Australasian Journal of Logic

Takeuti's Well-ordering Proof: An Accessible Reconstruction.

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Abstract

G. Genzten's 1938 proof of the consistency of pure arithmetic was hailed as a success for finitism and constructivism, but his proof requires induction along ordinal notations in Cantor normal form up to the first epsilon number, ε_0. This left the task of giving a finitisically acceptable proof of the well-ordering of those ordinal notations, without which Gentzen's proof could hardly be seen as a success for finitism. In his seminal book *Proof Theory* G. Takeuti provides such a proof. After a brief philosophical introduction, we provide a reconstruction of Takeuti's proof including corrections, comments, reorganization and notational adjustments for the sake of clarity. The result is a much longer, but much more tractable proof of the well-ordering of ordinal notations in Cantor normal form less than ε_0, that nevertheless follows Takeuti's strategy closely. We end with some more general comments about that proof strategy and the notion of accessibility more generally.

1 Introduction

In 1938 Gerhard Gentzen published a consistency proof of pure arithmetic, or what we today call first-order Peano arithmetic [9]. This was particularly important at the time because, despite Kurt Gödel's proof of the incompleteness of pure arithmetic at the beginning of that same decade [11], David Hilbert and Paul Bernays maintained a commitment to their finitist program [see 12]. The central aim of Hilbert's program was to show that there are finitistic consistency proofs for important mathematical theories, thus a fintistically acceptable consistency proof for pure arithmetic would be a key victory for finitism. Bernays certainly thought Gentzen's first consistency proof [8] fit the bill, stating in the final paragraph of [4]

> During the printing of this report the proof for the consistency of the full number theoretic formalism has been presented by G. Gentzen, using a method that conforms to the fundamental demands of the finite standpoint.

Presumably, this would have carried over to the second proof, as the most significant difference between the two proofs is the choice of ordinal notation systems. Notably, in fact, Akiyoshi & Takahashi [3] use modern proof-theoretic techniques to show that the second proof is a special case of the first. Nevertheless, the question of the finitistic acceptability of Gentzen's proofs was (and still is) in doubt. The reason for this is that Gentzen's proofs require induction on ordinal notations for ordinals less than the first epsilon number, ε_0. In the 1938 proof, Gentzen uses ordinal notations in Cantor normal form, thus what is at issue from a finitist perspective is whether the well-ordering of the ordinal notations in Cantor normal form for ordinals less than ε_0 can be established in a finitistically acceptable manner.

In his 1975 book *Proof Theory*, Gaisi Takeuti attempts to establish just that. That proof, and Takeuti's finitistic position and related consistency proofs, have recently been the subject of significant renewed interest. Darnell & Thomas-Bolduc [6] discuss whether and from which conceptions of finitism Takeuti's well-ordering proof is finitistically acceptable. They argue that Takeuti's proof conforms to what Takeuti himself terms the "Hilbert-Gentzen finitist standpoint", but that the finitistic acceptability of the proof ultimately depends on the philosophical motivations behind a given finitist standpoint. The Hilbert-Gentzen standpoint, very roughly, is a "natural extension" of Hilbert's standpoint [20, pp 100-101] that allows for operations on operations or *Gendakenexperimente*, so long as those operations are ultimately grounded in the concrete [6, p 177].

That standpoint is arguably close to Takeuti's own view, although pinning down Takeuti's philosophical position is a more complex task. However, that task has been taken on by Akiyoshi & Arana [1], who argue that Takeuti's philosophical standpoint was highly influenced by Nishida and the Kyoto school. Bringing in concepts related to that school, they have clarified the relationship between Takeuti's position and the positions of Hilbert, Bernays and Gentzen.[1] In particular, it appears that Takeuti's position is more liberal with respect to which operations are finitistically acceptable than Hilbert or more recent standpoints such as that of Tait [15, 16]. This is due to a different understanding, on Takeuti's part, of the relationship between the finite and the infinite.

The goal of this paper is to provide a reconstruction of Takeuti's well-ordering proof, which although interesting and original, is difficult to read, and contains a number of small errors and omissions.[2] In order to make the proof as clear as possible, we've renamed some variables to reduce ambiguity, and reordered some steps in the proof. This latter was needed because Takeuti has the habit, in this proof, of ordering the induction steps somewhat haphazardly. Additionally, we have filled in a couple of steps that Takeuti omitted in the original version. We indicate our own additions and corrections to the proof in footnotes and remarks. All of this has resulted in our version of the proof taking up roughly three times more space than the original, but we hope the increase in length brings with it an increase in clarity and makes the proof more accessible. In turn, we hope that this will help fuel progress on research related to Gentzen and Takeuti's consistency proofs in the vein of the papers just discussed.

Before diving into the proof in §3, we review the ordinal notation system, and the relevant aspects of Gentzen's consistency proof. At the end we briefly discuss a couple of comments Takeuti makes directly after his proof.

2 Induction on Ordinal Notations[3]

Georg Cantor [5] proved that every ordinal can be written as a sum of 0 and exponents of ω in the following way. For any ordinal, α,

[1] Akiyoshi and Arana indicate that they intend to further explore Takeuti's philosophical commitments in that regard and we are hopeful that such investigations will contribute to both our understanding of Takeuti's thought, as well as the space of modern constructivist positions more generally.

[2] It appears that [2] also contains a reconstruction or reformulation of Takeuti's well-ordering proof using modern techniques, however that paper is written in Japanese (which we are regrettably unable to read), and approaches the reconstruction from a different perspective than the present paper.

[3] This section has been adapted from §3 of Darnell & Thomas-Bolduc [6]. For an introduction to these methods and proofs requiring little background in proof theory, see [14].

$$\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \omega^{\beta_3} + \omega^{\beta_4} + \dots$$

α, β ordinals, the βs either in this form, or 0s, and

$$\beta_1 \geq \beta_2 \geq \beta_3 \geq \dots$$

Since we need only consider ordinals less the ε_0, and ε_0 is the first ordinal such that $\omega^\alpha = \alpha$ (i.e. it is the first epsilon number), it is guaranteed that $\alpha > \omega^{\beta_i}$ for all $i > 1$, and $\alpha = \omega^{\beta_1}$ only in the case where the $\omega^{\beta_{i>1}}$ are empty.

An ordinal expressed in this way is in *Cantor normal form*. For ease of notation we can write ω^0 as 1, $\omega^0 + \omega^0$ as 2, and so on for the natural numbers.[4] One further condition, that any '+0' terms are deleted, guarantees that each notation is unique.

We can then define the *natural sum* of two ordinals, expressed $\alpha \# \mu$, as a (possibly) new ordinal in Cantor normal form found by interleaving the monomials (terms of the form ω^{β_i}) so that the β's are decreasing.[5] It is the well-ordering of these notations that Gentzen used for the transfinite induction steps in his consistency proof for pure arithmetic.

Gentzen's general strategy[6] is to take an arbitrary proof in the sequent calculus with arithmetical initial sequents and the inference rule for (full) arithmetical induction (i.e. first-order Peano Arithmetic formulated in the sequent calculus) of the empty sequent and show that such a proof cannot exist.

To do this the 'end-part' of a proof is defined as the largest segment of the proof, looking up from the end-sequent, that contains only structural rules or inductions. The end-part is then pushed to the top of the proof, which can be done because the end-sequent contains no connectives, so any complex formulae will have to have been removed with a cut.

All inductions in the end part of the proof are replaced with sequences of cuts, and all inessential cuts (cuts on complex formulae) are reduced to essential cuts (cuts on atomic formulae). All of this is done in a principled way to a *regular* proof.[7]

[4]Gentzen includes only '1', but Takeuti makes use of this obvious notational extension so we have included it here for completeness.

[5]Note that it may be that $\beta_i = \beta_{i+1} = \ldots = \beta_{i+n}$ for some $i, n > 0$.

[6]See [13], and publications deriving therefrom for an expansion and other uses of these methods.

[7]A regular proof is one in which all of the non-eigen variables have been replaced with

The key to the proof, and the part that we are here interested in, is showing that the procedure terminates. Gentzen does this by assigning ordinal notations to each sequent and inference, and showing that each reduction step decreases the ordinal notation assigned to the end-sequent of the proof. Although the procedures for assigning ordinal notations to inductions and cuts are necessarily quite complex, the operations involved are just the stacking of finitely many ω exponents, the taking of natural sums, and ordinary arithmetical subtractions. What is left to prove, from a finitistic perspective, is the well-ordering of the ordinal notations.[8]

3 Takeuti's Proof

Takeuti proposes a concrete method for demonstrating that the ordinal notations in Cantor normal form (henceforth just *ordinals*) $<\varepsilon_0$ are well-ordered [20].[9] His demonstration consists in applications of a series of concrete methods, which he calls "eliminators". Eliminators are methods for taking any (concretely) given strictly decreasing sequence of ordinals and (concretely) constructing a new strictly decreasing sequence of ordinals such that if the latter contains no infinitely descending chains, neither does the former. Using eliminators, Takeuti gives a proof that the ordinals $<\varepsilon_0$ are well-ordered.

3.1 Introducing "Eliminators"

Takeuti begins his proof by supposing that the natural numbers are well-ordered. That is, for any strictly decreasing sequence of ordinals, S, that begins with some natural number n, the length of S is, at most, $n + 1$ [20, pp. 92–3].[10] Takeuti's *eliminators* capitalize on the well-ordering of the naturals and enable him to demonstrate that strictly decreasing sequences of ordinals beginning with any ordinal less than ε_0 must be finite.

0s and the eigenvariables have been replaced with appropriate arithmetic terms.

[8]From outside the finite standpoint it can easily be seen that these notations are well-ordered, because they are unique, and ε_0 is well-ordered by definition.

[9]All references are to the second edition.

[10]Takeuti takes this assumption to be uncontroversial because he sees it as an obvious consequence of his definitions of ordinals and interpretation of, '=', '+', and '<' between the ordinals [20, pp. 90–1].

The first eliminator that Takeuti introduces is the *1-eliminator* [20, p. 93]:[11] the (concrete) method he uses to show the well-ordering of ordinal notations $< \omega^2$. In order to introduce the 1-eliminator, we require two key notions. Let a_i be an arbitrary ordinal $<\varepsilon_0$. In Cantor normal form, the form of a_i is:

$$\omega^{\mu_1^i} + \omega^{\mu_2^i} + ... + \omega^{\mu_{n_i}^i} + k_i$$

where each $\mu_m^i > 0$, $\mu_{m-1}^i > \mu_m^i$, and k_i is a natural number (the i's are simply meant to index the given ordinal notations to a_i).

Definition 3.1. *[Adapted from 20, p. 93.] For any ordinal $a_i <\varepsilon_0$, the 1-major part of a_i is the part of the Cantor normal form of a_i that does not contain k_i,*

$$\underbrace{\omega^{\mu_1^i} + \omega^{\mu_2^i} + ... + \omega^{\mu_{n_i}^i}}_{\textit{1-major part of } a_i} + k_i.$$

For every ordinal $a_i <\varepsilon_0$, we write a_i^* for the 1-major part of a_i.

Definition 3.2. *[Adapted from 20, p. 93.] For any strictly decreasing sequence of ordinals $a_0 > a_1 > ...$, the sequence $a_0^* > a_1^* > ...$ is the 1-sequence corresponding to $a_0 > a_1 > ...$, just in case for any $i \geq 0$, every a_i^* in $a_0^* > a_1^* > ...$ is the 1-major part of a_i in $a_0 > a_1 > ...$.*

For any strictly decreasing sequence of ordinals S_j, we write S_j^* for the 1-sequence corresponding to S_j. For any strictly decreasing sequence of ordinals S, we say that S is a 1-sequence just in case there is a strictly decreasing sequence of ordinals S_j such that $S = S_j^*$ (i.e., S is the 1-sequence corresponding to S_j).

In accordance with Definitions 3.1 and 3.2, Takeuti introduces the 1-eliminator.

Definition 3.3. *[Adapted from 20, p. 93.] A 1-eliminator is a (concrete) method for constructing a* 1-sequence*, S_j^* from a (concretely) given decreasing sequence of ordinals S_j, such that the first ordinal in S_j^* is the* 1-major part *of the first ordinal in S_j, and if S_j^* is finite, then (it can be concretely shown that) S_j is also finite.*

[11] All subsequent eliminators (and their associated terminology) are analogous to the 1-eliminator.

Takeuti explains Definition 3.3 of the 1-eliminator by way of an illustration. Let,

$$(S_0) \qquad a_0 > a_1 > ...$$

be a strictly decreasing sequence of ordinals such that, each a_i in (S_0) is written Cantor normal form and a_0 is not a natural number. By Definition 3.3, a 1-eliminator is a method for taking (S_0) and (concretely) producing a decreasing 1-sequence:

$$(S_0^*) \qquad b_0 > b_1 > ...$$

that satisfies each of the following conditions:

(CI$_1$) $b_0 = a_0^*$ (i.e., b_0 is the 1-major part of a_0).

(CII$_1$) If (S_0^*) is finite, then (it can be concretely shown that) (S_0) is finite also.

Remark 3.1. *It is important to emphasize Takeuti's use of '>' in (S_0^*). This indicates that applying the 1-eliminator to (S_0) produces a* strictly *decreasing 1-sequence. Hence, (S_0^*) may not be the* very same sequence *as the sequence that would be produced simply by removing each k_i from every $a_i = \omega^{\mu_1^i} + \omega^{\mu_2^i} + ... + \omega^{\mu_{n_i}^i} + k_i$ in (S_0). (S_0^*) does not contain multiple occurrences of identical ordinals, whereas simply removing each k_i from every a_i in (S_0)* may *result in a sequence that contains multiple occurrences of identical ordinals. This feature of the decreasing sequences constructed with eliminators is extremely important for Takeuti's argument.*

Proposition 3.1. *The 1-sequence (S_0^*) that is produced by applying the 1-eliminator to (S_0) satisfies conditions (CI$_1$) and (CII$_1$).*

Proof Sketch. [Adapted from 20, p. 93.] By stipulation and Definition 3.1, each ordinal notation in (S_0) is identical with its 1-major part plus a given natural number. That is, $a_i = a_i^* + k_i$ for every a_i in (S_0). Hence, (S_0) can be written as:

$$a_0^* + k_0 > a_1^* + k_1 > ...$$

Applying a 1-eliminator to this sequence still produces the 1-sequence, (S_0^*) which satisfies (CI$_1$) by Definition 3.2. To show that (S_0^*) satisfies (CII$_1$), take some finite part of (S_0^*), say the sequence:

$$b_0 > b_1 > ... > b_m$$

where, $b_0 = a_0^*$ and $b_m = a_i^*$, for some i. So, either $a_i^* = a_{i+1}^* = ... = a_{i+p}^*$ for some p and a_{i+p} is the last term in (S_0) with a 1-major part, or $a_{i+p}^* > a_{i+p+1}^*$. If the former is the case, then stop. If the latter is the case, then make $b_{m+1} = a_{i+p+1}^*$ and repeat. If one arrives at a sequence of a_i^*'s such that $b_m = a_i^*$ and $a_i^* = a_{i+1}^* = ... = a_{i+p}^* = ...$, it follows that ($S_0$) must be finite. Since (S_0) is a strictly decreasing sequence, $a_i^* = a_{i+1}^* = ... = a_{i+p}^* = ...$ entails that $k_i > k_{i+1} > ... > k_{i+p} > ....$ Given the well-ordering of the natural numbers, the latter sequence must be finite. Hence, (S_0) must be finite. (S_0^*) is a strictly decreasing 1-sequence corresponding to (S_0) and so by Definition 3.2, if (S_0^*) is finite, then there is a (last) term, b_m in (S_0^*) such that $b_m = a_i^*$ and $a_i^* = a_{i+1}^* = ... = a_{i+p}^* = ...$ Therefore, if (S_0^*) is finite, so is (S_0). □

Takeuti uses the 1-eliminator to show that,

Theorem 3.1. *All decreasing sequences of ordinals that begin with an ordinal* $< \omega^2$ *must be finite.*

Proof. [Adapted from 20, p. 93.] Let,

$$(S_1) \qquad a_0 > a_1 > ...$$

be a strictly decreasing sequence of ordinals such that $a_0 < \omega^2$. Applying a 1-eliminator to (S_1) enables the construction of the 1-sequence:

$$(S_1^*) \qquad b_0 > b_1 > ...$$

such that (S_1^*) satisfies conditions (CI_1) and (CII_1)[12] and where $a_0 \geq b_0$. Since $a_0 < \omega^2$, $b_0 < \omega^2$. It follows that each ordinal b_i in (S_1^*) has the form $\omega \cdot k_i$ (where k_i is a natural number).[13] Accordingly, since a_0 is not a natural number, each b_i in (S_1^*) will be of the form $\omega \cdot k_i$ (where k_i is a natural number). Hence, (S_1^*) can be written as:

$$\omega \cdot k_0 > \omega \cdot k_1 > ...$$

[12]Where (CI_1) and (CII_1) are amended such that (S_0) is changed to (S_1) and (S_0^*) is changed to (S_1^*).

[13]This is clear because $\omega \cdot \omega = \omega^2$ and any ordinal $< \omega$ is a natural number.

It must then be the case that $k_0 > k_1 > ....$ Given the well-ordering of the natural numbers, $k_0 > k_1 > ...$ must be finite. Therefore, $b_0 > b_1 > ...$ must be finite. Since (S_1^*) satisfies (CI_1) and (CII_1), (S_1) must be finite. (S_1) was arbitrary, so this result generalizes which means that any decreasing sequence of ordinals which begins with an ordinal $< \omega^2$ must be finite. □

The proof of Theorem 3.1 nicely illustrates Takeuti's overall proof strategy for demonstrating the well-ordering of the ordinals $<\varepsilon_0$. However, the demonstration becomes significantly more complex as we progress through its successive stages. As such, it will be instructive to briefly recap. Takeuti begins by assuming the well-ordering of the ordinals less than ω (the naturals). He then defines an eliminator which, when combined with the already established well-ordering of the natural numbers, enables him to show that the sequence of ordinals whose limit is ω^2 is also well-ordered. Takeuti continues with this strategy such that, with each new eliminator, combined with the well-ordering of the sequences of ordinals already established, he is able to show that sequences of ordinals with higher and higher limits must be well-ordered, ending with a demonstration of the well-ordering of the ordinals $<\varepsilon_0$.

3.2 Proving the Ordinals $< \omega^\omega$ are Well-Ordered

Takeuti extends the above result to show that eliminators can be used to establish the well-ordering of the ordinals $< \omega^\omega$. To do this, Takeuti introduces a new (concrete) method which he calls the *n-eliminator*. The n-eliminator is an analogue of the 1-eliminator and is introduced using the relevant analogues of the notions given in Definitions 3.1 and 3.2.

Definition 3.4. *[Adapted from 20, p. 94.] For any ordinal $a_i <\varepsilon_0$ such that, $a_i = \omega^{\mu_1^i} + \ldots + \omega^{\mu_{m_i}^i} + \omega^{\mu_{m_i+p_i}^i} + \ldots + \omega^{\mu_{m_i+p_i+l_i}^i}$ where, $\mu_1^i > \ldots > \mu_{m_i}^i > \ldots > \mu_{m_i+p_i}^i > \ldots > \mu_{m_i+p_i+l_i}^i > 0$, and for any natural number $n \geq 1$, the n-major part of a_i is the part of the Cantor normal form of a_i that contains no monomial $< \omega^n$:*

$$\underbrace{\omega^{\mu_1^i} + \ldots + \omega^{\mu_{m_i}^i}}_{n\text{-major part of } a_i} + \omega^{\mu_{m_i+p_i}^i} + \ldots + \omega^{\mu_{m_i+p_i+l_i}^i}$$

where, $\omega^{\mu_1^i} \geq \omega^n, \ldots, \omega^{\mu_{m_i}^i} \geq \omega^n$ and $\omega^{\mu_{m_i+p_i}^i} < \omega^n, \ldots, \omega^{\mu_{m_i+p_i+l_i}^i} < \omega^n$.

For every ordinal $a_i <\varepsilon_0$, and for any given natural number $n \geq 1$, we write a'_i for the n-major part of a_i.

Definition 3.5. *[Adapted from 20, p. 94.] For any strictly decreasing sequence of ordinals $a_0 > a_1 > \ldots$, the sequence $a'_0 > a'_1 > \ldots$ is the n-sequence corresponding to $a_0 > a_1 > \ldots$, just in case for any $i \geq 0$, every a'_i in $a'_0 > a'_1 > \ldots$ is the n-major part of a_i in $a_0 > a_1 > \ldots$.*

For any decreasing sequence of ordinals S_j and any given $n \geq 1$, we write S'_j for the n-sequence corresponding to S_j. For any strictly decreasing sequence of ordinals S, and any $n \geq 1$, we say that S is an n-sequence just in case there is a decreasing sequence of ordinals S_j such that, $S = S'_j$ (i.e., S is the n-sequence corresponding to S_j for the given number $n \geq 1$).

Where the 1-eliminator is a method for producing a 1-sequence (a strictly decreasing sequence of ordinals where each ordinal in the sequence is $\geq \omega^1$), the n-eliminator is a method for producing an n-sequence (i.e. a strictly decreasing sequence of ordinals where each ordinal in the sequence is $\geq \omega^n$).

Definition 3.6. *[Adapted from 20, p. 94.] For any $n \geq 1$, an n-eliminator is a (concrete) method for constructing an n-sequence, S'_j from a (concretely) given decreasing sequence of ordinals S_j, such that, the first ordinal in S'_j is the n-major part of the first ordinal in S_j, and if S'_j is finite, then (it can be concretely shown that) S_j is also finite.*

Takeuti gives Definition 3.6 by way of an illustration. Choose $n \geq 1$ and let,

$$a_0 > a_1 > \ldots \tag{S_n}$$

be a (strictly) decreasing sequence of ordinals such that, every a_i in (S_n) is of the form, $a'_i + c_i$, and a_i is of the form

$$\underbrace{\omega^{\mu^i_1} + \ldots + \omega^{\mu^i_{m_i}}}_{a'_i/n\text{-major part of } a_i} + \underbrace{\omega^{\mu^i_{m_i+p_i}} + \ldots + \omega^{\mu^i_{m_i+p_i+l_i}}}_{c_i}$$

only if,

$$\omega^{\mu^i_1} \geq \omega^n, \ldots, \omega^{\mu^i_{m_i}} \geq \omega^n \text{ and } \omega^{\mu^i_{m_i+p_i}} < \omega^n, \ldots, \omega^{\mu^i_{m_i+p_i+l_i}} < \omega^n.$$

The n-eliminator is a method that takes (S_n) and concretely produces an

n-sequence:

(S'_n) $$b_0 > b_1 > ...$$

such that (S'_n) satisfies each of the following conditions:

(CI_n) $b_0 = a'_0$ (i.e., b_0 is the n-major part of a_0).

(CII_n) If (S'_n) is finite, then (it can be concretely shown that) (S_n) is finite also.

Proposition 3.2. *The n-sequence (S'_n) that is produced by applying the n-eliminator to (S_n) satisfies conditions (CI_n) and (CII_n).*

The demonstration of Proposition 3.2 proceeds in a manner similar to that of Proposition 3.1; however, demonstrating that the n-eliminator will produce an n-sequence that satisfies (CI_n) and (CII_n) requires introducing an induction hypothesis. Takeuti [20, p. 94] gives the following:

(IH) *Any descending sequence $d_0 > d_1 > ...$, with $d_0 < \omega^n$ is finite.*

Remark 3.2. *Takeuti does not include a demonstration in his original proof that for any given $n \geq 1$, the sequence $d_0 > d_1 > ...$ with $d_0 < \omega^n$ is finite; however, strictly speaking, it is needed given that the purpose of the proof is a demonstration of concreteness.*[14] *Demonstrating* (IH) *is not difficult. In a manner analogous to the proof sketch for Proposition 3.1 that the 1-eliminator satisfies* (CI_1) *and* (CII_1)*, show that the 2-eliminator satisfies conditions (CI_2) and (CII_2) (where, (CI_2) and (CII_2) are the respective appropriate analogues of* (CI_1) *and* (CII_1)*) by appealing to the well-ordering of the ordinals $< \omega^2$. Then use the 2-eliminator to prove the well-ordering of the ordinals up to ω^3. Next, show that the 3-eliminator satisfies (CI_3) and (CII_3) (where, (CI_3) and (CII_3) are the respective appropriate analogues of* (CI_1) *and* (CII_1)*) by appealing to the well-ordering of the ordinals up to ω^3, and likewise for ω^4. Continue in this way until reaching the $(n-1)$-eliminator. Use the established well-ordering of the ordinals up to ω^{n-1} to show that the $(n-1)$-eliminator satisfies (CI_{n-1}) and (CII_{n-1}). Then use*

[14] We suspect that Takeuti does not include the relevant demonstration because he takes it to be obvious.

the $(n-1)$-eliminator to prove the well-ordering of the ordinals $< \omega^n$. This procedure establishes that the induction hypothesis holds for any value of n in $n-2$ steps. For any n, $n-2$ is finite and so there is a concrete procedure for demonstrating that $d_0 > d_1 > \ldots$, with $d_0 < \omega^n$ is finite.[15]

Proof Sketch. [Adapted from 20, p. 94.] For the purposes of demonstrating Proposition 3.2, (IH) does the work that the well-ordering of $\mathbb{N}$ does in the demonstration of Proposition 3.1 and so, we proceed (roughly) as before. By Definition 3.5, $b_0 = a'_0$ and a'_0 is the n-major part of a_0 in (S_n). Thus, (S'_n) satisfies (CI_n). Suppose (S'_n) is finite:

$$b_0 > b_1 > \ldots > b_m$$

Make $b_m = a'_i$ (for some i). If $a'_i = a'_{i+1} = \ldots = a'_{i+p}$ (for some p) and a_{i+p} is the last term in (S_n) with a non-empty n-major part (i.e. $a_{i+p} = a'_{i+p} + c_{i+p}$ but possibly $a_{i+p+1} = c_{i+p+1}$), then stop. If $a'_{i+p} > a'_{i+p+1}$ (for some p), then make $b_m = a'_{i+p+1}$. Since (S'_n) is a strictly decreasing sequence of ordinals *constructed from* the n-major parts of (S_n),[16] and (by supposition) (S'_n) is finite, one must arrive at some string of a'_is such that $b_m = a'_i$ and $a'_i = \ldots = a'_{i+l}$ where a_{i+l} is the last term in (S_n) with a non-empty n-major part. Since (S_n) is a strictly decreasing sequence, if $a'_i = \ldots = a'_{i+l}$ and a_{i+l} is the last term in (S_n) with a non-empty n-major part, then

$$c_i > \ldots > c_{i+l} > \ldots .$$

By supposition and Definition 3.5,

$$c_i < \omega^n, \ldots, c_{i+l} < \omega^n, \ldots .$$

Hence, by (IH), the sequence $c_i > \ldots > c_{i+l} > \ldots$ must be finite. Hence, (S_n) is finite. Therefore, if (S'_n) is finite, then so is (S_n). Therefore, (S'_n) satisfies (CII_n). $\square$

Given Proposition 3.1, Takeuti uses the n-eliminator to prove that,

Lemma 3.2. *All decreasing sequences of ordinals that begin with an ordinal $< \omega^{n+1}$ are well-ordered.*

[15] See [6] for further discussion of this addition to Takeuti's proof.

[16] That is, (S'_n) is a decreasing sequence constructed by taking all and only the n-major parts of the ordinals in (S_n) and removing multiple occurrences of identical n-major parts.

Proof. [Adapted from 20, p. 94.] Let,

$$(S_{n1}) \qquad a_0 > a_1 > \ldots$$

be a decreasing sequence of ordinals such that $a_0 < \omega^{n+1}$. Applying the n-eliminator to (S_{n1}) produces an n-sequence,

$$(S'_{n1}) \qquad b_0 > b_1 > \ldots$$

such that (S'_{n1}) satisfies both (CI_n) and (CII_n).[17] Each a_i in (S_{n1}) has the form $a'_i + c_i$ (where a'_i is the n-major part of a_i). Hence, a_0 has the form $a'_0 + c_0$. (S'_{n1}) satisfies (CI_n) and (CII_n) and so, $b_0 = a'_0$. Therefore, $a_0 \geq b_0$, which entails that $b_0 < \omega^{n+1}$. Hence, each b_i in (S'_{n1}) must have the form $\omega^n \cdot k_i$ where k_i is a natural number.[18] Thus, (S'_{n1}) can be written as:

$$\omega^n \cdot k_0 > \omega^n \cdot k_1 > \ldots$$

Since $\omega^n = \omega^n$, it must be the case that,

$$k_0 > k_1 > \ldots.$$

Given that the natural numbers are well-ordered, $k_0 > k_1 > \ldots$ must be finite. Therefore, (S'_{n1}) must be finite. Therefore, (S_{n1}) is finite (because (S'_{n1}) satisfies condition (CII_n)). Since (S_{n1}) was arbitrary, this result generalizes. □

It is a consequence of Lemma 3.2 that,

Theorem 3.3. *All decreasing sequences of ordinals that begin with an ordinal $< \omega^\omega$ are well-ordered.*

Proof. [Adapted from 20, p. 94.] From Lemma 3.2, any (strictly) decreasing sequence of ordinals that begins with an ordinal $< \omega^{n+1}$ must be finite. Since any decreasing sequence of ordinals that begins with an ordinal $< \omega^{n+1}$ is finite, any decreasing sequence of ordinals which begins with an ordinal $< \omega^n$ is finite. Given any decreasing sequence of ordinals, $a_0 > a_1 > \ldots$, if $a_0 < \omega^\omega$, then (by definition of ω^ω) it follows that $a_0 < \omega^n$ for some natural number n.

[17] That is, $b_0 = a'_0$ where a'_0 is the n-major part of a_0 and if (S'_{n1}) is finite, then (it can be concretely shown that) (S_{n1}) is finite also.

[18] See footnote 13

Hence, $a_0 > a_1 > \ldots$ must be finite. Therefore, all decreasing sequences of ordinals that begin with an ordinal $< \omega^\omega$ are finite. Therefore, the ordinals $< \omega^\omega$ are well-ordered. $\square$

3.3 Proving that the Ordinals $<\varepsilon_0$ are Well-Ordered

Next, Takeuti argues that eliminators can be used to demonstrate that the ordinals $<\varepsilon_0$ are well-ordered [20, pp. 94–5]. Specifically, through the use of *(α, n)-eliminators*. (α, n)-eliminators are like the eliminators above, only they are defined for all ordinals $<\varepsilon_0$. The α is meant to range over all ordinals $<\varepsilon_0$ and the n is meant to range over all natural numbers > 0. Hence, (α, n)-eliminators are eliminators for all ordinals of the form:

$$\omega^\alpha \cdot n$$

Associated with (α, n)-eliminators are analogues of the concepts given in Definitions 3.1 and 3.4, and Definitions 3.2 and 3.5.

Definition 3.7. *[Adapted from 20, p. 94.] For any ordinal a_i, such that, a_i of the form $\omega^\alpha \cdot n$ where, α is an ordinal $<\varepsilon_0$ and n is a natural number, the (α, n)-major part of a_i is the part of the Cantor normal form of a_i that contains no monomial $< \omega^\alpha \cdot n$,*

$$\underbrace{\omega^{\mu_1^i} \cdot k_i^1 + \ldots + \omega^{\mu_{m_i}^i} \cdot k_i^m}_{(\alpha,\ n)\text{-major part of } a_i} + \omega^{\mu_{m_i+p_i}^i} \cdot k_i^{m+p} + \ldots + \omega^{\mu_{m_i+p_i+l_i}^i} \cdot k_i^{m+p+l}$$

where, $\omega^{\mu_1^i} \cdot k_i^1 \geq \omega^\alpha \cdot n, \ldots, \omega^{\mu_{m_i}^i} \cdot k_i^m \geq \omega^\alpha \cdot n$ and $\omega^{\mu_{m_i+p_i}^i} \cdot k_i^{m+p} < \omega^\alpha \cdot n, \ldots, \omega^{\mu_{m_i+p_i+l_i}^i} \cdot k_i^{m+p+l} < \omega^\alpha \cdot n$.

For every ordinal $a_i <\varepsilon_0$, and for any given ordinal $\alpha <\varepsilon_0$ and any given number $n \geq 1$, we write $a_i^\dagger$ for the (α, n)-major part of a_i.

Definition 3.8. *[Adapted from 20, p. 94.] For any strictly decreasing sequence of ordinals $a_0 > a_1 > \ldots$, the sequence $a_0^\dagger > a_1^\dagger > \ldots$ is the (α, n)-sequence corresponding to $a_0 > a_1 > \ldots$, just in case for any $i \geq 0$, every $a_i^\dagger$ in $a_0^\dagger > a_1^\dagger > \ldots$ is the (α, n)-major part of a_i in $a_0 > a_1 > \ldots$.*

For any strictly decreasing sequence of ordinals S_j and any given $\alpha <\varepsilon_0$ and $n \geq 1$, we write $S_j^\dagger$ for the (α, n)-sequence corresponding to S_j. For any strictly decreasing sequence of ordinals S, and any ordinal $\alpha < \varepsilon_0$ and $n \geq 1$, we say that S is an $(\alpha,\, n)$-sequence just in case there is a decreasing sequence of ordinals S_j such that $S = S_j^\dagger$ (i.e., S is the $(\alpha,\, n)$-sequence corresponding to S_j for the given ordinal $\alpha < \varepsilon_0$ and number $n \geq 1$).

Definition 3.9. *[Adapted from 20, pp. 94–95.] For any ordinal $\alpha < \varepsilon_0$ and any natural number $n \geq 1$, an $(\alpha,\, n)$-eliminator is a (concrete) method for constructing an $(\alpha,\, n)$-sequence $S_j^\dagger$ from a (concretely) given strictly decreasing sequence of ordinals S_j, such that the first ordinal in $S_j^\dagger$ is the $(\alpha,\, n)$-major part of the first ordinal in S_j, and if $S_j^\dagger$ is finite, then S_j is also finite.*

Takeuti gives Definition 3.9 by way of illustration. Choose an ordinal $\alpha < \varepsilon_0$ and an $n \geq 1$, and let,

$$(S_\alpha) \qquad\qquad a_0 > a_1 > \ldots$$

be a strictly decreasing sequence such that, for any $i \geq 0$, every a_i in (S_α) is of the form $a_i^\dagger + c_i$, and

$$\underbrace{\omega^{\mu_1^i} \cdot k_i^1 + \ldots + \omega^{\mu_{m_i}^i} \cdot k_i^m}_{a_i^\dagger = (\alpha,\, n)\text{-major part of } a_i} + \underbrace{\omega^{\mu_{m_i+p_i}^i} \cdot k_i^{m+p} + \ldots + \omega^{\mu_{m_i+p_i+l_i}^i} \cdot k_i^{m+p+l}}_{c_i}$$

only if,

$$\omega^{\mu_1^i} \cdot k_i^1 \geq \omega^\alpha \cdot n, \ldots, \omega^{\mu_{m_i}^i} \cdot k_i^m \geq \omega^\alpha \cdot n$$

and

$$\omega^{\mu_{m_i+p_i}^i} \cdot k_i^{m+p} < \omega^\alpha \cdot n, \ldots, \omega^{\mu_{m_i+p_i+l_i}^i} \cdot k_i^{m+p+l} < \omega^\alpha \cdot n.$$

The $(\alpha,\, n)$-eliminator is a method that takes (S_α) and concretely produces an $(\alpha,\, n)$-sequence,

$$(S_\alpha^\dagger) \qquad\qquad b_0 > b_1 > \ldots$$

such that $(S_\alpha^\dagger)$ satisfies each of the following conditions:

(CI_α) $\qquad$ $b_0 = a_0^\dagger$ (i.e., b_0 is the $(\alpha,\, n)$-major part of a_0).

(CII_α) If ($S^\dagger_\alpha$) is finite, then (it can be concretely shown that) (S_α) is finite also.

3.4 Defining an (α, n)-eliminator for all Ordinals $<\varepsilon_0$

If we assume that there is an (α, n)-eliminator for every ordinal $<\varepsilon_0$, it is relatively straightforward to demonstrate that the ordinals $<\varepsilon_0$ are well-ordered.

Lemma 3.4. *If for every ordinal $\omega^\alpha <\varepsilon_0$, an ($\alpha$, n)-eliminator is concretely defined, then for every strictly decreasing sequence of ordinals $a_0 > a_1 \ldots$ such that, $a_0 < \omega^{\alpha+1}$ it can be concretely shown that $a_0 > a_1 \ldots$ is well-ordered.*

Proof Sketch. Assume that for every ordinal $\omega^\alpha <\varepsilon_0$, there is an ($\alpha$, n)-eliminator. Let,

$$(S_{a_0}) \qquad a_0 > \ldots > a_h > \ldots$$

be an arbitrary strictly decreasing sequence of ordinals such that, $a_0 < \omega^{\alpha+1}$. Apply the appropriate (α, n)-eliminator to (S_{a_0}) to produce the (α, n)-sequence,

$$(S^\dagger_{a_0}) \qquad b_0 > \ldots > b_h > \ldots .$$

For any $i \geq 0$, every a_i in (S_{a_0}) has the form $a^\dagger_i + c_i$ (where, $a^\dagger_i$ is the (α, n)-major part of a_i). Hence, a_0 is of the form $a^\dagger_0 + c_0$. By Definition 3.9, ($S^\dagger_{a_0}$) satisfies (CI_α) and so, $b_0 = a^\dagger_0$. By Definition 3.7, $b_0 \leq a_0$ and so, $b_0 < \omega^{\alpha+1}$. Hence, for any $i \geq 0$, every b_i in ($S^\dagger_{a_0}$) has the form $\omega^\alpha \cdot k_i$ where k_i is a natural number. Thus, ($S^\dagger_{a_0}$) can be written as, $\omega^\alpha \cdot k_0 > \ldots \omega^\alpha \cdot k_h > \ldots$. Since $\omega^\alpha = \omega^\alpha$, $k_0 > \ldots > k_h > \ldots$. Since $\mathbb{N}$ is well-ordered, $k_0 > \ldots > k_h > \ldots$ is finite. Therefore, ($S^\dagger_{a_0}$) is finite. Since ($S^\dagger_{a_0}$) satisfies (CII_α), (S_{a_0}) is finite. □

Theorem 3.5. *If for every ordinal $<\varepsilon_0$ an (α, n)-eliminator can be concretely defined, it can be concretely shown that the ordinals $<\varepsilon_0$ are well-ordered.*

Proof. Assume that an (α, n)-eliminator can be concretely given for every ordinal $<\varepsilon_0$. For any given decreasing sequence of ordinals $a_0 > a_1 > \ldots$, if $a_0 <\varepsilon_0$, then (by definition of ε_0) $a_0 < \omega^\alpha$ for some ordinal $\alpha <\varepsilon_0$. Choose α. It follows from Lemma 3.4 that every strictly decreasing sequence of ordinals that begins with an ordinal $< \omega^{\alpha+1}$ is well-ordered. Hence, every strictly decreasing sequence of ordinals that begins with an ordinal $< \omega^{\alpha}$ is well-ordered. Therefore, $a_0 > a_1 > \ldots$ is well-ordered. Since α was arbitrary, the result generalizes: every strictly decreasing sequence of ordinals $a_0 > a_1 > \ldots$ such that $a_0 < \varepsilon_0$ is well-ordered. $\square$

Remark 3.3. *We use 'α-eliminator' and '(α, 1)-eliminator' interchangeably (for ordinals $\alpha < \varepsilon_0$) below. It is a consequence of Definition 3.9 that for every $\alpha < \varepsilon_0$, an (α, 1)-eliminator and an α-eliminator are the same eliminator (Takeuti makes this point in [20, p. 95.]).*

In order to demonstrate the antecedent of Theorem 3.5, Takeuti establishes four main preliminary results. First, that an (α, 1)-eliminator can be defined for all $\alpha < \varepsilon_0$ where a θ-eliminator has been defined for every $\theta < \alpha$ (Lemma 3.6, below). Second, for all α and any (given) n, an (α, n)-eliminator can be defined if a (θ, n)-eliminator has been defined for every $\theta < \alpha$ (Lemma 3.7, below). Third, that an ($\alpha \cdot \omega$, n)-eliminator (i.e., for $\omega^{\alpha+1} \cdot n$) can be defined from a given (α, n)-eliminator (Lemma 3.8, below). Fourth, that for any $\alpha < \varepsilon_0$ and any $n \geq 1$, an (α, $n+1$)-eliminator can be defined from a given (α, n)-eliminator (Lemma 3.9, below).

Lemma 3.6. *For any ordinal $\alpha < \varepsilon_0$, if for every ordinal $\theta < \alpha$, a (θ, 1)-eliminator is concretely given, an (α, 1)-eliminator can concretely constructed from the given (θ, 1)-eliminators.*

Proof Sketch. [Adapted from 20, p. 95–6.] For every $i \geq 0$, let $m_i < \omega$. Let,

$$(\theta_m) \qquad ... < \theta_{m_i} < ... < \theta$$

be an *increasing* sequence of ordinals with limit θ such that, for every m_i, there is a concrete method for obtaining θ_{m_i}. Suppose that, for every m_i, there is a θ_{m_i}-eliminator, g_{m_i}. Let,

$$(S_\theta) \qquad a_0 > a_1 > ...$$

be a (concretely given) strictly decreasing sequence of ordinals $< \varepsilon_0$. By Definition 3.9, a θ-eliminator is a method for (concretely) producing the θ-sequence,

$$(S^\dagger_\theta) \qquad b_0 > b_1 > ...$$

from (S_θ) such that, ($S^\dagger_\theta$) satisfies both (CI_α) and (CII_α) (where 'α' is replaced with 'θ'). Takeuti gives the following multi-step procedure for (concretely) producing ($S^\dagger_\theta$) from (S_θ) such that ($S^\dagger_\theta$) satisfies (CI_α) and (CII_α).

STEP 1: *A method for producing b_0 in ($S^\dagger_\theta$).* It is possible to write a_0 as $a^\dagger_0 + c_0$ where $a^\dagger_0$ is the θ-major part of a_0.[19] From this and Definition 3.7, it follows that there is an m_k such that, $c_0 < \omega^{\theta_{m_k}}$. Choose m_k. By Definition 3.7, if a_i has the form $a^\dagger_i + c_i$, each monomial in c_i is $< \omega^{\theta_{m_k}}$, and each monomial in $a^\dagger_i$ is $\geq \omega^{\theta_{m_k}}$, then $a^\dagger_i$ is the θ_{m_k}-major part of a_i. Hence, every a_i in (S_θ) can be written as $a^\dagger_i + c_i$, where $a^\dagger_i$ is the θ_{m_k}-major part of a_i. From this and by supposition, applying g_{m_k} to (S_θ) (concretely) produces a θ_{m_k}-sequence:

$$(S^\dagger_\theta 1) \qquad b_{1,0} > b_{1,1} > b_{1,2} > \ldots$$

such that ($S^\dagger_\theta 1$) satisfies both (CI_α) and (CII_α) where, 'α' is replaced with 'θ_{m_k}'.[20] Given (CI_α), $b_{1,0} = a^\dagger_0$. Since $a^\dagger_0$ is the θ-major part of a_0, the first ordinal in ($S^\dagger_\theta$) is $b_{1,0}$. Therefore, we let $b_0 = b_{1,0}$.

STEP 2: *A method for producing b_1 in ($S^\dagger_\theta$).* Consider the sequence $b_{1,1} > b_{1,2} > \ldots$ and suppose that $b_{1,1} \geq \omega^\theta$.

Remark 3.4. *Takeuti does not explicitly justify the supposition that $b_{1,1} \geq \omega^\theta$. The supposition is justified and motivated by the fact that if $b_{1,1} \ngeq \omega^\theta$, then the θ-sequence constructed from (S_θ) would contain only b_0. Given that each a_i in (S_θ) is in Cantor normal form, either the θ-major part of a_0 is greater than the θ_m-major part of a_1 (i.e. $b_{1,1}$) or they are identical. If the latter, then $b_{1,0} = b_{1,1}$ and so, $b_{1,1} \geq \omega^\theta$. If the former, then either $b_{1,1} \geq \omega^\theta$ or $b_{1,1} \ngeq \omega^\theta$. If $b_{1,1} \geq \omega^\theta$, then the supposition is true. If $b_{1,1} \ngeq \omega^\theta$, then each monomial in $b_{1,1}$ is $< \omega^\theta$. Hence, $b_{1,1}$ is not in ($S^\dagger_\theta$). Moreover, since ($S^\dagger_\theta 1$) is a strictly decreasing sequence, if $b_{1,1} \ngeq \omega^\theta$, then for all $b_{1,i} < b_{1,1}$, $b_{1,i} \ngeq \omega^\theta$. From this it follows that only $b_{1,0}$ is in ($S^\dagger_\theta$). Hence, if the supposition that $b_{1,1} \geq \omega^\theta$ is false, then ($S^\dagger_\theta$) would contain only b_0 and there is no need to proceed to Step 2.*

[19] That is, each monomial in $a^\dagger_0$ is $\geq \omega^\theta$.

[20] This is a consequence of the supposition that, for every m_i, there is a θ_{m_i}-eliminator, g_{m_i} and of Definition 3.9. θ_{m_k} is an ordinal $<\varepsilon_0$ and so g_{m_k} is the (α, n)-eliminator for: $\alpha = \theta_m$ and $n = 1$. Hence, ($S^\dagger_\theta 1$) must satisfy both (CI_α) and (CII_α), where α is replaced with θ_m, by Definition 3.9 (or more accurately, from the definition of the α-sequence produced by an application of an α-eliminator).

Repeat the procedure from Step 1 on the sequence $b_{1,1} > b_{1,2} > ....$ Write $b_{1,1}$ as $b^{\dagger}_{1,1} + c_{1,1}$ where, $b^{\dagger}_{1,1}$ is the θ-major part of $b_{1,1}$. From this and Definition 3.7, it follows that there must be some m_l such that, $c_{1,1} < \omega^{\theta_{m_l}}$. Hence, there is a θ_{m_l}-eliminator, g_{m_l} which when applied to $b_{1,1} > b_{1,2} > ...$ produces the θ_{m_l}-sequence:

$$(S^{\dagger}_{\theta}2) \qquad b_{2,1} > b_{2,2} > b_{2,3} > ...$$

such that, ($S^{\dagger}_{\theta}2$) satisfies both (CI_{α}) and (CII_{α}). By Definition 3.9, $b_{2,1} = b^{\dagger}_{1,1}$. Hence, make $b_{2,1} = b_1$.

Remark 3.5. *The portion of Takeuti's [20] proof that corresponds to the above* Step *2 makes $b_{1,0} = b^{\dagger}_{1,0} + c_{1,0}$ where, $b^{\dagger}_{1,0}$ is the θ-major part of $b_{1,0}$. Takeuti states that there must be an $m_l < \omega$ such that $c_{1,0} < \omega^{\theta_{m_l}}$, and defines g_{m_l} accordingly [20, p. 95]. This is an error. Since $b_{1,0}$* just is *the θ-major part of a_0, every monomial in $b_{1,0}$ is $\geq \omega^{\theta}$. By definition $b_{1,0} = b^{\dagger}_{1,0}$, hence $c_{1,0}$ must be empty.*

Step 3: *A method for producing b_2 in* ($S^{\dagger}_{\theta}$). Consider the sequence, $b_{2,2} > b_{2,3} > \ldots.$ As before, suppose that $b_{2,2} \geq \omega^{\theta}$ (for a justification and motivation for this supposition see, Remark 2). Repeat the procedure given in Steps 1 and 2 on the sequence, $b_{2,2} > b_{2,3} > ...$ to concretely produce the sequence:

$$(S^{\dagger}_{\theta}3) \qquad b_{3,2} > b_{3,3} > b_{3,4} > ...$$

such that it follows that $b_{3,2} = b_2$. Continuing this procedure (i.e., successively executing Steps appropriately analogous to the above) will eventually produce the target θ-sequence,

$$(S^{\dagger}_{\theta}) \qquad b_0 > b_1 > \ldots$$

Takeuti's multi-step procedure for (concretely) producing ($S^{\dagger}_{\theta}$) from (S_{θ}) constitutes the target θ-eliminator g, if ($S^{\dagger}_{\theta}$) satisfies both (CI_{α}) and (CII_{α}). The direct outcome of Step 1 is that b_0 is the θ-major part of a_0 in (S_{θ}) and so, ($S^{\dagger}_{\theta}$) satisfies (CI_{α}). To show that ($S^{\dagger}_{\theta}$) satisfies (CII_{α}), suppose that ($S^{\dagger}_{\theta}$) is finite:

$$b_0 > b_1 > ... > b_j$$

where, $b_j = b_{j+1,j}$ (given the method by which ($S_\theta^\dagger$) was constructed). From this it follows that the sequence which would have been created at STEP $j+1$:

$$(S_\theta^\dagger j+1) \qquad b_{j+1,j} > b_{j+1,j+1} > b_{j+1,j+2} > ...$$

is such that $b_{j+1,j+1} < \omega^\theta$.[21] It follows that, $b_{j+1,j+1} < \omega_{m'}^\theta$ for some m'. Hence, it is possible to construct a $\theta_{m'}$-eliminator, $g_{m'}$. Applying $g_{m'}$ to ($S_\theta^\dagger j+1$) produces the finite $\theta_{m'}$-sequence:

$$b_{j+1,j}^\dagger > \dots$$

which satisfies both (CI_α) and (CII_α) and where $b_{j+1,j} \geq b_{j+1,j}^\dagger$.[22] Hence, by Definition 3.8, ($S_\theta^\dagger j+1$) must be finite. Likewise, the sequence:

$$(S_\theta^\dagger j) \qquad b_{j,j-1} > b_{j,j} > b_{j,j+1} > ...$$

must be finite. These implications continue backwards for each sequence introduced at each step in Takeuti's multi-step procedure for (concretely) producing ($S_\theta^\dagger$) until it is demonstrated that (S_θ) $a_0 > a_1 > ...$ is finite. Therefore, if ($S_\theta^\dagger$) is finite, then (it can be concretely shown that) (S_θ) is finite also. Therefore, ($S_\theta^\dagger$) satisfies (CII_α). Therefore, Takeuti's multi-step procedure for (concretely) producing ($S_\theta^\dagger$) from (S_θ) constitutes the target θ-eliminator g. Since the original increasing sequence of ordinals (θ_m) was arbitrary, the above result generalizes: for any $\alpha < \varepsilon_0$, an (α, 1)-eliminator can be concretely defined if an eliminator has been defined for ordinals $< \alpha$. □

Next, Takeuti demonstrates Lemma 3.7 to show that the preceding result (Lemma 3.6) can be generalized for any given $\alpha < \varepsilon_0$ and $n \geq 1$. To do this, Takeuti uses following notion.

Definition 3.10. *[Adapted from 20, p. 95.] For any ordinal $\beta < \varepsilon_0$ and any $n \geq 0$, a (β, $n+1$)-eliminator is a method for taking any given (α, n)-eliminator and (concretely) constructing an ($\alpha \cdot \omega^\beta$, n)-eliminator.*

[21] This is because, if $b_{j+1,j+1} \geq \omega^\theta$, then b_{j+1} would be the last ordinal in ($S_\theta^\dagger$).

[22] Note: this diverges slightly from Takeuti's original [20, p. 96], but the result is the same.

Lemma 3.7. *For any $n \geq 1$ and for every $\alpha <\varepsilon_0$, if for every $\theta < \alpha$, a (θ, n)-eliminator is concretely given, then an (α, n)-eliminator can be concretely constructed using the given (θ, n)-eliminators.*

Proof Sketch. [Adapted from 20, p. 96.] For every $i \geq 0$, let $m_i^* < \omega$. Let,

$$(\theta_{m^*}) \qquad \ldots < \theta_{m_i^*} < \ldots < \theta$$

be an *increasing* sequence of ordinals with limit $\theta <\varepsilon_0$ and such that for every m_i^*, there is a concrete method for obtaining $\theta_{m_i^*}$. Further, suppose that there is a concretely given ($\theta_{m_i^*}$, $j+1$)-eliminator ($g_{m_i^*}$) for every $\theta_{m_i^*} < \theta$. From this, it is possible to define a (θ, $j+1$)-eliminator, g, by induction on j. Takeuti considers three cases:

$$\text{(Case 1)} \qquad j = 0.$$

Since $j = 0$, $n = 1$ and g is a (θ, 1)-eliminator. Hence, g is defined in a manner exactly analogous to that developed in the proof sketch of Lemma 3.6.

$$\text{(Case 2)} \qquad j = l + 1$$

where, l is a natural number ≥ 0 and l is substituted for j in g. That is, make g a (θ, $l+1$)-eliminator or, in other words, g is a (θ, j)-eliminator and $n = j$.

Remark 3.6. *The need for* (Case 2) *may not be immediately obvious; however,* (Case 2) *is needed. It behaves like an induction hypothesis that will allow for the construction of the (θ, $j+1$)-eliminator needed in* (Case 3) *below.*[23]

Takeuti shows that g can be defined for $j = l + 1$ as follows. Again, consider the increasing sequence of ordinals (θ_{m^*}), with limit θ. By the original supposition, there is a concretely given ($\theta_{m_i^*}$, $j+1$)-eliminator $g_{m_i^*}$ for every $\theta_{m_i^*} < \theta$ in (θ_{m^*}). Since 'j' was an arbitrary natural number, we can say that there is a concretely given ($\theta_{m_i^*}$, $l+1$)-eliminator for every $\theta_{m_i^*} < \theta$. Hence, there is a concretely given ($\theta_{m_i^*}$, j)-eliminator $g'_{m_i^*}$, for every $\theta_{m_i^*} < \theta$.

[23]Note that more (or more detailed) steps are needed than might have been expected to ensure that the demonstration is concrete.

Takeuti claims that there is, therefore, an operation, k_n such that applying k_n to $g'_{m_i^*}$ concretely produces the target (θ, j)-eliminator, g.

Remark 3.7. *Takeuti does not describe k_n in any more detail than this. Constructing the operation k_n is analogous to the procedure outlined in the proof sketch of Lemma 3.6.*

Since the sequence (θ_{m^*}) was arbitrary, this conclusion generalizes.

$$j > 0 \tag{Case 3}$$

In order to show that g can be defined for $j > 0$, Takeuti makes use of the (β, $n+1$)-eliminator. For for every $i \geq 0$, let $m_i < \omega$. Let,

$$\ldots < \beta_{m_i} < \ldots < \beta \tag{β_m}$$

be an increasing sequence of ordinals such that the limit of (β_m) is β. As in (Case 2), Takeuti supposes that there is a (β_{m_i}, $j+1$)-eliminator g_{m_i} concretely given for every $\beta_{m_i} < \beta$ in (β_m). Let p be a (concretely) given (α, n)-eliminator for some $\alpha < \theta$. By Definition 3.10, g_{m_i} can be applied to p and will concretely produce an ($\alpha \cdot \omega^{\beta_{m_i}}$, j)-eliminator, $g_{m_i}(p)$. The sequences (θ_{m^*}) and (β_m) are arbitrary and so, Takeuti sets $\alpha \cdot \omega^{\beta_{m_i}}$ as $\theta_{m_i^*}$ (for every β_{m_i} and $\theta_{m_i^*}$) and $\alpha \cdot \omega^{\beta}$ as θ, which makes $g_{m_i}(p)$ the same as the eliminator $g'_{m_i^*}$ from (Case 2). Hence, applying the earlier operation k_n (from (Case 2)) to each concretely given $g'_{m_i^*}$ for the sequence (θ_{m^*}) is a method which constitutes (or defines) an ($\alpha \cdot \omega^{\beta}$, j)-eliminator, q.[24]. By Definition 3.10, the method outlined above for constructing q from p constitutes a (β, $n+1$)-eliminator. More specifically, the method defines the target (θ, $j+1$)-eliminator, g. Therefore, for any $n \geq 1$ and any given $\alpha < \varepsilon_0$, if a (θ, n)-eliminator is concretely given for every $\theta < \alpha$, an (α, n)-eliminator can be concretely constructed from the given (θ, n)-eliminators. □

Lemma 3.8. *For any ordinal $\alpha < \varepsilon_0$ and any number $n \geq 1$ such that an (α, n)-eliminator has been concretely given, an ($\alpha \cdot \omega$, n)-eliminator can be concretely constructed from the concretely given (α, n)-eliminator.*

Proof Sketch. [Adapted from 20, p. 96.] Let,

$$\ldots < \theta < \ldots < \theta \cdot \omega \tag{θ_ω}$$

[24] This is because that same method defines a (θ, n)-eliminator, and $\theta = \alpha \cdot \omega^{\beta}$.

be an *increasing* sequence of ordinals such that the limit of (θ_ω) is $\theta \cdot \omega$ and where, for every ordinal $\theta' < \theta \cdot \omega$ in (θ_ω), there is a concrete method for obtaining θ'. For some $\theta < \theta \cdot \omega$ in (θ_ω) and a given $n \geq 0$, assume that g is a $(\theta,\, n+1)$-eliminator that has been concretely given. Takeuti offers the following two stage method for constructing a $(\theta \cdot \omega,\, n+1)$-eliminator from the given $(\theta,\, n+1)$-eliminator, g.

STAGE 1: *From g, concretely construct a $(\theta \cdot m,\, n+1)$-eliminator for all $m < \omega$.* To do this, first suppose that for some $\alpha < \theta$, an $(\alpha,\, n)$-eliminator, f, has been concretely given. By Definition 3.10, g is a $(\beta,\, n+1)$-eliminator and so, applying g to f (concretely) gives the $(\alpha \cdot \omega^\theta,\, n)$-eliminator, $g(f)$. By Definition 3.9, $g(f)$ is an $(\alpha,\, n)$-eliminator and so, g can be applied to $g(f)$. By Definition 3.10, applying g to $g(f)$ concretely gives the $(\alpha \cdot \omega^\theta \cdot \omega^\theta,\, n)$-eliminator, $g(g(f))$. By Definition 3.9, $g(g(f))$ is an $(\alpha,\, n)$-eliminator and so, g can also be applied to $g(g(f))$. By Definition 3.10, applying g to $g(g(f))$ concretely gives the $(\alpha \cdot \omega^\theta \cdot \omega^\theta \cdot \omega^\theta,\, n)$-eliminator, $g(g(g(f)))$. For any $m < \omega$, by Definitions 3.9 and 3.10, this procedure can be iterated m many times to concretely give the:

$$(\alpha \cdot \underbrace{\omega^\theta \cdot \omega^\theta \cdot \ldots \cdot \omega^\theta}_{m}, n) - \text{eliminator}.$$

For all $m < \omega$,

$$\alpha \cdot \underbrace{\omega^\theta \cdot \omega^\theta \cdot \ldots \cdot \omega^\theta}_{m} = \alpha \cdot \omega^{\theta \cdot m}.$$

Therefore, for all $m < \omega$, the

$$(\alpha \cdot \omega^{\theta \cdot m}, n) - \text{eliminator}$$

can be concretely constructed from g.

STAGE 2: *Generalize this result to show that for every $\theta' < \theta \cdot \omega$, a $(\theta',\, n+1)$-eliminator can be concretely constructed.*

Claim 3.1. *For any ordinal $\theta < \theta \cdot \omega$ and any $n \geq 1$, if an $(\theta,\, n)$-eliminator can be concretely constructed, then for any ordinal $\theta^* < \theta$, a $(\theta^*,\, n)$-eliminator can be concretely constructed.*

Remark 3.8. *For brevity, we do not provide a detailed demonstration of Claim 3.1 here. However, the method for demonstrating Claim 3.1 is straightforward. Choose an ordinal $\alpha < \varepsilon_0$ such that an $(\alpha,\, n)$-eliminator can be* *concretely constructed and such that it has been established that an (α', n)-eliminator can be concretely constructed for any $\alpha' < \alpha$ (given Propositions 3.1 and 3.2, methods outlined above, etc.). From Lemma 3.6 or Lemma 3.7, and α and the concretely given (α, n)-eliminator (and (α', n)-eliminators), construct an (α^S, n)-eliminator for the relevant successor of α, α^S. Repeat this process in a manner analogous to the procedure outlined in Remark 3.2 until the relevant (θ^*, n)-eliminator is concretely constructed.*

If $\theta = \theta'$, it follows trivially by our initial assumption that a (θ', $n+1$)-eliminator can be concretely constructed from g. Now consider the remaining two cases.

(Case 1) $$\theta < \theta' < \theta \cdot \omega.$$

Since $\theta' < \theta \cdot \omega$, it follows (by definition of $\theta \cdot \omega$ that for some $m < \omega$ and $\alpha < \theta$, there is an $\alpha \cdot \omega^{\theta \cdot m} < \theta \cdot \omega$ such that $\theta' < \alpha \cdot \omega^{\theta \cdot m}$. From Claim 3.1, an ($\alpha$, n)-eliminator can be concretely constructed for all $\alpha < \theta$. Since the original choice of $\alpha < \theta$ was arbitrary, the above result can be generalized to: for every $\alpha < \theta$ and for all $m < \omega$, an ($\alpha \cdot \omega^{\theta \cdot m}$, n)-eliminator can be concretely constructed from g. From this and Claim 3.1, it follows that a (θ', n)-eliminator can be concretely constructed.

Remark 3.9. *The relevant (θ', n)-eliminator can be constructed directly from g using the procedure outlined in Remark 3.8.*

Remark 3.10. *For all $m < \omega$, an ($\alpha \cdot \omega^{\theta \cdot m}$, n)-eliminator can be concretely constructed from g. The original sequence (θ_ω) was arbitrary. Thus, a procedure analogous to the one that Takeuti gives for (Case 3) in the proof sketch of Lemma 3.7 could be used (instead of appealing to Claim 3.1) to construct the relevant (θ', n)-eliminator from g.*

(Case 2) $$\theta' < \theta < \theta \cdot \omega$$

By assumption, for a given $n \geq 0$, g is a concretely given (θ, $n+1$)-eliminator. Hence, from Claim 3.1, there is an $1 \leq n' = (n+1)$, such that a (θ', n')-eliminator can be concretely given.

It follows from the initial assumption that g is concretely given and (Case 1) and (Case 2) that a (θ^*, n)-eliminator can be concretely constructed for every $\theta^* < \theta \cdot \omega$. Hence, Lemmas 3.6–3.7 entail that it is possible to construct a $(\theta \cdot \omega,\ n+1)$-eliminator from g. Since our initial sequence (θ_ω) was arbitrary, this result generalizes. An $(\alpha \cdot \omega,\ n)$-eliminator can be concretely constructed from a concretely given $(\alpha,\ n)$-eliminator. □

Lemma 3.9. *For any $n \geq 1$ and any ordinal $\alpha < \varepsilon_0$ such that an $(\alpha,\ n)$-eliminator has been concretely given, an $(\alpha,\ n+1)$-eliminator can be concretely constructed from the concretely given $(\alpha,\ n)$-eliminator.*

Proof Sketch. Takeuti demonstrates Lemma 3.9 by showing that a $(1,\ m{+}1)$-eliminator can be constructed for all $m \geq 0$. He proceeds by induction on m. Consider the following three cases:

(1) $m = 0$

Takeuti stipulates that the 1-eliminator can be taken as a $(1,\ 1)$-eliminator. Hence, it has already been shown that a $(1,\ m+1)$-eliminator can be constructed, where $m = 0$.

(2) $m = 1$

Takeuti reduces the construction of a $(1,\ 2)$-eliminator to the construction of an $(\alpha+\alpha)$-eliminator from an $(\alpha,\ 1)$-eliminator.[25] An $(\alpha+\alpha)$-eliminator is a method for constructing, from a given decreasing sequence of ordinals S, an $(\alpha+\alpha)$-sequence, $S^{\alpha\alpha}$ satisfying an analogue of Definition 3.8. That is, the first ordinal in $S^{\alpha\alpha}$, is the $(\alpha+\alpha)$-major part[26] of the first ordinal in S, and if $S^{\alpha\alpha}$ is finite, then (it can be concretely shown that) S is also finite.

Takeuti's task now is to give a method for generating an $(\alpha+\alpha)$-sequence (that satisfies $(\mathrm{CI}_{\alpha\alpha})$ and $(\mathrm{CII}_{\alpha\alpha})$) from an α-eliminator. He begins by considering a given decreasing sequence of ordinals:

$$(S_m) \qquad a_0 > a_1 > \ldots$$

Next, apply an α-eliminator to S_m to obtain the α-sequence:

$$(S_m^\dagger) \qquad b_0 > b_1 > \ldots$$

where $(S_m^\dagger)$ satisfies (CI_α) and (CII_α). That is, b_0 is the α-major part of a_0 and if $(S_m^\dagger)$ is finite then (it can be concretely shown that) (S_m) is also finite.

[25] This is because $\alpha \cdot 2 = \alpha + \alpha$

[26] The definition of this is analogous to the definition of an α-major part.

By Definition 3.8, each monomial in each b_i in $(S^\dagger_m)$ is $\geq \alpha$. Thus, each b_i in $(S^\dagger_m)$ can be written in the form $\omega^\alpha \cdot c_i$. Since $\omega^\alpha = \omega^\alpha$, if the sequence:

$$(S^*_m) \qquad c_0 > c_1 > ...$$

is finite, then it follows that $(S^\dagger_m)$ is finite. Now apply an α-eliminator to (S^*_m) to obtain the α-sequence:

$$(S^{*\dagger}_m) \qquad d_0 > d_1 > ...$$

where $(S^{*\dagger}_m)$ satisfies both (CI_α) and (CII_α)[27] From $(S^{*\dagger}_m)$, it is straightforward to construct an $(\alpha + \alpha)$-sequence:

$$(S^{\alpha\alpha}_m) \qquad \omega^\alpha \cdot d_0 > \omega^\alpha \cdot d_1 > ...$$

Thus, $(S^{\alpha\alpha}_m)$ is the $(\alpha + \alpha)$-sequence that satisfies $(\mathrm{CI}_{\alpha\alpha})$ and $(\mathrm{CII}_{\alpha\alpha})$ for the starting sequence, (S_m). It can be shown that $(S^{\alpha\alpha}_m)$ satisfies condition $(\mathrm{CI}_{\alpha\alpha})$ for (S_m). In virtue of its construction, d_0 is the α-major part of c_0. Hence,

$$\omega^\alpha \cdot d_0 = \omega^\alpha \cdot (\text{the } \alpha\text{-major part of } c_0)$$

By stipulation, $b_0 = \omega^\alpha \cdot c_0$. Hence,

$$\omega^\alpha \cdot d_0 = \text{the } (\alpha + \alpha)\text{-major part of } b_0$$

In virtue of its construction, b_0 is the α-major part of a_0. Thus, $a_0 = b_0 + e_0$, where $e_0 < \omega^\alpha$. Therefore, a_0 can be written as, $(\omega^\alpha \cdot c_0) + e_0$. When written in this way, it is clear that:

$$\omega^\alpha \cdot d_0 = \text{the } (\alpha + \alpha)\text{-major part of } a_0$$

Therefore, $(S^{\alpha\alpha}_m)$ satisfies condition $(\mathrm{CI}_{\alpha\alpha})$ for (S_m).

It can also be shown that $(S^{\alpha\alpha}_m)$ satisfies $(\mathrm{CII}_{\alpha\alpha})$. Suppose $(S^{\alpha\alpha}_m)$ is finite. It follows that $d_0 > d_1 > ...$ must be finite. Since $d_0 > d_1 > ...$ is an α-sequence which satisfies (CI_α) and (CII_α) for $c_0 > c_1 > ...$, $c_0 > c_1 > ...$ must be finite. By stipulation, every b_i in $(S^\dagger_m)$ is written as $\omega^\alpha \cdot c_i$. Hence, it follows from the fact that $c_0 > c_1 > ...$ is finite that $b_0 > b_1 > ...$ is finite. Since $b_0 > b_1 > ...$ is an α-sequence which satisfies (CI_α) and (CII_α) for (S_m), $a_0 > a_1 > ...$ must be finite. Therefore, if $(S^{\alpha\alpha}_m)$ is finite, then (it can be concretely shown that) (S_m) is finite also. Therefore, $(S^{\alpha\alpha}_m)$ satisfies

[27] That is, b_0 is the α major part of c_0 and if $(S^{*\dagger}_m)$ is finite, then so is (S^*_m).

both ($\mathrm{CI}_{\alpha\alpha}$) and ($\mathrm{CII}_{\alpha\alpha}$) for ($S_m$), and the above method for constructing an $(\alpha+\alpha)$-sequence constitutes an $(\alpha+\alpha)$-eliminator. Hence, a $(1,\, m+1)$-eliminator can be constructed, where $m = 1$.

(3) $m > 1$

Let f be an $(\alpha,\, m)$-eliminator. Since $m > 1$, $m = n + 1$ (where $n \geq 1$). Hence, f is an $(\alpha,\, n+1)$-eliminator. As was demonstrated in the proof sketch of Lemma 3.8, applying f to itself enables the concrete construction of an $(\alpha \cdot \omega,\, m)$-eliminator. Takeuti concludes that a $(1,\, m+1)$-eliminator can be constructed where $m > 1$. Therefore, (1), (2) and (3) provide a means to construct a $(1,\, m+1)$-eliminator for all $m \geq 0$. □

Theorem 3.10. *An $(\alpha,\, n)$-eliminator can be concretely constructed for every ordinal $<\varepsilon_0$.*

Proof Sketch. [Adapted from 20, p. 97.] From Lemmas 3.6–3.8, Takeuti concludes that an $(\alpha,\, n)$-eliminator can be constructed for every α of the form:

$$\omega^{\cdot^{\cdot^{\cdot^{\omega}}}} \}m$$

The construction of the relevant $(\alpha,\, n)$-eliminator is done by induction on m. This, together with Lemma 3.9 entail that an $(\alpha,\, n)$-eliminator can be constructed for all ordinals $<\varepsilon_0$. For instance, where $m = 0$, α is ω^0 (i.e. $\alpha = 1$). Let f be a $(1,\, n)$-eliminator and let g be an (already defined) $(\alpha,\, n+1)$-eliminator. Applying g to f (concretely) produces a $(1 \cdot \omega^\alpha,\, n)$-eliminator, $g(f)$. $g(f)$ is equivalent to the target $(\omega^\alpha,\, n)$-eliminator. □

3.5 A Proof that the Ordinals $<\varepsilon_0$ are Well-Ordered

Now, it can be concretely shown that,

Theorem 3.11. *The ordinals $<\varepsilon_0$ are well-ordered.*

Proof Sketch. [Adapted from 20, p. 95.] The demonstration is analogous to the demonstration of Theorem 3.3. Let,

$$(S_\alpha) \qquad a_0 > a_1 > \ldots$$

be a decreasing sequence of ordinals such that, $a_0 < \omega^{\alpha+1}$. Now apply an (α, n)-eliminator to S_α to (concretely) construct an α-sequence:

$$(S_\alpha^\dagger) \qquad b_0 > b_1 > ...$$

where ($S_\alpha^\dagger$) satisfies both (CI_α) and (CII_α). Then, b_0 is the α-major part of a_0 and if ($S_\alpha^\dagger$) is finite, then (it can be concretely shown that) (S_α) is finite. Since, b_0 is the α-major part of a_0, $a_0 = b_0 + e_0$. Hence, $b_0 < \omega^{\alpha+1}$. It follows that each b_i can be written as $\omega^\alpha \cdot k_i$ where k_i is a natural number. Thus, ($S_\alpha^\dagger$) can be written as:

$$\omega^\alpha \cdot k_0 > \omega^\alpha \cdot k_1 > ...$$

Since $\omega^\alpha = \omega^\alpha$, this sequence is finite if $k_0 > k_1 > ...$ is finite. Given that the natural numbers are well-ordered, $k_0 > k_1 > ...$ must be finite, so $b_0 > b_1 > ...$ must be finite. By (CII_1), it follows that $a_0 > a_1 > ...$ must be finite too. (S_α) is arbitrary so this result generalizes such that any decreasing sequence of ordinals which begins with an ordinal $< \omega^{\alpha+1}$ must be finite. Therefore, the ordinals $< \omega^{\alpha+1}$ are well-ordered. From this it trivially follows that the ordinals $< \omega^\alpha$ are well-ordered.

The fact that the ordinals $< \omega^\alpha$ are well-ordered entails that the ordinals $<\varepsilon_0$ are well-ordered. Given any decreasing sequence of ordinals, $a_0 > a_1 > ...$, if $a_0 <\varepsilon_0$, then since α ranges over all ordinals $<\varepsilon_0$, it follows that $a_0 < \omega^\alpha$ for some α. Hence, $a_0 > a_1 > ...$ must be finite. Therefore, all decreasing sequences of ordinals which begin with an ordinal $<\varepsilon_0$ are finite and thus the ordinals $<\varepsilon_0$ are well-ordered. □

This ends (our reconstruction of) Takeuti's proof of the well-ordering of the ordinal notations in Cantor normal form $<\varepsilon_0$.

4 Final Thoughts

Immediately after the well-ordering proof, Takeuti brings up a couple of things that are worth briefly mentioning here. The first is that he takes the method of eliminators to be a precise, concrete way to show that the ordinal notations for ordinal less than ε_0 are *accessible*. His gloss on accessibility is the following:

> We say an ordinal μ is accessible if it has been demonstrated that every strictly decreasing sequence starting with μ is finite. More precisely, we consider the notion of accessibility only when we have actually seen, or demonstrated constructively, that a given ordinal is accessible. [20, p. 98]

This more common notion is a helpful way to think about which forms of induction might be acceptable from certain constructive or finitist standpoints [see 19, for an early discussion].

It is also closely related to Takeuti's method of ordinal diagrams [see e.g. 17, 18, 20, Ch. 5] which he uses to establish the accessibility of ordinals required for consistency proofs, although the question of whether the well-ordering of ordinal diagrams can be constructively established is much less certain than in the present case [see 1, p. 5].

In a similar vein, Takeuti asserts that the method of eliminators can be extended [p. 97]. This suggests that more transfinite induction might be available to the finitist or constructionist who accepts the proof we've presented. Further such speculation is beyond the scope of this paper, but we do explore the extent of the applicability of the method of eliminators in our [7].

As noted above, it is our hope that our presentation of Takeuti's proof in this paper facilitates future research around these topics, as well as others that have not occurred to us.

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