How to rewrite the semantical conditions, for example, for a binary truth-functional operator $\circ$ into an inference rule? Kooi and Tamminga suggest to find an inference scheme Γ/A such that $\Gamma \models A$ iff $x \circ y = z$, where x, y, z are truth values. If one finds such inference schemes for all the possible equations of the form $x \circ y = z$, then one gets a set of sound and complete inference rules

[2]The reasons to choose this fragment are as follows. First, we need negations to define the notion of a canonic valuation in our completeness proof. Second, the rules for all other connectives can be obtained by correspondence analysis with the help of negations.

for $\circ$. In the case of our logic, we will need to consider proof construction rules of the following form: if $B_1, \ldots, B_m, \Gamma_1/A_1, \ldots, \Gamma_l/A_l$ is provable, then C is provable as well. Moreover, we will deal with n-ary operators $\circledcirc_1, \ldots, \circledcirc_k$. In contrast to [18, 39, 28, 26], we present all the rules in a general way, i.e. we introduce one equivalence from which all the rules for all the equations of the form $\circledcirc(x_1, \ldots, x_n) = y$ are extracted.

First, we introduce a natural deduction system $\mathfrak{ND}^{\neg\sim}_{\mathbf{LIRA}}$ for the $\{\neg, \sim\}$-fragment of **LIRA**. Second, we present a natural deduction system $\mathfrak{ND}^{\neg\sim\circledcirc}_{\mathbf{LIRA}}$ for each extension of the $\{\neg, \sim\}$-fragment of **LIRA** by any truth-functional n-ary operators $\circledcirc_1, \ldots, \circledcirc_k$ (we denote such an extension by $\mathbf{LIRA}_{(\circledcirc)_k}$). As a particular case, we obtain a natural deduction system for **LIRA** itself. Let $\mathscr{L}_{(\circledcirc)_k}$ be an extension of $\mathscr{L}$ by n-ary operators $\circledcirc_1, \ldots, \circledcirc_k$. Let $\mathscr{F}_{(\circledcirc)_k}$ be the set of all formulas of the language $\mathscr{L}_{(\circledcirc)_k}$.

Let us introduce the natural deduction system $\mathfrak{ND}^{\neg\sim}_{\mathbf{LIRA}}$ which has the following inference and proof construction rules:

$$(EM_1)\ \frac{\begin{matrix}[A] & [\neg A]\\ B & B\end{matrix}}{B}, \quad (EM_2)\ \frac{\begin{matrix}[A] & [\sim A]\\ B & B\end{matrix}}{B}, \quad (EM_3)\ \frac{\begin{matrix}[A] & [\sim\neg A]\\ B & B\end{matrix}}{B}, \quad (EM_4)\ \frac{\begin{matrix}[\neg A] & [\sim A]\\ B & B\end{matrix}}{B},$$

$$(EFQ)\ \frac{A \quad \neg A \quad \sim A \quad \sim\neg A}{B}, \quad (\neg\neg I)\ \frac{A}{\neg\neg A}, \quad (\sim\sim I)\ \frac{A}{\sim\sim A},$$

$$(\sim\neg\neg I)\ \frac{\sim A}{\sim\neg\neg A}, \quad (\neg\sim\sim I)\ \frac{\neg A}{\neg\sim\sim A}, \quad (\neg\sim I)\ \frac{\sim\neg A}{\neg\sim A}, \quad (\sim\neg I)\ \frac{\neg\sim A}{\sim\neg A}.$$

Definition 4.1. The notion of an inference of a formula C from a set of formulas Γ in $\mathfrak{ND}^{\neg\sim}_{\mathbf{LIRA}}$ is understood in the spirit of [11, p. 366] and [35] as a finite non-empty sequence of formulas which satisfies the following conditions:

- all formulas are premises (elements of Γ) or assumptions or follow from the previous ones via $\mathfrak{ND}^{\neg\sim}_{\mathbf{LIRA}}$-inference rules,
- by applying the rule (EM_i) each formula starting from the assumption A' until the formula B, inclusively, as well as each formula starting from the assumption A'' until the formula B, inclusively, is discarded from the inference, where $i \in \{1, \ldots, 4\}$ and $A' = A$ for $i \in \{1, 2, 3\}$, $A' = \neg A$ for $i = 4$, $A'' = \neg A$ for $i = 1$, $A'' = \sim A$ for $i \in \{2, 4\}$, and $A'' = \sim\neg A$ for $i = 3$.

A proof of a formula C is its inference from the empty set of premises.

An example of an inference in $\mathfrak{ND}^{\neg\sim}_{\mathbf{LIRA}}$ is presented in Figure 1.

Proposition 4.2. *The following inference rules are derivable in* $\mathfrak{ND}^{\neg\sim}_{\mathbf{LIRA}}$:

$$(\neg\neg E)\ \frac{\neg\neg A}{A}, \quad (\sim\sim E)\ \frac{\sim\sim A}{A}, \quad (\sim\neg\neg E)\ \frac{\sim\neg\neg A}{\sim A}, \quad (\neg\sim\sim E)\ \frac{\neg\sim\sim A}{\neg A}.$$

Proof. As an example, we show the derivability of $(\sim\sim E)$ in Figure 1. □

Definition 4.3. For each formula A and each $X \in \mathscr{V}_4$, A^X and $A^{\overline{X}}$, respectively, are called X– and $\overline{X}$–images of A and are defined as follows:

$$A^X = \begin{cases} A & \text{iff} \quad X = \mathtt{T1};\\ \sim A & \text{iff} \quad X = \mathtt{T0};\\ \neg A & \text{iff} \quad X = \mathtt{F1};\\ \sim\neg A & \text{iff} \quad X = \mathtt{F0}; \end{cases} \qquad A^{\overline{X}} = \begin{cases} A & \text{iff} \quad X = \mathtt{F0};\\ \sim A & \text{iff} \quad X = \mathtt{F1};\\ \neg A & \text{iff} \quad X = \mathtt{T0};\\ \sim\neg A & \text{iff} \quad X = \mathtt{T1}. \end{cases}$$

1	$\sim\sim p$	premise
2	p	assumption
3	$\neg p$	assumption
4	$\neg\sim\sim p$	$(\neg\sim\sim I)$: 3
5	$\sim\neg\sim p$	$(\sim\neg I)$: 4
6	p	assumption
7	$\sim p$	assumption
8	p	assumption
9	$\sim\neg p$	assumption
10	$\neg\sim p$	$(\neg\sim I)$: 9
11	p	(EFQ): 7, 10, 1, 5
12	p	(EM_3): 8, 11 [8], [9–11]
13	p	(EM_2): 6, 12 [6], [7–12]
14	p	(EM_1): 2, 13 [2], [3–13]

Figure 1: An inference of p from $\sim\sim p$ in $\mathfrak{ND}^{\neg\sim}_{\mathbf{LIRA}}$.

The subsequent corollaries follow from Definition 4.3.

Corollary 4.4. *For each formula A, each valuation V, and each $X \in \mathscr{V}_4$, it holds that $V(A^X) = \mathtt{T1}$ iff $V(A) = X$.*

Corollary 4.5. *For each formula A, each valuation V, and each $X \in \mathscr{V}_4$, it holds that $V(A^{\overline{X}}) = \mathtt{F0}$ iff $V(A) = X$.*

Definition 4.6 (An auxiliary entailment relation)**.** For each formulas $A_1, \ldots, A_m, B_1, \ldots, B_l, C$ it holds that

$$A_1, \ldots, A_m, B_1 \Rightarrow C, \ldots, B_l \Rightarrow C \models C \quad \text{iff}$$

if $V(A_i) \neq \mathtt{F0}$ and it holds that $V(B_j) \neq \mathtt{F0}$ implies $V(C) \neq \mathtt{F0}$, then $V(C) \neq \mathtt{F0}$, for each $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, l\}$.

Remark 4.7. An inference scheme $A_1, \ldots, A_m, B_1 \Rightarrow C, \ldots, B_l \Rightarrow C/C$ produces the following proof construction rule:

$$\frac{A_1 \quad \ldots \quad A_m \quad \begin{matrix}[B_1] \\ C\end{matrix} \quad \ldots \quad \begin{matrix}[B_l] \\ C\end{matrix}}{C}.$$

Let us introduce an adaptation for our case of a generalized version [28, Definition 4.1] of Kooi and Tamminga's original definition of single entry correspondence [18, 39, Definition 2.1, Definition 1].

Definition 4.8 (Generalized single entry correspondence)**.** Let $\mathbf{L} = \mathbf{LIRA}_{(\circledcirc)_k}$, $A_1, \ldots, A_m, B_1, \ldots, B_l$, $C_1, \ldots, C_j \in \mathscr{F}_{(\circledcirc)_k}$, $x_1, \ldots, x_n, y \in \mathscr{V}_4$, and $1 \leqslant i \leqslant j$. Let I_i/C_i be an inference scheme of the type $A_1, \ldots, A_m, B_1 \Rightarrow C_i, \ldots, B_l \Rightarrow C_i/C_i$ or $A_1, \ldots, A_m/C_i$. Then a truth table entry $\circledcirc(x_1, \ldots, x_n) = y$ is characterized by inference schemes $I_1/C_1, \ldots, I_j/C_j$, if

$$\circledcirc(x_1, \ldots, x_n) = y \text{ if and only if } I_1 \models_{\mathbf{L}} C_1, \ldots, I_j \models_{\mathbf{L}} C_j.$$

Theorem 4.9. *Let* $\mathbf{L} = \mathbf{LIRA}_{(\circledcirc)_k}$. *For each* $X_1, \ldots, X_n, Y \in \mathscr{V}_4$, *it holds that*

$$\circledcirc(X_1, \ldots, X_n) = Y \;\; \textit{iff} \;\; \begin{array}{l} \circledcirc(A_1, \ldots, A_n)^{\overline{Y}}, A_1^{\overline{X_1}} \Rightarrow B, \ldots, A_n^{\overline{X_n}} \Rightarrow B \models_{\mathbf{L}} B, \\ \textit{for each formulas } A_1, \ldots, A_n, B. \end{array}$$

Proof. Suppose there are formulas $A_1, \ldots, A_n, B$ such that $\circledcirc(A_1, \ldots, A_n)^{\overline{Y}}$, $A_1^{\overline{X_1}} \Rightarrow B$, $\ldots$, $A_n^{\overline{X_n}} \Rightarrow B \not\models B$. Then there is a valuation V such that $V(\circledcirc(A_1, \ldots, A_n)^{\overline{Y}}) \neq \mathtt{F0}$ and it holds that $V(A_1^{\overline{X_1}}) \neq \mathtt{F0}$ implies $V(B) \neq \mathtt{F0}$, $\ldots$, $V(A_n^{\overline{X_n}}) \neq \mathtt{F0}$ implies $V(B) \neq \mathtt{F0}$, while $V(B) = \mathtt{F0}$. Then $V(A_1^{\overline{X_1}}) = \mathtt{F0}$, $\ldots$, $V(A_n^{\overline{X_n}}) = \mathtt{F0}$. By Corollary 4.5, $V(A_1) = X_1$, $\ldots$, $V(A_n) = X_n$, and $V(\circledcirc(A_1, \ldots, A_n)) \neq Y$. But then $\circledcirc(X_1, \ldots, X_n) \neq Y$.

Suppose $\circledcirc(X_1, \ldots, X_n) \neq Y$. Then there are formulas $A_1, \ldots, A_n$ and a valuation V such that $V(A_1) = X_1, \ldots, V(A_n) = X_n$ while $V(\circledcirc(X_1, \ldots, X_n)) \neq Y$. By Corollary 4.5, $V(A_1^{\overline{X_1}}) = \mathtt{F0}$, $\ldots$, $V(A_n^{\overline{X_n}}) = \mathtt{F0}$, $V(\circledcirc(A_1, \ldots, A_n)^{\overline{Y}}) \neq \mathtt{F0}$. Let B be a formula such that $V(B) = \mathtt{F0}$. Thus, it holds that $A_1^{\overline{X_1}} \Rightarrow B, \ldots, A_n^{\overline{X_n}} \Rightarrow B$. Then $\circledcirc(A_1, \ldots, A_n)^{\overline{Y}}$, $A_1^{\overline{X_1}} \Rightarrow B$, $\ldots$, $A_n^{\overline{X_n}} \Rightarrow B \not\models B$. □

Remark 4.10. Theorem 4.9 produces proof construction rules of the following form:

$$R_{\circledcirc}(X_1, \ldots, X_n, Y) \;\; \frac{\circledcirc(A_1, \ldots, A_n)^{\overline{Y}} \quad \begin{array}{c}[A_1^{\overline{X_1}}]\\ B\end{array} \quad \begin{array}{c}\ldots\\ \ldots\end{array} \quad \begin{array}{c}[A_n^{\overline{X_n}}]\\ B\end{array}}{B}.$$

In order to obtain the definition of an inference of a formula C from a set of formulas Γ in $\mathfrak{ND}_{\mathbf{LIRA}}^{\neg\sim\circledcirc}$ one should extend Definition 4.1 by the following point:

- by applying the rule $R_{\circledcirc}(X_1, \ldots, X_n, Y)$ each formula starting from the assumption $A_i^{\overline{X_i}}$ ($1 \leqslant i \leqslant n$) until the formula B, inclusively, is discarded from the inference.

Let us mention one strange feature of Theorem 4.9: it produces elimination rules only. It raises a question: what to do, if one wants to introduce the formula $\circledcirc(A_1, \ldots, A_n)$? The answer is to use the rules (EM_1)–(EM_4) or (EFQ). These rules are two-faced: on the one hand, they eliminate formulas with negations, on the other hand, they introduce the formula B which, in particular, can be of the form $\circledcirc(A_1, \ldots, A_n)$. Let us remark that one may find examples of sequent calculus with the left rules only (which are sequent-style versions of natural deduction elimination rules) for some of their connectives. Negri and Von Plato [25] provide the method of the transformation of the axioms into sequent rules. Being applied to a particular logic this method may produce the left rules only, as it was, for example, in the case of Suszko's [38] logic **SCI** which was considered by Chlebowski [10]. Interestingly, Chlebowski presented also a version of sequent calculus for **SCI** with the right rules only. But in contrast to the calculus with the left rules, this one does not enjoy cut elimination theorem.

Theorem 4.11 (Soundness)**.** *Let* $\mathbf{L} = \mathbf{LIRA}_{(\circledcirc)_k}$. *For each set of formulas* Γ *and each formula* A, *it holds that* $\Gamma \vdash_{\mathbf{L}} A$ *implies* $\Gamma \models_{\mathbf{L}} A$.

Proof. By induction on the depth of derivation. In the case of the rules for $\circledcirc$ use Theorem 4.9. □

Definition 4.12. A set of formulas Γ is said to be an $\mathbf{LIRA}_{(\circledcirc)_k}$-theory iff the following conditions hold, for each $X, X_1, \ldots, X_n \in \mathscr{V}_4$ and each formulas $A, A_1, \ldots, A_n$:

(Γ_1) $\Gamma \neq \mathscr{F}_{(\circledcirc)_k}$;
(Γ_2) $\Gamma \vdash_{\mathbf{L}} A$ implies $A \in \Gamma$, where $\mathbf{L} = \mathbf{LIRA}_{(\circledcirc)_k}$;
(Γ_3) $A \in \Gamma$ or $\neg A \in \Gamma$;
(Γ_4) $A \in \Gamma$ or $\sim A \in \Gamma$;
(Γ_5) $A \in \Gamma$ or $\sim\neg A \in \Gamma$;
(Γ_6) $\neg A \in \Gamma$ or $\sim A \in \Gamma$;
(Γ_7) if $\circledcirc(A_1, \ldots, A_n)^{\overline{X}} \in \Gamma$ and $\circledcirc(X_1, \ldots, X_n) = X$, then $A_1^{\overline{X}_1} \in \Gamma$ or $\ldots$ or $A_n^{\overline{X}_n} \in \Gamma$.

Definition 4.13. An interpretation function of a formula A in a set of formulas Γ is defined as follows:

$$I(A,\Gamma) = \begin{cases} \texttt{T1} & \text{iff } A \in \Gamma, \neg A \in \Gamma, \sim A \in \Gamma, \sim\neg A \notin \Gamma; \\ \texttt{T0} & \text{iff } A \in \Gamma, \neg A \notin \Gamma, \sim A \in \Gamma, \sim\neg A \in \Gamma; \\ \texttt{F1} & \text{iff } A \in \Gamma, \neg A \in \Gamma, \sim A \notin \Gamma, \sim\neg A \in \Gamma; \\ \texttt{F0} & \text{iff } A \notin \Gamma, \neg A \in \Gamma, \sim A \in \Gamma, \sim\neg A \in \Gamma; \\ \emptyset & \text{otherwise.} \end{cases}$$

Lemma 4.14. *For each* $\mathbf{LIRA}_{(\circledcirc)_k}$*-theory* Γ *and each formulas* $A, A_1, \ldots, A_n$*, the following equations hold*:

(1) $I(A,\Gamma) \neq \emptyset$;
(2) $I(\neg A,\Gamma) = \neg I(A,\Gamma)$;
(3) $I(\sim A,\Gamma) = \sim I(A,\Gamma)$;
(4) $I(\circledcirc(A_1, \ldots, A_n),\Gamma) = \circledcirc(I(A_1,\Gamma), \ldots, I(A_n,\Gamma))$.

Proof. (1) Suppose $I(A,\Gamma) = \emptyset$. Suppose $A \in \Gamma$, $\neg A \in \Gamma$, $\sim A \in \Gamma$, and $\sim\neg A \in \Gamma$. Then, by the rule (*EFQ*), $B \in \Gamma$. Thus, $\Gamma = \mathscr{F}_{(\circledcirc)_k}$ which contradicts (Γ_1). Suppose $A \notin \Gamma$ and $\neg A \notin \Gamma$. However, by (Γ_3), $A \in \Gamma$ or $\neg A \in \Gamma$. Contradiction. The other cases are considered similarly. Use (Γ_4)–(Γ_6). Thus, $I(A,\Gamma) \neq \emptyset$.

(2) Suppose $I(A,\Gamma) = \texttt{T1}$. Then $A \in \Gamma, \neg A \in \Gamma, \sim A \in \Gamma, \sim\neg A \notin \Gamma$. By ($\Gamma_6$), $\neg\neg A \in \Gamma$ or $\sim\neg A \in \Gamma$. Since $\sim\neg A \notin \Gamma$, $\neg\neg A \in \Gamma$. By (Γ_3), $\sim\neg A \in \Gamma$ or $\neg\sim\neg A \in \Gamma$. Since $\sim\neg A \notin \Gamma$, $\neg\sim\neg A \in \Gamma$. By the rule ($\sim\neg I$), $\sim\neg\neg A \in \Gamma$. Thus, $I(\neg A,\Gamma) = \texttt{F1} = \neg\texttt{T1} = \neg I(A,\Gamma)$.

Suppose $I(A,\Gamma) = \texttt{T0}$. Then $A \in \Gamma, \neg A \notin \Gamma, \sim A \in \Gamma, \sim\neg A \in \Gamma$. By ($\Gamma_3$), $\neg A \in \Gamma$ or $\neg\neg A \in \Gamma$. By (Γ_5), $\neg A \in \Gamma$ or $\sim\neg\neg A \in \Gamma$. Since $\neg A \notin \Gamma$, $\neg\neg A \in \Gamma$ and $\sim\neg\neg A \in \Gamma$. Thus, $I(\neg A,\Gamma) = \texttt{F0} = \neg\texttt{T0} = \neg I(A,\Gamma)$.

Suppose $I(A,\Gamma) = \texttt{F1}$. Then $A \in \Gamma, \neg A \in \Gamma, \sim A \notin \Gamma, \sim\neg A \in \Gamma$. By the rule ($\neg\neg I$), $\neg\neg A \in \Gamma$. Suppose $\sim\neg\neg A \in \Gamma$. By the rule (*EFQ*), $B \in \Gamma$, i.e. $\Gamma = \mathscr{F}_{(\circledcirc)_k}$, which contradicts ($\Gamma_1$). Hence, $\sim\neg\neg A \notin \Gamma$. Thus, $I(\neg A,\Gamma) = \texttt{T1} = \neg\texttt{F1} = \neg I(A,\Gamma)$.

Suppose $I(A,\Gamma) = \texttt{F0}$. Then $A \notin \Gamma, \neg A \in \Gamma, \sim A \in \Gamma, \sim\neg A \in \Gamma$. By the rule ($\sim\neg\neg I$), $\sim\neg\neg A \in \Gamma$. Suppose $\neg\neg A \in \Gamma$. By the rule (*EFQ*), $B \in \Gamma$, which contradicts (Γ_1). Hence, $\neg\neg A \notin \Gamma$. Thus, $I(\neg A,\Gamma) = \texttt{T0} = \neg\texttt{F0} = \neg I(A,\Gamma)$.

(3) Suppose $I(A,\Gamma) = \texttt{T1}$. Then $A \in \Gamma, \neg A \in \Gamma, \sim A \in \Gamma, \sim\neg A \notin \Gamma$. By ($\Gamma_6$), $\neg\sim A \in \Gamma$ or $\sim\sim A \in \Gamma$. Since $\sim\neg A \notin \Gamma$, $\sim\sim A \in \Gamma$. By the rule ($\neg\sim\sim I$), $\neg\sim\sim A \in \Gamma$. By the rule ($\sim\neg I$), $\sim\neg\sim A \in \Gamma$. Thus, $I(\sim A,\Gamma) = \texttt{T0} = \sim\texttt{T1} = \sim I(A,\Gamma)$.

Suppose $I(A,\Gamma) = \texttt{T0}$. Then $A \in \Gamma, \neg A \notin \Gamma, \sim A \in \Gamma, \sim\neg A \in \Gamma$. By the rule ($\sim\sim I$), $\sim\sim A \in \Gamma$. By the rule (*EFQ*), $\sim\neg\sim A \notin \Gamma$. Thus, $I(\sim A,\Gamma) = \texttt{T1} = \sim\texttt{T0} = \sim I(A,\Gamma)$.

Suppose $I(A,\Gamma) = \texttt{F1}$. Then $A \in \Gamma, \neg A \in \Gamma, \sim A \notin \Gamma, \sim\neg A \in \Gamma$. By (Γ_4), $\sim A \in \Gamma$ or $\sim\sim A \in \Gamma$. Thus, $\sim\sim A \in \Gamma$. By the rules $(\neg\sim\sim I)$ and $(\sim\neg I)$, $\sim\neg\sim A \in \Gamma$. Thus, $I(\sim A,\Gamma) = \texttt{F0} = \sim\texttt{F1} = \sim I(A,\Gamma)$.

Suppose $I(A,\Gamma) = \texttt{F0}$. Then $A \notin \Gamma, \neg A \in \Gamma, \sim A \in \Gamma, \sim\neg A \in \Gamma$. By the rules $(\neg\sim\sim I)$ and $(\sim\neg I)$, $\sim\neg\sim A \in \Gamma$. By the rule (EFQ), $\sim\sim A \notin \Gamma$. Thus, $I(\sim A,\Gamma) = \texttt{F1} = \sim\texttt{F0} = \sim I(A,\Gamma)$.

(4) Suppose $X, X_1, \ldots, X_n \in \mathscr{V}_4$ and $\circledcirc(X_1,\ldots,X_n) = X$. Suppose $I(A_1,\Gamma) = X_1, \ldots, I(A_n,\Gamma) = X_n$. As follows from Definitions 4.3 and 4.13, $I(C,\Gamma) = Y$ iff $C^{\overline{Y}} \notin \Gamma$, for each formula C and each $Y \in \mathscr{V}_4$. Therefore, $A_1^{\overline{X_1}} \notin \Gamma, \ldots, A_n^{\overline{X_n}} \notin \Gamma$. Suppose $\circledcirc(A_1,\ldots,A_n)^{\overline{X}} \in \Gamma$. Then, by (Γ_7), $A_1^{\overline{X}_1} \in \Gamma$ or $\ldots$ or $A_n^{\overline{X}_n} \in \Gamma$. Contradiction. Thus, $\circledcirc(A_1,\ldots,A_n)^{\overline{X}} \notin \Gamma$. Then $I(\circledcirc(A_1,\ldots,A_n),\Gamma) = X$. Hence, $I(\circledcirc(A_1,\ldots,A_n),\Gamma) = \circledcirc(X_1,\ldots,X_n) = \circledcirc(I(A_1,\Gamma),\ldots,I(A_n,\Gamma))$. □

Definition 4.15. For each $\mathbf{LIRA}_{(\circledcirc)_k}$-theory Γ and each $P \in \mathcal{P}$, a function V^c_Γ is said to be a *canonic valuation* iff $V^c_\Gamma(P) = I(P,\Gamma)$.

Lemma 4.16. *For each $\mathbf{LIRA}_{(\circledcirc)_k}$-theory Γ, each formula A, and each canonic valuation V^c_Γ, it holds that $V^c_\Gamma(A) = I(A,\Gamma)$.*

Proof. By induction on A, using Lemma 4.14. □

Lemma 4.17 (Lindenbaum)**.** *Let $\mathbf{L} = \mathbf{LIRA}_{(\circledcirc)_k}$. For each set of formulas Γ and each formula A, it holds that if $\Gamma \nvdash_{\mathbf{L}} A$, then there is an $\mathbf{LIRA}_{(\circledcirc)_k}$-theory Δ such that $\Gamma \subseteq \Delta$ and $\Delta \nvdash_{\mathbf{L}} A$.*

Proof. In the standard way. See, for example, [18]. □

Theorem 4.18 (Completeness)**.** *Let $\mathbf{L} = \mathbf{LIRA}_{(\circledcirc)_k}$. For each set of formulas Γ and each formula A, it holds that $\Gamma \models_{\mathbf{L}} A$ implies $\Gamma \vdash_{\mathbf{L}} A$.*

Proof. By contraposition. Suppose $\Gamma \nvdash_{\mathbf{L}} A$. Then, by Lemma 4.17, there is an $\mathbf{LIRA}_{(\circledcirc)_k}$-theory Δ such that $\Gamma \subseteq \Delta$ and $\Delta \nvdash A$. By Lemma 4.16, $V^c_\Delta(G) \neq \texttt{F0}$ (for each $G \in \Gamma$), while $V^c_\Delta(A) = \texttt{F0}$, where V^c_Δ is a canonic valuation. Thus, $\Gamma \not\models_{\mathbf{L}} A$. □

Corollary 4.19 (Adequacy)**.** *Let $\mathbf{L} = \mathbf{LIRA}_{(\circledcirc)_k}$. For each set of formulas Γ and each formula A, it holds that $\Gamma \models_{\mathbf{L}} A$ iff $\Gamma \vdash_{\mathbf{L}} A$.*

Proof. Follows from Theorems 4.11 and 4.18. □

5 Generalized formalization of LRA

The first axiomatization of **LRA** was presented by Kubyshkina and Zaitsev [16] in the $\{\neg,\sim,\wedge,\vee\}$-language in the form of binary consequence system. In [27], correspondence analysis was applied for **LRA**. As a result two natural deduction systems were obtained. The former one is built in the $\{\neg,\sim,\wedge,\vee\}$-language for **LRA** itself. The latter one is developed in the $\{\neg,\sim,\wedge,\vee,\star_1,\ldots,\star_m, \circ_1,\ldots,\circ_l\}$-language for the extension of **LRA** by tabular unary operators $\star_1,\ldots,\star_m$ and binary ones $\circ_1,\ldots,\circ_l$. In this section, we generalize this result. First, we consider n-ary operators $\circledcirc_1,\ldots,\circledcirc_k$ (not only unary and binary ones as in [27]). Second, we supply the $\{\neg,\sim\}$-fragment of **LRA** with them (not **LRA** itself as in [27]). Thus, more logics are covered. The extension of the $\{\neg,\sim\}$-fragment of **LRA** by n-ary operators $\circledcirc_1,\ldots,\circledcirc_k$ will be denoted by $\mathbf{LRA}_{(\circledcirc)_k}$.

Inference and proof construction rules of a natural deduction system $\mathfrak{ND}^{\neg\sim}_{\mathbf{LRA}}$ for the $\{\neg,\sim\}$-fragment of **LRA** are as follows: $(\neg\neg I)$, $(\sim\sim I)$, $(\sim\neg\neg I)$, $(\neg\sim\sim I)$, $(\neg\sim I)$, $(\sim\neg I)$ as well as the subsequent ones

$$(EM)\ \frac{\begin{matrix}[A] & [\neg A] & [\sim A] & [\sim\neg A]\\ B & B & B & B\end{matrix}}{B},\quad (EFQ_1)\ \frac{A\quad \neg A}{B},$$

$$(EFQ_2)\ \frac{A\quad \sim A}{B},\quad (EFQ_3)\ \frac{A\quad \sim\neg A}{B},\quad (EFQ_4)\ \frac{\neg A\quad \sim A}{B}.$$

Proposition 5.1. *The rules* $(\neg\neg E)$, $(\sim\sim E)$, $(\sim\neg\neg E)$, *and* $(\neg\sim\sim E)$ *are derivable in* $\mathfrak{ND}^{\neg\sim}_{\mathbf{LRA}}$.

Proof. Left for the reader. □

Theorem 5.2. *Let* $\mathbf{L} = \mathbf{LRA}_{(\circledcirc)_k}$. *For each* $X_1, \ldots, X_n, Y \in \mathscr{V}_4$, *it holds that*

$$\circledcirc(X_1,\ldots,X_n) = Y \quad \textit{iff} \quad \begin{array}{l} A_1^{X_1},\ldots,A_n^{X_n} \models_{\mathbf{L}} \circledcirc(A_1,\ldots,A_n)^Y, \\ \textit{for each formulas } A_1,\ldots,A_n. \end{array}$$

Proof. Suppose there are $A_1, \ldots, A_n \in \mathscr{F}_{(\circledcirc)_k}$ such that $A_1^{X_1}, \ldots, A_n^{X_n} \models_{\mathbf{L}} \circledcirc(A_1, \ldots, A_n)^Y$. Then there is a valuation V such that $V(\circledcirc(A_1, \ldots, A_n)^Y) \neq \mathtt{T1}$ and $V(A_1^{X_1}) = \mathtt{T1}$, ..., $V(A_n^{X_n}) = \mathtt{T1}$. By Corollary 4.5, $V(A_1) = X_1$, ..., $V(A_n) = X_n$, and $V(\circledcirc(A_1, \ldots, A_n)) \neq Y$. But then $\circledcirc(X_1, \ldots, X_n) \neq Y$.

Suppose $\circledcirc(X_1, \ldots, X_n) \neq Y$. Then there are formulas $A_1, \ldots, A_n$ and a valuation V such that $V(A_1) = X_1, \ldots, V(A_n) = X_n$ while $V(\circledcirc(X_1, \ldots, X_n)) \neq Y$. By Corollary 4.5, $V(A_1^{X_1}) = \mathtt{T1}$, ..., $V(A_n^{X_1}) = \mathtt{T1}$, $V(\circledcirc(A_1, \ldots, A_n)^Y) \neq \mathtt{T1}$. Then $A_1^{X_1}, \ldots, A_n^{X_n} \not\models_{\mathbf{L}} \circledcirc(A_1, \ldots, A_n)^Y$. □

Remark 5.3. Theorem 5.2 produces inference rules of the following form:

$$R_{\circledcirc}(X_1,\ldots,X_n,Y)\ \ \frac{A_1^{X_1}\ \ldots\ A_n^{X_n}}{\circledcirc(A_1,\ldots,A_n)^Y}$$

In contrast to Theorem 4.9, Theorem 5.2 produces introduction rules only. To eliminate the formulas of the form $\circledcirc(A_1, \ldots, A_n)$ one may use the rules (EM) and (EFQ_1)–(EFQ_4).

$\mathfrak{ND}^{\neg\sim}_{\mathbf{LRA}}$ should be extended by the rules based on the inference schemes from Theorem 5.2 to obtain natural deduction system $\mathfrak{ND}^{\neg\sim\circledcirc}_{\mathbf{LRA}}$ for $\mathbf{LRA}_{(\circledcirc)_k}$. The notion of inference in these systems is defined similarly to the case of $\mathfrak{ND}^{\neg\sim}_{\mathbf{LIRA}}$ and $\mathfrak{ND}^{\neg\sim\circledcirc}_{\mathbf{LIRA}}$ (see Definition 4.1).

Theorem 5.4 (Adequacy)**.** *Let* $\mathbf{L} = \mathbf{LRA}_{(\circledcirc)_k}$. *For each set of formulas* Γ *and each formula* A, *it holds that* $\Gamma \models_{\mathbf{L}} A$ *iff* $\Gamma \vdash_{\mathbf{L}} A$.

Proof. Similarly to Theorems 4.11 and 4.18. Use the following definition of an interpretation function of a formula A in a set of formulas Γ:

$$I(A,\Gamma) = \begin{cases} \mathtt{T1} & \text{iff } A \in \Gamma, \neg A \notin \Gamma, \sim A \notin \Gamma, \sim\neg A \notin \Gamma; \\ \mathtt{T0} & \text{iff } A \notin \Gamma, \neg A \notin \Gamma, \sim A \in \Gamma, \sim\neg A \notin \Gamma; \\ \mathtt{F1} & \text{iff } A \notin \Gamma, \neg A \in \Gamma, \sim A \notin \Gamma, \sim\neg A \notin \Gamma; \\ \mathtt{F0} & \text{iff } A \notin \Gamma, \neg A \notin \Gamma, \sim A \notin \Gamma, \sim\neg A \in \Gamma; \\ \emptyset & \text{otherwise.} \end{cases}$$

□

6 Some possible truth-functional extensions of LRA and LIRA

In this section, we describe several truth-functional operators which can be added to **LRA** or **LIRA** themselves or to their $\{\neg, \sim\}$-fragments. To be sure, the resulting logics can be formalized via correspondence analysis.

Since **LIRA** and **LRA** have two negations, conjunction, and disjunction, it is natural to think about an implication for them. We suggest the class of implications which may be added to these logics. This class consists of four-valued versions of Tomova's [40, 15] three-valued natural implications.[3]

Definition 6.1. An implication $\rightarrow$ is said to be natural iff the following conditions hold, for each $x, y \in \mathscr{V}_4$:

(1) (a) $\rightarrow$ coincides with classical implication on the set $\{\mathtt{T1}, \mathtt{F1}\}$ (for the case of **LRA**);
(b) $\rightarrow$ coincides with classical implication on the set $\{\mathtt{T1}, \mathtt{F0}\}$ (for the case of **LIRA**);

(2) $\rightarrow$ satisfies the modus ponens principle, i.e.

(a) $x = \mathtt{T1}$ and $x \rightarrow y = \mathtt{T1}$ implies $y = \mathtt{T1}$ (for the case of **LRA**),
(b) $x \neq \mathtt{F0}$ and $x \rightarrow y \neq \mathtt{F0}$ implies $y \neq \mathtt{F0}$ (for the case of **LIRA**).

(3) (a) $x \leqslant y$ implies $x \rightarrow y = \mathtt{T1}$ (for the case of **LRA**),
(b) $x \leqslant' y$ implies $x \rightarrow y \neq \mathtt{F0}$ (for the case of **LIRA**).

Thus, for the case of **LRA** and **LIRA**, respectively, we have 576 and 34992 natural implications which are as follows:

LRA

$\rightarrow$	T1	T0	F0	F1
T1	T1	$\mathtt{C}_1$	$\mathtt{C}_2$	F1
T0	T1	T1	$\mathtt{A}_1$	$\mathtt{A}_2$
F0	T1	T1	T1	$\mathtt{A}_3$
F1	T1	T1	T1	T1

where $\mathtt{A}_1, \mathtt{A}_2, \mathtt{A}_3 \in \mathscr{V}_4$ and $\mathtt{C}_1, \mathtt{C}_2 \in \{\mathtt{T0}, \mathtt{F1}, \mathtt{F0}\}$.

LIRA

$\rightarrow$	T1	T0	F1	F0
T1	T1	$\mathtt{A}_1$	$\mathtt{A}_2$	F0
T0	$\mathtt{D}_1$	$\mathtt{D}_2$	$\mathtt{A}_3$	F0
F1	$\mathtt{D}_3$	$\mathtt{D}_4$	$\mathtt{D}_5$	F0
F0	T1	$\mathtt{D}_6$	$\mathtt{D}_7$	T1

where $\mathtt{A}_1, \mathtt{A}_2, \mathtt{A}_3 \in \mathscr{V}_4$ and $\mathtt{D}_i \in \{\mathtt{T1}, \mathtt{T0}, \mathtt{F1}\}$, $1 \leqslant i \leqslant 7$.

Let us stress that inference rules for all these implications can be found due to correspondence analysis. In the rest of this section, we are going to discuss in more details several well-known implications. We start with the ones which could be added to **LRA** (all of them, except $\rightarrow_3$, are natural).

$\rightarrow_1$	T1	T0	F0	F1
T1	T1	T0	F0	F1
T0	T1	T1	T0	F0
F0	T1	T1	T1	T0
F1	T1	T1	T1	T1

$\rightarrow_2$	T1	T0	F0	F1
T1	T1	T0	F0	F1
T0	T1	T1	F0	F1
F0	T1	T1	T1	F1
F1	T1	T1	T1	T1

$\rightarrow_3$	T1	T0	F0	F1
T1	T1	F1	F1	F1
T0	T1	T1	F1	F1
F0	T1	F1	T1	F1
F1	T1	T1	T1	T1

[3] It were Méndez and Robles [24] who first adapted Tomova's definition for the four-valued case. They dealt with the only designated value (as in the case of **LRA**), but with partially ordered truth values (recall that in our case they are linearly ordered). As a result, Méndez and Robles [24] obtained 2304 natural implications. Let us mention their recent papers [31, 30], where they present Hilbert-style calculi and Routley-Meyer semantics for Tomova's three-valued natural logics. Some of Tomova's three-valued natural implications are natural in the sense of Avron's paper [2] as well.

$\to_4$	T1	T0	F0	F1
T1	T1	F1	F1	F1
T0	T1	T1	F1	F1
F0	T1	T1	T1	F1
F1	T1	T1	T1	T1

$\to_5$	T1	T0	F0	F1
T1	T1	F1	F1	F1
T0	T1	T1	T1	T1
F0	T1	T1	T1	T1
F1	T1	T1	T1	T1

$\to_6$	T1	T0	F0	F1
T1	T1	T0	F0	F1
T0	T1	T1	T1	T1
F0	T1	T1	T1	T1
F1	T1	T1	T1	T1

The implication $\to_1$ is Łukasiewisz's [23, 22], from his logic $\mathbf{Ł_4}$. The implication $\to_2$ is Gödel's, from his logic $\mathbf{G_4}$ [13].[4] The implication $\to_3$ is Smiley's. He found a characteristic matrix for Anderson and Belnap's $\mathbf{E_{fde}}$ and sent in the mail correspondence to Anderson and Belnap. Then this matrix was published by Anderson and Belnap in [1] with the reference to personal communication with Smiley. The connective $\to_4$ is a four-valued version of Rescher's three-valued implication [29]. The implication $\to_5$ is due to Sette and Carnielli and their four-valued logic $\mathbf{I^1_4}$ which is a generalization of their logic $\mathbf{I^1}$ [34]. The implication $\to_6$ satisfies Rosser and Turquette's [32] conditions of the classicality of the implication in the four-valued case with one designated value.[5]

Let us consider some remarkable natural implications which could be added to **LIRA**.

$\to_7$	T1	T0	F1	F0
T1	T1	T0	F1	F0
T0	T1	T1	F1	F0
F1	T1	T1	T1	F0
F0	T1	T1	T1	T1

$\to_8$	T1	T0	F1	F0
T1	T1	F0	F0	F0
T0	T1	T1	F0	F0
F1	T1	T1	T1	F0
F0	T1	T1	T1	T1

$\to_9$	T1	T0	F1	F0
T1	T1	T1	T1	F0
T0	T1	T1	T1	F0
F1	T1	T1	T1	F0
F0	T1	T1	T1	T1

$\to_{10}$	T1	T0	F1	F0
T1	T1	T0	F1	F0
T0	T1	T0	F1	F0
F1	T1	T0	F1	F0
F0	T1	T1	T1	T1

The implication $\to_7$ is yet another version of Gödel's from $\mathbf{G_4}$. In contrast to the case of **LRA** (see footnote 4), now we associate $1/3$ with F1 and 0 with F0. Similarly, $\to_8$ is yet another version of Rescher's implication. The implication $\to_9$ is from Carnielli and Lima-Marques' [9] four-valued logic $\mathbf{P^1_4}$ which is a modification of Sette's three-valued logic $\mathbf{P^1}$ [33]. The implication $\to_{10}$ satisfies Rosser and Turquette's conditions of the classicality of the implication in the four-valued case with three designated values.

7 Modal extensions of LIRA

7.1 Kubyshkina and Zaitsev's modal operators for LRA

Kubyshkina and Zaitsev [16] consider relational models of the form $\langle W, R, V\rangle$, where W is a non-empty set of worlds, R is a reflexive, transitive, and symmetric (viz., universal) binary relation on

[4]Notice that $\mathbf{Ł_4}$ and $\mathbf{G_4}$ have the following truth values: 1, $2/3$, $1/3$, 0. We associate 1 with T1, $2/3$ with T0, $1/3$ with F0, and 0 with F1.

[5]Let $\mathscr{D}$ be the set of designated values. Rosser and Turquette [32] showed that the set of tautologies of the many-valued logic coincides with the set of tautologies of classical logic, if the connectives of a many-valued logic satisfies the following conditions: $V(\neg A) \in \mathscr{D}$ iff $v(A) \notin \mathscr{D}$, $V(A \to B) \notin \mathscr{D}$ iff $v(A) \in \mathscr{D}$ and $v(B) \notin \mathscr{D}$, $V(A \wedge B) \in \mathscr{D}$ iff $v(A) \in \mathscr{D}$ and $V(B) \in \mathscr{D}$, $V(A \vee B) \notin \mathscr{D}$ iff $v(A) \notin \mathscr{D}$ and $V(B) \notin \mathscr{D}$.

W, and V is a valuation defined as a function from[6] $\mathscr{L}^M \times W$ into $\mathscr{V}_4$ such that it preserves the above mentioned conditions for the truth-functional operators and, for any $A \in \mathscr{F}^M$ and $x \in W$, we have:

- $V(\Box A, w) = \texttt{T1}$ iff $\forall_{u \in R[w]} V(A, u) = \texttt{T1}$,
- $V(\Box A, w) = \texttt{T0}$ iff $\forall_{u \in R[w]} (V(A, u) = \texttt{T1}$ or $V(A, u) = \texttt{T0})$ and $\exists_{u \in R[w]} V(A, u) = \texttt{T0}$,
- $V(\Box A, w) = \texttt{F1}$ iff $\exists_{u \in R[w]} V(A, u) = \texttt{F1}$,
- $V(\Box A, w) = \texttt{F0}$ iff $\forall_{u \in R[w]} V(A, u) \neq \texttt{F1}$ and $\exists_{u \in R[w]} V(A, u) = \texttt{F0}$;

- ∘ $V(\Diamond^{\neg} A, w) = \texttt{T1}$ iff $\exists_{u \in R[w]} V(A, u) = \texttt{T1}$,
- ∘ $V(\Diamond^{\neg} A, w) = \texttt{T0}$ iff $\forall_{u \in R[w]} V(A, u) \neq \texttt{T1}$ and $\exists_{u \in R[w]} V(A, u) = \texttt{T0}$,
- ∘ $V(\Diamond^{\neg} A, w) = \texttt{F1}$ iff $\forall_{u \in R[w]} V(A, u) = \texttt{F1}$,
- ∘ $V(\Diamond^{\neg} A, w) = \texttt{F0}$ iff $\forall_{u \in R[w]} (V(A, u) = \texttt{F1}$ or $V(A, u) = \texttt{F0})$ and $\exists_{u \in R[w]} V(A, u) = \texttt{F0}$.

Kubyshkina and Zaitsev use the name $\mathbf{LRA}^{\Diamond\Box}$ for an extension of **LRA** by these operators. Let us use the abbreviation $\mathbf{LRA}^{\mathbf{K}}$ for the logic which has an arbitrary accessibility relation R. As they notice, $\Diamond^{\neg} A$ is equivalent to $\neg\Box\neg A$. Moreover, it is easy to show that $\Box A$ is equivalent to $\neg\Diamond^{\neg}\neg A$. It is possible to introduce two new possibility operators as follows: $\Diamond^{\sim} A := {\sim}\Box{\sim} A$ and $\Diamond^{C} A := {\sim}\neg\Box{\sim}\neg A$ (C stands for classical, since a combination of $\sim$ and $\neg$ is classical negation):

- $V(\Diamond^{\sim} A, w) = \texttt{T1}$ iff $\forall_{u \in R[w]} (V(A, u) = \texttt{T1}$ or $V(A, u) = \texttt{T0})$ and $\exists_{u \in R[w]} V(A, u) = \texttt{T1}$,
- $V(\Diamond^{\sim} A, w) = \texttt{T0}$ iff $\forall_{u \in R[w]} V(A, u) = \texttt{T0}$,
- $V(\Diamond^{\sim} A, w) = \texttt{F1}$ iff $\forall_{u \in R[w]} V(A, u) \neq \texttt{F0}$ and $\exists_{u \in R[w]} V(A, u) = \texttt{F1}$,
- $V(\Diamond^{\sim} A, w) = \texttt{F0}$ iff $\exists_{u \in R[w]} V(A, u) = \texttt{F0}$.

- ∘ $V(\Diamond^{C} A, w) = \texttt{T1}$ iff $\forall_{u \in R[w]} (V(A, u) \neq \texttt{T0})$ and $\exists_{u \in R[w]} V(A, u) = \texttt{T1}$,
- ∘ $V(\Diamond^{C} A, w) = \texttt{T0}$ iff $\exists_{u \in R[w]} V(A, u) = \texttt{T0}$,
- ∘ $V(\Diamond^{C} A, w) = \texttt{F1}$ iff $\forall_{u \in R[w]} (V(A, u) = \texttt{F1}$ or $V(A, u) = \texttt{F0})$ and $\exists_{u \in R[w]} V(A, u) = \texttt{F1}$,
- ∘ $V(\Diamond^{C} A, w) = \texttt{F0}$ iff $\forall_{u \in R[w]} V(A, u) = \texttt{F0}$.

By a routine check, we obtain the following propositions which hold for $\mathbf{LRA}^{\mathbf{K}}$ and $\mathbf{LRA}^{\Diamond\Box}$.

Proposition 7.1. *For each $A \in \mathscr{F}^M$, it holds that*

(1) $\Box A = \neg\Diamond^{\neg}\neg A = {\sim}\Diamond^{\sim}{\sim} A = {\sim}\neg\Diamond^{C}{\sim}\neg A$;

(2) $\Diamond^{\neg} A = {\sim}\neg\Diamond^{\sim}{\sim}\neg A = {\sim}\Diamond^{C}{\sim} A$;

(3) $\Diamond^{\sim} A = {\sim}\neg\Diamond^{\neg}{\sim}\neg A = \neg\Diamond^{C}\neg A$;

(4) $\Diamond^{C} A = {\sim}\Diamond^{\neg}{\sim} A = \neg\Diamond^{\sim}\neg A$.

Proposition 7.2. *For each $A \in \mathscr{F}^M$, it holds that*

(1) $\neg\Box A = \Diamond^{\neg}\neg A$, ${\sim}\Box A = \Diamond^{\sim}{\sim} A$, ${\sim}\neg\Box A = \Diamond^{C}{\sim}\neg A$;

(2) $\Box\neg A = \neg\Diamond^{\neg} A$, $\Box{\sim} A = {\sim}\Diamond^{\sim} A$, $\Box{\sim}\neg A = {\sim}\neg\Diamond^{C} A$;

(3) ${\sim}\Diamond^{\neg} A = \Diamond^{C}{\sim} A$, $\neg\Diamond^{\sim} A = \Diamond^{C}\neg A$, $\neg\Diamond^{C} A = \Diamond^{\sim}\neg A$, ${\sim}\Diamond^{C} A = \Diamond^{\neg}{\sim} A$;

(4) ${\sim}\neg\Diamond^{\neg} A = \Diamond^{\sim}{\sim}\neg A$, ${\sim}\neg\Diamond^{\sim} A = \Diamond^{\neg}{\sim}\neg A$.

[6] We write $\mathscr{L}^M$ for an extension of $\mathscr{L}$ by $\Box$. The operator $\Diamond^{\neg}$ can be introduced as $\neg\Box\neg$ (for this reason we also change its original denotation, i.e. $\Diamond$, which has a sense, since **LRA** has several negations). We denote via $\mathscr{F}^M$ the set of all $\mathscr{L}^M$-formulas. We will also use notation $\mathscr{L}^M_{(\circledcirc)_k}$ and $\mathscr{F}^M_{(\circledcirc)_k}$ when we enrich the language $\mathscr{L}_{(\circledcirc)_k}$ by $\Box$. We put $R[w] := \{u \in W \mid wRu\}$.

7.2 Modal operators for LIRA

Let us consider the following rule:

$$(\Box I)\ \frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$$

It is a standard introduction rule for $\Box$ in linear-type natural deduction systems. One may consult Garson's book [12] for its usage in the modal logics which have classical logic as its propositional basis. If $\Gamma = \emptyset$, then it is just the well-known Gödel's rule. Unfortunately, if we enrich **LIRA** by Kubyshkina and Zaitsev's $\Box$, then $(\Box I)$ is not valid.[7] It is not surprising and it is not a drawback of Kubyshkina and Zaitsev's $\Box$, since their definition of $\Box$ was designed for **LRA** and its entailment relation, while **LIRA** has a different set of designated values. Let us pinpoint the case $V(\Box A, w) = \texttt{F1}$. Then $V(A, u) = \texttt{F1}$, for some $u \in R[w]$. It is possible that in some other element of $R[w]$, say z, $V(A, z) = \texttt{F0}$. It is not a problem for **LRA**, since both F1 and F0 are not designated values. But in **LIRA**, F1 is designated, while F0 is not. We think that this fact has made it possible to find a countermodel for $(\Box I)$ in **LIRA** extended by $\Box$ (see a footnote 7). Thus, in order to have a necessity operator in **LIRA** with the standard introduction rule for it we need to change the clause for F1 in Kubyshkina and Zaitsev's definition of $\Box$. We need to require that $V(A, z) \neq \texttt{F0}$, for all $z \in R[w]$. As a consequence, we need to change the clause F0 as well, where it is required that $V(A, z) \neq \texttt{F0}$, for all $z \in R[w]$. As a result, we have a new necessity operator, let us call it $\boxdot$ (do not confuse it with Boolos' [7] 'boxdot' operator understood as 'provable and true'), which truth conditions are as follows:

- $V(\boxdot A, w) = \texttt{T1}$ iff $\forall_{u\in R[w]} V(A, u) = \texttt{T1}$,
- $V(\boxdot A, w) = \texttt{T0}$ iff $\forall_{u\in R[w]}(V(A, u) = \texttt{T1}$ or $V(A, u) = \texttt{T0})$ and $\exists_{u\in R[w]} V(A, u) = \texttt{T0}$,
- $V(\boxdot A, w) = \texttt{F1}$ iff $\forall_{u\in R[w]} V(A, u) \neq \texttt{F0}$ and $\exists_{u\in R[w]} V(A, u) = \texttt{F1}$,
- $V(\boxdot A, w) = \texttt{F0}$ iff $\exists_{u\in R[w]} V(A, u) = \texttt{F0}$.

We introduce the possibility operators, using the equations $\blacklozenge^{\neg} A := \neg\boxdot\neg A$, $\blacklozenge^{\sim} A := {\sim}\boxdot{\sim} A$, and $\blacklozenge^{C} A := {\sim}\neg\boxdot{\sim}\neg A$ as follows:

- ○ $V(\blacklozenge^{\neg} A, w) = \texttt{T1}$ iff $\forall_{u\in R[w]} V(A, u) \neq \texttt{T0}$ and $\exists_{u\in R[w]} V(A, u) = \texttt{T1}$,
- ○ $V(\blacklozenge^{\neg} A, w) = \texttt{T0}$ iff $\exists_{u\in R[w]} V(A, u) = \texttt{T0}$,
- ○ $V(\blacklozenge^{\neg} A, w) = \texttt{F1}$ iff $\forall_{u\in R[w]} V(A, u) = \texttt{F1}$,
- ○ $V(\blacklozenge^{\neg} A, w) = \texttt{F0}$ iff $\forall_{u\in R[w]}(V(A, u) = \texttt{F1}$ or $V(A, u) = \texttt{F0})$ and $\exists_{u\in R[w]} V(A, u) = \texttt{F0}$.

- $V(\blacklozenge^{\sim} A, w) = \texttt{T1}$ iff $\forall_{u\in R[w]}(V(A, u) = \texttt{T1}$ or $V(A, u) = \texttt{T0})$ and $\exists_{u\in R[w]} V(A, u) = \texttt{T1}$,
- $V(\blacklozenge^{\sim} A, w) = \texttt{T0}$ iff $\forall_{u\in R[w]} V(A, u) = \texttt{T0}$,
- $V(\blacklozenge^{\sim} A, w) = \texttt{F1}$ iff $\exists_{u\in R[w]} V(A, u) = \texttt{F1}$,
- $V(\blacklozenge^{\sim} A, w) = \texttt{F0}$ iff $\forall_{u\in R[w]} V(A, u) \neq \texttt{F1}$ and $\exists_{u\in R[w]} V(A, u) = \texttt{F0}$.

- ○ $V(\blacklozenge^{C} A, w) = \texttt{T1}$ iff $\exists_{u\in R[w]} V(A, u) = \texttt{T1}$,
- ○ $V(\blacklozenge^{C} A, w) = \texttt{T0}$ iff $\exists_{u\in R[w]} V(A, u) = \texttt{T0}$ and $\forall_{u\in R[w]} V(A, u) \neq \texttt{T1}$,

[7] Consider its particular case, for some $A, B \in \mathscr{F}^M$: $B \vdash A / \Box B \vdash \Box A$. A countermodel is as follows: $\langle W, R, V\rangle$, where $W = \{w, u, z\}$, wRu, wRz, $V(A, w) \neq \texttt{F0}$, $V(A, u) = \texttt{F0}$, $V(A, z) = \texttt{T1}$, $V(B, w) \neq \texttt{F0}$, $V(B, u) = \texttt{F0}$, and $V(B, z) = \texttt{F1}$. Then, for each $x \in W$, $V(B, x) \neq \texttt{F0}$ implies $V(A, x) \neq \texttt{F0}$. But $V(\Box B, w) = \texttt{F1}$ and $V(\Box A, w) = \texttt{F0}$.

- $V(\blacklozenge^{C} A, w) = \texttt{F1}$ iff $\forall_{u \in R[w]}(V(A,u) = \texttt{F1}$ or $V(A,u) = \texttt{F0})$ and $\exists_{u \in R[w]} V(A,u) = \texttt{F1}$,
- $V(\blacklozenge^{C} A, w) = \texttt{F0}$ iff $\forall_{u \in R[w]} V(A,u) = \texttt{F0}$.

Let $\mathbf{LIRA}^{\mathbf{K}}$ and $\mathbf{LIRA}^{\mathbf{K}}_{(\circledcirc)_k}$, respectively, be extensions of **LIRA** and $\mathbf{LIRA}_{(\circledcirc)_k}$ by $\boxdot$ such that there are no restrictions on the relation R. In the other words, $\mathbf{LIRA}^{\mathbf{K}}$ and $\mathbf{LIRA}^{\mathbf{K}}_{(\circledcirc)_k}$, respectively, are an **LIRA**- and $\mathbf{LIRA}_{(\circledcirc)_k}$-based versions of the basic normal modal logic **K**. It is easy to show that the equalities from Propositions 7.1 and 7.2 hold for $\boxdot$, $\blacklozenge^{\neg}$, $\blacklozenge^{\sim}$, and $\blacklozenge^{C}$ for $\mathbf{LIRA}^{\mathbf{K}}$ and $\mathbf{LIRA}^{\mathbf{K}}_{(\circledcirc)_k}$.

7.3 A natural deduction system for K-style modal extension of LIRA

For each $\Gamma \subseteq \mathscr{F}^{M}_{(\circledcirc)_k}$, we use the following notation: $\boxdot\Gamma = \{\boxdot G \mid G \in \Gamma\}$, $\neg\boxdot\Gamma = \{\neg\boxdot G \mid \neg G \in \Gamma\}$, and $\sim\boxdot\Gamma = \{\sim\boxdot G \mid \sim G \in \Gamma\}$. Let us introduce a natural deduction system for $\mathbf{LIRA}^{\mathbf{K}}_{(\circledcirc)_k}$ (we do it in the spirit of one of the approaches to natural deduction systems for classical modal logics described in, for example, Indrzejczak's book [14]). It has modal subderivations ($\boxdot$-subderivations) of the following form:

$$\begin{array}{l|l} 1 & \boxdot \\ \hline 2 & \vdots \end{array}$$

and it extends the natural deduction system for $\mathbf{LIRA}_{(\circledcirc)_k}$ by the following rules:

$$(\boxdot I)\ \frac{\Gamma \vdash A}{\boxdot\Gamma \vdash \boxdot A}, \quad (\neg\boxdot I)\ \frac{\Gamma \vdash \neg A}{\boxdot\Gamma \vdash \neg\boxdot A}, \quad (\sim\boxdot I)\ \frac{\Gamma \vdash \sim A}{\boxdot\Gamma \vdash \sim\boxdot A},$$

and the rule (*Reit*) of reiteration which works as follows:

- each formula which appears in a derivation can be added to its *non*-modal subderivation;
- each *non*-modal formula which appears in a derivation can be added to its $\boxdot$-subderivation;
- for each formula of the form $\boxdot A$ which appears in a derivation, the formula A can be added to its $\boxdot$-subderivation.

Let us illustrate these cases as follows (from left to right), where A' is a non-modal formula:

1	A		1	A'		1	$\boxdot A$	
2	$\quad B$		2	$\quad \boxdot$		2	$\quad \boxdot$	
3	$\quad A$	(*Reit*): 1	3	$\quad A'$	(*Reit*): 1	3	$\quad A$	(*Reit*): 1

Let us illustrate applications of the rules $(\boxdot I)$, $(\neg\boxdot I)$, and $(\sim\boxdot I)$ as follows:

1	$\boxdot G$		1	$\boxdot G$		1	$\boxdot G$	
2	$\quad \boxdot$		2	$\quad \boxdot$		2	$\quad \boxdot$	
3	$\quad G$	(*Reit*): 1	3	$\quad G$	(*Reit*): 1	3	$\quad G$	(*Reit*): 1
4	$\quad \vdots$		4	$\quad \vdots$		4	$\quad \vdots$	
5	$\quad A$		5	$\quad \neg A$		5	$\quad \sim A$	
6	$\boxdot A$	$(\boxdot I)$: 2–5	6	$\neg\boxdot A$	$(\neg\boxdot I)$: 2–5	6	$\sim\boxdot A$	$(\sim\boxdot I)$: 2–5

Clearly, by applying the rule ($\boxdot I$) (resp. ($\neg\boxdot I$), ($\sim\boxdot I$)) all formulas starting from the formula G until the formula A (resp. $\neg A$, $\sim A$), inclusively, are discarded from the inference.

Lemma 7.3. *All the rules of a natural deduction system for* $\mathbf{LIRA}^{\mathbf{K}}_{(\circledcirc)_k}$ *are sound.*

Proof. As an example, we show the soundness of the rule ($\neg\boxdot I$). Consider a relational model $\langle W, R, V\rangle$. Suppose $\Gamma \models \neg A$. Then ($*$) for each $x \in W$, if $V(G, x) \neq \mathtt{F0}$ (for each $G \in \Gamma$), then $V(\neg A, x) \neq \mathtt{F0}$. Suppose $\boxdot\Gamma \not\models \neg\boxdot A$. Consider $w \in W$ such that $V(\boxdot G, w) \neq \mathtt{F0}$ (for each $G \in \Gamma$) and $V(\neg\boxdot A, w) = \mathtt{F0}$. Then ($\star$) for each $G \in \Gamma$, $\forall_{u\in R[w]} V(G, u) \neq \mathtt{F0}$. Moreover, $V(\boxdot A, w) = \mathtt{T0}$. Thus, $\forall_{u\in R[w]}(V(A, u) = \mathtt{T1}$ or $V(A, u) = \mathtt{T0})$ and $\exists_{u\in R[w]} V(A, u) = \mathtt{T0}$. Consider $u \in W$ such that $u \in R[w]$ and $V(A, u) = \mathtt{T0}$. Then $V(\neg A, u) = \mathtt{F0}$. Using ($*$), we have $V(G, u) = \mathtt{F0}$ (for some $G \in \Gamma$). However, by ($\star$), $V(G, u) \neq \mathtt{F0}$, for each $G \in \Gamma$. Contradiction. Thus, $\boxdot\Gamma \models \neg\boxdot A$. □

Theorem 7.4. *Let* $\mathbf{L}$ *be* $\mathbf{LIRA}^{\mathbf{K}}_{(\circledcirc)_k}$*. For each* $\Gamma \subseteq \mathscr{F}^M_{(\circledcirc)_k}$ *and* $A \in \mathscr{F}^M_{(\circledcirc)_k}$*, it holds that if* $\Gamma \vdash_{\mathbf{L}} A$*, then* $\Gamma \models_{\mathbf{L}} A$*.*

Proof. By induction on the length of the derivation. Use Theorem 4.11 and Lemma 7.3. □

Let us present a completeness proof for $\mathbf{LIRA}^{\mathbf{K}}_{(\circledcirc)_k}$ (we follow Garson's book [12], where the strategy for modal logics with classical propositional basis is described).

Definition 7.5. Let $\mathbf{L}$ be $\mathbf{LIRA}^{\mathbf{K}}_{(\circledcirc)_k}$. Then the canonic model for $\mathbf{L}$ is a triple $\langle W_{\mathbf{L}}, R_{\mathbf{L}}, V_{\mathbf{L}}\rangle$, where

(1) $W_{\mathbf{L}}$ is the set of all $\mathbf{LIRA}^{\mathbf{L}}_{(\circledcirc)_k}$-theories,

(2) $\Gamma R_{\mathbf{L}}\Delta$ (where $\Gamma, \Delta \in W_{\mathbf{L}}$) iff for each $A \in \mathscr{F}^M_{(\circledcirc)_k}$, $\boxdot A \in \Gamma$ implies $A \in \Delta$, $\neg\boxdot A \in \Gamma$ implies $\neg A \in \Delta$, and $\sim\boxdot A \in \Gamma$ implies $\sim A \in \Delta$,

(3) for each $A \in \mathscr{F}^M_{(\circledcirc)_k}$ and $\Gamma \in W_{\mathbf{L}}$, $V_{\mathbf{L}}(A, \Gamma) = V^c_\Gamma(A)$ (see Definition 4.15, note that we extend the valuation $V^c_\Gamma(A)$ on the set $\mathscr{F}^M_{(\circledcirc)_k}$).

Lemma 7.6. *The canonic model for* $\mathbf{LIRA}^{\mathbf{K}}_{(\circledcirc)_k}$ *is* $\mathbf{LIRA}^{\mathbf{K}}_{(\circledcirc)_k}$*-model.*

Proof. The propositional case follows from lemmas 4.14 and 4.16. Let us consider the case of formulas of the form $\boxdot A$. We need to prove the following equivalences (where $R_{\mathbf{L}}[\Gamma] := \{\Delta \in W_{\mathbf{L}} \mid \Gamma R_{\mathbf{L}}\Delta\}$):

- $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{T1}$ iff $\forall_{\Delta\in R_{\mathbf{L}}[\Gamma]} V_{\mathbf{L}}(A, \Delta) = \mathtt{T1}$,
- $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{T0}$ iff $\forall_{\Delta\in R_{\mathbf{L}}[\Gamma]}(V_{\mathbf{L}}(A, \Delta) = \mathtt{T1}$ or $V_{\mathbf{L}}(A, u) = \mathtt{T0})$ as well as $\exists_{\Delta\in R_{\mathbf{L}}[\Gamma]} V_{\mathbf{L}}(A, \Delta) = \mathtt{T0}$,
- $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{F1}$ iff $\forall_{\Delta\in R_{\mathbf{L}}[\Gamma]} V_{\mathbf{L}}(A, \Delta) \neq \mathtt{F0}$ and $\exists_{\Delta\in R_{\mathbf{L}}[\Gamma]} V_{\mathbf{L}}(A, \Delta) = \mathtt{F1}$,
- $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{F0}$ iff $\exists_{\Delta\in R_{\mathbf{L}}[\Gamma]} V_{\mathbf{L}}(A, \Delta) = \mathtt{F0}$.

We need to show the following statements (where $\equiv_{df}$ means 'equivalent by the definition'):

- ∘ $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{T1} \equiv_{df} \big(\boxdot A \in \Gamma, \neg\boxdot A \in \Gamma, \sim\boxdot A \in \Gamma, \sim\neg\boxdot A \notin \Gamma\big)$ iff $\forall_{\Delta\in R_{\mathbf{L}}[\Gamma]}\big(A \in \Delta, \neg A \in \Delta, \sim A \in \Delta, \sim\neg A \notin \Delta\big)$,
- ∘ $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{T0} \equiv_{df} \big(\boxdot A \in \Gamma, \neg\boxdot A \notin \Gamma, \sim\boxdot A \in \Gamma, \sim\neg\boxdot A \in \Gamma\big)$ iff $\forall_{\Delta\in R_{\mathbf{L}}[\Gamma]}\big((A \in \Delta, \neg A \in \Delta, \sim A \in \Delta, \sim\neg A \notin \Delta)$ or $(A \in \Delta, \neg A \notin \Delta, \sim A \in \Delta, \sim\neg A \in \Delta)\big)$ as well as $\exists_{\Delta\in R_{\mathbf{L}}[\Gamma]}\big(A \in \Delta, \neg A \notin \Delta, \sim A \in \Delta, \sim\neg A \in \Delta\big)$,

- $\circ$ $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{F1} \equiv_{df} \big(\boxdot A \in \Gamma, \neg\boxdot A \in \Gamma, {\sim}\boxdot A \notin \Gamma, {\sim}\neg\boxdot A \in \Gamma\big)$ iff $\forall_{\Delta \in R_{\mathbf{L}}[\Gamma]}\big(A \in \Delta$ or $\neg A \notin \Delta$ or ${\sim}A \notin \Delta$ or ${\sim}\neg A \notin \Delta\big)$ as well as $\exists_{\Delta \in R_{\mathbf{L}}[\Gamma]}\big(A \in \Delta, \neg A \in \Delta, {\sim}A \notin \Delta, {\sim}\neg A \in \Delta\big)$,
- $\circ$ $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{F0} \equiv_{df} \big(\boxdot A \notin \Gamma, \neg\boxdot A \in \Gamma, {\sim}\boxdot A \in \Gamma, {\sim}\neg\boxdot A \in \Gamma\big)$ iff $\exists_{\Delta \in R_{\mathbf{L}}[\Gamma]}\big(A \notin \Delta, \neg A \in \Delta, {\sim}A \in \Delta, {\sim}\neg A \in \Delta\big)$.

Let us define the following set Φ, for each $B \in \mathscr{F}^{M}_{(\circledcirc)_k}$ and the above mentioned Γ:

(Φ_1) $B \in \Phi$ iff $\boxdot B \in \Gamma$,

(Φ_2) $\neg B \in \Phi$ iff $\neg\boxdot B \in \Gamma$,

(Φ_3) ${\sim}B \in \Phi$ iff ${\sim}\boxdot B \in \Gamma$.

We put $\Psi = \boxdot\Phi \cup \neg\boxdot\Phi \cup {\sim}\boxdot\Phi$. Let us prove several auxiliary propositions.

Proposition 7.7. *Γ is an extension of Ψ.*

Proof. Follows from the definition of Ψ. □

Proposition 7.8. *Consider the above mentioned A, Γ, and Φ as well as some finite subset Ω of Φ and an arbitrary $C \in \mathscr{F}^{M}_{(\circledcirc)_k}$. Then:*

(1) *if $\boxdot A \notin \Gamma$, then $\Omega \cup \{\neg A, {\sim}A, {\sim}\neg A\} \nvdash C$;*

(2) *if $\neg\boxdot A \notin \Gamma$, then $\Omega \cup \{A, {\sim}A, {\sim}\neg A\} \nvdash C$;*

(3) *if ${\sim}\boxdot A \notin \Gamma$, then $\Omega \cup \{A, \neg A, {\sim}\neg A\} \nvdash C$.*

Proof. First of all, we show that the following rules are derivable in the natural deduction system for $\mathbf{LIRA}^{\mathbf{K}}_{(\circledcirc)_k}$:

$$(R_1)\ \frac{\Omega \cup \{\neg A, {\sim}A, {\sim}\neg A\} \vdash C}{\Omega \vdash A}, \quad (R_2)\ \frac{\Omega \cup \{A, {\sim}A, {\sim}\neg A\} \vdash C}{\Omega \vdash \neg A},$$

$$(R_3)\ \frac{\Omega \cup \{A, \neg A, {\sim}\neg A\} \vdash C}{\Omega \vdash {\sim}A}.$$

As an example, we present the derivability of (R_2) on Figure 2 (in this proof, we assume that $\Omega = \emptyset$ for the space reasons only).

(1) Suppose $\boxdot A \notin \Gamma$. Then, since Γ is $\mathbf{LIRA}^{\mathbf{K}}_{(\circledcirc)_k}$-theory, $\Gamma \nvdash \boxdot A$. Suppose $\Omega \cup \{\neg A, {\sim}A, {\sim}\neg A\} \vdash C$. Then, by the rule (R_1), $\Omega \vdash A$. By the rule $(\boxdot I)$, $\boxdot\Omega \vdash \boxdot A$. Since $\Omega \subseteq \Phi$, $\boxdot\Omega \subseteq \boxdot\Phi$. Moreover, $\boxdot\Phi \subseteq \Psi$. By Proposition 7.7, Γ is an extension of Ψ. Thus, $\Gamma \vdash \boxdot A$. Contradiction. Hence, $\Omega \cup \{\neg A, {\sim}A, {\sim}\neg A\} \nvdash C$.

(2) Suppose $\neg\boxdot A \notin \Gamma$. Then $\Gamma \nvdash \neg\boxdot A$. Suppose $\Omega \cup \{A, {\sim}A, {\sim}\neg A\} \vdash C$. By the rule (R_2), $\Omega \vdash \neg A$. By the rule $(\neg\boxdot I)$, $\boxdot\Omega \vdash \neg\boxdot A$. Thus, $\Gamma \vdash \neg\boxdot A$. Contradiction. Hence, $\Omega \cup \{A, {\sim}A, {\sim}\neg A\} \nvdash C$.

(3) Suppose ${\sim}\boxdot A \notin \Gamma$. Then $\Gamma \nvdash {\sim}\boxdot A$. Suppose $\Omega \cup \{A, \neg A, {\sim}\neg A\} \vdash C$. By the rule (R_3), $\Omega \vdash {\sim}A$. By the rule $({\sim}\boxdot I)$, $\boxdot\Omega \vdash {\sim}\boxdot A$. Thus, $\Gamma \vdash {\sim}\boxdot A$. Contradiction. Hence, $\Omega \cup \{A, \neg A, {\sim}\neg A\} \nvdash C$. □

Proposition 7.9. *Consider the above mentioned Γ, Δ, Φ, and A. Then:*

(1) *if Δ is an extension of $\Phi \cup \{\neg A, {\sim}A, {\sim}\neg A\}$, then $\Gamma R_{\mathbf{L}} \Delta$;*

(2) *if Δ is an extension of $\Phi \cup \{A, {\sim}A, {\sim}\neg A\}$, then $\Gamma R_{\mathbf{L}} \Delta$;*

1	$\neg A$	assumption
2	$\sim A$	assumption
3	$\neg A$	assumption
4	$\sim\neg A$	assumption
5	A	assumption
6	$\neg A$	by the condition: 2, 4, 5
7	$\neg A$	assumption
8	$\neg A$	(EM_1): 6, 7 [5–6], [7]
9	$\neg A$	(EM_2): 3, 8 [3], [4–8]
10	$\neg A$	(EM_4): 1, 9 [1], [2–9]

Figure 2: An inference of $\vdash \neg A$ from $A, \sim A, \sim\neg A \vdash C$ in $\mathfrak{ND}^{\neg\sim}_{\mathbf{DLRA}}$.

(3) *if* Δ *is an extension of* $\Phi \cup \{A, \neg A, \sim\neg A\}$*, then* $\Gamma R_{\mathbf{L}} \Delta$.

Proof. (1) Suppose Δ is an extension of $\Phi \cup \{\neg A, \sim A, \sim\neg A\}$. Then, by the definition $R_{\mathbf{L}}$, we need to show that $\boxdot A \in \Gamma$ implies $A \in \Delta$, $\neg\boxdot A \in \Gamma$ implies $\neg A \in \Delta$, and $\sim\boxdot A \in \Gamma$ implies $\sim A \in \Delta$. Suppose $\boxdot A \in \Gamma$. Then $A \in \Phi$. Since $\Phi \cup \{\neg A, \sim A, \sim\neg A\} \subseteq \Delta$, we have $A \in \Delta$. Suppose $\neg\boxdot A \in \Gamma$. Then $\neg A \in \Phi$. Thus, $\neg A \in \Delta$. Suppose $\sim\boxdot A \in \Gamma$. Then $\sim A \in \Phi$. Thus, $\sim A \in \Delta$. Therefore, it holds that $\Gamma R_{\mathbf{L}} \Delta$.

The cases (2) and (3) are proved similarly. □

Suppose $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{T1}$, i.e. $\boxdot A \in \Gamma, \neg\boxdot A \in \Gamma, \sim\boxdot A \in \Gamma, \sim\neg\boxdot A \notin \Gamma$. Consider an arbitrary Δ such that $\Gamma R_{\mathbf{L}} \Delta$. Then, by the definition of $R_{\mathbf{L}}$, we have $A \in \Delta$, $\neg A \in \Delta$, and $\sim A \in \Delta$. By the rule (EFQ), $\sim\neg A \notin \Delta$.

Suppose $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{T0}$, i.e. $\boxdot A \in \Gamma, \neg\boxdot A \notin \Gamma, \sim\boxdot A \in \Gamma, \sim\neg\boxdot A \in \Gamma$. Consider an arbitrary Δ such that $\Gamma R_{\mathbf{L}} \Delta$. Then, by the definition of $R_{\mathbf{L}}$, we have $A \in \Delta$ and $\sim A \in \Delta$. By the rule (EFQ), $\neg A \in \Delta$ implies $\sim\neg A \notin \Delta$ as well as $\sim\neg A \in \Delta$ implies $\neg A \notin \Delta$. Thus, $(A \in \Delta, \neg A \in \Delta, \sim A \in \Delta, \sim\neg A \notin \Delta)$ or $(A \in \Delta, \neg A \notin \Delta, \sim A \in \Delta, \sim\neg A \in \Delta)$. We need also show that $\exists_{\Delta \in R_{\mathbf{L}}[\Gamma]}\big(A \in \Delta, \neg A \notin \Delta, \sim A \in \Delta, \sim\neg A \in \Delta\big)$. Since $\neg\boxdot A \notin \Gamma$, by Proposition 7.8, $\Omega \cup \{A, \sim A, \sim\neg A\} \nvdash C$. By Lindenbaum's lemma (see Lemma 4.17 which holds without any changes for modal extensions of $\mathbf{LIRA}_{(\circledcirc)_k}$), there is an $\mathbf{LIRA}^{\mathbf{L}}_{(\circledcirc)_k}$-theory Δ such that $\Omega \cup \{A, \sim A, \sim\neg A\} \subseteq \Delta$ and $\Delta \nvdash C$. Clearly, $\Delta \in W_{\mathbf{L}}$. Since $\Omega \subseteq \Phi$, $\Phi \cup \{A, \sim A, \sim\neg A\} \subseteq \Delta$. By Proposition 7.9, $\Gamma R_{\mathbf{L}} \Delta$. Moreover, $A \in \Delta$, $\sim A \in \Delta$, and $\sim\neg A \in \Delta$. However, $\neg A \notin \Delta$ (by the rule (EFQ)). Thus, we have found $\Delta \in W_{\mathbf{L}}$ such that $\Gamma R_{\mathbf{L}} \Delta$ and $A \in \Delta, \neg A \notin \Delta, \sim A \in \Delta, \sim\neg A \in \Delta$.

Suppose $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{F1}$, i.e. $\boxdot A \in \Gamma, \neg\boxdot A \in \Gamma, \sim\boxdot A \notin \Gamma, \sim\neg\boxdot A \in \Gamma$. Consider an arbitrary Δ such that $\Gamma R_{\mathbf{L}} \Delta$. Then, by the definition of $R_{\mathbf{L}}$, we have $A \in \Delta$. Thus, $A \in \Delta$ or $\neg A \notin \Delta$ or $\sim A \notin \Delta$ or $\sim\neg A \notin \Delta$. Since $\sim\boxdot A \notin \Gamma$, by Proposition 7.8, $\Omega \cup \{A, \neg A, \sim\neg A\} \nvdash C$. By Lindenbaum's lemma, there is $\Delta \in W_{\mathbf{L}}$ such that $\Omega \cup \{A, \neg A, \sim\neg A\} \subseteq \Delta$ and $\Delta \nvdash C$. Moreover, $\Phi \cup \{A, \neg A, \sim\neg A\} \subseteq \Delta$. By Proposition 7.9, $\Gamma R_{\mathbf{L}} \Delta$. Additionally, $A \in \Delta$, $\neg A \in \Delta$, and $\sim\neg A \in \Delta$. However, $\sim A \notin \Delta$ (by the rule (EFQ)).

Suppose $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{F0}$, i.e. $\boxdot A \notin \Gamma, \neg\boxdot A \in \Gamma, \sim\boxdot A \in \Gamma, \sim\neg\boxdot A \in \Gamma$. Since $\boxdot A \notin \Gamma$, by Proposition 7.8, $\Omega \cup \{\neg A, \sim A, \sim\neg A\} \nvdash C$. By Lindenbaum's lemma, there is $\Delta \in W_{\mathbf{L}}$ such that

$\Omega \cup \{\neg A, \sim A, \sim\neg A\} \subseteq \Delta$ and $\Delta \nvdash C$. Moreover, $\Phi \cup \{\neg A, \sim A, \sim\neg A\} \subseteq \Delta$. By Proposition 7.9, $\Gamma R_{\mathbf{L}} \Delta$. Additionally, $\neg A \in \Delta$, $\sim A \in \Delta$, and $\sim\neg A \in \Delta$. However, $A \notin \Delta$ (by the rule (*EFQ*)).

Now we need to show that the equivalences which have been already proved from right to left are also valid from left to right.

Suppose $\forall_{\Delta \in R_{\mathbf{L}}[\Gamma]} \big(A \in \Delta, \neg A \in \Delta, \sim A \in \Delta, \sim\neg A \notin \Delta\big)$. Hence, $\forall_{\Delta \in R_{\mathbf{L}}[\Gamma]} V_{\mathbf{L}}(A, \Delta) = \mathtt{T1}$. Suppose $V_{\mathbf{L}}(\boxdot A, \Gamma) \neq \mathtt{T1}$. Suppose $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{T0}$. Then, in particular, it holds that $\exists_{\Delta \in R_{\mathbf{L}}[\Gamma]} V_{\mathbf{L}}(A, \Delta) = \mathtt{T0}$, which cannot be the case, since $\forall_{\Delta \in R_{\mathbf{L}}[\Gamma]} V_{\mathbf{L}}(A, \Delta) = \mathtt{T1}$. If $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{F1}$, then we have, in particular, $\exists_{\Delta \in R_{\mathbf{L}}[\Gamma]} V_{\mathbf{L}}(A, \Delta) = \mathtt{F1}$. Contradiction. If $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{F0}$, then $\exists_{\Delta \in R_{\mathbf{L}}[\Gamma]} V_{\mathbf{L}}(A, \Delta) = \mathtt{F0}$. Contradiction. Thus, $V_{\mathbf{L}}(\boxdot A, \Gamma) = \mathtt{T1}$.

The other cases are considered similarly. □

Theorem 7.10. *Let* $\mathbf{L}$ *be* $\mathbf{LIRA}^{\mathbf{K}}_{(\circledcirc)_k}$. *For each* $\Gamma \subseteq \mathscr{F}^{M}_{(\circledcirc)_k}$ *and* $A \in \mathscr{F}^{M}_{(\circledcirc)_k}$, *it holds that if* $\Gamma \models_{\mathbf{L}} A$, *then* $\Gamma \vdash_{\mathbf{L}} A$.

Proof. By contraposition. Suppose $\Gamma \nvdash_{\mathbf{L}} A$. By Lindenbaum's lemma, there is $\mathbf{LIRA}^{\mathbf{L}}_{(\circledcirc)_k}$-theory Δ such that $\Gamma \subseteq \Delta$ and $\Delta \nvdash A$. Consider a canonic model $\langle W_{\mathbf{L}}, R_{\mathbf{L}}, V_{\mathbf{L}}\rangle$. By Lemma 7.6, it is $\mathbf{L}$-model. Since Δ is $\mathbf{LIRA}^{\mathbf{L}}_{(\circledcirc)_k}$-theory, $\Delta \in W_{\mathbf{L}}$. Moreover, $A \notin \Delta$. By (Γ_3), (Γ_4), and (Γ_5), respectively, $\neg A \in \Delta$, $\sim A \in \Delta$, and $\sim\neg A \in \Delta$. Thus, $V_{\mathbf{L}}(A, \Delta) = \mathtt{F0}$. Since $\Gamma \subseteq \Delta$, $G \in \Delta$, for each $G \in \Gamma$. Hence, $V_{\mathbf{L}}(G, \Delta) \neq \mathtt{F0}$. Therefore, $\Gamma \not\models_{\mathbf{L}} A$. □

Corollary 7.11. *Let* $\mathbf{L}$ *be* $\mathbf{LIRA}^{\mathbf{K}}_{(\circledcirc)_k}$. *For each* $\Gamma \subseteq \mathscr{F}^{M}_{(\circledcirc)_k}$ *and* $A \in \mathscr{F}^{M}_{(\circledcirc)_k}$, *it holds that* $\Gamma \models_{\mathbf{L}} A$ *iff* $\Gamma \vdash_{\mathbf{L}} A$.

Proof. Follows from Theorems 7.4 and 7.10. □

8 Conclusion

An obvious task for future research is a formalization of modal extensions of **LRA** and $\mathbf{LRA}_{(\circledcirc)_k}$. However, it seems that natural deduction systems are not the most suitable framework for it, since rules of the following form are required in a completeness proof: if $\Gamma, \neg A \vdash B$, then $\Gamma \vdash A$ or $\Gamma \vdash \sim A$ or $\Gamma \vdash \sim\neg A$. Such rules are quite strange for natural deduction systems in contradistinction to hypersequent calculi. We believe that this topic requires a separate research paper.

Let us mention Kubyshkina's recent paper [17], where fifteen-valued logic **GLRA**, a generalization of **LRA**, is presented. The entailment relation is defined via the preservation of the only designated value. Similarly, we can define logic **GLIRA** which will have the only *non*-designated value.

Yet another topic of further investigations is a presentation of automated proof-searching by the methods of [26] for **LIRA**, $\mathbf{LIRA}_{(\circledcirc)_k}$, and their modal extensions.

One more topic for further research is an examination of first-order extensions of **LRA** and **LIRA**. It is well-known that modalities and quantifiers have in some sense a similar behavior. Thus, we can try to define suitable quantifiers for the concerned logics taking into account the above mentioned modalities. For example, we can define universal quantifier for **LIRA** as follows:[8]

- $V(\forall\alpha A, \mathfrak{a}) = \mathtt{T1}$ iff $V(A^{\alpha}_{\beta}, \mathfrak{a}) = \mathtt{T1}$, for each $\beta \in U$,

[8] To be sure, we need to extend the language of **LIRA** at least by the set of predicate symbols $\{P_n \mid n \in \mathbb{N}\}$, the set of variables $\{v_n \mid n \in \mathbb{N}\}$, and universal quantifier itself. Following Smullyan [37], we consider a model $\langle U, J\rangle$, where U is a universe of individuals (a non-empty set of constants) and J is an interpretation function. For each formula A, we write A^{α}_{β} for the result of substituting $\beta \in U$ for every occurrence of a variable α in A. We write $\mathfrak{a}$ for a first-order assignment.

- $V(\forall\alpha A, \mathfrak{a}) = \mathtt{T0}$ iff $V(A^{\alpha}_{\beta}, \mathfrak{a}) = \mathtt{T1}$ or $V(A^{\alpha}_{\beta}, \mathfrak{a}) = \mathtt{T0}$, for each $\beta \in U$, and $V(A^{\alpha}_{\beta}, \mathfrak{a}) = \mathtt{T0}$, for some $\beta \in U$,
- $V(\forall\alpha A, \mathfrak{a}) = \mathtt{F1}$ iff $V(A^{\alpha}_{\beta}, \mathfrak{a}) \neq \mathtt{F0}$, for each $\beta \in U$, and $V(A^{\alpha}_{\beta}, \mathfrak{a}) = \mathtt{F1}$, for some $\beta \in U$,
- $V(\forall\alpha A, \mathfrak{a}) = \mathtt{F0}$ iff $V(A^{\alpha}_{\beta}, \mathfrak{a}) = \mathtt{F0}$, for some $\beta \in U$.

Then one may put $\exists^{\neg}\alpha A := \neg\forall\alpha\neg A$, $\exists^{\sim}\alpha A := {\sim}\forall\alpha{\sim}A$, and $\exists^{C}\alpha A := {\sim}\neg\forall\alpha{\sim}\neg A$. For the case of **LRA** this definition of universal quantifier may be adapted as follows (the cases for T1 and T0 are the same):

- ○ $V(\forall\alpha A, \mathfrak{a}) = \mathtt{F1}$ iff $V(A^{\alpha}_{\beta}, \mathfrak{a}) = \mathtt{F1}$, for some $\beta \in U$.
- ○ $V(\forall\alpha A, \mathfrak{a}) = \mathtt{F0}$ iff $V(A^{\alpha}_{\beta}, \mathfrak{a}) \neq \mathtt{F1}$, for each $\beta \in U$, and $V(A^{\alpha}_{\beta}, \mathfrak{a}) = \mathtt{F0}$, for some $\beta \in U$.

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